Bangor University

DOCTOR OF PHILOSOPHY

Chance-constrained and nonlinear goal programming

El-Dash, Afaf Aly

Award date:
1984

Awarding institution:
Bangor University

CHANCE-CONSTRAINED AND
NONLINEAR GOAL PROGRAMMING

AFAF ALY EL-DASH

B.Sc. Applied Statistics, 1976

M.Sc. Operations Research, 1979

Faculty of Economics and Political Science,

Cairo University, Egypt.

Thesis Submitted to the University of Wales

In Support of the Application for the

Degree of Philosophiae Doctor

Department of Applied Mathematics

and Computation,

University College of North Wales,

Bangor.

1984

ACKNOWLEDGEMENTS

The candidate wishes to express her thanks to her supervisor, Dr. J.B. Hughes, for his help and encouragement.

The candidate is also grateful to Prof. N.M. Gargas and Prof A.Y. Mostafa, Faculty of Economics and Political Sciences, Cairo University, for their help in the first stage of the research.

ABSTRACT

In this thesis the chance-constrained linear goal programming approach is developed to cover the following cases when the parameters have non-negative distributions: the exponential and the chi-square distributions.

Case 1, when the right hand side coefficients are exponential or chi-square random variables.

Case 2, when the input coefficients are exponential or chi-square random variables.

The following have been achieved:

For Case 1

1. We have developed a method for constructing deterministic linear goal programs equivalent to the original probabilistic linear goal programs.
2. We have given a probabilistic interpretation to the deviational random variables and the deviational random variable levels.

For Case 2

3. We have developed a method for constructing deterministic nonlinear goal programs through the definition of the probabilistic deviational variables.
4. We have transformed the equivalent deterministic nonlinear goal programs into equivalent signomial goal programs.
5. We have developed a computational algorithm for solving nonlinear goal programs generally and, more particularly, deterministic nonlinear goal programs equivalent to chance-constrained goal programs.

6. We have proved that Sengupta's transformation for obtaining deterministic programs equivalent to chance-constrained programs does not lead to solvable programs.
7. We have formulated and solved a practical application - namely that of finding the "optimal distribution of exports and imports to the marine ports" using the methods and the algorithm presented in the thesis.

The methods can be used when a program has mixed goals, some with right hand side coefficients or input coefficients that are exponential or chi-square random variables; others, deterministic, that is without random variable parameters.

TABLE OF CONTENTS

INTRODUCTION

In many applications of mathematical programming to real world problems the decision-maker has to deal with multi-objectives and goals which are often conflicting and competitive.

Linear goal programming is one of the techniques capable of solving these problems. Additionally, most of the problems where linear goal programming is applied to economics, certain parameters such as prices, supplies and demands which are non-negative random variables with probability distributions. In such cases, when some or all of the parameters are random variables, we have probabilistic linear goal programming problems.

Up to now, most of the area of probabilistic linear goal programming, which is very closely related to non-linear goal programming, has not been researched, and the studies presented in this area are unwieldy or complex. Moreover, the techniques for solving probabilistic linear programming problems when the parameters are non-negative random variables have not been established completely.

As far as the author is aware, there have been only two attempts, both due to Ignizio, to employ nonlinear programming methods to solve nonlinear goal programming problems.

The objective of this research is to develop a chance-constrained goal programming approach for solving problems when the linear goals have non-negatively distributed parameters.

We present two methods to transform probabilistic linear goal programs (models) into equivalent deterministic linear or nonlinear goal programs when the right hand side or the input coefficient of the goals have exponential and chi-square distributions.

For the first time, the condensed geometric programming technique is employed to develop a "sequential double condensed geometric goal programming" algorithm to solve the equivalent deterministic nonlinear goal programs and also nonlinear goal programs in general.

Some numerical examples are presented to demonstrate the methods and the algorithm.

Finally, the problem faced by many emerging countries of optimizing the distribution of exports and imports on their marine ports is formulated and the method of solution is illustrated by an example.

CHAPTER I

GOAL PROGRAMMING (G P)

1.1 Introduction

The technique of Goal Programming (G P) is one of several possible techniques used for solving problems with multiobjectives. In the linear case, it is an extension of linear programming[1] (L P) [51] .

G P allows the solution of problems having, simultaneously, a system of complex objectives (conflicting and competitive) rather than a single objective. The G P technique is not the ultimate technique for all multiple objective decision problems. It requires that the decision maker be capable of defining, quantifying and ordering the objectives, or selecting the optimum approach to obtain the priorities and weights [51, 38, 37, 54] .

1.2 Literature Survey and Formulation

This section presents the fundamental concepts of G P and the standard form of the G P model (program) through an account of the historical development of G P . These concepts and formulation play an important part in the following chapters.

---

[1] Some authors (e.g. [51]) consider that linear G P is an extension of L P , while others [37, 53, 38] consider that L P is a special case of linear G P . For particular cases, Markowski [53] was able to prove by duality theory that L P is a special case of linear G P but the converse is not true.

The concept of GP was first introduced by Charnes and Cooper (1955) as an issue [7] for unsolved LP problems. In (1961), they used the name GP in their book [8] : "Linear Programming". Their approach was to use deviational variables to transform objectives and constraints into goals [1] in a standard form [2] and hence optimization becomes an attempt to minimize these deviations. The linear multiple objective problem becomes a conventional LP problem where the single objective function is a linear function of the deviational variables. The formulation is as follows:

$$\text{minimize} \quad a = g(d^-, d^+) \tag{1.1}$$

$$\text{subject to} \quad \sum_{j=1}^{N} a_{ij}x_j + d_i^- - d_i^+ = b_i \qquad j = 1, 2, \ldots N \quad i = 1, 2, \ldots M \tag{1.2}$$

$$x_j, d_i^-, d_i^+ \geq 0 \qquad j = 1, 2, \ldots N \quad i = 1, 2, \ldots M \tag{1.3}$$

---

[1] A goal is a mathematical function of the decision variables which represents the combination of an objective with a target (i.e., right hand side) value. The mathematical form of a goal is either: $f(x) \leq b$ or $f(x) \geq b$ or $f(x) = b$, where x is the vector of decision variables. A constraint has the same mathematical appearance as a goal. However, the difference between a goal and a constraint is that a goal implies some flexibility, whereas a constraint, at least in the mathematical sense is absolute or inflexible [38 , page 26].

[2] The standard form of a goal is obtained by adding the deviational variables to the left hand side of a goal and transforming inequalities to equations. Hence, the goal becomes equivalent to an equality constraint.

where

x_j : decision variables, $j = 1, 2, \ldots N$;

a_{ij} : constants representing input coefficients, $i = 1, 2, \ldots M$, $j = 1, 2, \ldots N$;

b_i : constants representing target values (aspiration levels), $i = 1, 2, \ldots M$;

d^-_i, d^+_i : non-negative deviational variables which represent under and over achievement respectively of the i^{th} goal, i.e.

$$d^-_i = b_i - \sum_{j=1}^{N} a_{ij}x_j \qquad i = 1, 2, \ldots M \tag{1.4}$$

$$d^+_i = \sum_{j=1}^{N} a_{ij}x_j - b_i \qquad i = 1, 2, \ldots M \tag{1.5}$$

and

$$d^-_i \cdot d^+_i = 0 \quad \text{for all } i = 1, 2, \ldots M \tag{1.6}$$

$g(d^-, d^+)$: linear function of the deviational variables d^-, d^+ where d^-, d^+ are the vectors of deviational variable.

The constraint set (1.2) is the standard form for a goal set.

Ijiri (1965) used a generalized inverse approach [39] to study GP problems and introduced the notion of "preemptive priority factors" to treat multiple goals according to their importance, assigning weights to goals of the same priority level. Accordingly, the formulation of a GP model (program) becomes:

Find $x = (x_1, x_2, \ldots x_N)$

So as to minimize:

$$a = \{P_1[g_1(d^-, d^+)], P_2[g_2(d^-, d^+)], \ldots, P_k[g_k(d^-, d^+)], \ldots P_K[g_K(d^-, d^+)]\}$$

$$K \leq M \tag{1.7}$$

subject to

$$\sum_{j=1}^{N} a_{ij}x_j + d_i^- - d_i^+ = b_i \qquad i = 1, 2, \ldots M \tag{1.8}$$

$$x_j, d_i^-, d_i^+ \geq 0 \qquad \text{for all } i = 1, 2, \ldots M \quad j = 1, 2, \ldots N \tag{1.9}$$

where

P_k is the priority level associated with $g_k(d^-, d^+)$;

P_{k-1} is more important than P_k, for all $k = 2, 3, \ldots K$;

$g_k(d^-, d^+)$ is a linear function of the weighted deviational variables at the k^{th} priority level.

Although Ijiri reinforced and refined the concept of GP and developed it as a distinct mathematical programming technique, the generalized inverse approach is efficient for attacking problems of multiple goals only if the variables involved in the problem are not required to be non-negative[1]. If the non-negative constraints are critical in the solution, then it is better to use some other approaches. Further, the approach of generalized inverse is not considered to be a practical one for solving real world GP program, in particular, when priorities and weights of goals are used in large size problems.

Contini, B. (1968) suggested a form of chance-constrained goal programming (CCGP) when the parameters b_i have

---

[1] The non-negativity condition is very important for economic problems.

normal distributions [16] . Contini's work and its drawbacks will be discussed in Section 2.5.

In the text by Lee (1972), a multiphase simplex algorithm, referred to as a modified simplex procedure[1] , was presented [50, 51] . In order to find an optimal compromise among conflicting goals with priorities, he used a multicriterion simplex algorithm with lexicographical minimization of the weighted sum of the deviations from the aspiration levels (b_i) . Lee's text did much to popularize G P and its potential for solving several types of problems with applications in the real world.

More recent texts by Ignizio (1976, 1982) make use of an achievement function which is an ordered vector expressing the level of achievement of each set of goals with a priority scheme. The generalization of Lee's formulation, using Ignizio's notation, is referred to as the generalized G P program and its formulation is as follows [37, 38] :

Find $x = (x_1, x_2, \ldots x_N)$

so as to

$$\text{lexico-min} \quad a = \{[g_1(d^-, d^+)], [g_2(d^-, d^+)], \ldots [g_k(d^-, d^+)], \ldots [g_K(d^-, d^+)]\}$$

$$K \le M \tag{1.10}$$

subject to

$$f_i(x) + d^-_i - d^+_i = b_i \qquad i = 1,2,\ldots M \tag{1.11}$$

$$x_j, d^-_i, d^+_i \ge 0 \qquad i = 1,2,\ldots M, \quad j = 1,2,\ldots N \tag{1.12}$$

---

[1] At that time, the solution of G P problems by the simplex method had not been thoroughly discussed in the literature.

where $f_i(x)$ is a function (linear or nonlinear) of decision variables and $g_k(d^-, d^+)$ is a function (linear or nonlinear) of deviational variables and in linear GP programs each $f_i(x)$ and $g_k(d^-, d^+)$ for all $i = 1,2,\ldots M$ and $k = 1,2,\ldots K$ must be a linear function. Ignizio has further modified the existing methods for solving single objective nonlinear programming problems (Griffith and Stewart [32] and pattern search [36] method) to solve GP programs when the $f_i(x)$ are nonlinear functions (Chapter 5 contains all the details about nonlinear GP). Ignizio also presented the sequential linear GP approach (SLGP) , which is the original approach to the lexicographic GP program and treats it as a series of LP programs (see Section 1.3). Dauer and Kruger (1977) presented "an iterative GP" method [19] . This method is a generalization of the SLGP approach, and can be used to solve integral and nonlinear GP programs and, in turn, probabilstic GP programs. We will present this method in the next section.

Markowski (1980) presented the theory and methodologies of linear GP duality [53] .

Since the standard form of goals are equality constraints with deviational variables d^- , d^+ (as in equations (1.11)), the weights may be associated with d^- , d^+ in an achievement function or in the constraints. Widhelm, W.B. (1981) presented three models: Minsum, Minmax and Maxmin. The basic difference between the three models is in the form of the achievement function, but in each of them weights are associated with d^- , d^+ in the constraints. He suggested [86] a norming correction method for ~~solving~~ Formulating these models.

Sometimes, the assignment of preemptive priorities and weights causes problems for decision-makers [1]. There are many approaches for dealing with this problem. The "nondominated solution set" is one of the most important approaches to deal with this problem. But this approach suffers from a primary disadvantage in that the number of efficient extreme points is enormous even for modest size problems [37, 38] .

Lately, some approaches were presented to provide a link between GP and interactive approaches [38] such as: Interactive Goal P (IGP) and Sequential Information Generator for Multiple Objective Problems (SIGMoP) . The disadvantage of the SIGMoP approach is that it is possible to construct an inconsistent constraint set.

Masud and Hwang (1981) avoided this disadvantage of SIGMoP in their approach [54] "interactive sequential Goal programming (ISGP) , which combines and extends attractive features of both GP and interactive solution approaches of multiple objective decision making problems.

But most of the recent literature on GP consists of accounts of applications in many various fields [52, 47, 37, 53, 38] , such as manpower planning, production planning, transportation, inventory, health care systems, agriculture planning, allocation of library funds, insurance agency management.

---

[1] This depends on the nature of the problem and the decision-maker [37] . In many real world problems, prior assignment of preemptive priorities is considered an advantage of the GP technique and not a disadvantage or handicap for the solution of those problems.

1.3 Sequential Goal Programming Algorithm

In this section we present again the sequential GP algorithm due to Dauer and Kruger [19, 20] because it is capable of solving linear or nonlinear GP problems generally and CCGP problems in particular (see Section 5.8) by incorporating in it a corresponding optimization algorithm. This algorithm is based on first decomposing a goal program to K "single-objective" subprograms, according to their priority levels; and then solving a series of subprograms such that the solution of the subprogram associated with priority level k, $k = 2, 3, \ldots K$, includes the optimum solution of the subprogram associated with priority level $(k - 1)$ as a constraint. Let the subprogram associated with the priority level k have the following form:

$$\text{minimize} \quad a_k = g_k(d^-, d^+) \tag{1.13}$$

subject to

$$f_i(x) + d^-_i - d^+_i = b_i \quad \text{for } i \in P_k \tag{1.14}$$

$$x, d^-, d^+ \geq 0 \tag{1.15}$$

(see the program (1.10)-(1.12)).

That is, we are minimizing the k^{th} term of the achievement function subject only to those goals in priority level k (i.e., $i \in P_k$).

The procedure of algorithm is as follows:

Step 1

set $k = 1$.

Step 2

Form the program associated with priority level 1 only, as in (1.13)-(1.15). The resultant program is a conventional (single-objective) program and may be solved by an appropriate optimization algorithm.

Step 3

Solve the single objective program associated with priority level k. Let the optimal solution to this program be given as a_k^* where, a_k^* is the optimal value of $g_k(d^-, d^+)$.

Step 4

Set $k = k + 1$. If $k > K$ go to Step 7.

Step 5

Form the equivalent single objective program for the next priority level (level k). This program is given by:

$$\text{minimize} \quad a_k = g_k(d^-, d^+) \tag{1.16}$$

subject to

$$f_t(x) + d_t^- - d_t^+ = b_t \tag{1.17}$$

$$g_s(d^-, d^+) = a_s^* \tag{1.18}$$

$$x, d^-, d^+ \geq 0 \tag{1.19}$$

where

$s = 1, 2, \ldots k-1$

t = set of subscripts associated with those goals included in priority levels $1, 2, \ldots k$.

Step 6

Go to Step 3.

Step 7

The solution vector x^* associated with the last single objective program solved, is the optimal vector for the original goal program.

CHAPTER 2

PROBABILISTIC PROGRAMMING (P P)

2.1 Introduction

In this chapter we present a brief account of the works introduced to study and apply probabilistic linear goal programming (P L G P). The drawbacks of these works are determined and analysed (section 2.5).

We also give the most important factors to choose the chance constrained programming approach (C C P) to study P L G P in the next chapters. Therefore the fundamental concepts of P P are given (section 2.2), and, in section 2.4, the formulation and properties of the C C P model are presented as a necessary part of the study of C C G P .

2.2 Probabilistic Programming Technique

P P technique is a technique which deals with the theories and methods of mathematical programming, in which random variation of the parameters (coefficients) are incorporated into the models. The random variation of the parameters may arise from several sources, depending on the type of problem and the type of decisions arrived at [62] .

In the classical situation, these coefficients are assumed to be completely known, but, if one wants to be more realistic, then this assumption must be relaxed [77] . Tintner (1941) distinguished between subjective risk and subjective uncertainty. He considered that to be a subjective risk when "there exists a probability distribution of anticipation which is itself known with certainty" and subjective uncertainty when "there is a priori probability of the probability distributions themselves." [75] .

In this dissertation, we deal with problems of the first kind, where the probability distributions of the random variable parameters are known.

2.3 Probabilistic Linear Programming (P L P)

A L P problem is said to be a P L P problem if one or more of the parameters is known only by its probability distribution.

These problems can be solved by one of the following principal approaches[1] :

(1) stochastic linear programming (S L P) , [65,63,62,77,76,69];

(2) linear programming under uncertainty which, in some special cases, is called two stages programming under uncertainty [17,83,18,62,77,76,78,84,82,33,85,79] ; and

(3) C C P [62, 77, 76] , which will be discussed in detail in the next section.

These three approaches have the following characteristics in common:

First, the initial probability distributions of the parameters are incorporated to convert a P L P model into deterministic form.

---

[1] There are other approaches such as transition probability programming, probabilistic sensitivity analysis, ... etc. [64] . These approaches are considered to be less general than the approaches mentioned above.

Second, a set of decision rules having some optimality properties are defined. Methods of incorporating probability distributions and specifying decision rules are of course different in the different approaches [62, 77] . If the initial distribution of the parameters is either unknown or incompletely specified, the problem of characterizing the optimal decision variables becomes much more complicated. Such problems come under the headings of decision rules under uncertainty and simulation techniques [76] .

2.4 Chance-Constrained Programming

An ordinary LP model is said to be a chance-constrained programming model if its linear constraints are associated with a set of probability measures indicating the extent of violation of the constraints.

If the general form of an ordinary LP is as follows:

$$\text{maximize} \quad z = \sum_{j=1}^{N} c_j x_j \tag{2.1}$$

$$\text{subject to} \quad \sum_{j=1}^{N} a_{ij} x_j \leq b_i \qquad i = 1,2,\ldots,M \tag{2.2}$$

$$x_j \geq 0 \qquad j = 1,2,\ldots,N \tag{2.3}$$

where

x_j are decision variables, $j = 1, 2, \ldots, N$ and a_{ij}, b_i, c_j are constants for $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$, the problem is then to choose a set of values for the variables x_j, $j = 1, 2, \ldots, N$, so that:

(a) they satisfy all the constraints (2.2), (2.3) and

(b) they make $\sum_{j=1}^{N} c_j x_j$ a maximum in accordance with the given criterion elements c_j, $j = 1, 2, \ldots, N$.

A C C P formulation would replace the problem (2.1) - (2.3) with the following problem [10, 62] :

$$\text{optimize} \quad f(c\,,x) \tag{2.4}$$

$$\text{subject to} \quad P_r\left(\sum_{j=1}^{N} a_{ij}x_j \le b_i\right) = \gamma_i \qquad i = 1,2,\ldots, M \tag{2.5}$$

$$x_j \ge 0 \qquad j = 1,2,\ldots, N \tag{2.6}$$

$$0 \le \gamma_i \le 1 \qquad i = 1,2,\ldots, M \tag{2.7}$$

where "P_r" means probability. Here a_{ij}, b_i, c_j for $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$ are not necessarily constants and, in general, some or all of them are random variables. γ_i, $i = 1, 2, \ldots, M$ are preassigned constants called "Tolerance measures" where γ_i indicates the extent to which the i^{th} inequality is satisfied (i.e. the extent to which there are no violations of the i^{th} inequality). In other words, $0 \le 1 - \gamma_i \le 1$ indicates a probability measure of the extent to which violations of the i^{th} constraint are permitted. Thus, an element $0 \le \gamma_i \le 1$, is associated with a constraint

$$\sum_{j=1}^{N} a_{ij}x_j \le b_i \quad \text{to give}$$

$$P_r\left(\sum_{j=1}^{N} a_{ij}x_j \le b_i\right) = \gamma_i \tag{2.8}$$

when deciding upon an objective, there is a fairly wide range of reasonable choices to be considered for the form of (2.4) as a replacement for (2.1) .

From the above, we conclude that CCP approach is important in studying PLGP problems because:

(1) CCP allows the constraints to be violated with preassigned probabilities. This assumption is in accord with the assumptions of GP.

(2) The present assumptions or objectives of other PLP approaches (see section 2.3) are not in accord with GP.

Furthermore, the CCP model has two desirable properties [80] : (a) it leads to an equivalent linear or nonlinear deterministic program that has the same size as the deterministic version; and

(b) the only information required about each uncertain element is the γ_i fractile for the unconditional distribution.

CCP was first presented by Charnes and Cooper (1958) to solve the scheduling of the production of heating oil, which is an important and complex management problem [11]. Also, in (1959), they presented new conceptual and analytical framework for problems of temporal planning under uncertainty [9]. In (1963) they developed different kinds of decision rules and optimizing objectives that may be used so that, under certain conditions, an equivalent deterministic programming problems can be achieved in the sense that all random elements have been eliminated [12, 10].

In the last few years, the CCP approach has been generalized in several directions and applied to various industrial and economic problems [57, 13, 14, 66, 45, 67, 70, 43, 44, 72]. For economic problems, most of them have non-negatively distributed parameters; in this field

Sengupta presented some studies [70, 66, 68] . But, up to the present, there are many areas in this field that have not been researched.

We will present in Chapters 3 and 4 an analytical study of CCGP with non-negatively distributed parameters (chi-square and exponential distributions) from various aspects.

2.5 Probabilistic Linear Goal Programming (PLGP)

Up to now, there are many areas of GP which have not been completely researched, such as PLGP and nonlinear GP[1] which are very closely related (as will be shown in sections 3.4 and 4.4). The LGP model becomes a PLGP model when some or all of the parameters are random variables. The PLGP technique is one of the most important techniques for optimal decision-making under uncertainty, where there are many problems in the practical application of GP having random variable parameters. Unfortunately, the studies presented in this area (PLGP) are unwieldy or complex [16, 50, 43, 44, 45] .

Now, we present briefly, the studies that have been introduced and determine and analyse most of their drawbacks about which more research is needed.

Charnes, Cooper, Neihaus and Sholtz (1968) have jointly developed a manpower planning model which considers the effects of Markov processes from period to period. [15]

---

[1] And other areas such as dynamic GP , post optimality analysis of GP , ... [37 Chapter 9, 50 Chapter 7] .

Contini (1968) used a generalized inverse method [39] to study CCGP, when the vector of the targets values b (b is vector of b_i, $i = 1,2,\ldots,M$, see section 1.2) represents random variables having a normal distribution. He considered b_i as endogenous variables and the decision variables x (x is vector of x_j, $j = 1,2,\ldots,N$) as exogenous variables [16], i.e.

$$Ax + u = b \tag{2.9}$$

where the elements of u have $N(O,\Sigma)$, and the matrix A is constant. But, Contini's approach, however, suffers from many drawbacks. The most important of these drawbacks being:

(1) often, the form of equations (2.9) are not realistic for applied economic problems, in that:

a) most real economic problems can not be put in this form.

b) usually, most economic parameters are non-negative, and in turn, the normality assumptions are not valid for most applied economic problems.

(2) a generalized inverse method was used to FoRM the resultant models, although this method is not efficient for economic problems (see section 1.2).

(3) it is impossible to use this approach when the elements of the matrix A are random variables.

(4) it is very difficult or often impossible to use this approach when priorities and weights are to be considered.

Lee (1972) presented two examples to study the effects of uncertainty on the GP models [50] whilst keeping the simplex algorithm. To some extent Lee's approach resembles the piecewise linear approximation approach of El-maghraby [26, 27].

The results using Lee's approach showed by contrast, when a non-PGP of these examples are solved using the expected values (of random variable parameters), the results are more "reasonable". In addition, El-maghraby's approach is too cumbersome to work with if the size of the problem becomes large.

Keown & Martin (1977), Keown (1978) and Keown & Taylor III (1980) presented three attempts to form CCGP models for working capital management [43], bank liquidity management [44] and capital budgeting in the production area [45] respectively. The above attempts suffer from the following fundamental disadvantages:

(1) in each attempt, the normal distribution is used as the approximate distribution of the random variable parameters despite the fact that some of these parameters have non-negative distirbutions (e.g. the future demand for certain products, the level of cash balances, ..., etc.) and which therefore are best approximated by non-negative distributions [56, 62].

(2) in each attempt the deviational random variables were considered as deviational deterministic variables and they did not distinguish between the values of deviation variables and their bounds.

In chapters 3 and 4, the disadvantages of PLGP studies noted in this section will be treated by replacing the assumption of normality by the non-negativity assumption about the distributions of parameters (exponential and chi-square distributions are used) and presenting a probabilistic interpretation of the deviational variables.

2.6 Conclusion

In this chapter we have determined and analysed the drawbacks of the P L.G P studies that have been presented and indicated the points about which more research is needed. Also the effective factors to use C C P approach to study P L G P have been given.

CHAPTER 3

CHANCE CONSTRAINED GOAL PROGRAMMING (C C G P) WITH EXPONENTIALLY DISTRIBUTED PARAMETERS

3.1 Introduction

The present chapter deals with the approach of C C G P when the goals have exponentially distributed parameters.

In Section 3.3, we present a method to transform probabilistic goal programs into the deterministic goal programs when the right hand side coefficients b_i, $i = 1,2,\ldots,M$ have exponential distributions (Case 1). In addition, a probabilistic interpretation of deviational random variables will be given and deviational random variable levels will be defined.

In Section 3.4, by a method similar to that mentioned above, we form the transformed deterministic goal programs and define the probabilistic deviational variables when some or all of the input coefficients a_{ij}, $i = 1,2,\ldots,M$; $j = 1,2,\ldots,N$ have exponential distributions (Cases 2 and 3 respectively). In addition the equivalent signomial programs are presented.

3.2 Exponentially Distributed Parameters

In this chapter, we consider the right hand side coefficients or input coefficients to be exponentially distributed random variables.

The main reasons for choosing the exponential distribution as the non-negative distribution for the coefficients are:

1. it is used for a wide class of economic models involving non-negative prices, input coefficients and non-negative resource vectors [62].
2. it is related to the chi-square distribution [42,21].
3. under certain conditions, it provides a limiting distribution for a wide class of non-negative variables by a limit theorem [71,66] (just as the normal distribution provides a limiting distribution for many distributions under the central limit theorem.)

3.3 Case 1: The Right Hand Side Coefficients (b_i)

In this section, we consider the goal set (see Section 1.2):

$$\sum_{j=1}^{N} a_{ij}x_j \le b_i \qquad i = 1,2,3,\ldots,m \qquad (3.1)$$

$$\sum_{j=1}^{N} a_{ij}x_j \ge b_i \qquad i = m+1,m+2,\ldots,M \qquad (3.2)$$

where $x_j \ge 0$, $\quad j = 1,2,\ldots,N$

x_j, $j = 1,2,\ldots,N$ are the decision variables;

a_{ij}, $i = 1,2,\ldots,M$; $j = 1,2,\ldots,N$ are constants;

and b_i, $i = 1,2,\ldots,M$ are mutually independent random variables, having exponential distribution with two-parameters[1] (α_i,σ_i). The density fuction of b_i is

$$f(b_i) = \frac{1}{\sigma_i} e^{-(b_i-\alpha_i)/\sigma_i} \qquad b_i \ge \alpha_i \ge 0 \qquad (3.3)$$

with mean, $E(b_i) = \alpha_i + \sigma_i \qquad i = 1,2,\ldots,m \qquad (3.4)$

---

[1] The disadvantage of the single parameter exponential distribution is that its density function has its mode at the origin, $b_i = 0$. This can be avoided by hypothesizing a two-parameter exponential distribution [62].

and variance, $\quad var(b_i) = \sigma_i^2 \qquad i = 1,2, \ldots M \qquad (3.5)$

Now, we present a method to determine the optimum values of the x's , namely those which satisfy the goals (3.1), (3.2) to the fullest possible extent according to their priorities with probabilities that are greater than or equal to preassigned probabilities [1] (i.e. tolerance measures).

Our method is developed as follows:

<u>First</u>: the deviational random variables.

The goal set (3.1) and (3.2) can be formed in the standard form by adding non-negative deviational random variables $\tilde{d}_i^-$, $\tilde{d}_i^+$ for $i = 1,2, \ldots, m, m+1, \ldots, M$ (see section 1.2) :

$$\sum_{j=1}^{N} a_{ij}x_j + \tilde{d}_i^- - \tilde{d}_i^+ = b_i \qquad i = 1,2,\ldots,m,m+1,\ldots,M \qquad (3.6)$$

such that

$$\tilde{d}_i^- = \max \left\{ 0, b_i - \sum_{j=1}^{N} a_{ij}x_j \right\} \quad i = 1,2,\ldots,m,m+1,\ldots,M \qquad (3.7)$$

$$\tilde{d}_i^+ = \max \left\{ 0, \sum_{j=1}^{N} a_{ij}x_j - b_i \right\} \quad i = 1,2,\ldots,m,m+1,\ldots,M \qquad (3.8)$$

$$P_r(\tilde{d}_i^- > 0 \cap \tilde{d}_i^+ \geq 0) = 0 \qquad i = 1,2,\ldots,m,m+1,\ldots,M \qquad (3.9)$$

and

$$P_r(\tilde{d}_i^- > 0 \cup \tilde{d}_i^+ > 0) = P_r(\tilde{d}_i^- > 0) + P_r(\tilde{d}_i^+ > 0) = 1$$
$$i = 1,2,\ldots,m,m+1,\ldots,M \qquad (3.10)$$

---

[1] These probabilities are assigned by the decision-maker according to the implicit cost of such an assignment.

Second: the chance-goal set.

Since b_i, $i = 1,2,\ldots,m,m+1,\ldots,M$ are random variables, then, from Section 2.4, the goals (3.1), (3.2) may also be reformed using the following chance-goal set:

$$P_r\left(\sum_{j=1}^{N} a_{ij}x_j \le b_i\right) = \gamma_i \qquad i = 1,2,\ldots,m \qquad (3.11)$$

$$P_r\left(\sum_{j=1}^{N} a_{ij}x_j \ge b_i\right) = \gamma_i \qquad i = m+1,m+2,\ldots,M \qquad (3.12)$$

where $0 \le \gamma_i \le 1$ for all $i = 1,2,\ldots,m,m+1,\ldots,M$. The γ_i are preassigned constants called tolerance measures, in the sense that the probability that the i^{th} goal is satisfied is equal to γ_i or, in other words the probability that the i^{th} goal is not satisfied is equal to $(1-\gamma_i)$.

Equations (3.11), (3.12) are equivalent to:

$$1 - F_i\left(\sum_{j=1}^{N} a_{ij}x_j\right) = \gamma_i \qquad i = 1,2,\ldots,m \qquad (3.13)$$

$$F_i\left(\sum_{j=1}^{N} a_{ij}x_j\right) = \gamma_i \qquad i = m+1,m+2,\ldots,M \qquad (3.14)$$

where F_i is the cumulative function of the exponential variable b_i.

Hence

$$\sum_{j=1}^{N} a_{ij}x_j = F_i^{-1}(1-\gamma_i) \qquad i = 1,2,\ldots,m \qquad (3.15)$$

or

$$\sum_{j=1}^{N} a_{ij}x_j = -\sigma_i \ln \gamma_i + \alpha_i \qquad i = 1,2,\ldots,m \qquad (3.16)$$

and

$$\sum_{j=1}^{N} a_{ij}x_j = F_i^{-1}(\gamma_i) \qquad i = m+1,m+2,\ldots,M \qquad (3.17)$$

or

$$\sum_{j=1}^{N} a_{ij}x_j = -\sigma_i \ln (1-\gamma_i) + \alpha_i \qquad i = m+1,m+2,\ldots,M \tag{3.18}$$

where F_i^{-1} is the inverse function of the cumulative function F_i .

Third: the deviational random variable levels.

After the chance-goal set (3.11), (3.12) has been converted to the deterministic goal set (3.16), (3.18). It can be reformed in standard form by adding deviational random variable levels d_i^- , d_i^+ for all $i = 1,2,\ldots,m,m+1,\ldots M$, such that

$$\sum_{j=1}^{N} a_{ij}x_j + d_i^- - d_i^+ = -\sigma_i \ln \gamma_i + \alpha_i \qquad i = 1,2,\ldots m \tag{3.19}$$

$$\sum_{j=1}^{N} a_{ij}x_j + d_i^- - d_i^+ = -\sigma_i \ln (1-\gamma_i) + \alpha_i \qquad i = m+1,\ldots M \tag{3.20}$$

where

$$d_i^- = \begin{cases} \max\,[0,(-\sigma_i \ln\gamma_i + \alpha_i) - \sum_{j=1}^{N} a_{ij}x_j] & i = 1,2,\ldots,m \quad (3.21) \\ \max\,[0,(-\sigma_i \ln(1-\gamma_i) + \alpha_i) - \sum_{j=1}^{N} a_{ij}x_j] & i = m+1,m+2,\ldots,M \quad (3.22) \end{cases}$$

$$d_i^+ = \begin{cases} \max\,[0,[\sum_{j=1}^{N} a_{ij}x_j - (-\sigma_i \ln\gamma_i + \alpha_i)] & i=1,2,\ldots,m \quad (3.23) \\ \max\,[0,[\sum_{j=1}^{N} a_{ij}x_j - (-\sigma_i \ln(1-\gamma_i) + \alpha_i)] \,, & i = m+1,m+2,\ldots,M \quad (3.24) \end{cases}$$

and

$$d_i^- , d_i^+ \geq 0 \quad , \quad d_i^- \cdot d_i^+ = 0$$

for all $i = 1,2,\ldots,m,m+1,\ldots,M$ (3.25)

The definitions of $\tilde{d}_i^-$, $\tilde{d}_i^+$ in (3.7), (3.8) and of d_i^- , d_i^+ in (3.21) - (3.25) show that:

(1) The d_i^- are the lower levels of the negative deviational random variables $\tilde{d}_i^-$ with probability γ_i for $i = 1,2,\ldots,m$ and $(1-\gamma_i)$ for $i = m+1,m+2,\ldots,M$ if and only if d_i^+ is equal to zero for all $i = 1,2,\ldots,m,m+1,\ldots,M$, i.e.

$$P_r\,(\tilde{d}_i^- \geq d_i^-) \begin{cases} = \gamma_i & \text{if } d_i^+ = 0,\ i = 1,2,\ldots,m \qquad (3.26) \\ = 1-\gamma_i & \text{if } d_i^+ = 0,\ i = m+1,m+2,\ldots,M \qquad (3.27) \end{cases}$$

or equivalently,

$$P_r(\tilde{d}_i^- < d_i^-) \begin{cases} = 0 & \text{if } d_i^- = 0,\ i = 1,2,\ldots,m,\ldots M \qquad (3.28) \\ < 1-\gamma_i & \text{if } d_i^- > 0,\ i = 1,2,\ldots,m \qquad (3.29) \\ < \gamma_i & \text{if } d_i^- > 0,\ i = m+1,\ldots,M \qquad (3.30) \end{cases}$$

<u>Definition 3.1</u>

$P_r(\tilde{d}_i^- < d_i^-)$ is a monotonic increasing function of d_i^- for all $i = 1,2,\ldots,m,m+1,\ldots,M$ and is defined for $d_i^- \geq 0$. This follows immediately from the definition of a cumulative distribution function.

(2) The d_i^+ are the lower levels of the positive deviational random variables $\tilde{d}_i^+$ with probability $1-\gamma_i$ for $i = 1,2,\ldots,m$ and γ_i for $i = m+1,\ldots,M$ if and only if d_i^- is equal to zero, i.e.,

$$P_r(\tilde{d}_i^+ \geq d_i^+) \begin{cases} = 1-\gamma_i & \text{if } d_i^- = 0,\ i = 1,2,\ldots,m \quad (3.31) \\ = \gamma_i & \text{if } d_i^- = 0,\ i = m+1,\ldots,M \quad (3.32) \end{cases}$$

or equivalently

$$P_r(\tilde{d}_i^+ < d_i^+) \begin{cases} = 0 & \text{if } d_i^+ = 0,\ i = 1,2,\ldots,m,m+1,\ldots,M \quad (3.33) \\ < \gamma i & \text{if } d_i^+ > 0,\ i = 1,2,\ldots,m \quad (3.34) \\ < 1-\gamma_i & \text{if } d_i^+ > 0,\ i = m+1,\ldots,M \quad (3.35) \end{cases}$$

Definition 3.2

$P_r(\tilde{d}_i^+ < d_i^+)$ is a monotonic increasing function of d_i^+ for all $i = 1,2,\ldots,m,m+1,\ldots,M$ and is defined for all $d_i^+ \geq 0$. This follows immediately from the definition of a cumulative distribution function also.

Lemma 3.1

The i^{th} goal in the goal set (3.1), (3.2), $i = 1,2,\ldots,m,m+1,\ldots,M$ is satisfied with probability greater than or equal to γ_i if and only if:

$$d_i^+ = 0 \quad , \quad i = 1,2,\ldots,m$$

and

$$d_i^- = 0 \quad , \quad i = m+1,m+2,\ldots,M$$

Proof

(1) If $d_i^+ = 0$, $i = 1,2,\ldots,m$.

From (3.26)

$$P_r(\tilde{d}_i^- \geq d_i^-) = \gamma_i \qquad d_i^- > 0 \qquad (3.36)$$

Since $P_r(\tilde{d}_i^- < d_i^-)$ is a monotonic increasing function of d_i^- , (definition 3.1) (3.37)

From (3.36), (3.37), then

$$P_r(\tilde{d}_i^- > 0) = P_r(\tilde{d}_i^- \geq d_i^-) + P_r(\tilde{d}_i^- < d_i^-) \geq \gamma_i \quad (3.38)$$

(2) If $d_i^- = 0$, $i = m+1, m+2, \ldots, M$

From (3.32)

$$P_r(\tilde{d}_i^+ \geq d_i^+) = \gamma_i \qquad d_i^+ > 0 \quad (3.39)$$

Since $P_r(\tilde{d}_i^+ < d_i^+)$ is a monotonic increasing function (definition 3.2) (3.40)

From (3.39), (3.40), then

$$P_r(\tilde{d}^+ > 0) = P_r(\tilde{d}_i^+ \geq d_i^+) + P_r(\tilde{d}_i^+ < d_i^+) \geq \gamma_i \quad (3.41)$$

Q.E.D.

Fourth: the transformed deterministic goal program.

In lemma 3.1, it was shown that the i^{th} goal of the goal set (3.1), (3.2), $i = 1,2,\ldots,m,m+1,\ldots,M$, is satisfied with probability greater than or equal to γ_i when $d_i^+ = 0$ for $i = 1,2,\ldots,m$ and $d_i^- = 0$ for $i = m+1,\ldots,M$.

Since from (3.10):

$$P_r(\tilde{d}_i^- > d_i^-) + P(\tilde{d}_i^- < d_i^-) + P(\tilde{d}_i^+ \geq d_i^+) + P(\tilde{d}_i^+ < d_i^+) = 1 \quad (3.42)$$

in the case $d_i^+ > 0$ for all $i = 1,2,\ldots,m$ and $d_i^- > 0$ for all $i = m+1,m+2,\ldots,M$, then the i^{th} goal $i = 1,2,\ldots,m,m+1,\ldots,M$, is satisfied to the fullest possible extent when d_i^+ for $i = 1,2,\ldots,m$ and d_i^- for $i = m+1,m+2,\ldots,M$ are a minimum because $P_r(\tilde{d}_i^- < d_i^-)$ and $P_r(\tilde{d}_i^+ < d_i^+)$ are monotonic increasing functions of d_i^- , d_i^+

respectively (definitions 3.1 and 3.2) .

From above, we can determine the values of x's those which satisfy the goals (3.1), (3.2) to the fullest possible extent according to their priorities with probabilities greater than or equal to γ_i , $i = 1,2,\ldots,m,m+1,\ldots,M$ by solving the following transformed deterministic goal program:

Find $x = (x_1, x_2, \ldots, x_N)$

so as to

$$\text{lexico-min} \quad a = \{[g_1(d^-,d^+)],[g_2(d^-.d^+)], \ldots,[g_k(d^-,d^+)], \ldots,[g_K(d^-,d^+)]\} \quad K \le M \tag{3.43}$$

subject to

$$\sum_{j=1}^{N} a_{ij}x_j + d_i^- - d_i^+ = -\sigma_i \ln\gamma_i + \alpha_i \qquad i = 1,2,\ldots, m \tag{3.44}$$

$$\sum_{j=1}^{N} a_{ij}x_j + d_i^- - d_i^+ = -\sigma_i \ln(1-\gamma_i) + \alpha_i \quad i = m+1,m+2,\ldots,M \tag{3.45}$$

$$x_j \, , \, d_i^- \, , \, d_i^+ \ge 0 \qquad i = 1,2,\ldots,M \tag{3.46}$$
$$j = 1,2,\ldots,N$$

and

$$g_k(d^-,d^+) = \sum_{i\in P_k} d_i^+ + \sum_{i'\in P_k} d_{i'}^- , \qquad i = 1,2,\ldots,m \tag{3.47}$$
$$i' = m+1,\ldots,M$$
$$k = 1,2,\ldots,K$$

where P_k is the k^{th} priority level.

It is worth noting that:

(1) The terms of the achievement function a (3.43) are linear functions of the lower levels of the deviational random variables $\tilde{d}_i^-$, $\tilde{d}_i^+$, $i = 1,2,\ldots,M$ which were defined in (3.26) - (3.35).

(2) The goal set (3.44), (3.45) are linear constraints. Consequently, the above program can be solved either by a multiphase algorithm or by a sequential linear algorithm [38,37,50] .

3.4 The Input Coefficients (a_{ij})

In this section, we consider the input coefficients a_{ij} , $i = 1,2,\ldots,M$, $j = 1,2,\ldots,N$ of the goals:

$$\sum_{j=1}^{N} a_{ij}x_j \le b_i \qquad i = 1,2,\ldots,m \tag{3.48}$$

$$\sum_{j=1}^{N} a_{ij}x_j \ge b_i \qquad i = m+1,m+2,\ldots,M \tag{3.49}$$

to be random variables having exponential distributions. Two cases are presented. In the first, only some of the a_{ij}'s of the i^{th} goal are exponentially distributed random variables (Case 2); in the second, all of the a_{ij}'s of the i^{th} goal are exponentially distributed random variables (Case 3).

3.4.1 Case 2: Some of the a_{ij}'s have exponential distributions

We consider the goals (3.48), (3.49)

$$\sum_{j=1}^{N} a_{ij}x_j \le b_i \qquad i = 1,2,\ldots,m \tag{3.50}$$

$$\sum_{j=1}^{N} a_{ij}x_j \ge b_i \qquad i = m+1,m+2,\ldots,M \tag{3.51}$$

where

b_i are constants for $i = 1,2,\ldots,m,m+1,\ldots,M$;

x_j are the decision variables for $j = 1,2,\ldots,N$; and

a_{ij} are constants for $i = 1,2,\ldots,m,m+1,\ldots,M$; $j = n+1,n+2,\ldots,N$, and mutually independent random variables for $i = 1,2,\ldots,M$; $j = 1,2,\ldots,n$ $(n < N)$, having exponential distributions with mean $(\alpha_{ij} + \sigma_{ij})$, variance σ_{ij}^2 .

The density function of a_{ij} is :

$$f(a_{ij}) = \frac{1}{\sigma_{ij}} e^{-(a_{ij}-\alpha_{ij})/\sigma_{ij}} \qquad \begin{array}{l} a_{ij} \geq \alpha_{ij} \geq 0 \\ i = 1,2,\ldots,M \\ j = 1,2,\ldots,n \end{array} \tag{3.52}$$

By a method similar to that presented in Case 1, we construct a deterministic goal program to determine the optimum values of the x's namely those which satisfy the goals (3.50), (3.51) to the fullest possible extent according to their priorities with probabilities greater than or equal to the pre-assigned probabilities (γ_i). This method is developed as follows:

<u>First</u>: The deviational random variables.

The goal set (3.50), (3.51) can be reformed in standard form by adding [illegible] non-negative random variables $\tilde{d}^-$, $\tilde{d}^+$:

$$\sum_{j=1}^{N} a_{ij}x_j + \tilde{d}_i^- - \tilde{d}_i^+ = b_i \qquad \begin{array}{l} i = 1,2,\ldots, M \\ j = 1,2,\ldots, N \end{array} \tag{3.53}$$

where $\tilde{d}_i^-$, $\tilde{d}_i^+$, $i = 1,2,\ldots, M$ are defined in the same way as for Case 1 by equations (3.7) - (3.10).

<u>Second</u>: The chance-goal set.

Since, for $i = 1,2,\ldots,m,m+1,\ldots,M$, $j = 1,2,\ldots,n$ $(n < N)$ the a_{ij} are random variables, the goal set (3.50), (3.51) can be expressed as the following chance-goal set:

$$P_r\left(\sum_{j=1}^{N} a_{ij}x_j \le b_i\right) = \gamma_i \quad i = 1,2,3,\ldots,m \tag{3.54}$$

$$P_r\left(\sum_{j=1}^{N} a_{ij}x_j \ge b_i\right) = \gamma_i \quad i = m+1,m+2,\ldots,M \tag{3.55}$$

where γ_i are preassigned constants such that

$0 \le \gamma_i \le 1$ for all $i = 1,2,\ldots,m,m+1,\ldots,M$.

The goals (3.54), (3.55) are equivalent to:

$$1 - P_r\left(\sum_{j=1}^{n} a_{ij}x_j \ge b_i - \sum_{j=n+1}^{N} a_{ij}x_j\right) = \gamma_i \quad i = 1,2,\ldots,m \tag{3.56}$$

$$P_r\left(\sum_{j=1}^{n} a_{ij}x_j \ge b_i - \sum_{j=n+1}^{N} a_{ij}x_j\right) = \gamma_i \quad i = m+1,m+2,\ldots,M \tag{3.57}$$

to transform goals (5.56), (5.57) to deterministic goals, we first transform the variable $\left(\sum_{j=1}^{n} a_{ij}x_j\right)$, $i = 1,2,\ldots, M$ into a weighted finite sum of random variables w_{ij} plus a deterministic term. Each of the variables w_{ij} has a chi-square (χ^2) distribution with two degrees of freedom [67]. Since the variables a_{ij}, $i = 1,2,\ldots,M$, $j = 1,2,\ldots,n$ have exponential distributions and $x_j \ge 0$, then:

$$\sum_{j=1}^{n} a_{ij}x_j = \frac{1}{2}\left\{\sum_{j=1}^{n}[2(a_{ij}-\alpha_{ij})/\sigma_{ij}]\sigma_{ij}x_j\right\} + \sum_{j=1}^{n}\alpha_{ij}x_j$$

$$= \frac{1}{2}\left[\sum_{j=1}^{n}\sigma_{ij}x_j\,w_{ij} + \sum_{j=1}^{n}2\alpha_{ij}x_j\right] \tag{3.58}$$

where

$$w_{ij} = [2(a_{ij}-\alpha_{ij})/\sigma_{ij}] \sim \chi^2(2) \tag{3.59}$$

to obtain the equivalent deterministic goals we use a result due to Box [6] which gives the exact distribution of a weighted sum of χ^2 distributed variables.

Box's result is given in the following theorem.

Theorem 3.1

If $\chi^2(s_j)$ is a chi-square distributed variable with s_j degrees of freedom and λ_j is a constant, the exact distribution of $y = \sum_{j=1}^{n} \lambda_j \chi^2(s_j)$, where the $s_j = 2g_j$ are even integers, is a weighted finite sum of χ^2 distributions and given by:

$$P_r(y > y_o) = \sum_{j=1}^{n} \sum_{t=1}^{g_j} \eta_{jt} P_r[\chi^2(2t) > y_o / \lambda_j] \qquad (3.59)$$

In (3.59), each η_{jt} is a constant involving only the λ's and is given by:

$$\eta_{j(g_j-h)} = ({}_{j(h)}^{\prime\mu} / h!)\, \eta_{j(g_j)} \qquad h \geq 0 \qquad (3.60)$$

$$\eta_{j(g_j)} = \prod_{d \neq j}^{n} \left[\frac{\lambda_j}{\lambda_j - \lambda_d} \right]^{g_j} \qquad (3.61)$$

Using David & Kendall's tables of symmetric function [21,42] which gives the moments ${}_{j(h)}^{\prime\mu}$ in terms of cumulants $K_{j(h)}$, we can determine $\eta_{j(g_j-h)}$, where

$$K_{j(h)} = (h-1)! \sum_{d \neq j}^{n} [g_d (\frac{-\lambda_d}{\lambda_j - \lambda_d})^h], \qquad h \geq 1 \qquad (3.62)$$

Proof: [6, page 291].

Substituting transformation (3.58) in (3.56), (3.57) yields:

$$1 - P_r \left\{ \sum_{j=1}^{n} \sigma_{ij} x_j w_{ij} \geq 2(b_i - \sum_{j=1}^{n} \alpha_{ij} x_j - \sum_{j=n+1}^{N} a_{ij} x_j) \right\} = \gamma_i$$

$$i = 1,2,\ldots,m \qquad (3.63)$$

$$P_r\left\{\sum_{j=1}^{n}\sigma_{ij}x_jw_{ij} \geq 2(b_i - \sum_{j=1}^{n}\alpha_{ij}x_j - \sum_{j=n+1}^{N}a_{ij}x_j)\right\} = \gamma_i$$

$$i = m+1, m+2, \ldots, M \qquad (3.64)$$

By applying theorem 3.1, equations (3.63), (3.64) are equivalent to:

$$1 - \sum_{j=1}^{n}\eta_{ij}P_r\left\{\chi^2(2) \geq \frac{2(b_i - \sum_{j=1}^{n}\alpha_{ij}x_j - \sum_{j=n+1}^{N}a_{ij}x_j)}{\sigma_{ij}x_j}\right\} = \gamma_i$$

$$i = 1, 2, \ldots, m \qquad (3.65)$$

$$\sum_{j=1}^{n}\eta_{ij}P_r\left\{\chi^2(2) \geq \frac{2(b_i - \sum_{j=1}^{n}\alpha_{ij}x_j - \sum_{j=n+1}^{N}a_{ij}x_j)}{\sigma_{ij}x_j}\right\} = \gamma_i$$

$$i = m+1, m+2, \ldots, M \qquad (3.66)$$

where

$$\eta_{ij} = \prod_{d\neq j}^{n}\left(\frac{\sigma_{ij}x_j}{\sigma_{ij}x_j - \sigma_{id}x_d}\right) \qquad i = 1, 2, \ldots, M \qquad (3.67)$$

Since

$$P_r\left[\chi^2(2) \geq \frac{2(b_i - \sum_{j=1}^{n}\alpha_{ij}x_j - \sum_{j=n+1}^{N}a_{ij}x_j)}{\sigma_{ij}x_j}\right]$$

$$= e^{-(b_i - \sum_{j=1}^{n}\alpha_{ij}x_j - \sum_{j=n+1}^{N}a_{ij}x_j)/\sigma_{ij}x_j} \qquad (3.68)$$

on substituting (3.67), (3.68) in (3.65), (3.66), we obtain the deterministic goals:

$$1 - \sum_{j=1}^{n}\prod_{d\neq j}^{n}\left(1 - \frac{\sigma_{id}x_d}{\sigma_i\ x_j}\right)^{-1} e^{-(b_i - \sum_{j=1}^{n}\alpha_{ij}x_j - \sum_{j=n+1}^{N}a_{ij}x_j)/\sigma_{ij}x_j}$$

$$= \gamma_i \qquad i = 1, 2, \ldots, m \qquad (3.69)$$

$$\sum_{j=1}^{n} \prod_{d\neq j}^{n} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} e^{-(b_i - \sum_{j=1}^{n} \alpha_{ij}x_j - \sum_{j=n+1}^{N} a_{ij}x_j) / \sigma_{ij}x_j}$$

$$= \gamma_i \qquad i=m+1,m+2,\ldots,M \qquad (3.70)$$

Third: The probabilistic deviational variables.

Goals (3.69), (3.70) can be reformed in the standard form by adding deviational variables d_i^-, d_i^+ for $i = 1,2,\ldots,m,m+1,\ldots,M$, as follows:

$$1 - \left\{ \sum_{j=1}^{n} \prod_{d\neq j}^{n} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} e^{-(b_i - \sum_{j=1}^{n} \alpha_{ij}x_j - \sum_{j=n+1}^{N} a_{ij}x_j) / \sigma_{ij}x_j} \right\}$$

$$+ d_i^- - d_i^+ = \gamma_i \qquad i = 1,2,\ldots,m \qquad (3.71)$$

$$\left\{ \sum_{j=1}^{n} \prod_{d\neq j}^{n} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} e^{-(b_i - \sum_{j=1}^{n} \alpha_{ij}x_j - \sum_{j=n+1}^{N} a_{ij}x_j) / \sigma_{ij}x_j} \right\}$$

$$+ d_i^- - d_i^+ = \gamma_i \qquad i = m+1,m+2,\ldots,M \qquad (3.72)$$

where

$$d_i^- = \begin{cases} \max\left[0, \gamma_i - P_r\left(\sum_{j=1}^{N} a_{ij}x_j \le b_i\right)\right] & i = 1,2,\ldots,m \qquad (3.73) \\ \max\left[0, \gamma_i - P_r\left(\sum_{j=1}^{N} a_{ij}x_j \ge b_i\right)\right] & i = m+1,m+2,\ldots,m \qquad (3.79) \end{cases}$$

$$d_i^+ = \begin{cases} \max\left[0, P_r\left(\sum_{j=1}^{N} a_{ij}x_j \le b_i\right) - \gamma_i\right] & i = 1,2,\ldots,m \qquad (3.75) \\ \max\left[0, P_r\left(\sum_{j=1}^{N} a_{ij}x_j \ge b_i\right) - \gamma_i\right] & i = m+1,m+2,\ldots,M \qquad (3.76) \end{cases}$$

and

$$0 \le d_i \ , \ d_i^+ \le 1 \ , \ d_i^- \cdot d_i^+ = 0$$

for all $i = 1,2,\ldots,m,m+1,\ldots,M$ (3.77)

From the above we conclude

Result 3.1

The i^{th} goal is satisfied with probability greater than or equal to γ_i if and only if $d_i^- = 0$ and $d_i^+ \ge 0$, $i = 1,2,\ldots,m,m+1,\ldots,M$, i.e.,

$$\left.\begin{array}{l} P_r\left(\sum_{j=1}^{N} a_{ij}x_j \le b_i\right) \\ \text{or} \quad P_r(\tilde{d}_i \ge 0) \end{array}\right\} = \gamma_i + d_i^+ \qquad i=1,2,\ldots,m$$

and

$$\left.\begin{array}{l} P_r\left(\sum_{j=1}^{N} a_{ij}x_j \ge b_i\right) \\ \text{or} \quad P_r(\tilde{d}_i^+ \ge 0) \end{array}\right\} = \gamma_i + d_i^+ \qquad i=m+1,m+2,\ldots,M$$

Result 3.2

The i^{th} goal is satisfied with probability is less than or equal to γ_i if and only if $d_i^- \ge 0$ and

$d_i^+ = 0$, $i = 1,2,3,\ldots,m,m+1,M$, i.e.,

$$\left.\begin{array}{l} P_r\left(\sum_{j=1}^{N} a_{ij}x_j \le \hat{b}_i\right) \\ \text{or} \quad P_r(\tilde{d}_i^- \ge 0) \end{array}\right\} = \gamma_i - d_i^- \qquad i = 1,2,\ldots,m$$

and

$$\left.\begin{array}{l} P_r\left(\sum_{j=1}^{N} a_{ij}x_j \geq b_i\right) \\ \text{or} \quad P_r(\tilde{d}_i^+ \geq 0) \end{array}\right\} = \gamma_i - d_i^- \quad i = m+1, m+2, \ldots, M$$

<u>Fourth</u>: The transformed deterministic goal program.

From results 3.1 , we can determine the optimum values of x's i.e. those which satisfy the goals (3.50), (3.51)) to the fullest possible extent according to their priorities with probabilities greater than or equal to the preassigned probabilities γ_i , $(i = 1,2,\ldots,M)$ by solving the transformed deterministic goal program.

Find $x = (x_1, x_2, \ldots, x_N)$

So as to

$$\text{lexico-min} \quad a = \left\{ [g_1(d^-)], [g_2(d^-)], \ldots, [g_k(d^-)], \ldots, [g_K(d^-)] \right\}$$

$$K \leq M \tag{3.78}$$

subject to

$$1 - \left\{ \sum_{j=1}^{n} \prod_{d \neq j}^{n} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} e^{-(b_i - \sum_{j=1}^{n} \alpha_{ij}x_j - \sum_{j=n+1}^{N} a_{ij}x_j)/\sigma_{ij}x_j} \right\}$$

$$+ d_i^- - d_i^+ = \gamma_i \qquad i = 1,2,\ldots,m \tag{3.79}$$

$$\left\{ \sum_{j=1}^{n} \prod_{d \neq j}^{n} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} e^{-(b_i - \sum_{j=1}^{n} \alpha_{ij}x_j - \sum_{j=n+1}^{N} a_{ij}x_j)/\sigma_{ij}x_j} \right\}$$

$$+ d_i^- - d_i^+ = \gamma_i \qquad i = m+1, m+2, \ldots, M \tag{3.80}$$

$$x_j \geq 0 \qquad j = 1,2,\ldots,N \tag{3.81}$$

$$0 \leq d_i^- , d_i^- \leq 1 \ , \ d_i^- \cdot d_i^+ = 0 \quad \text{for all } i = 1,2,\ldots,m,m+1,\ldots,M$$

(3.81)

where

$$g_k(d^-) = \sum_{i\in P_k} d_i^- \quad i = 1,2,\ldots,m,m+1,\ldots,M \qquad k = 1,2,\ldots,K \tag{3.83}$$

3.4.2 The equivalent signomial program

In subsection 3.4.1, it was shown that the set (3.79), (3.80) of the program (3.79),(3.80) consists of very complicated nonlinear constraints. But they can be transformed to standard signomial form (see definition 5.3 [3,24]) as follows:

(1) For each of the goals (3.79), (3.80)

$$\gamma_i^{-1}-\gamma_i^{-1}\left\{\sum_{j=1}^{n}\prod_{d\neq j}^{n}\left(1-\frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} e^{-(b_i-\sum_{j=1}^{n}\alpha_{ij}x_j-\sum_{j=n+1}^{N}a_{ij}x_j)/\sigma_{ij}x_j}\right\}$$

$$+\gamma_i^{-1}d_i^- - \gamma_i^{-1}d_i^+ = 1 \qquad i = 1,2,\ldots,m \tag{3.84}$$

or

$$\gamma_i^{-1}\left\{\sum_{j=1}^{n}\prod_{d\neq j}^{n}\left(1-\frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} e^{-(b_i-\sum_{j=1}^{n}\alpha_{ij}x_j-\sum_{n=n+1}^{N}a_{ij}x_j)\,/\,\sigma_{ij}x_j}\right\}$$

$$+\gamma_i^{-1}d_i^- - \gamma_i^{-1}d_i^+ = 1 \qquad i = m+1,m+2,\ldots,M \tag{3.85}$$

define additional variables z_{ij} , z'_{ij} , where

$$z_{ij} = \frac{b_i - \sum_{j=1}^{n}\alpha_{ij}x_j - \sum_{j=n+1}^{N}a_{ij}x_j}{\sigma_{ij}x_j}, \quad z_{ij}\geq 0, \quad i=1,2,\ldots,m \quad j=1,2,\ldots,n \tag{3.86}$$

$$z'_{ij} = \frac{b_i - \sum_{j=1}^{n}\alpha_{ij}x_j - \sum_{j=n+1}^{N}a_{ij}x_j}{\sigma_{ij}x_j} + c, \quad z'_{ij}\geq 0, \quad i=m+1,m+2,\ldots,M \quad j=1,2,\ldots,N \tag{3.87}$$

and c is a large positive constant. Then goals (3.84), (3.85) can be replaced by the two following sets of goals and constraints:

$$\gamma_i^{-1} - \gamma_i^{-1} \left\{ \sum_{j=1}^{n} \prod_{d \neq j}^{n} \left(1 - \frac{\sigma_{id} x_d}{\sigma_{ij} x_j}\right)^{-1} e^{-z_{ij}} \right\} + \gamma_i^{-1} d_i^- - \gamma_i^{-1} d_i^+ = 1$$

$$i = 1,2,\ldots,m \qquad (3.88)$$

$$b_i^{-1} \sigma_{ij} x_j z_{ij} + b_i^{-1} \sum_{j=1}^{n} \alpha_{ij} x_j + b_i^{-1} \sum_{j=n+1}^{N} a_{ij} x_j = 1$$

$$i = 1,2,\ldots,m \qquad (3.89)$$

$$j = 1,2,\ldots,n$$

or

$$\gamma_i^{-1} \left\{ \sum_{j=1}^{n} \prod_{d \neq j}^{n} \left(1 - \frac{\sigma_{id} x_d}{\sigma_{ij} x_j}\right)^{-1} e^{-z'_{ij}} \right\} + \gamma_i^{-1} d_i^- - \gamma_i^{-1} d_i^+ = 1$$

$$i = m+1,m+2,\ldots,M \qquad (3.90)$$

$$b_i^{-1} \sigma_{ij} x_j z'_{ij} + b_i^{-1} \sum_{j=1}^{n} \alpha_{ij} x_j + b_i^{-1} \sum_{j=n+1}^{N} a_{ij} x_j - b_i^{-1} c\, \sigma_{ij} x_j = 1$$

$$i = m+1,m+2,\ldots,M \qquad (3.91)$$

$$j = 1,2,\ldots,n$$

(2) Since

$$e^{-z_{ij}} \simeq [1 - z_{ij}\,\phi^{-1}]^{\phi} = \beta_{ij}^{\phi} \qquad i=1,2,\ldots,m \qquad (3.93)$$

$$j=1,2,\ldots,M$$

and

$$e^{-z'_{ij}} \simeq [1 - z'_{ij}\phi^{-1}]^{\phi} = \beta'^{\phi}_{ij} \qquad i=m+1,m+2,\ldots,M \qquad (3.94)$$

$$j=1,2,\ldots,n$$

where

$$\beta_{ij} = 1 - z_{ij}\,\phi^{-1} \qquad i=1,2,\ldots,m \qquad (3.95)$$

$$j=1,2,\ldots,n$$

$$\beta'_{ij} = 1 - z'_{ij}\ \phi^{-1} \qquad i=m+1,m+2,\ldots,M \qquad (3.96)$$
$$j=1,2,\ldots,n$$
$$\phi \to \infty$$

(see Appendix A)

Using (3.93), (3.94), goals (3.88) or (3.90) can be replaced by the two following sets of goals and constraints:

$$\gamma_i^{-1} - \gamma_i^{-1} \left\{ \sum_{j=1}^{n} \prod_{d\neq j}^{n} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} \beta_{ij}^{\phi} \right\} + \gamma_i^{-1}\, d_i^- - \gamma_i^{-1}.d_i^+ = 1$$

$$i=1,2,\ldots,m \qquad (3.97)$$

$$\beta_{ij} + z_{ij}\ \phi^{-1} = 1 \qquad i=1,2,\ldots,m \qquad (3.98)$$
$$j=1,3,\ldots,n$$

or

$$\gamma_i^{-1} \left\{ \sum_{j=1}^{n} \prod_{d\neq j}^{n} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} \beta_{ij}'^{\phi} \right\} + \gamma_i^{-1}\, d_i^- - \gamma_i^{-1}.d_i^+ = 1$$
$$i=m+1,m+2,\ldots,M \qquad (3.99)$$

$$\beta'_{ij} + z'_{ij}\ \phi^{-1} = 1 \qquad i=m+1,\ldots,M \qquad (3.100)$$
$$j=1,2,\ldots,n$$

and $\quad \phi \to \infty$

(3) By means of the above transformations, goals (3.84), (3.85) can be replaced by the following three sets of signomial constraints :

$$\left.\begin{array}{ll}
\gamma_i^{-1} - \gamma_i^{-1} \sum_{j=1}^{n} \prod_{d\neq j}^{n} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} \beta_{ij}^{\phi} + \gamma_i^{-1}d_i^- - \gamma_i^{-1}d_i^+ = 1 & (3.101) \\
b_i^{-1}\sigma_{ij}x_j z_{ij} + b_i^{-1} \sum_{j=1}^{n} \alpha_{ij}x_j + b_i^{-1} \sum_{j=n+1}^{N} a_{ij}x_j = 1 & (3.102) \\
 & i=1,2,\ldots m \\
 & j=1,2,\ldots n \\
\beta_{ij} + z_{ij}\ \phi^{-1} = 1 & (3.103)
\end{array}\right\}$$

or

$$\gamma_i^{-1}\sum_{j=1}^{n}\prod_{d\neq j}^{n}\left(1-\frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1}\beta_{ij}^{1\phi}+\gamma_i^{-1}d_i^{-}-\gamma_i^{-1}d_i^{+}=1 \qquad (3.104)$$

$$b_i^{-1}\,\sigma_{ij}x_jz'_{ij}+b_i^{-1}\sum_{j=1}^{n}\alpha_{ij}x_j+b^{-1}\sum_{j=n+1}^{N}a_{ij}x_j-b_i^{-1}c\,\sigma_{ij}x_j=1 \qquad (3.105)$$

$$\beta'_{ij}+z'_{ij}\,\phi^{-1}=1 \qquad (3.106)$$

$i=m+1,m+2,\ldots,M$
$j=1,2,\ldots,n$

and $\phi \to \infty$

Constraints (3.102), (3.103) or (3.105), (3.106) are in standard signomial form [1]. On carrying out the summation in the left hand side of (3.101) or (3.104), constraints (3.101) or (3.104) are also seen to be in standard signomial form (see example 3.1, section 3.5).

It is worth noting that:

(a) constraints (3.102), (3.103) or (3.105), (3.106) are rigid constraints related with goal sets (3.101) or (3.104) respectively.

(b) the transformation to signomial form leads to a goal set consisting of M goals in standard form and a constraint set consisting of $2nM$ rigid constraints of N decision variables, $2nM$ additional variables and $2M$ deviational variables instead of a goal set consisting of M goals of N decision variables and $2M$ deviational variables.

---

[1] The constraints(3.102), (3.103) and (3.106) are in posynomial form (see definition 5.3 [3,24]).

(4) Hence, program (3.78)-(3.82) is equivalent to the following signomial program:

Find $x = (x_1, x_2, \ldots, x_N)$

So as to

$$\text{lexico-min } a = \left\{[g_1(d^-)], [g_2(d^-)], \ldots, [g_k(d^-)], \ldots, [g_K(d^-)]\right\}$$

$$K \le M \tag{3.107}$$

subject to

$$\gamma_i^{-1} - \gamma_i^{-1}\left\{\sum_{j=1}^{m} \prod_{d\neq j}^{n} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} \beta_{ij}^{\phi}\right\} + \gamma_i^{-1} d_i^- - \gamma_i^{-1} d_i^+ = 1$$

$$i=1,2,\ldots,m \tag{3.108}$$

$$b_i^{-1}\sigma_{ij}x_jz_{ij} + b_i^{-1}\sum_{j=1}^{n}\alpha_{ij}x_j + b_i^{-1}\sum_{j=n+1}^{N} a_{ij}x_j = 1 \qquad i=1,2,\ldots,m \quad j=1,2,\ldots,n \tag{3.109}$$

$$\beta_{ij} + z_{ij}\,\phi^{-1} = 1 \tag{3.110}$$

$$\gamma_i^{-1}\left\{\sum_{j=1}^{n} \prod_{d\neq j}^{n} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} \beta_{ij}^{\prime\phi}\right\} + \gamma_i^{-1}d_i^- - \gamma_i^{-1}d_i^+ = 1$$

$$i=m+1,m+2,\ldots,M \tag{3.111}$$

$$b_i^{-1}\sigma_{ij}x_jz'_{ij} + b_i^{-1}\sum_{j=1}^{n}\alpha_{ij}x_j + b_i^{-1}\sum_{j=n+1}^{N} a_{ij}x_j - b_i^{-1}c\sigma_{ij}x_j = 1 \tag{3.112}$$

$$\beta'_{ij} + z'_{ij}\,\phi^{-1} = 1 \tag{3.113}$$

$$i=m+1,m+2,\ldots,M \qquad j=1,2,\ldots,n$$

$$x_j, z_{ij}, z'_{ij}, \beta_{ij}, \beta'_{ij} \ge 0 \qquad i=1,2,\ldots,M \quad j=1,2,\ldots,N \tag{3.114}$$

$$0 \le d_i^-,\ d_i^+ \le 1,\ d_i^-.d_i^+ = 0 \qquad i=1,2,\ldots,m,m+1,\ldots,M \tag{3.115}$$

$$g_k(d^-) = \sum_{i \in P_k} d_i^- \qquad \begin{array}{l} i=1,2,\ldots,m,m+1,\ldots,M \\ k=1,2,\ldots,K \end{array} \qquad (3.116)$$

and

c is a large positive constant.

The above program can be solved by the algorithm presented in section 5.8 for solving nonlinear goal programs.

In section 3.5, we present a simple numerical example to illustrate the various steps in arriving at the transformed deterministic goal program and transform it to the equivalent signomial program.

3.4.3 Case 3: all a_{ij}'s have exponential distributions

If we consider Case 2, when all the a_{ij}'s for $i = 1,2,\ldots,m,m+1,\ldots,M$, $j = 1,2,\ldots,n,n+1,\ldots,N$ have exponential distributions, then this case is equivalent to Case 2 with $n = N$. In turn, the transformed deterministic goal program is :

Find $x = (x_1,x_2,\ldots,x_N)$

So as to

$$\text{lexico-min} \quad a = \left\{ [g_1(d^-)],[g_2(d^-)],\ldots,[g_k(d^-)],\ldots,[g_K(d^-)] \right\}$$

$$K \le M \qquad (3.117)$$

subject to

$$1 - \left\{ \sum_{j=1}^{N} \prod_{d \ne j}^{N} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} e^{-(b_i - \sum_{j=1}^{N} \alpha_{ij}x_j)/\sigma_{ij}x_j} \right\} + d_i^- - d_i^+ = \gamma_i$$

$$i = 1,2,\ldots,m \qquad (3.118)$$

$$\left\{\sum_{j=1}^{N} \prod_{d\neq j}^{N} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} e^{-(b_i - \sum_{j=1}^{N} \alpha_{ij}x_j)/\sigma_{ij}x_j}\right\} + d_i^- - d_i^+ = \gamma_i$$

$$i=m+1,m+2,\ldots,M \qquad (3.119)$$

$$x_j \geq 0 \qquad i=1,2,\ldots,N$$

$$0 \leq d_i^-, \; d_i^+ \leq 1, \quad d_i^-.d_i^+ = 0$$

$$i=1,2,\ldots,m,m+1,\ldots,M \qquad (3.120)$$

$$g_k(d^-) = \sum_{i\in P_k} d_i^- \qquad i=1,2,\ldots,m,m+1,\ldots,M \qquad (3.121)$$

$$k = 1,2,\ldots,K$$

By applying the transformations set out in subsection 3.4.2; program (3.117), (3.120) is equivalent to the following signomial program:

Find $x = (x_1,x_2,\ldots,x_N)$

So as to

$$\text{lexico-min} \quad a = \left\{[g_1(d^-)],[g_2(d^-)],\ldots,[g_k(d^-)],\ldots,[g_K(d^-)]\right\}$$

$$K \leq M \qquad (3.122)$$

subject to

$$\gamma_i^{-1}-\gamma_i^{-1} \sum_{j=1}^{N} \prod_{d\neq j}^{N} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} \beta_{ij}^{\phi} + \gamma_i^{-1}d_i^- - \gamma_i^{-1}d_i^+ = 1$$

$$i=1,2,\ldots,m \qquad (3.123)$$

$$\left.\begin{aligned} & b_i^{-1}\sigma_{ij}x_j z_{ij} + b_i^{-1} \sum_{j=1}^{N} \alpha_{ij}x_j = 1 \\ & \beta_{ij} + z_{ij}\,\phi^{-1} = 1 \end{aligned}\right\} \quad \begin{aligned} & i=1,2,\ldots,m \qquad (3.124) \\ & j=1,2,\ldots,N \qquad (3.125) \end{aligned}$$

$$\gamma_i^{-1}-\gamma_i^{-1} \sum_{j=1}^{N} \prod_{d\neq j}^{N} \left(1 - \frac{\sigma_{id}x_d}{\sigma_{ij}x_j}\right)^{-1} \beta_{ij}^{\prime\phi} + \gamma_i^{-1} d_i^- - \gamma_i^{-1} d_i^+ = 1$$

$$i=m+1,m+2,\ldots,M \qquad (3.126)$$

$$\left.\begin{array}{l} b_i^{-1}\sigma_{ij}x_j z'_{ij} + b_i^{-1} \sum_{j=1}^{N} \alpha_{ij}x_j - b_i^{-1}\, c\, \sigma_{ij}x_j = 1 \\ \beta'_{ij} + z'_{ij}\, \phi^{-1} = 1 \end{array}\right\} \begin{array}{l} i=m+1,m+2,\ldots,M \\ j=1,2,\ldots,N \end{array} \quad \begin{array}{r} (3.127) \\ \\ (3.128) \end{array}$$

$$x_j,\ z_{ij},\ z'_{ij},\ \beta_{ij},\ \beta'_{ij} \geq 0 \qquad \begin{array}{l} i=1,2,\ldots,m,m+1,\ldots,N \\ j=1,2,\ldots,N \end{array} \qquad (3.129)$$

$$0 \leq d_i^-,\ d_i^+ \leq 1,\ d_i^- . d_i^+ = 0 \qquad i=1,2,\ldots,m,m+1,\ldots,M \qquad (3.130)$$

where

$$g_k(d^-) = \sum_{i \in P_k} d_i^- \qquad \begin{array}{l} i=1,2,\ldots,m,m+1,\ldots,M \\ k=1,2,\ldots,K \end{array} \qquad (3.131)$$

$\phi \rightarrow \infty$

Programs (3.117)-(3.120) and (3.122)-(3.130) are considered to be special cases respectively of programs (3.78)-(3.82), (3.108)-(3.115) and in turn they have the same properties as programs (3.78)-(3.82), (3.108)-(3.115) of subsection 3.4.2. The program (3.122)-(3.130) can also be solved by the algorithm presented in section 5.8.

3.5 A Numberical Example

If we want to determine x's which satisfy the following goals:

$$a_{11}x_1 + a_{12}x_2 + 3x_3 \leq 25 \qquad (3.132)$$

$$2x_1 + x_2 + x_3 \leq b_2 \qquad (3.133)$$

$$x_1 + x_2 \geq b_3 \qquad (3.134)$$

to the fullest possible extent, with probabilities greater than or equal to: $\gamma_1 = .55$, $\gamma_2 = .70$, $\gamma_3 = .70$ respectively, such that goals (3.133), (3.134) have first priority and (3.132) has second priority, where: a_{11}, a_{12}, b_2 and b_3 have exponential distributions with parameters

$$\left.\begin{array}{l} (\alpha_{11} = 3 \ , \ \sigma_{11} = 1) \\ (\alpha_{12} = 4 \ , \ \sigma_{12} = 1) \\ (\alpha_2 = 9 \ , \ \sigma_2 = 3) \\ (\alpha_3 = 4 \ , \ \sigma_3 = 2) \end{array}\right\} \qquad (3.135)$$

respectively.

<u>Solution</u>

<u>Step 1</u>

Transform probabilistic goals (3.132)-(3.134) to deterministic goals in standard form as follows:

(1) From (3.71) the following goal corresponds to goal (3.132):

$$1 - \left\{(1 - \frac{x_2}{x_1})^{-1} e^{-(25-3x_1-4x_2-3x_3)/x_1} + (1 - \frac{x_1}{x_2})^{-1} e^{-(25-3x_1-4x_2-3x_3)/x_2}\right\} + d_1^- - d_1^+ = .55 \qquad (3.136)$$

(2) From (3.19),(3.20) the following goals correspond to goals (3.133),(3.134):

$$2x_1 + x_2 + x_3 + d_2^- - d_2^+ = -3 \ln (.70) + 9 \qquad (3.137)$$

$$x_1 + x_2 + d_3^- - d_3^+ = -2 \ln (.30) + 4 \qquad (3.138)$$

Step 2

The transformed deterministic goal program (see fourth page 26 and 35) is:

Find $x = (x_1,x_2,x_3)$

So as to

$$\text{lexico-min} \quad a = \{(d_2^+ + d_3^-),(d_1^-)\} \tag{3.139}$$

subject to

$$2x_1 + x_2 + x_3 + d_2^- - d_2^+ = 10.07 \tag{3.140}$$

$$x_1 + x_2 + d_3^- - d_3^+ = 6.408 \tag{3.141}$$

$$1 - \left\{\left(\frac{x_1}{x_1-x_2}\right) e^{-(25-3x_1-4x_2-3x_3)/x_1} + \left(\frac{x_2}{x_2-x_1}\right) e^{-(25-3x_1-4x_2-3x_3)/x_2}\right\} + d_1^- - d_1^+ = .55 \tag{3.142}$$

$$x_j, d_i^-, d_i^+ \geq 0 \qquad j=1,2,3 \tag{3.143}$$

$$d_i^- . d_i^+ = 0 \qquad i=1,2,3$$

$$0 \leq d_1^- \leq .55 \quad , \quad 0 \leq d_1^+ \leq .45 \tag{3.144}$$

Step 3

The following signomial goal programing is equivalent to program (3.139)-(3.144) (see subsection 3.4.2):

Find $x = (x_1,x_2,x_3)$

So as to:

$$\text{lexico-min} \quad a = \{(d_2^+ + d_3^-) , (d_1^-)\} \tag{3.145}$$

subject to

$$2x_1 + x_2 + x_3 + d_2^- - d_2^+ = 10.07 \tag{3.146}$$

$$x_1 + x_2 + d_3^- - d_3^+ = 6.408 \tag{3.147}$$

$$1 - \left\{ \frac{x_1\beta_{11}^{\phi} - x_2\beta_{12}^{\phi}}{x_1 - x_2} \right\} + d_1^- - d_1^+ = .55 \tag{3.148}$$

$$.04x_1z_{11} + .12x_1 + .16x_2 + .12x_3 = 1 \tag{3.149}$$

$$.04x_2z_{12} + .12x_1 + .16x_2 + .12x_3 = 1 \tag{3.150}$$

$$\beta_{11} + z_{11}\,\phi^{-1} = 1 \tag{3.151}$$

$$\beta_{12} + z_{12}\,\phi^{-1} = 1 \tag{3.152}$$

$$\left.\begin{array}{l} x_j, z_{ij}, \beta_{ij}, d_i^-, d_i^+ \geq 0 \\ d_i^- \cdot d_i^+ = 0 \end{array}\right. \quad \begin{array}{l} i=1,2,3 \\ j=1,2,3 \end{array} \tag{3.153}$$

$$0 \leq d_1^- \leq .55 \quad , \quad 0 \leq d_1^+ \leq .45 \tag{3.154}$$

where

$$z_{ij} = \frac{b_i - \sum_{j=1}^{z} \alpha_{ij}x_j - a_{iz}x_z}{\sigma_{ij}x_j} \tag{3.155}$$

$$\beta_{ij} = 1 - z_{ij}\,\phi^{-1} \tag{3.156}$$

$i=1, j=1,2$

and

$$\phi \rightarrow \infty$$

Step 4

By using the algorithm which is presented in section 5.8, the global solution to the above program is :

$x_1 = 3.204$, $x_2 = 3.204$, $x_3 = 0$

$d_1^- = 0$, $d_1^+ = .45$

$d_2^- = .458$, $d_2^+ = 0$

$d_3^- = 0$, $d_3^+ = 0$

(the detailed solution is given in Appendix D.)

3.6 Conclusion

In this chapter, the approach of CCGP has been presented when the goals have exponentially distributed parameters.

Two cases have been considered:

The first, when the right hand side coefficients have exponential distributions. In this case:

(1) We have developed a method to construct the transformed deterministic linear goal program.

(2) The probabilistic interpretation of the deviational random variables and the deviational random variables levels have been introduced.

The second, when some or all input coefficients have exponential distributions. In this case:

(3) a method similar to that in (1) has been developed to construct the transformed deterministic nonlinear goal programs;

(4) the probabilistic deviational variables have been defined; and

(5) the signomial programs equivalent to the transformed deterministic nonlinear goal programs have been presented. These can be solved by the algorithm presented in section 5.8.

The procedures of these methods have been clarified by two numerical examples. In addition, our methods allow the goal set to contain a mix of probabilistic goals, some of them have right hand side exponentially distributed variables and the others have input which are exponentially distributed variables and of course, deterministic goals aLso, as shown in examples 3.1 and 4.1.

CHAPTER 4

CCGP With Chi-Square Distributed Paramaters

4.1 Introduction

In this chapter we consider CCGP approach when the goals have chi-square distributed parameters. Using the methods presented in Chapter 3, we present the transformed deterministic goal programs when:

(i) the b_i's have χ^2 distributions (Case 4, Section 4.3), or

(ii) some or all of the a_{ij}'s have χ^2 distributions (Cases 5, 6 respectively, Section 4.4).

The signomial programs equivalent to the transformed deterministic goal programs of Cases 5 and 6 are presented also.

In addition, in Section 4.5 we prove that Sengupta's transformation (for obtaining deterministic programs when the a_{ij}'s have χ^2 distributions) does not lead to a solvable program.

4.2 Chi-Square Distributed Parameters

In this chapter, we consider the following two cases - first when the b_i's have χ^2 distributions and second when some or all of the a_{ij}'s have χ^2 distributions.

We consider parameters having χ^2 distributions for the following reasons:

(1) it is known that a χ^2 distribution arises when considering the sum of squares of independent random variables, each of which comes from a normal population with zero mean and unit variance. However, if each of the random variables comes from a normal population with non-zero mean and constant variance, then the resulting distribution of the squares of the independent variables defines a non-central χ^2 distribution [62, 41, 49]. Statistical tables of non-central χ^2 variables are available [30,58]. In addition, the non-central χ^2 distribution may be closely approximated by a central χ^2 distribution [61, 42].

(2) a χ^2 distribution is closely related to other non-negative continuous distributions (e.g. the exponential and gamma distributions), that have been used frequently in operational research [62].

(3) χ^2 variables have the well-known reproductive property that a sum of independent χ^2 variables also has a χ^2 distribution.

(4) the ratio of two χ^2 variables is distributed like Fisher's (F) distribution, for which standard statistical tables are available [30].

4.3 Case 4: The right hand side coefficients (b_i's)

We investigate the implications of replacing the assumption that the b_i's have exponential distributions, Case 1, with the assumption that they have χ^2.

If $b_i \sim \chi^2(s_i)$ with density function

$$f(b_i) = \frac{1}{\Gamma\frac{s_i}{2}} \left(\frac{1}{2}\right)^{\frac{s_i}{2}} (b_i)^{\frac{s_i}{2}-1} e^{-\frac{b_i}{2}} \quad b_i \geq 0 \qquad (4.1)$$

$$i=1,2,\ldots M$$

then, the goals

$$\sum_{j=1}^{N} a_{ij}x_j \leq b_i \qquad i = 1,2,\ldots,m \qquad (4.2)$$

$$\sum_{j=1}^{N} a_{ij}x_j \geq b_i \qquad i = m+1,m+2,\ldots,M \qquad (4.3)$$

can be reformed as the chance-goal set:

$$P_r\left(\sum_{j=1}^{N} a_{ij}x_j \leq b_i\right) = \gamma_i \qquad i = 1,2,\ldots,m \qquad (4.4)$$

$$P_r\left(\sum_{j=1}^{N} a_{ij}x_j \geq b_i\right) = \gamma_i \qquad i = m+1,m+2,\ldots,M \qquad (4.5)$$

where

a_{ij} are constants for all $i = 1,2,\ldots,m,m+1,\ldots,M$, $j = 1,2,\ldots,N$;

γ_i are preassigned constants where $0 \leq \gamma_i \leq 1$,

$i = 1,2,\ldots,m,m+1,\ldots,M$

The goals (4.4), (4.5) are equivalent to:

$$\sum_{j=1}^{N} a_{ij}x_j = F^{-1}(1-\gamma_i) \qquad i = 1,2,\ldots,m \qquad (4.6)$$

$$\sum_{j=1}^{N} a_{ij}x_j = F^{-1}(\gamma_i) \qquad i = m+1,m+2,\ldots,M \qquad (4.7)$$

where F^{-1} is the inverse function of the cumulative function of a χ^2 random variable with s_i degfees of freedom.

Since the γ_i, $i = 1,2,\ldots,m,m+1,\ldots,M$ are constants, then $F^{-1}(1-\gamma_i)$ and $F^{-1}(\gamma_i)$ are constants also and can be calculated from statistical tables [30]. We can reform goals

(4.6), (4.7) in standard form for goals by adding the deviational random variable levels d_i^- , d_i^+ of the deviational random variables $\tilde{d}_i^-$, $\tilde{d}_i^+$ respectively, where d_i^- , d_i^+ , $\tilde{d}_i^-$ and $\tilde{d}_i^+$ are defined in the same way and have the same probabilistic interpretation as in Case 1. This is done as follows:

$$\sum_{j=1}^{N} a_{ij}x_j + d_i^- - d_i^+ = F^{-1}(1-\gamma_i) \qquad i = 1,2,\ldots,m \qquad (4.8)$$

$$\sum_{j=1}^{N} a_{ij}x_j + d_i^- - d_i^+ = F^{-1}(\gamma_i) \qquad i = m+1,m+2,\ldots,M \qquad (4.9)$$

We can determine the optimum values of the x's namely those which satisfy the goals (4.2), (4.3) to the fullest possible extent according to their priorities with probabilities greater than or equal to preassigned values (see Fourth, page 26) by solving the following transformed deterministic goal program:

Find $x = (x_1.x_2,\ldots,x_N)$

So as to

$$\text{lexico-min} \quad a = \{[g_1(d^-,d^+)],[g_2(d^-,d^+)],\ldots,[g_k(d^-,d^+)],\ldots$$
$$\ldots,[g_K(d^-,d^+)]\} \qquad K \le M \qquad (4.10)$$

subject to

$$\sum_{j=1}^{N} a_{ij}x_j + d_i^- - d_i^+ = F^{-1}(1-\gamma_i) \qquad i = 1,2,\ldots,m \qquad (4.11)$$

$$\sum_{j=1}^{N} a_{ij}x_j + d_i^- - d_i^+ = F^{-1}(\gamma_i) \qquad i=m+1,m+2,\ldots,M \qquad (4.12)$$

$$x_j \, , \, d_i^- \, , \, d_i^+ \ge 0 \qquad i = 1,2,\ldots,M \qquad (4.13)$$
$$j = 1,2,\ldots,N$$

and

$$g_k(d^-,d^+) = \sum_{i\epsilon P_k} d_i^+ + \sum_{i'\epsilon P_k} d_i^- \qquad (4.14)$$

$$i = 1,2,\ldots,m$$
$$i' = m+1,m+2,\ldots,M$$
$$k = 1,2,\ldots,K$$

The above program is a deterministic linear goal program and can be solved using either a multiphase algorithm or a sequential linear algorithm [38, 37, 50] .

4.4 The Input Coefficients (a_{ij}'s)

In this section, we consider the case where some or all of the quantities a_{ij} the input coefficients of the goals have χ^2 distributions with s_{ij} degrees of freedom. The consequences of replacing the assumption of Section 3.4, that the a_{ij}'s have exponential distributions with the assumption that they have χ^2 distributions are investigated below.

4.4.1 Case 5: Some of the a_{ij}'s have chi-square distributions

We assume that, a_{ij} has a χ^2 distribution with s_{ij} degrees of freedom and that its density function is given by

$$f(a_{ij}) = \frac{1}{\Gamma \frac{s_{ij}}{2}} \left(\frac{1}{2} \right)^{\frac{s_{ij}}{2}} a_{ij}^{\frac{a_{ij}}{2} - 1} e^{-\frac{a_{ij}}{2}} \qquad a_{ij} \geq 0 \qquad (4.15)$$

$$i = 1,2,\ldots,m,m+1,\ldots,M$$
$$j = 1,2,\ldots,n \text{ and } n < N$$

Since the quantities a_{ij} for $i = 1,2,\ldots,M$, $n = 1,2,\ldots,n$ $(n < N)$ are χ^2 random variables, the goals:

$$\sum_{j=1}^{N} a_{ij}x_j \le b_i \qquad i = 1,2,\ldots,m \qquad (4.16)$$

$$\sum_{j=1}^{N} a_{ij}x_j \ge b_i \qquad i = m+1,m+2,\ldots,M \qquad (4.17)$$

will be replaced by the following chance-goal set

$$P_r\left(\sum_{j=1}^{N} a_{ij}x_j \le b_i\right) = \gamma_i \qquad i = 1,2,\ldots,m \qquad (4.18)$$

$$P_r\left(\sum_{j=1}^{N} a_{ij}x_j \ge b_i\right) = \gamma_i \qquad i = m+1,m+2,\ldots,M \qquad (4.19)$$

where

b_i and a_{ij} for $i = 1,2,\ldots,m,m+1,\ldots,M$, $j = n+1,n+2,\ldots,N$.

are constants;

x_j for $j = 1,2,\ldots,N$ are decision variables; and

γ_i for $i = 1,2,\ldots,m,m+1,\ldots,M$ are preassigned constants

where $0 \le \gamma_i \le 1$.

Goals (4.18), (4.19) are equivalent to:

$$1 - P_r\left(\sum_{j=1}^{n} a_{ij}x_j \ge b_i - \sum_{j=n+1}^{N} a_{ij}x_j\right) = \gamma_i \qquad i = 1,2,\ldots,m \qquad (4.20)$$

$$P_r\left(\sum_{j=1}^{m} a_{ij}x_j \ge b_i - \sum_{j=n+1}^{N} a_{ij}x_j\right) = \gamma_i \qquad i = m+1,m+2,\ldots,M \qquad (4.21)$$

let $s_{ij} = 2g_{ij}$ [1] (4.22)

[1] When s_{ij} is not an even integer, it may be approximated by an even integer [68] .
For applied problems, if s_{ij} is odd, it can be approximated by $s_{ij}-1$ or $s_{ij}+1$ and the choice between $s_{ij}-1$ and $s_{ij}+1$ is closely related to tests of hypotheses and significance levels of the mean s_{ij} of a_{ij} .

where g_{ij} is an integer number.

Chance-goal set (4.20), (4.21) can be transformed to a deterministic goal set by applying theorem 3.1, page 31 as follows:

$$1 - \sum_{j=1}^{n} \sum_{t=1}^{g_{ij}} \eta_{ijt} \, P_r(\chi^2(2t) \geq \frac{b_i - \sum_{j=n+1}^{N} a_{ij}x_j}{x_j}) = \gamma_i$$

$$i=1,2,\ldots,m \qquad (4.23)$$

$$\sum_{j=1}^{n} \sum_{t=1}^{g_{ij}} \eta_{ijt} \, P_r(\chi^2(2t) \geq \frac{b_i - \sum_{j=n+1}^{N} a_{ij}x_j}{x_j}) = \gamma_i$$

$$i=m+1,m+2,\ldots,M \qquad (4.24)$$

or

$$1 - \sum_{j=1}^{n} \sum_{g_{ij}-h=1}^{g_{ij}} \eta_{ij(g_{ij}-h)} \, P_r\{\chi^2(2(g_{ij}-h)) \geq x_j^{-1} b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij}x_j\} = \gamma_i$$

$$i=1,2,3,\ldots,m \qquad (4.25)$$

$$\sum_{j=1}^{n} \sum_{g_{ij}-h=1}^{g_{ij}} \eta_{ij(g_{ij}-h)} \, P_r\{\chi^2(2(g_{ij}-h)) \geq x_j^{-1} b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij}x_j\} = \gamma_i$$

$$i=m+1,m+2,\ldots,M \qquad (4.26)$$

where

$$\eta_{ij(g_{ij}-h)} = ({}'\mu_{ij(h)} / h!) \, \eta_{ij(g_{ij})} \qquad h \geq 0 \qquad (4.27)$$

$$\eta_{ij(g_{ij})} = \prod_{d \neq j}^{n} \left(\frac{x_j}{x_j - x_d} \right)^{g_{ij}} \qquad i=1,2,\ldots,m,m+1,\ldots,M \quad j=1,2,\ldots,n \qquad (4.28)$$

and

$$P_r\left\{\chi^2(2(g_{ij}-h) \geq x_j^{-1}b_i - x_j^{-1}\sum_{j=n+1}^{N} a_{ij}x_j\right\} =$$

$$\frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!}\left\{e^{-\frac{1}{2}(x_j^{-1}b_i-x_j^{-1}\sum_{j=n+1}^{N}a_{ij}x_j)}\left[\right.\right.$$

$$(x_j^{-1}b_i - x_j^{-1}\sum_{j=n+1}^{N} a_{ij}x_j)^{g_{ij}-h-1}$$

$$+\sum_{t=1}^{g_{ij}-h-1} 2^t(g_{ij}-h-1)(g_{ij}-h-2)\ldots(g_{ij}-h-t)(x_j^{-1}b_i$$

$$\left.\left.- x_j^{-1}\sum_{j=n+1}^{N} a_{ij}x_j)^{g_{ij}-h-t-1}\right]\right\} \qquad (4.29)$$

(see Appendix B).

Taking account of (4.27), (4.28) and (4.29), the equations (4.25) and (4.26) yield:

$$1 - \sum_{j=1}^{n}\sum_{g_{ij}-h=1}^{g_{ij}}\frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!}\left[({}'\mu_{ij(h)} / h!)\prod_{d\neq j}^{n}(1-\frac{x_d}{x_j})^{-g_{ij}}\right]\left[\right.$$

$$e^{-\frac{1}{2}(x_j^{-1}b_i-\sum_{j=n+1}^{N}a_{ij}x_j)}\left((x_j^{-1}b_i-x_j^{-1}\sum_{j=n+1}^{N}a_{ij}x_j)^{g_{ij}-h-1}\right.$$

$$+\sum_{t=1}^{g_{ij}-h-1} 2^t(g_{ij}-h-1)(g_{ij}-h-2),\ldots,(g_{ij}-h-t)(x_j^{-1}b_i -$$

$$\left.\left. x_j^{-1}\sum_{j=n+1}^{N}a_{ij}x_j)^{g_{ij}-h-t-1}\right)\right] = \gamma_i$$

$$i=1,2,\ldots,m \qquad (4.30)$$

and

$$\sum_{j=1}^{n} \sum_{g_{ij}-h=1}^{g_{ij}} \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!} \left[({}'\mu_{ij(h)} / h!) \prod_{d \neq j}^{n} (1 - \frac{x_d}{x_j})^{-g_{ij}} \right] \Big[$$

$$e^{-\frac{1}{2}(x_j^{-1} b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij} x_j)} \cdot ((x_j^{-1} b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij} x_j)^{g_{ij}-h-1} +$$

$$\sum_{t=1}^{g_{ij}-h-1} 2^t (g_{ij}-h-1)(g_{ij}-h-2)\ldots(g_{ij}-h-t)($$

$$x_j^{-1} b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij} x_j \,)^{g_{ij}-h-t-1} \quad) \Big] \quad = \gamma_i$$

$$i=m+1,m+2,\ldots,M \qquad (4.31)$$

Goals (4.30), (4.31) can be formulated in standard form for goals by adding the probabilistic deviational variables d_i^-, d_i^+ for $i = 1,2,\ldots,m,m+1,\ldots,M$, where d_i^-, d_i^+ are defined in the same way and have the same probabilistic interpretation as in Case 2.

When this is done, we have

$$1 - \sum_{j=1}^{n} \sum_{g_{ij}-h=1}^{g_{ij}} \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!} \left[({}'\mu_{ij(h)} / h!) \prod_{d \neq j}^{n} (1 - \frac{x_d}{x_j})^{-g_{ij}} \right] \Big[$$

$$e^{-\frac{1}{2}(x_j^{-1} b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij} x_j)} ((x_j^{-1} b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij} x_j)^{g_{ij}-h-1} +$$

$$\sum_{t=1}^{g_{ij}-h-1} 2^t (g_{ij}-h-1)(g_{ij}-h-2)\ldots(g_{ij}-h-t)($$

$$x_j^{-1} b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij} x_j)^{g_{ij}-h-t-1} \quad) \Big] + d_i^- - d_i^+ \; = \; \gamma_i$$

$$i=1,2,\ldots,m \qquad (4.32)$$

and

$$\sum_{j=1}^{n} \sum_{g_{ij}-h=1}^{g_{ij}} \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!} \left[('\mu_{ij(h)} / h!) \prod_{d \neq j}^{n} (1 - \frac{x_d}{x_j})^{-g_{ij}} \right] \Big[$$

$$e^{-\frac{1}{2}(x_j^{-1} b_i - \sum_{j=n+1}^{N} a_{ij} x_j)} \left((x_j^{-1} b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij} x_j)^{g_{ij}-h-1} \right. +$$

$$\sum_{t=1}^{g_{ij}-h-1} 2^{t} (g_{ij}-h-1)(g_{ij}-h-2)\ldots(g_{ij}-h-t)($$

$$\left. x_j^{-1} b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij} x_j)^{g_{ij}-h-t-1} \right) \Big] + d_i^{-} - d_i^{+} = \gamma_i$$

$$i = m+1, m+2, \ldots, M \qquad (4.33)$$

where the $'\mu_{ig(h)}$ are determined using the cumulants $k_{ij(h)}$,

$$k_{ij(h)} = (h-1)! \sum_{d \neq j}^{n} \left[g_{ij} (1 - \frac{x_j}{x_d})^{-h} \right] \qquad h > 0 \qquad (4.34)$$

(see theorem 3.1)

and

$$'\mu_{ij(o)} = 1 \qquad i = 1, 2, \ldots, m, m+1, \ldots, M \qquad (4.35)$$
$$j = 1, 2, \ldots, n$$

Hence, we can determine the optimum values of the x's namely those which satisfy the goals (4.16), (4.17) to the fullest possible extent according to their priorities with probabilities greater than or equal to preassigned values (see Fourth, page 35), by solving the following transformed deterministic goal program:

Find $x = (x_1, x_2, \ldots, x_N)$

so as to

$$\text{lexico-min } a = \left\{ [g_1(d^-)], [g_2(d^-)], \ldots, [g_k(d^-)], \ldots [g_K(d^-)] \right\}$$

$$K \leq M \qquad (4.36)$$

subject to

$$1 - \sum_{j=1}^{N} \sum_{g_{ij}-h-1}^{g_{ij}} \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!} \left[({}'\mu_{ij(h)} / h!) \prod_{d \neq j}^{n} (1 - \frac{x_d}{x_j})^{-g_{ij}} \right] \Bigg[$$

$$e^{-\frac{1}{2}(x_j^{-1}b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij}x_j)} \left((x_j^{-1}b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij}x_j)^{g_{ij}-h-t-1} \right) +$$

$$\sum_{t=1}^{g_{ij}-h-1} 2^t (g_{ij}-h-1)(g_{ij}-h-2)\ldots(g_{ij}-h-t)($$

$$x_j^{-1}b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij}x_j)^{g_{ij}-h-t-1}) \Bigg] + d_i^- - d_i^+ = \gamma_i$$

$$i=1,2,\ldots,m \qquad (4.37)$$

$$\sum_{j=1}^{n} \sum_{g_{ij}-h=1}^{g_{ij}} \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!} \left[({}'\mu_{ij(h)} / h!) \prod_{d \neq j}^{n} (1 - \frac{x_d}{x_j})^{-g_{ij}} \right] \Bigg[$$

$$e^{-\frac{1}{2}(x_j^{-1}b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij}x_j)} \left((x_j^{-1}b_i - x_j^{-1} \sum_{j=n+1}^{N} a_{ij}x_j)^{g_{ih}-h-1} \right. +$$

$$\sum_{t=1}^{g_{ij}-h-1} 2^t (g_{ij}-h-1)(g_{ij}-h-2)\ldots(g_{ij}-h-t)($$

$$x_j^{-1}b_i - x_j^{-1} \sum_{n=n+1}^{N} a_{ij}x_j)^{g_{ij}-h-t-1}) \Bigg] + d_i^- - d_i^+ = \gamma_i$$

$$i=m+1,m+2,\ldots,M \qquad (4.38)$$

$$x_j \geq 0 \qquad j=1,2,\ldots,N \qquad (4.39)$$

$$0 \leq d_i^-,\ d_i^+ \leq 1\ ,\ d_i^- \cdot d_i^+ = 0 \qquad i=1,2,\ldots,m,m,\ldots,M \qquad (4.40)$$

and

$$g_k(d^-) = \sum_{i \in P_k} d_i^- \qquad i=1,2,\ldots,m,m+1,\ldots,M \qquad (4.41)$$

$$k=1,2,\ldots,K$$

4.4.2 The Equivalent Signomial Program

Constraints (4.37), (4.38) are very complicated non-linear constraints but they can be transformed to standard signomial form using the same method that was used in subsection 3.4.2. When this is done, program (4.36)-(4.40) is equivalent to the following signomial program:

Find $x = (x_1, x_2, \ldots, x_N)$

so as to

$$\text{lexico-min} \quad a = \left\{[g_1(d^-)],[g_2(d^-)],\ldots,[g_k(d^-)],\ldots,[g_K(d^-)]\right\} \qquad K \le M \tag{4.42}$$

subject to

$$\gamma_i^{-1} - \gamma_i^{-1}\left\{\sum_{j=1}^{n}\sum_{g_{ij}-h=1}^{g_{ij}} \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!}\left[\left({}'\mu_{ij(h)}/h!\right)\prod_{d\neq j}^{n}\left(1-\frac{x_d}{x_j}\right)^{-g_{ij}}\right]\left[\beta_{ij}^{\phi}\left(z_{ij}^{g_{ij}-h-1} + \sum_{t=1}^{g_{ij}-h-1} 2^t (g_{ij}-h-1)(g_{ij}-h-2)\ldots(g_{ij}-h-t)\, z_{ij}^{g_{ij}-h-t-1}\right)\right]\right\} + \gamma_i^{-1}d_i^- - \gamma_i^{-1}d_i^+ = 1$$

$$i=1,2,\ldots,m \tag{4.43}$$

$$\left.\begin{array}{ll} b_i^{-1}x_j\, z_{ij} + b_i^{-1}\sum_{j=n+1}^{N} a_{ij}x_j = 1 & (4.44) \\ \beta_{ij} + \frac{1}{2} z_{ij}\,\phi^{-1} = 1 & (4.45) \end{array}\right\} \quad i=1,2,\ldots,m;\ j=1,2,\ldots,n$$

$$\gamma_i^{-1}\left\{\sum_{j=1}^{n}\sum_{g_{ij}-h=1}^{g_{ij}} \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!}\left[\left({}'\mu_{ij(h)}/h!\right)\prod_{d\neq j}^{n}\left(1-\frac{x_d}{x_j}\right)^{-g_{ij}}\right]\left[{}'\beta_{ij}^{\phi}\left({z'}_{ij}^{g_{ij}-h-1} + \sum_{t=1}^{g_{ij}-h-1} 2^t (g_{ij}-h-1)(g_{ij}-h-2)\ldots(g_{ij}-h-t)\, {z'}_{ij}^{g_{ij}-h-t-1}\right)\right]\right\} + \gamma_i^{-1}d_i^- - \gamma_i^{-1}d_i^+ = 1$$

$$i=m+1,m+2,\ldots,M \tag{4.46}$$

$$\left.\begin{array}{ll} b_i^{-1}x_j\ z'_{ij} + b_i^{-1}\sum_{j=n+1}^{N} a_{ij}x_j - b_i^{-1}c\,x_j = 1 & (4.47) \\ \beta'_{ij} + \frac{1}{2} z'_{ij}\ \phi^{-1} = 1 & (4.48) \end{array}\right\}\quad \begin{array}{l} i=m+1,m+2,\ldots,M \\ j=1,2,\ldots,n \end{array}$$

$$x_j,\ z_{ij},\ z'_{ij},\ \beta_{ij},\ \beta'_{ij} \geq 0 \qquad \begin{array}{l} i=1,2,\ldots,m,m+1,\ldots,M \\ j=1,2,\ldots,N \end{array} \qquad (4.49)$$

$$0 \leq d_i^-,\ d_i^+ \leq 1 \qquad d_i^-.d_i^+ = 0 \qquad i=1,2,\ldots,m,m+1,\ldots,M \qquad (4.50)$$

where

$$\left.\begin{array}{ll} z_{ij} = \dfrac{b_i - \sum_{j=n+1}^{N} a_{ij}x_j}{x_j} & (4.51) \\ \beta_{ij} = 1 - \frac{1}{2} z_{ij}\ \phi^{-1} & (4.52) \end{array}\right\}\quad \begin{array}{l} i=1,2,\ldots,m \\ j=1,2,\ldots,n \end{array}$$

$$\left.\begin{array}{ll} z'_{ij} = \dfrac{b_i - \sum_{j=n+1}^{N} a_{ij}x_j}{x_j} + c & (4.53) \\ \beta'_{ij} = 1 - \frac{1}{2} z'_{ij}\ \phi^{-1} & (4.54) \end{array}\right\}\quad \begin{array}{l} i=m+1,m+2,\ldots,M \\ j=1,2,\ldots,n \end{array}$$

c is a large positive constant and $\phi \to \infty$.

Constraints (4.44), (4.45), (4.47) and (4.48) are in standard signomial form [1]. Also, on substituting in (4.43) and (4.46) for $'\mu_{ij(h)}$ as functions of x_j and carrying out the summation in the left hand side, the constraints (4.43) and (4.46) are also seen to be in standard signomial form.

---

[1] (4.44), (4.45) and (4.48) are in posynomial form (see definition 5.3).

It is worth noting that the constraint set (4.43) - (4.48) have the same properties as those of the constraint set (3.108)-(3.115) stated in page 39 . Also the above signomial program can be solved by the algorithm presented in Section 5.8.

4.4.3 Case 6: All a_{ij}'s Have Chi-square Distributions

To consider the particular case when all a_{ij}'s for $i = 1,2,\ldots,m,m+1,\ldots,M$; $j = 1,2,\ldots,n,n+1,\ldots,N$, have χ^2 distributions, we note that this case is equivalent to Case 5 with $n = N$. Hence the transformed deterministic goal program is:

Find $x = (x_1,x_2,\ldots,x_N)$

so as to

$$\text{lexico-min} \quad a = \left\{[g_1(d^-)],[g_2(d^-)],\ldots,[g_k(d^-)], \ldots,[g_K(d^-)]\right\}$$

$$K \le M \tag{4.55}$$

subject to

$$1 - \left\{ \sum_{j=1}^{N} \sum_{g_{ij}-h=1}^{g_{ij}} \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!} \left[({}^{\prime}\mu_{ij(h)} / h!) \prod_{d \neq j}^{N} (1 - \frac{x_d}{x_j})^{-g_{ij}} \right] \left[e^{-\frac{1}{2}b_i x_j^{-1}} \left((b_i - x_j^{-1})^{g_{ij}-h-1} + \sum_{t=1}^{g_{ij}-h-1} 2^t (g_{ij}-h-1)(g_{ij}-h-2)\ldots \right.\right.\right.$$

$$\left.\left.\left. \ldots (g_{ij}-h-t)(g_i x_j^{-1})^{g_{ij}-h-t-1} \right) \right] \right\} + d_i^- - d_i^+ = \gamma_i$$

$$i=1,2,\ldots,m \tag{4.56}$$

$$\Big\{ \sum_{j=1}^{N} \sum_{g_j-h=1}^{g_{ij}} \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!} \quad C'^{\mu}_{ij(h)} / h! \prod_{d\neq j}^{N} (1 - \frac{x_d}{x_j})^{-g_{ij}} \Big] \Big[$$

$$e^{-\frac{1}{2}b_i x_j^{-1}} \Big((b_i x_j^{-1})^{g_{ij}-h-1} + \sum_{t=1}^{g_{ij}-h-1} 2^t (g_{ij}-h-1)(g_{ij}-h-2)\,..$$

$$..(g_{ij}-h-t)(b_i x_j^{-1})^{g_{ij}-h-t-1} \Big) \Big] \Big\} + d_i^- - d_i^+ = \gamma_i$$

$$i=m+1,m+2,\ldots,M \qquad (4.57)$$

$$x_j \geq 0 \qquad j=1,2,\ldots,N \qquad (4.58)$$

$$0 \leq d_i^-,\ d_i^+ \leq 1,\ d_i^-.d_i^+ = 0 \qquad i=1,2,\ldots,m,m+1,\ldots,M \qquad (4.59)$$

and

$$g_k(d^-) = \sum_{i\in P_k} d_i^- \qquad i=1,2,\ldots,m,m+1,\ldots,M \qquad (4.60)$$

$$k=1,2,\ldots,K$$

By applying the set of transformations set out in sub section 4.4.2., the above program is equivalent to the following signomial program:

Find $x = (x_1, x_2, \ldots, x_N)$

so as go

lexico-min $a = \{[g_1(d^-)],[g_2(d^-)],\ldots,[g_k(d^-)],\ldots,[g_K(d^-)]\}$

$$K \leq M \qquad (4.61)$$

subject to

$$\gamma_i^{-1} - \gamma_i^{-1} \Big(\sum_{j=1}^{N} \sum_{g_{ij}-h=1}^{g_{ij}} \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!} \Big[C'^{\mu}_{ij(h)} / h!) \prod_{d\neq j}^{N} (1 - \frac{x_d}{x_j})^{-g_{ij}} \Big] \Big[$$

$$\beta_{ij}^{\phi} \Big((b_i x_j^{-1})^{g_{ij}-h-1} + \sum_{t=1}^{g_{ij}-h-1} 2^t (g_{ij}-h-1)(g_{ij}-h-2)\,..$$

$$..(g_{ij}-h-t)(b_i x_j^{-1})^{g_{ij}-h-t-1} \Big) \Big] \Big\} + \gamma_i^{-1} d_i^- - \gamma_i^{-1} d_i^+ = 1$$

$$i=1,2,\ldots,m \qquad (4.62)$$

$$\gamma_i^{-1} \left\{ \sum_{j=1}^{N} \sum_{g_{ij}-h=1}^{g_{ij}} \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!} \left[(\mu'_{ij(h)} / h!) \prod_{d \neq j}^{N} (1 - \frac{x_d}{x_j})^{-g_{ij}} \right] \right[$$

$$\beta_{ij}^{\phi} \; (\; (b_i x_j^{-1})^{g_{ij}-h-1} + \sum_{t=1}^{g_{ij}-h-1} 2^t (g_{ij}-h-1)(g_{ij}-h-2) \; ..$$

$$..(g_{ij}-h-t)(b_i x_j^{-1})^{g_{ij}-h-t-1} \;) \Big] \Big\} + \gamma_i^{-1} d_i^- - \gamma_i^{-1} d_i^+ = 1$$

$$i=m+1, m+2, \ldots, M \qquad (4.63)$$

$$\beta_{ij} + \frac{1}{2} x_j^{-1} \phi^{-1} = 1 \qquad (4.64)$$

$$x_j, \; \beta_{ij} \geq 0 \qquad i=1,2,\ldots,m,m+1,\ldots,M \quad j=1,2,\ldots,N \qquad (4.65)$$

$$0 \leq d_i^-, \; d_i^+ \leq 1, \; d_i^- . d_i^+ = 0 \qquad (4.66)$$

where

$$g_k(d^-) = \sum_{i \in P_k} d_i^- \qquad i=1,2,\ldots,m,m+1,\ldots,M \quad k=1,2,\ldots,K \qquad (4.67)$$

$$\beta_{ij} = 1 - \frac{1}{2} b_i x_j^{-1} \phi^{-1} \qquad i=1,2,\ldots,m,m+1,\ldots,M \quad j=1,2,\ldots,N \qquad (4.68)$$

(see Appendix A)

and $\phi \to \infty$.

4.5 The Approximate Distribution of $\sum_{j=1}^{n} a_{ij} x_j$

In the previous section (Cases 5 and 6) our transformed deterministic goal programs were obtained by using the exact distributions of the variables $\sum_j a_{kj} x_j$, $k = 1,2,\ldots,m,m+1,\ldots,M$ where $a_{kj} \sim \chi^2(s_{kj})$ and the x_j are decision variables. In this section, we prove that Sengupta's transformation [62,70,67] in which the exact distribution of $\sum_j a_{kj}x_j$ is approximated by a central χ^2 distribution whose first two moments agree with those of the distribution of $\sum_j a_{kj}x_j$, does not lead to a solvable program because the parameters of the approximate distribution depend on the decision variables x_j as will be shown below.

Since $a_{kj} \sim \chi^2(s_{kj})$

and $\quad E(a_{kj}) = s_{kj} \quad , \quad \text{variance } (a_{kj}) = 2\, s_{kj} \qquad (4.69)$

then a_{kj} can be written as the square of a normally distributed variable [42, page 380]:

$$a_{kj}x_j = \eta^2_{kj} = (n_{kj}r_j)^2 , \quad \text{where} \quad r_j = \sqrt{x_j} \geq 0 \qquad (4.70)$$

$$k=1,2,\ldots,m,m+1,\ldots,M$$
$$j=1,2,\ldots,n$$

and the n_{kj} are independent normal variables with finite means and variances, then the input coefficient a_{kj} has the distribution of n^2_{kj} for $k=1,2,\ldots,m,m+1,\ldots,M$; $j=1,2,\ldots,n$.

Since the r_j's are non-stochastic decision variables and the n_{kj}'s are assumed to be independent, then the η_{kj} are independent normal variables with expectations m_{kj} and variances v_{kj} , i.e.

$$\begin{aligned} m_{kj} &= E(\eta_{kj}) = \sqrt{x_j}\, E(m_{kj}) \\ &= \sqrt{x_j}\, A_{kj} \end{aligned} \qquad (4.71)$$

$$\begin{aligned} v_{kj} &= \text{variance } (\eta_{kj}) = E(\eta_{kj} - m_{kj})^2 = x_j \text{ variance } (n_{kj}) \\ &= x_j\, B_{kj} \end{aligned} \qquad (4.72)$$

A_{kj} and B_{kj} are constants such that:

$$A_{kj} \approx (4\, s_{kj}^4 - s_{kj}^3 - 28\, s_{kj}^2 + 10\, s_{kj} + 42) \,/\, 4\, s_{kj}^3 \sqrt{s_{kj}}$$

$$B_{kj} \approx s_{kj} - [(4\, s_{kj}^4 - s_{kj}^3 - 28\, s_{kj}^2 + 10\, s_{kj} + 42) \,/\, 4\, s_{kj}^3 \sqrt{s_{kj}}\,]^2$$

(see Appendix C).

We have now

$$P_k = \sum_j \bar{a}_{kj} \bar{x}_j = \sum_{j=1}^{n} \psi_{kj}^2 = \sum_{j=1}^{n} V_{kj}(q_{kj} + \bar{m}_{kj})^2 \tag{4.73}$$

$k=1,2,\ldots,m,m+1,\ldots,M$

$j=1,2,\ldots,n$

where

$$\bar{m}_{kj} = m_{kj} \,/\, \sqrt{V_{kj}} = A_{kj} \,/\, \sqrt{B_{kj}} \tag{4.75}$$

$k=1,2,\ldots,m,m+1,\ldots,M$

$j=1,2,\ldots,n$

$$q_{kj} = (\psi_{kj} - m_{kj}) \,/\, \sqrt{V_{kj}} \tag{4.76}$$

$k=1,2,\ldots,m,m+1,\ldots,M$

$j=1,2,\ldots,n$

We note that q_{kj} follows the standard normal distribution (i.e., $q_{kj} \sim N(0,1)$) .

Hence, the characteristic function of P_k $\phi_{P_k}(t)$ is given by:

$$\phi_{P_k}(t) = E(e^{itP_k}) = \prod_{j=1}^{n} [(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{itV_{kj}(q_{kj}+\bar{m}_{kj})^2 - \frac{1}{2} q_{kj}^2} \, d\, q_{kj}\,] \tag{4.77}$$

since the integral in (4.77) is equal to

$$[2\pi/(1 - 2it\, V_{kj}\,)]^{\frac{1}{2}} \; e^{it\, V_{kj}\, \bar{m}_{kj}^2 \,/(1-2it\, V_{kj}\,)} \tag{4.78}$$

then:

$$\phi_{P_k}(t) = \left[\prod_{j=1}^{n} (1 - 2it\, V_{kj})^{-\frac{1}{2}} \right] e^{\sum_{j=1}^{n} \frac{it\, V_{kj}\, \bar{m}_{kj}^2}{(1 - 2it\, V_{kj})}}$$

$$i = \sqrt{-1} \qquad (4.79)$$

From (4.79) all the moments of the distribution of P_k can be derived since the T-moment ($'\mu_T$) is [42] :

$$'\mu_T = (-i)^T \left[\frac{d^T\, \Phi(t)}{d\, t^T} \right]_{t=o} \qquad (4.80)$$

We note that the characteristic function (4.79) of P_k is closely related to that of a non-central χ^2 distributed variable [41] .

Sengupta suggested approximating P_k by a central χ^2 distributed variable P_k' using the first two moments from the characteristic function (4.79).

$$'\mu_1 = \text{mean}(P_k) = \sum_{j=1}^{n} V_{kj} + V_{kj}\, \bar{m}_{kj}^2$$

$$= \sum_{j=1}^{n} x_j \, (A_{kj}^2 + B_{kj}) \qquad (4.81)$$

$$'\mu_2 = \text{variance}\ (P_k) = 2 \sum_{j=1}^{n} V_{kj}^2 + 2\, V_{kj}^2\, \bar{m}_{kj}^2$$

$$= 2 \sum_{j=1}^{n} x_j^2 \, (2\, A_{kj}^2\, B_{kj} + B_{kj}^2) \qquad (4.82)$$

if we define the variable P_k' such that,

$$P_k' \sim \chi^2(s_k)$$

then the first two moments of P_k' are:

$$'\mu_1 = s_k \quad . \quad '\mu_2 = 2\, s_k \qquad (4.83)$$

By equating the first two moments of P_k with those of $\rho_k P_k'$, where ρ_k is to be determined, we have from (4.81), (4.82) and (4.83)

$$\sum_{j=1}^{n} V_{kj} + V_{kj}\,\bar{m}_{kj}^2 = \rho_k s_k , \tag{4.84}$$

$$2 \sum_{j=1}^{n} V_{kj}^2 + 2\,V_{kj}^2\,\bar{m}_{kj}^2 = 2\,\rho_k^2 s_k \tag{4.85}$$

Hence P_k/ρ_k is approximately $\chi^2(s_k)$ with

$$\rho_k = \frac{\sum_{j=1}^{n} V_{kj}^2 + 2\,V_{kj}^2\,\bar{m}_{kj}^2}{\sum_{j=1}^{n} V_{kj} + V_{kj}\,\bar{m}_{kj}^2} = \frac{\sum_{j=1}^{n} x_j^2(2\,A_{kj}^2\,B_{kj} + B_{kj}^2)}{\sum_{j=1}^{n} x_j(A_{kj}^2 + B_{kj})} \tag{4.86}$$

$$s_k = \frac{(\sum_{j=1}^{n} V_{kj} + V_{kj}\,\bar{m}_{kj}^2)^2}{(\sum_{j=1}^{n} V_{kj}^2 + 2\,V_{kj}^2\,\bar{m}_{kj}^2)} = \frac{\left[\sum_{j=1}^{n} x_j(A_{kj}^2 + B_{kj})\right]^2}{\left[\sum_{j=1}^{n} x_j^2(2\,A_{kj}^2\,B_{kj} + B_{kj}^2)\right]} \tag{4.87}$$

From (4.87), we find that s_k (the parameter of the approximate distribution of $P_k = \sum_{j=1}^{n} a_{kj}x_k$) is a function of the unknown decision variables x_j, $j=1,2,\ldots,n$ and consequently, it is impossible to transform the chance-goal set (4.18), (4.19) into deterministic goals by using the above approximation.

4.6 A Numerical Example

Suppose we want to determine x_1, x_2 satisfying to the fullest possible extent

$$a_{11}\,x_1 + a_{12}\,x_2 \quad \le 20 \tag{4.88}$$

$$x_1 + x_2 \qquad \ge b_2 \tag{4.89}$$

with probabilities greater than or equal to:

$$\gamma_1 = .75 \quad , \quad \gamma_2 = .50 \quad \text{respectively,}$$

where a_{11}, a_{12}, b_2 have $\chi^2(2)$, $\chi^2(4)$, $\chi^2(10)$ distributions respectively and goal (4.88) has first priority and (4.89) has second priority.

Solution

Step 1

transform probabilistic goals (4.88), (4.89) to deterministic goals in standard form as follows:

1) From (4.56) the following goal corresponds to goal (4.88):

$$1 - \left\{ \left(\frac{x_1}{x_1-x_2} \right) e^{-10/x_1} + \left(\frac{2x_1}{x_1-x_2}\right)\left(\frac{x_2}{x_2-x_1}\right)^2 e^{-10/x_2} + \left(\frac{x_2}{x_2-x_1}\right)^2 e^{-10/x_2} \left(1 + \frac{10}{x_2}\right) \right\} + d_1^- - d_1^+ = .75 \tag{4.90}$$

where

$$\mu_{11(0)} = {}'\mu_{12(0)} = 1 \tag{4.91}$$

and

$$'\mu_{12(1)} = K_{12(1)} = (1-1)! \sum_{d\neq 2}^{2} g_{12} \left(\frac{x_1-x_2}{x_1} \right)^{-1}$$

$$= 2\left(\frac{x_1}{x_1-x_2} \right) \tag{4.92}$$

2) From (4.9) the following goal corresponds to goal (4.89):

$$x_1 + x_2 + d_2^- - d_2^+ = F^{-1}(.50) = 9.34 \tag{4.93}$$

Step 2

since the goal (4.88) has the first priority and (4.89) has the second priority, then our transformed deterministic goal program is:

Find $x = (x_1, x_2)$

so as to

$$\text{lexico-min: } a = \{ (d_1^-), (d_2^-) \} \tag{4.94}$$

subject to

$$1 - \left\{ \left(\frac{x_1}{x_1 - x_2}\right) e^{-10/x_1} + \left(\frac{2x_1}{x_1 - x_2}\right) \left(\frac{x_2}{x_2 - x_1}\right)^2 e^{-10/x_2} + \left(\frac{x_2}{x_2 - x_1}\right)^2 e^{-10/x_2} \left(1 + \frac{10}{x_2}\right) \right\} + d_1^- - d_1^+ = .75 \tag{4.95}$$

$$x_1 + x_2 + d_2^- - d_2^+ = 9.34 \tag{4.96}$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+ \geq 0 \tag{4.97}$$

Step 3

from sub-section 4.4.3, the following signomial goal program is equivalent to program (4.94)-(4.97):

Find $x = (x_1, x_2)$

so as to

$$\text{lexico-min } a = \{ (d_1^-), (d_2^-) \} \tag{4.98}$$

subject to

$$1 - \left\{ \beta_{11}^{\phi} \left(\frac{x_1}{x_1 - x_2}\right) + \beta_{12}^{\phi} \left(\frac{x_2}{x_1 - x_2}\right)^2 \left[\left(\frac{2x_1}{x_1 - x_2}\right) + \left(1 + \frac{10}{x_2}\right) \right] \right\} + d_1^- - d_1^+ = .75 \tag{4.99}$$

$$\beta_{11} + 10\ \phi^{-1}\ x_1^{-1} = 1 \tag{4.100}$$

$$\beta_{12} + 10\ \phi^{-1}\ x_2^{-1} = 1 \tag{4.101}$$

$$x_1 + x_2 + d_2^- - d_2^+ = 9.34 \tag{4.102}$$

$$x_1,\ x_2,\ d_1^-,\ d_1^+,\ d_2^-,\ d_2^+,\ \beta_{11},\ \beta_{12} \geq 0 \tag{4.103}$$

and $\phi \to \infty$.

Step 4

using the algorithm presented in Section 5.8; the global solution is:

$x_1 = 3.34$ $\qquad$ $x_2 = 6$

$d_1^- = 0$ $\qquad$ $d_1^+ = .19$

$d_2^- = 0$ $\qquad$ $d_2^+ = 0$

The detailed solution is given in Appendix E.

4.7 Conclusion

Using the method presented in Chapter 3, in this chapter, we have presented:

(1) the transformed deterministic linear goal program when the right hand side coefficients of the goals have χ^2 distributions,

(2) the transformed deterministic non-linear goal programs when some or all of the input coefficients have χ^2 distributions,

(3) the signomial programs equivalent to the transformed deterministic non-linear goal programs,

(4) a numerical example to illustrate the various steps in arriving at the transformed deterministic goal program and transforming it to the equivalent signomial program when the goal set contains a mix of probabilistic goals (see Section 3.6),

(5) Sengupta's transformation to obtain an approximate distribution for $\sum_{j=1}^{n} a_{ij}x_j$ when $a_{ij} \sim \chi^2$, and proved that this transformation does not lead to a solvable program.

CHAPTER 5

NONLINEAR GOAL PROGRAMMING

5.1 Introduction

It was shown in Chapters 3 and 4 that CCGP study is closely related to nonlinear GP. As yet, there are no special nonlinear programming methods for solving nonlinear goal programs. The field of nonlinear programming has concentrated on the solution of problems with a single objective function. Additionally, there is, in general, no way to guarantee finding the global optimum for a given problem unless that problem is of a very special form. Experience in single objective nonlinear programming has indicated that [37] :

(i) a particular method may perform well on one problem but poorly on a slight modification of that problem;

(ii) the results obtained by any method are highly dependent on the starting point or points used to initiate the search;

(iii) one can only hope to obtain a local optimum unless the problem is of a very special form.

The only attempt to employ the methods for nonlinear single objective programs to solve nonlinear goal programs was presented by Ignizio (see next section).

In this chapter we employ, for the first time, a condensed geometric programming technique [3] to solve nonlinear goal programs.

The formulation of subprograms of a goal program as generalized geometric programs, and a "sequential double condensed geometric goal programming" algorithm are presented in Section 5.7 and 5.8 respectively. An illustrative numerical example which demonstrates the formulation and the procedures of the algorithm is presented in Section 5.9. Our algorithm is constructed by combining a "sequential goal programming" algorithm (which was given in Section 1.3) with a "double condensed geometric programming algorithm (which is given in Section 5.6). Therefore, the condensed algorithms are necessary for the double condensed geometric algorithm given in Section 5.5.

The effective factors which lead us to use a double condensed algorithm are given in Section 5.3. Also, the fundamental concepts of the theory of geometric programming and the technique of condensation, which are the basis of the double condensed geometric algorithm, are given in Sections 5.3 and 5.4.

In the next section, modified Ignizio methods to solve nonlinear goal programs and their most important drawbacks are presented briefly. It will then be possible to compare those methods with the algorithm given in Section 5.8.

5.2 The Existing Modified Methods for Solving Nonlinear Goal Programs

Ignizio modified both the Griffith & Stewart and the pattern search techniques to solve nonlinear goal programs.

We present below a brief outline of these methods.

1. The modified Griffith & Stewart method [32, 4] :

This method is based on transforming a nonlinear function into a linear function by using the Taylor series of the function about a given point and ignoring all terms of higher order than the first, and then using the modified simplex algorithm [50,37] .

This method suffers from some of the drawbacks listed in Section 5.1 and, in addition, has a set of drawbacks peculiar to the method itself:

(1) as yet, there are no proofs of convergence to a local or a global solution when this method is used;

(2) the linear approximation, as mentioned above, is only a "good" approximation to the nonlinear function in the "neighborhood" about the starting point (initial point);

(3) one must employ either a numerical or an exact method of differentiation in the performance of the algorithm (which in turn implies that the problem must be amenable to such methods).

2. The modified search method [36, 4] :

This method avoids the third drawback of the above method. It is based on an extension of the search method of Hooke & Jeeves which is one of a class of search techniques known as accelerated search methods. Such methods increase their search step size if previous searches have been successful and maintain or decrease the step size otherwise. The pattern search method is based on constructing sequential patterns, which contain a number of trial points. In each trial we perturb each of the decision variables and evaluate the achievement function.

This method, also, suffers from drawbacks, the most important of these are:

(1) as yet, there are no proofs of convergence to a local or a global solution by this method;

(2) it depends on the perturbation step size, as yet there is no certain method of obtaining the best perturbation step size;

(3) there is no effective rule to terminate the search;

(4) if the starting point is a local optimal point, then the pattern search will not progress.

5.3 Geometric Programming

Geometric programming is considered a relatively new [1] technique, developed for solving nonlinear programming problems. Geometric programming algorithms have recently been improved so that they now provide powerful tools for solving nonlinear programming problems in general.

The original mathematical development of geometric programming used the arithmetic-geometric mean inequality relationship between sums and products of positive values [3]. This section provides the fundamental definitions and concepts of geometric programming theory, and a summary of the existing methods used in practice for solving generalized geometric programs.

---

[1] The first work on geometric programming was carried out by Zener in the early sixties, later generalized by Duffin, Peterson, Passy, Avriel, Dembo and others [29, 3, 24].

5.3.1 Definitions and Background

Definition 5.1: Feasible points or feasible solutions; feasible regions.

A feasible point or a feasible solution is a point that satisfies a particular set of constraints. The feasible region of a set of constraints is the set of its feasible points [4] .

Definition 5.2: Consistent constraints.

A set of constraints is said to be consistent if it has at least one feasible solution [23] .

Definition 5.3: Posynomial and signomial functions.

A real values positive function $p(x)$ is called a posynomial if it is given by:

$$\begin{aligned} p(x) &= \sum_{t=1}^{T} p_t(x) \\ &= \sum_{t=1}^{T} c_t \, x_1^{a_{t1}} \, x_2^{a_{t2}} \cdots x_N^{a_{tN}} \\ &= \sum_{t=1}^{T} c_t \prod_{j=1}^{N} x_j^{a_{tj}} \end{aligned} \tag{5.1}$$

and $x_j , c_t > 0$ $\quad j=1,2,\ldots,N$

$\quad t=1,2,\ldots,T$

where the terms $p_t(x)$ are called monomials or single-term posynomials, the exponents a_{tj} are arbitrary real constants and the coefficients c_t are positive constants.

When the coefficients c_t are not restricted to positive values, the above functions (5.1) are called signomials or generalized posynomials. A signomial may be considered as the difference between two posynomials [3] .

<u>Definition 5.4</u>: A regular geometric program.

A regular geometric program is defined as the following primal problem in the variables x :

$$\text{minimize} \quad g_o(x) \tag{5.2}$$

subject to

$$g_i(x) \le 1 \qquad i=1,2,\ldots,M \tag{5.3}$$

$$x_j > 0 \qquad j=1,2,\ldots,N \tag{5.4}$$

where

$$g_i(x) = \sum_{t=1}^{T_i} P_{it}(x)$$

$$= \sum_{t=1}^{T_i} c_{it} \prod_{j=1}^{N} x_j^{a_{itj}}$$

$$c_{it} > 0 \qquad i=o,1,2,\ldots,M$$

Let $z_j = \ln x_j$, then the above primal program may be transformed into an equivalent convex program [24,23] .

<u>Definition 5.5</u>: A dual geometric program.

Associated with every primal or regular geometric program is a dual geometric program and vice versa. A dual program is defined as the following linearly constrained nonlinear mathematical programming problem in the variables ω [3] :

$$\text{maximize} \quad d(\omega) = \prod_{i=0}^{M} \prod_{t=1}^{T_i} \left[\frac{c_{it}\omega_{io}}{\omega_{it}} \right]^{\omega_{it}} \tag{5.5}$$

subject to

a normality condition: $$\sum_{t=1}^{T_o} \omega_{ot} = 1 \tag{5.6}$$

and orthogonality conditions: $$\sum_{i=1}^{M} \sum_{t=1}^{T_i} \omega_{it} \, a_{itj} = 0 \qquad j=1,2,\ldots,N \tag{5.7}$$

$$\omega_{io} = \sum_{t=1}^{T_i} \omega_{it} \qquad i=1,2,\ldots,M \tag{5.8}$$

note that there are exactly $(N+1)$ independent dual constraint equalities, and exactly T independent dual variables ω ; one for each term of the primal problem,

$$T = \sum_{i=o}^{M} T_i \tag{5.9}$$

<u>Definition 5.6</u>: the degree of difficulty.

The degree of difficulty of a regular geometric programming problem (primal problem) is defined by the relation:

$$\begin{aligned} \text{degree} &= \text{the number of terms - the number of decision} \\ &\quad \text{variables - 1} \\ &= T - (N + 1) \end{aligned} \tag{5.10}$$

if the primal problem has zero degree of difficulty, the global solution of the dual problem and hence the global solution of the primal problem is obtained by solving the system of linear equations (5.6) and (5.7).

If the problem has degree of difficulty greater than zero, the corresponding system of linear equations has no single solution [24] .

<u>Definition 5.7</u>: Tight and loose constraints.

An inequality constraint, $g(\hat{x}) \le 0$, is said to be tight at a given point $\hat{x}$ if it becomes an equality $g(\hat{x}) = 0$, at that point. It is said to be loose if it becomes a strict inequality, $g(\hat{x}) < 0$, at that point.

If a primal constraint is loose at optimality, then all dual variables associated with that constraint will be zero at optimality [3, theorem 3.2]. In this case we cannot obtain the global solution of the dual problem and in turn of the primal problem.

Definition 5.8: A generalized geometric program.

The following primal program:

$$\text{minimize} \quad g_o(x) \tag{5.11}$$

subject to

$$g_i(x) \leq \sigma_i \qquad i=1,2,\ldots,M \tag{5.12}$$

$$x_j > 0 \qquad j=1,2,\ldots,N \tag{5.13}$$

where
$$g_i(x) = \sum_{t=1}^{T_i} \sigma_{it} c_{it} \prod_{j=1}^{N} x_j^{a_{itj}}$$

$$i=o,1,\ldots,M \tag{5.14}$$

and $\sigma_{it} = \pm 1$,

such that $c_{it} > 0$

is called a generalized geometric program. When σ_{it} equals $+1$, for all i, t, then the program (5.11)-(5.14) is a regular geometric program.

We can rewrite the above program in the following form:

$$\text{minimize:} \quad p_o(\bar{x}) - Q_o(\bar{x}) \tag{5.15}$$

subject to

$$p_i(\bar{x}) - Q_i(\bar{x}) \leq 1 \qquad i=1,2,\ldots,M \tag{5.16}$$

$$x_j > 0 \qquad j=1,2,\ldots,N \tag{5.17}$$

where $p_i(\bar{x})$ and $Q_i(\bar{x})$, $i=o,1,2,\ldots,M$ are posynomials (see Definition 5.3).

Definition 5.9: A quasidual program.

Corresponding to the primal program in Definition 5.8 (the generalized program) there exists a quasidual program defined as the following linearly constrained nonlinear program in the variables ω :

$$\text{Maximize} \quad d(\omega) = \sigma_o \left\{ \prod_{i=o}^{M} \prod_{t=1}^{T_i} \left[\frac{c_{it}\ \omega_{io}}{\omega_{it}} \right]^{\sigma_{it}\omega_{it}} \right\}^{\sigma_o} \tag{5.18}$$

subject to

a generalized normality condition: $\sum_{t=1}^{T_o} \sigma_{ot}\ \omega_{ot} = \sigma_o$ (5.19)

and the orthogonality conditions:

$$\sum_{i=o}^{M} \sum_{t=1}^{T_o} \sigma_{it}\ a_{itj}\ \omega_{it} = 0 \qquad j=1,2,\ldots,N \tag{5.20}$$

$$\omega_{io} = \sigma_i \sum_{t=1}^{T_i} \sigma_{it}\omega_{it} \geq 0 \qquad i=o,1,\ldots,M \tag{5.21}$$

$$\omega_{it} \geq 0 , \qquad t=1,2,\ldots,T_i \quad i=o,1,2,\ldots,M \tag{5.22}$$

where

$$\sigma_o = +1 \qquad \text{if} \quad g_o(x^*) > 0 \tag{5.23}$$

$$\sigma_o = -1 \qquad \text{if} \quad g_o(x^*) < 0 \tag{5.24}$$

and x^* is a stationary point of the generalized program (5.11) - (5.14) [23] .

the value of σ_o will usually be known in advance for most problems. Since the orthogonality conditions are homogeneous, changing the sign of σ_o simply reverses the signs of all other quasidual variables ω . Hence, a wrong initial guess for σ_o will only cause all the quadidual variables ω to have the wrong sign (all will be negative) but they will be correct in absolute value.

5.3.2 The Existing Methods used in Practice for Solving Generalized Geometric Programs.

The three principal methods used in practice for solving generalized geometric (i.e., signomial) programs are:

1. a method based on duality theory;
2. a method based on partial condensation; and
3. a method based on double condensation.

The first method is based on duality theory, where one can work with the linearly constrained quasidual program instead of attempting the direct solution of the primal program. Passy, Wilde, Blau & Wilde, Duffin & Peterson and others [3] have made attempts at generalizing some of the prototype concepts and theorems of regular geometric programming in order to include programs with negative as well as positive terms. They have found that most of the important prototype theorems are not valid in the more general setting [24,3,23].

The second method was presented by Avriel & Williams [1] and is based on approximating a generalized program by a sequence of regular programs where the sequence of optimal solutions of the regular programs converges to a local minimum of the generalized program (except in pathological cases. The details are given in Section 5.5). This method forms the basis of the third method. Similar algorithms to the Avriel & Williams algorithm have been developed independently by Broverman & Felerowicz & McWhirter, Pascual and Ben-Israel [23], but for somewhat smaller classes of programs and without convergence proofs.

The third method is due to Avriel, Dembo and Passy [2,29,23]. It is a combination of the Avriel & Williams algorithm (the second method) and a cutting plane algorithm [40], by double condensation of all primal inequality constraints, in which all the constraints are ultimately condensed into monomials (single-term posynomials). The details of this method are given in Section 5.6.

By using algorithms developed for the first method, we can obtain a stationary point for the quasidual program when the degree of difficulty is small [3,23], which is also a stationary point for the primal program. In order to guarantee that this stationary point is a local minimum, higher order conditions should be checked [87,74]. Also, to guarantee a global minimum one must find the smallest of the primal local minima. Passy & Wilde [60] called this procedure pseudominimization.

However, these algorithms will in general fail to find a stationary point for the primal program in those cases where some or all of the constraints of the primal program are loose at the solution.

The algorithms developed for the second method will be subject to the shortcomings associated with the solution of regular geometric programs, namely large degree of difficulty and loose constraints (see Definitions 5.6 and 5.7).

The third method avoids the shortcomings of the second method by solving each regular program of a sequence of regular programs by the cutting plane algorithm.

Additionally, a "better" local minimum of the generalized program may be obtained by using the Phase 1 algorithm of this method. We give details of this in Subsection 5.6.2.

5.4 A Condensed Geometric Programming Technique

This technique is constructed on a particular type of transformation based upon the arithmetic-geometric mean inequality. It was called condensation by Duffin who first suggested the technique [3] . The basic underlying principle of condensation is to approximate a multiterm posynomial function by a monomial or a single-term function. Later, we will see that this concept becomes very useful since the logarithmic transformation of a single term, multivariable function results in an equation linear in the logarithms of the primal variables.

The objective of this section is to present a cutting plane algorithm to solve regular geometric programs. Therefore, to begin with, the definitions and theorems related to the method of condensation and properties of condensed posynomials will be presented. Then we will demonstrate how condensation is used to approximate a regular geometric program by a linear program.

5.4.1 Definitions and Theorems

Definition 5.10: the arithmetic-geometric inequality.

If $u_1, u_2, \ldots, u_n$ are arbitrary non-negative numbers and $\delta_1, \delta_2, \ldots, \delta_n$ are arbitrary positive weights satisfying [2] :

a normality condition: $$\sum_{i=1}^{n} \delta_i = 1 \qquad (5.25)$$

then $$\sum_{i=1}^{n} u_i \geq \prod_{i=1}^{n} \left(\frac{u_i}{\delta_i} \right)^{\delta_i} \qquad (5.26)$$

Definition 5.11: regularity conditions.

A set of constraints $g_i(x) \leq 1$, $i=1,2,\ldots,M$ is said to be regular whenever [1] :

(1) the feasible set $x = \{x | g_i(x) \leq 1, \; i=1,2,\ldots,M\}$ is compact and nonempty.

(2) for each $\hat{x}$ such that $I(\hat{x}) = \{i | g_i(\hat{x}) = 1\} \neq \emptyset$, the cone generated by the vectors $\nabla g_i(\hat{x})$, $i \in I(\hat{x})$, is a pointed cone, i.e., the origin is not contained in the convex hull of $\nabla g_i(\hat{x})$, $i \in I(\hat{x})$.

Condition 1 is easily satisfied for generalized geometric programming problems by placing upper and lower bounds on each decision variable. It follows that the feasible set is compact and nonempty. Condition 2 is included to rule out the possibility of singularities occurring on the boundary of the constraints set. A generalized geometric problem possessing an optimal solution which is positive will satisfy condition 2.

Condition 2 can always be satisfied by adding a large positive constant to the primal objective function [3,23].

Definition 5.12: quasi-minimum.

The vector x^* is said to be a quasi-minimum of the problem:

$$\begin{aligned} &\text{minimize} \quad g_o(x) \;, \\ &\text{subject to} \\ &\qquad g_i(x) \geq 0 \;, \qquad i=1,2,\ldots,M \end{aligned} \tag{5.27}$$

where $g_o, g_1, \ldots, g_M$ are real-valued continuously differentiable functions, if x^* satisfies $g_i(x^*) \geq 0$ for $i=1,2,\ldots,M$ and the necessary conditions for a minimum [48]. i.e., a quasi-minimum is a point x^* which satisfies necessary conditions for a local minimum. Alternatively, we can say that a point that it not a quasi-minimum cannot be a local minimum.

If x^* is a quasi-minimum, then $g_o(x^*)$ is said to be a quasi-minimal value [1] .

<u>Lemma 5.1</u>:

Suppose that the constraint set (5.27) is regular and let $g_o(x)$ be a non-constant affine function and $B(x)$ the boundary of the feasible set x . If x^* is a quasi-minimum of problem (5.27) then $x^* \in B(x)$.

<u>Proof</u>: [1, page 113] .

<u>Definition 5.13</u>: stable and unstable quasi-minimum.

Suppose that the constraint set of problem (5.27) is regular, then $x^* \in B(x)$ is called a stable (unstable) quasi-minimum [1] if and only if $\nabla g_o(x^*)$ is (is not) contained in the interior of the cone generated by the vectors $\nabla g_i(x^*)$, $i \in I(x^*)$, where $I(x^*)$ is the index set for which $g_i(x^*) = 0$.

<u>Theorem 5.1</u>:

If the constraint set of (5.27) is regular and $X^* \in B(x)$ is a stable quasi-minimum, then x^* is a local minimum of problem (5.27).

<u>Proof</u>: [1, page 134] .

<u>Definition 5.14</u>: condensed posynomials.

For the set of weights δ such that

$$\sum_{t=1}^{T} \delta_t = 1 , \quad \delta_t \geq 0 \tag{5.28}$$

the arithmetic-geometric inequality (see Definition 5.10) takes the form

$$\sum_{t=1}^{T} u_t \geq \prod_{t=1}^{T} \left(\frac{u_t}{\delta_t} \right)^{\delta_t} \tag{5.29}$$

consider the posynomial $g(x)$,

$$g(x) = \sum_{t=1}^{T} u_t(x) = \sum_{t=1}^{T} c_t \prod_{j=1}^{N} x_j^{a_{tj}} \tag{5.30}$$

We define the condensed posynomial $\bar{g}(x,\bar{x})$, formed at the point $\bar{x} > 0$ as:

$$\bar{g}(x,\bar{x}) = \prod_{t=1}^{T} \left(\frac{u_t(x)}{\delta_t(\bar{x})} \right)^{\delta_t(\bar{x})}$$

$$= \theta(\bar{x}) \prod_{j=1}^{N} x_j^{\phi_j(\bar{x})} \tag{5.31}$$

where

$$\theta(\bar{x}) = \prod_{t=1}^{T} \left(\frac{c_t}{\delta_t(\bar{x})} \right)^{\delta_t(\bar{x})} \tag{5.32}$$

$$\phi_j(\bar{x}) = \sum_{t=1}^{T} a_{tj}\delta_t(\bar{x}) \qquad j=1,2,\ldots,N \tag{5.33}$$

It is easily observed that $\bar{g}(x,\bar{x})$ is a monomial for given $\bar{x} > 0$. We will choose the set of weights $\delta_t(\bar{x})$ such that:

$$\delta_t(\bar{x}) = \frac{u_t(\bar{x})}{g(\bar{x})} , \qquad t=1,2,\ldots,T \tag{5.34}$$

As a direct consequence of the arithmetic geometric inequality (5.29), we have that:

$$g(x) \geq \bar{g}(x,\bar{x}) . \tag{5.35}$$

It is possible to arrive at the identical approximating function (condensed posynomial) using a completely different approach. In that approach, we approximate a posynomial function by a first order Taylor-Series [23].

Properties of condensed posynomials

The following lemma gives the relationship between condensed and regular posynomials.

Lemma 5.2:

If $g(x)$ is any posynomial function and $\bar{g}(x,\bar{x})$ is the condensation of $g(x)$ at the point $\bar{x}$, then:

(a) $\bar{g}(x,\bar{x}) = g(x)$ if $x = \bar{x}$ (5.36)

(b) $\dfrac{\partial \bar{g}(x,\bar{x})}{\partial x_j} = \dfrac{\partial g(\bar{x})}{\partial x_j}$ $j=1,2,\ldots,N$ (5.37)

(c) $g(x) \geq \bar{g}(x,\bar{x})$ for all $x > 0$ (5.38)

Proof: [23 page 31].

5.4.2 Linearizing Geometric Programs Using Condensation

In this subsection we demonstrate how a regular geometric program may be approximated by a linear program using condensation. Consider the regular geometric program specified in Definition 5.4. We can transform it into an equivalent program with a linear objective function. Instead of minimizing $\hat{g}_o(x)$ we may define an additional variable, x_o, such that

$$x_o \geq \hat{g}_o(x) \tag{5.39}$$

and then minimize x_o. From inequality (5.39), $\hat{g}_o(x)$ provides a positive lower bound on the variable x_o and therefore inequality (5.39) will be satisfied as a strict equality at the optimal solution since x_o is being minimized let $$g_o(x) = \frac{\hat{g}(x)}{x_o} \le 1 \tag{5.40}$$

where $\hat{g}_o(x)$ is the objective function of the regular program of Definition 5.4. Define the set x as:

$$x = \{x_j \mid 0 < x_j^{LB} \le x_j \le x_j^{UB} \ , \quad j=o,1,2,\ldots,N\} \tag{5.41}$$

Here x_j^{UB} and x_j^{LB} are upper and lower bounds on the variables x_j respectively.

We will refer to the following program as gp.

<u>gp</u>: minimize x_o (5.42)

subject to

$$g_i(x) \le 1 \qquad i=o,1,2,\ldots,M \tag{5.43}$$

$$0 < x_j^{LB} \le x_j \le x_j^{UB} \qquad j=o,1,2,\ldots,N \tag{5.44}$$

where $g_i(x)$ are posynomials for $i=o,1,2,\ldots,M$. gp is equivalent to the regular geometric program in Definition 5.4 in the sense that the optimal solution to both programs is the same, provided that the variable bounds are chosen in such a way as not to be active at the optimal solution.

Consider the condensed program, $\overline{gp}(\bar{x})$, obtained by condensing all the posynomial constraints of gp to monomials at the point $\bar{x}$.

<u>$\overline{gp}(\bar{x})$</u> : minimize x_o (5.45)

subject to

$$g_i(x,\bar{x}) = \theta_i(\bar{x}) \prod_{j=o}^{N} x_j^{\phi_{ij}(\bar{x})} \le 1 \tag{5.46}$$

$$i=o,1,\ldots,M$$

$$0 < x_j^{LB} \le x_j \le x_j^{UB} \qquad j=o,1,2,\ldots,N \tag{5.47}$$

It follows from inequality (5.38) that a point, x_F , which satisfies the gp constraints (5.43) will also satisfy the constraints (5.46) of $\overline{gp}(\bar{x})$, i.e.,

$$\bar{g}_i(x_F,\bar{x}) \le g_i(x_F) \le 1 \qquad i=o,1,2,\ldots,M \qquad (5.48)$$

In general the converse will not be true. This implies that the feasible set of gp is entirely contained in $\overline{gp}(\bar{x})$ and therefore the solution of $\overline{gp}$ will generally not be a feasible point for gp . In fact it can be shown by using inequalities (5.39) and (5.48) that

$$x_o^*(gp) \ge x_o^*(\overline{gp}) \qquad (5.49)$$

where $x_o^*(gp)$ and $x_o^*(\overline{gp})$ are the optimal solutions of gp and $\overline{gp}(\bar{x})$ respectively.

$\overline{gp}(\bar{x})$ will now be shown to be equivalent to a linear program.

The natural logarithmic function, $F(Y) = \ln Y$, is monotomic increasing and defined for $Y > 0$. Therefore, the following program will be equivalent to $\overline{gp}(\bar{x})$:

$$\text{minimize} \quad \ln x_o \qquad (5.50)$$

subject to

$$\ln \bar{g}_i(x,\bar{x}) = \ln \theta_i(\bar{x}) + \sum_{j=o}^{N} \phi_{ij}(\bar{x}) \ln x_j \le 0$$

$$i=o,1,2,\ldots,M \qquad (5.51)$$

$$\ln x_j^{LB} \le \ln x_j \le \ln x_j^{UB} \qquad j=o,1,2,\ldots,N \qquad (5.52)$$

This program is a linear program in the variable $\ln x_j$, $j=o,1,2,\ldots,N$. However, it is not in a form suitable for direct application of the simplex method since the variables $\ln x_j$ may take on negative values. We therefore define new variables such that

$$z_j = \ln x_j - \ln x_j^{LB} \tag{5.53}$$

$$z_j^{UB} = \ln x_j^{UB} - \ln x_j^{LB} \tag{5.54}$$

and set z such that

$$z = \{ z_j \mid 0 \le z_j \le z_j^{UB}, \quad j=o,1,2,\ldots,N \} \tag{5.55}$$

Substituting the above in (5.50)-(5.52) gives the following program

$$\text{minimize} \quad z_o + \ln x_o^{LB} \tag{5.56}$$

subject to

$$\bar{g}_i(z,\bar{x}) = \ln \theta_i(\bar{x}) + \sum_{j=1}^{N} \phi_{ij}(\bar{x}) \ln x_j^{LB}$$

$$+ \sum_{j=o}^{N} \phi_{ij}(\bar{x}) z_j \le 0 \qquad i=o,1,\ldots,M \tag{5.57}$$

$$0 \le z_j \le z_j^{UB} \qquad j=o,1,2,\ldots,N \tag{5.58}$$

We note that:

(a) $\ln \theta_i(\bar{x}) + \sum_{j=o}^{N} \phi_{ij}(\bar{x}) \ln x_j^{LB} = \ln \bar{g}_i(x^{LB},\bar{x})$ (5.59)

is a constant

(b) $\ln x_o^{LB}$ is a constant.

We may rewrite the above program (5.56)-(5.58) as follows:

$$\text{minimize} \quad z_o \tag{5.60}$$

subject to

$$\sum_{j=o}^{N} \phi_{ij}(\bar{x}) z_j \le -\ln g_i(x^{LB},\bar{x}) \qquad i=o,1,\ldots,M \tag{5.61}$$

$$0 \le z_j \le z_j^{UB} \tag{5.62}$$

We will refer to this program as $LP(\bar{x})$ since it is a regular linear program in the upper bound variables z_j and is constructed about the point $\bar{x}$. $LP(\bar{x})$ is solved efficiently using a modified version of the dual simplex method [81], which accounts for upper-bounded variables.

5.4.3. A Cutting Plane Algorithm for Solving a Regular Geometric Program (gp)

In this subsection a cutting plane algorithm is presented for solving the regular geometric program gp. This algorithm is based on Kelley's algorithm [40] for convex programs and was presented a second time by Dembo [22,23,2] to solve regular geometric programs. As mentioned previously in Definition 5.4, a regular geometric program may be transformed into an equivalent convex program which therefore makes it amenable to any of the methods available for convex programs such as the cutting plane algorithm. Although noted for its poor convergence characteristics [88], it has the following advantages for our particular problem:

(1) The convergence of the cutting plane algorithm is satisfied where:

(a) the constrained minimum value of the objective function of the gp is positive,

(b) the gp constraint set is compact (since there are upper and lower bounds on each variable).

(2) Using suitable transformations, the problem to be solved at each iteration is a linear program as described below.

Consider the gp

$$\text{minimize} \quad x_o \tag{5.63}$$

subject to

$$g_i(x) \le 1 \qquad i=o,1,2,\ldots,M \tag{5.64}$$

$$0 < x_j^{LB} \le x_j \le x_j^{UB} \qquad j=o,1,2,\ldots,N \tag{5.65}$$

The algorithm proceeds as follows:

Step 1 Using an arbitrary starting point, x^o, linearize the gp as described in subsection 5.4.2 and form $LP(x^o)$.

set $m = 1$

Step 2 Solve $LP(x^{m-1})$. Call the solution z^m and compute x^m by equation (5.53).

Step 3 Evaluate the gp constraints at x^m.

(a) If $g_i(x^m) \le 1 + \epsilon \quad i=1,1,2,\ldots,M$ (5.66)

where ϵ is some small predetermined positive number, then x^m is optimal.

(b) Otherwise define

$$g_\ell(x) = \max_F \left\{ g_F(x^m) : g_F(x^m) > 1 \right\} \tag{5.67}$$

Step 4 Condense $g_\ell(x)$ at x^m (as in Definition 5.14) to obtain $\overline{g_\ell}(x,x^m)$ which in turn is transformed into the linear constraint

$$\overline{g_\ell}(z,\bar{x}) \le 0 \tag{5.68}$$

Add this constraint to the tableau of $LP(x^{m-1})$ and name the new problem $LP(x^m)$. Set $m = m+1$; return to Step 2.

In Step 1 the gp is approximated by a linear program $LP(x^o)$, for which highly efficient algorithms have been developed [18]. If the point x^m, obtained by solving $LP(x^{m-1})$, lies outside the region described by $g_i(x) \le 1$, $i=o,1,\ldots,M$, then Step 4 generates a modified LP problem that excludes $z^m (z^m = \ln x^m - \ln x^{LB})$ from its feasible region. Thus a series of LP's with progressively smaller feasible regions are solved until a point x^m is obtained which satisfies (5.66), at which stage the algorithm terminates.

This type of algorithm is known as a "cutting-plane" algorithm and the constraints generated in Step 4 are known as "cuts", since they cut off part of the feasible region of the approximating linear program at each iteration.

In order to see that this cut does not cut off any section of the feasible region of gp, we observe from inequalities (5.48) that for any point, x_F, feasible for gp we have

$$g_\ell(x_F,x^m) \le g_\ell(x_F) \le 1 \tag{5.69}$$

i.e., x_F will also be feasible for the cut. At each iteration m of the above algorithm we are required to solve the linear programming problem $LP(x^{m-1})$. However, the problem $LP(x^m)$ solved at iteration $m+1$, differs from $LP(x^{m-1})$ only in that it has an additional constraint. Use of the dual simplex method for bounded variables [81] enables the transformation from $LP(x^{m-1})$ to $LP(x^m)$ to be carried out in such a manner that the optimal solution to $LP(x^{m-1})$ is used as the starting point for the solution of $LP(x^m)$. Thus, only a modest amount of computation is required in moving from one iteration to the next (all the details about the computational advantages of the dual simplex method for bounded variables to solve a sequence of $LP(x^m)$ are given in [23]).

5.5 A Partially Condensed Method for Solving Generalized Geometric Programs

In this section we present a partially condensed method as one of the methods used in practice for solving generalized geometric programs, since it forms the basis of sections 5.6 and 5.8. We consider the generalized geometric program as in Definition 5.8:

$$\text{minimize} \quad P_o(\bar{x}) - Q_o(\bar{x}) \tag{5.70}$$

subject to

$$P_i(\bar{x}) - Q_i(\bar{x}) \leq 1 \qquad i=1,2,\ldots,M \tag{5.71}$$

$$x_j > 0 \qquad j=1,2,\ldots,N \tag{5.72}$$

The above program is equivalent to the following program which will be referred to as ggp .

$$\underline{ggp}: \quad \text{minimize} \quad x_o \tag{5.73}$$

subject to

$$\frac{P_i(x)}{1+Q_i(x)} \leq 1 \qquad i=o,1,2,\ldots,M \tag{5.74}$$

$$0 < x_j^{LB} \leq x_j \leq x_j^{UB} \qquad j=o,1,2,\ldots,N \tag{5.75}$$

(see subsection 5.4.2).

It is noted that ggp has an unconstant affine objective function and in turn, a quasi-minimum of ggp belongs to the boundary of the feasible set x (see Lemma 5.1).

Avriel & Williams referred to the above program as a 'complementary geometric program' and their version differs from g g p in that no bounds are placed on the variables x_j. However, for convergence of their algorithm (given below) it is required that the feasible set x be compact and bounding the variables as above is one way of guaranteeing this.

Let $Q_i(x,x^{(p)})$ denote the monomial obtained by condensing the posynomial $(1 + Q_i(x))$ at the point $x^{(p)}$. The following program obtained by substituting $Q_i(x,x^{(p)})$ for $(1 + Q_i(x))$ in g g p, will be referred as $gp^{(p)}$.

$\underline{gp^{(p)}}$: minimize x_o (5.76)

subject to

$$\frac{P_i(x)}{Q_i(x,x^{(p)})} \le 1 \qquad i=o,1,2,\ldots,M \tag{5.77}$$

$$0 < x_j^{LB} \le x_j \le x_j^{UB} \qquad j=o,1,2,\ldots,N \tag{5.78}$$

This program has the following interesting properties [2] :

(1) $gp^{(p)}$ is a regular geometric program, since the functions $(P_i(x) / Q_i(x,x^{(p)}))$ are posynomials.

(2) Any point x_F satisfying the constraints of $gp^{(p)}$ will satisfy the constraints of g g p . This can be observed by the condensation inequality (5.38)

$$\frac{P_i(x_F)}{1+Q_i(x_F)} \le \frac{P_i(x_F)}{Q_i(x_F,x^{(p)})} \le 1 \tag{5.79}$$

(3) Inequality (5.79) implies that the feasible set of $gp^{(p)}$ is entirely contained in g g p and therefore the optimal solution to $gp^{(p)}$ will be a feasible but not necessarily an optimal point for g g p . Under the regularity conditions

(see Definition 5.11) Avriel & Williams proved that the sequence of optimal solutions to $gp^{(p)}$ problems, where $gp^{(o)}$ is constructed using a point feasible for ggp, and $gp^{(p)}$. $p = 1, \ldots$ is constructed using the optimal solution to $gp^{(p-1)}$, converges to a quasi-minimum [1, theorem 5.3], which is a local minimum of ggp (except in pathological cases, when a quasi-minima is an unstable minimum (see Definition 5.13)). Duffin & Peterson [23] speculate, however, that convergence to an unstable minimum will be rare, owing to roundoff errors in computer arithmetic.

5.5.1 The Avriel & Williams Algorithm

Step 1 Construct $gp^{(o)}$, as described in (5.76)-(5.78), using the point $x^{(o)}$ which is a feasible solution to ggp

iteration i

Step 2 Let $x^{(i)}$ be any optimal solution to $gp^{(i-1)}$.

Step 3 Construct $gp^{(i)}$ using the point $x^{(i)}$

Put $i = i+1$

Step 4 Repeat steps 2 and 3 until convergence is obtained.

It is noted that at each iteration a regular geometric program is solved and therefore the algorithm may be used in conjunction with any algorithm for solving regular geometric problems.

5.5.2 Termination of the Avriel & Williams Algorithm

Dembo [23] suggested the following simple criterion to terminate the above algorithm:

Stop when we obtain $x_o^{(i+1)}$ such that

$$\left[\frac{x_o^{(i)} - x_o^{(i+1)}}{x_o^{(i)}} \right] \leq \epsilon \tag{5.80}$$

where ϵ is some small positive number. Other, different, criteria could also be used to obtain convergence. But these criteria are complicated and from the point of view of computational efficiency one would probably solve the program using the above criterion and then test to see if the solution obtained $(x^{*(i+1)})$ is in fact a local minimum. The necessary conditions for a local minimum are those of Kuhn-Tucker [48,74]. If these conditions are not satisfied by the above solution then the solution procedure should be continued using a smaller value for ϵ in (5.80).

Sufficiency may be tested for by using the second order conditions described in Wilde and Beightler ([87] , page 52).

5.6 A Double Condensed Method for Solving Generalized Geometric Programs

5.6.1 Phase 2 Algorithm

As mentioned previously (see subsection 5.3.2), this method provides a complete algorithm for solving ggp by combining the Avriel & Williams algorithm (see subsection 5.5.1) with the cutting plane algorithm (see subsection 5.4.3). This is done by double condensation of the generalized program since (i) a generalized program is condensed to a regular program then (ii) a regular program is condensed to a monomial program which is equivalent to a linear program.

However, Avriel & Dembo and Passy found that there is possibly a more efficient way of combining the above two algorithms whereby convergence to a ggp solution is accelerated. This acceleration technique is based on the following observations pertaining to the above two algorithms:

(1) The sequence of optimal solutions of $gp^{(p)}$ programs is feasible for ggp and thus each such solution $x_0^*(gp^{(p)})$ is greater than or equal to the optimal solution to the ggp, i.e.

$$x_0^*(gp^{(p)}) \geq x_0^*(ggp) \qquad (5.81)$$

(2) The sequence of optimal solutions of $Lp(x^m)$ programs (cutting plane iterations) converging to a particular $gp^{(p)}$ solution is not feasible for the $gp^{(p)}$ and thus

$$x_0^*(Lp(x^m)) \leq x_0^*(gp^{(p)}) \qquad (5.82)$$

At some stage, during the course of proceding to a solution of $gp^{(p)}$, the current optimal solution $x_0^*(Lp(x^m))$ may be feasible for ggp. This point may have a lower objective function value than the solution to $gp^{(p)}$ itself and usually it will serve as a 'better' point than the $gp^{(p)}$ optimum, for the formation of $gp^{(p+1)}$. Hence, this algorithm with the acceleration technique proceeds by the following steps:

Let $x^{m,p}$ indicate an optimal solution of $gp^{(p)}$ obtained after m cutting plane iteration.

<u>Step 1</u> Set $p = 1$

<u>iteration p</u>

<u>Step 2</u> Using the point $x^{o,p}$, which is a feasible solution to ggp to construct $gp^{(o,p)}$ as described in Section 5.5.

<u>Step 3</u> Linearize $gp^{(o,p)}$ and form $Lp(x^{o,p})$ as described in subsection 5.4.2.

<u>Step 4</u> Set $m = 1$

<u>Step 5</u> Solve $Lp(x^{m-1,p})$. Call the solution $x^{m,p}$.

<u>Step 6</u> Evaluate the ggp constraints at $x^{m,p}$:

(1) if $g_i(x^{m,p}) > 1 + \epsilon$ for any value of i, $i=1,2,\ldots,M$

define $g_\ell(x) = \max\limits_F\{g_F(x^{m,p}) : g_F(x^{m,p}) > 1\}$

(2) if $g_i(x^{m,p}) \le 1 + \epsilon$ and $\left[\dfrac{x_o^{m',p-1} - x_o^{m,p}}{x_o^{m',p-1}}\right]$[1] $> \epsilon$ for all $i=1,2,\ldots,M$,

the convergence criterion is not satisfied.

In that case, put $p = p + 1$,

$$x^{o,p} = x^{m,p-1}, \text{ go to step 2.}$$

(3) if $g_i(x^{m,p}) \le 1 + \epsilon$ and $\left[\dfrac{x_o^{m',p} - x_o^{m,p}}{x_o^{m',p}}\right] \le \epsilon$,

the convergence criterion is satisfied.

In that case, test the point $x^{m,p}$:

(a) if $x^{m,p}$ satisfies the necessary conditions for a local minimum, go to step 8.

(b) if $x^{m,p}$ does not satisfy the necessary conditions of a local minimum a smaller value of ϵ is chosen in the convergence criterion and put $p = p + 1$, $x^{o,p} = x^{m,p-1}$, go to step 2.

---

[1] m' is the number of cutting plane iterations needed to obtain the optimal solution of $gp^{(p-1)}$.

<u>Step 7</u> Condense $g_\ell(x)$ at $x^{m,p}$ (see Definition 5.14) to obtain $g_\ell(x,x^{m,p})$ which, in turn, is transformed into the linear constraint

$$\bar{g}_\ell(z,x^{m,p}) \le 0 \;.$$

Add this constraint to $Lp(x^{m-1,p})$, name the new Lp program $Lp(x^{m,p})$, put $m = m + 1$, go to step 5.

<u>Step 8</u> $x^{m,p}$ is the optimal solution [1] of ggp, stop.

For reasons that will be made obvious in the next subsection, the above algorithm to be referred to as the phase 2 algorithm.

5.6.2 Phase 1 Algorithm

The phase 2 algorithm of the previous subsection requires a starting point which is a feasible solution to the ggp constraints:

$$\frac{P_i(x)}{1 + Q_i(x)} \le 1 \qquad i=o,1,2,\ldots,M \qquad (5.83)$$

In many cases, determining a value for x which satisfies (5.83) for all $i=o,1,2,\ldots,M$, may be as difficult as the solution of the ggp itself. The authors of [2] determined the sufficient condition to obtain a feasible solution point to (5.83) (see the theorem given below). They presented a phase 1 [2] algorithm which, under their condition, is guaranteed to yield a solution of (5.83). Unfortunately, in general, the

---

[1] Except in pathological cases, $x^{m,p}$ is unstable point (see Section 5.5).

[2] Because of the similarities between this algorithm and the corresponding Lp algorithm for finding an initial feasible solution, they called it the phase 1 algorithm.

conditions necessary to ensure convergence of phase 1 do not hold the condition referred to above is sufficient for identifying a feasible solution point, but not necessary, i.e., there are points satisfying inequalities (5.83) which do not satisfy that condition.

Consider the following generalized geometric program, called ggp(w), formed from the ggp problem of Section 5.5:

<u>ggp(w)</u> minimize $\prod_{i=o}^{M} w_i$ (5.84)

subject to

$$\frac{P_i(x)}{1 + Q_i(x)} \le w_i \qquad i=o,1,2,\ldots,M \qquad (5.85)$$

$$w_i \ge 1 \qquad i=o,1,2,\ldots,M \qquad (5.86)$$

$$0 < x_j^{LB} \le x_j \le x_j^{UB} \qquad j=o,1,2,\ldots,N \qquad (5.87)$$

The reason for introducing ggp(w) is made obvious by the theorem below.

<u>Theorem 5.2</u>

The point $x = x^*$ satisfies inequalities (5.83) if and only if the optimal solution to ggp(w) is (x^*,w^*), where $\prod_{i=o}^{M} w_i^* = 1$.

<u>Proof:</u> [23, page 61]

If the local minimum of ggp(w) is equal to the global minimum, then, the solution of ggp(w) using the phase 2 algorithm may be guaranteed to yield a feasible point to inequalities (5.83). In cases where ggp(w) has more than one minimizing point, we know the desired solution is one of them. However, convergence to this particular solution is not guaranteed.

The steps of phase 1 are summarized as follows:

<u>Step 1</u> Let X^o be any point satisfying

$$0 < x_j^{LB} \leq x_j \leq x_j^{UB} \qquad j=o,1,2,\ldots,N \tag{5.88}$$

define

$$w_i^o = \max\left\{ \frac{P_i(x^o)}{1 + Q_i(x^o)} , 1 \right\} \qquad i=o,1,2,\ldots,M \tag{5.89}$$

The point (x^o,w^o) , where $w^o = (w_o^o, w_1^o, \ldots, w_M^o)$, is thus a feasible solution for $ggp(w)$.

<u>Step 2</u> Consider the value of w_i^o for $i=o,1,2,\ldots,M$:

(1) if $w_i^o = 1$ for all i , then phase 1 terminates, and x^o solves (5.83).

(2) if for at least one value of i , $w_i^o > 1$, then we solve $ggp(w)$ using the phase 2 algorithm with initial point (x^o,w^o).

<u>Step 3</u> Examine the optimal solution (x^*,w^*) to $ggp(w)$

(1) if $w_i^* = 1$ for all i , then x^* will be a solution to (5.83),

(2) if for some i , $w_i^* > 1$, then the algorithm has failed to converge to a global solution of $ggp(w)$.

There is one further application of the phase 1 algorithm. Assume that during the course of seeking a solution to a ggp the phase 2 algorithm converged to a local but not global minimum of the problem. We could attempt to improve on this solution by constraining the objective function to a value less than that attained previously and solving the resulting problem using the phase 1 algorithm. If phase 1 converges to a feasible solution of the restricted problem this solution may be used as a starting value for the phase 2 algorithm, which will then converge to a "better" local minimum

5.7 The Formulation of Subprograms of a Goal Program as Generalized Geometric Programs

In the next section an algorithm will be presented to solve a nonlinear goal program in a sequence of nonlinear subprograms (see Section 1.3), each of them having a single objective (i.e. single objective function). This algorithm requires the subprograms of a goal program to be formed as generalized geometric programs. In this section we discuss some of the difficulties which are encountered in formulating the subprograms of a nonlinear goal program as generalized geometric programs.

5.7.1 Equality Goal Set

From Section 1.2 the general goal program:

Find $\quad x = (x_1, x_2, \ldots x_N)$

so as to

$$\text{lexico - min} \quad a = \{[g_1(d^-,d^+)], \ldots, [g_k(d^-,d^+)], \ldots [g_K(d^-,d^+)]\} \qquad K \leq M \tag{5.89}$$

subject to

$$f_i(x) + d_i^- - d_i^+ = b_i \qquad i=1,2,\ldots,M \tag{5.90}$$

$$x_j, d_i^-, d_i^+ \geq 0 \qquad j=1,2,\ldots,N \quad i=1,2,\ldots,M \tag{5.91}$$

Where x_j and d_i^-, d_i^+ are decision variables and deviational variables respectively. It is shown that the goal set in the standard form (5.90) are equality constraints. Since the only constraints allowed in formulating a generalized geometric program are inequality constraints; therefore we must convert the equality goal constraints to inequality constraints before formulating subprograms of the goal program as generalized programs.

Proposition 5.1 If:

(i) the i^{th} goal of gaol set (5.90) is:

$$f_i(x) + d_i^- - d_i^+ = b_i \;; \tag{5.92}$$

(ii) d_i^- is included in an achievement function (5.89) and d_i^+ is not included; and

(iii) d_i^- is minimum in the optimum solution, then the goal (5.92) is equivalent to:

$$f_i(x) + d_i^- \geq b_i \tag{5.93}$$

Proof: the proof follows immediately from the definition of the deviational variables d_i^- , d_i^+ (see Section 1.2).

Results 5.1

Since d_i^- is a minimum in the optimal solution, then in the optimal solution:

(1) if $d_i^- > 0$ and $f_i(x) < b_i$ then $d_i^+ = 0$ (5.94)

(2) if $d_i^- = 0$ and $f_i(x) = b_i$ then $d_i^+ = 0$ (5.95)

(3) if $d_i^- = 0$ and $f_i(x) > b_i$ then $d_i^+ > 0$ and $d_i^+ = f_i(x) - b_i$ (5.96)

Proposition 5.2

Let the goal (5.92), if

(i) d_i^+ is included in an achievement function (5.89) and d_i^- is not included, and

(ii) d_i^+ is minimum in the optimal solution,

then the goal (5.92) is equivalent to:

$$f_i(x) - d_i^+ \leq b_i \tag{5.97}$$

Proof: The proof follows immediately from the definition of the deviational variables also.

Results 5.2

Since d_i^+ is a minimum in the optimal solution, then in the optimal solution:

(1) if $d_i^+ > 0$ and $f_i(x) > b_i$ then $d_i^- = 0$ (5.98)

(2) if $d_i^+ = 0$ and $f_i(x) = b_i$ then $d_i^- = 0$ (5.99)

(3) if $d_i^+ = 0$ and $f_i(x) < b_i$ then $d_i^- > 0$ and

$$d_i^- = b_i - f_i(x) \tag{5.100}$$

5.7.2 Equality Constraints Related with Goal Set

There are some special nonlinear goal programs which have equality constraints related to one or more goals in standard form and which do not represent goals (i.e. do not include deviational variables d^-, d^+) as in programs: 3.107 - 3.115, 3.122 - 3.130, 4.42 - 4.50 and 4.61 - 4.66 .

Since an equality constraint $g(x) = 1$ is exactly equivalent to the pair of inequality constraints $g(x) \geq 1$ and $g(x) \leq 1$, any equality constraint can be replaced by two inequality constraints [3] . This however has two main disadvantages [23] :

(1) The size of the problem is greatly increased.

(2) Numerical difficulties may result since the above approach generally leads to two rows of the linear program (see the cutting plane algorithm) having identical coefficients.

Generally one of these inequalities is redundant and the equality constraint may be replaced by one inequality constraint which should be tight at the optimal solution. If the incorrect sense of the inequality is chosen (i.e. the constraint is loose at the optimum) then the problem must be solved again using the opposite sense of the inequality. Choosing the correct sense of the inequality may be accomplished if the equality has some interpretation by means of which one can replace it by an inequality using logic based on the nature of the problem.

For equality constraints which do not have such interpretations as (3.112) and (3.113); we must consider the equality constraint replaced by inequalities written both ways [5] . The entire problem must then be solved using both forms of the constraint and the correct sense of the inequality deduced from the computer output (see Appendix D, E).

5.7.3 Bounding Problem Variables

In accordance with the requirements to form generalized geometric programs, all problem variables must be bounded from above and below by positive bounds. For some applications, most of the variables will be bounded by physical considerations. However, when no accurate bounds on the variables are available artificial ones must be assumed. This must be done with caution and optimal solutions examined to see if any variables are at their bounds. If some variables are on their artificial bounds at the optimum then these bounds have been incorrectly chosen the problem must be solved again with a less restricting set of bounds.

Bounds of the form x_j, d_i^-, $d_i^+ \geq 0$ for all $j=1,2,\ldots,N$, $i=1,2,\ldots,M$ may be replaced by x_j, d_i^-, $d_i^+ \geq \epsilon$, where ϵ is some small positive number, in order to ensure positivity of the variables. The problem is then solved and if the solution contains variables x_j, d_i^-, d_i^+ such that x_j or d_i^- or $d_i^+ = \epsilon$ then these variables may be assumed to have an optimal value of zero [23]. The correct choice of a value for ϵ depends on the problem being solved, however in most cases [22] $\epsilon = 10^{-6}$ was found to be suitable (see Section 5.9).

5.8 A Sequential Double Condensed Geometric Goal Programming Algorithm

In Section 1.3 an algorithm for solving a general goal program by solving a series of single objective programming subprograms was given. Section 5.3 gave an efficient algorithm to solve a generalized geometric program as a nonlinear single objective program. Thus by simply combining the above two algorithms, we have a complete algorithm for solving nonlinear goal programs.

Let the general goal program (see Section 1.3) be:

Find $\quad x = (x_1, x_2, \ldots, x_N)$

so as to

$$\text{lexico-min} : a = \{[g_1(d^-,d^+)], \ldots, [g_k(d^-,d^+)], \ldots, [g_K(d^-,d^+)]\} \quad K \leq M \tag{5.101}$$

$$\text{subject to} \quad G_i : f_i(x) + d_i^- - d_i^+ = b_i \quad i=1,2,\ldots,M \tag{5.102}$$

$$x_j, d_i^-, d_i^+ \geq 0 \quad j=1,2,\ldots,N;\ i=1,2,\ldots,M \tag{5.103}$$

Using Propositions 5.1 and 5.2, the above program is equivalent to the following program.

Find $\quad x = (x_1, x_2, \ldots, x_N)$

so as to

$$\text{lexico-min} : a = \Big\{[g_1(d^-,d^+)], [g_2(d^-,d^+)], \ldots, [g_k(d^-,d^+)], \ldots [g_K(d^-,d^+)]\Big\} \qquad K \leq M \tag{5.104}$$

subject to

$$f_i(x) + d_i^- \geq b_i \qquad i=1,2,\ldots,m \tag{5.105}$$

$$f_{i'}(x) - d_{i'}^+ \leq b_{i'} \qquad i'=m+1,m+2,\ldots,M \tag{5.106}$$

$$x,\ d^-,\ d^+ \geq 0 \tag{5.107}$$

From the above program, the subprogram associated with priority level k (see Section 1.3) is:

minimize $\quad a_k = g_k(d^-,d^+)$ $\qquad$ (5.108)

subject to

$$f_t(x) + d_t^- \geq b_t \tag{5.109}$$

$$f_{t'}(x) - d_{t'}^+ \leq b_{t'} \tag{5.110}$$

$$g_s(d^-,d^+) = a_s^* \qquad s=1,2,\ldots,k-1 \tag{5.111}$$

$$x,\ d^-, d^+ \geq 0 \tag{5.112}$$

where

t, t' belong to the set of subscripts associated with those goals included in priority levels $1,2,3,\ldots,k$.

Since equality constraints (5.111) represent the accomplished levels of goals $1,2,\ldots,k-1$, it is correct to say that:

$$g_s(d^-,d^+) \leq a_s^* \tag{5.113}$$

In turn, the above program is equivalent to the following program:

$$\min \qquad a_k = g_k(d^-,d^+) \tag{5.114}$$

subject to

$$f_t(x) + d_t^- \geq b_t \tag{5.115}$$

$$f_{t'}(x) - d_{t'}^+ \geq b_{t'} \tag{5.116}$$

$$g_s(d^-,d^+) \leq a_s^* \qquad s=1,2,\ldots,k-1 \tag{5.117}$$

$$x,\ d^-,\ d^+ \geq 0 \tag{5.118}$$

We denote the deviational variables vector of dimension M, by d such that:

$$d = \{d_t^-, d_{t'}^+ \geq 0 \mid t=1,2,\ldots,m;t'=m+1,m+2,\ldots,M\} \tag{5.119}$$

and define the decision-deviational variable set (x,d) by:

$$(x,d) = \left\{ x_j \mid 0 < x_j^{LB} \leq x_j \leq x_j^{UB},\ \text{for all } j \text{ and } \epsilon \leq d \leq d^{UB} \text{ for all } d \text{ where } \epsilon \to 0 \right\}$$

Now, program (5.114)-(5.118) is equivalent to the following generalized geometric program $g\,g\,p$ (see Section 5.5) and will be referred to as $(g\,g\,p)_k$:

(5.120)

$(g\,g\,p)_k$ minimize d_o

subject to

$$G_{tk} = \frac{P_{tk}(x,d)}{1 + Q_{tk}(x,d)} \leq 1 \qquad t=o,1,2,\ldots \tag{5.121}$$

$$0 \leq x^{LB} \leq x \leq x^{UB} \tag{5.122}$$

$$\epsilon \leq d \leq d^{UB} \tag{5.123}$$

where $t = 1,2,\ldots$ indicates the set of subscripts associated with the constraints of the k^{th} sub-program. When $t=o$, the constraint is:

$$g_{ok} = \frac{g_k(d^-,d^+)}{d_o} \le 1 \quad , \text{ where:}$$

$$d_o \ge g_k(d^-,d^+)$$

(see subsection 5.4.2).

Now, our algorithm proceeds as follows:

<u>Step 1</u> Set $k = 1$

<u>Step 2</u> Establish $(g\,g\,p)_k$ as in (5.120)-(5.123)

<u>Step 3</u> Find a feasible solution point to (5.121), by guessing or by the phase 1 algorithm (see subsection 5.6.2).

<u>Step 4</u> Solve $(g\,g\,p)_k$ by the phase 2 algorithm (see subsection 5.6.1) and obtain a local minimum solution to $(g\,g\,p)_k$.

<u>Step 5</u> Use phase 1 to obtain a "better" local minimum to $(g\,g\,p)_k$:

(1) if possible find a "better" local minimum point. We consider this point an optimal solution $(x^*,d^*)_k$ and a_k^* is the optimal value of $g_k(d^-,d^+)$;

(2) if it is impossible to find a "better" local minimum point, we consider the local point found in Step 4 as the optimal solution point $(x^*,d^*)_k$.

<u>Step 6</u> Set $k = k + 1$. If $k > K$ go to step 9.

<u>Step 7</u> Establish $(g\,g\,p)_k$.

<u>Step 8</u> Go to step 3

<u>Step 9</u> The solution $(x^*,d^*)_k$ is the optimal solution for the original nonlinear goal program.

This algorithm has the following properties:

(1) By using this algorithm, we guarantee to obtain a local or a better local minimum point for each of the subprograms. In turn, it gives detailed information about the accomplishment for each objective according to their priorities.

(2) The double condensed method does not suffer from the drawbacks of the Griffith & Stewart and the pattern search methods (see Section 5.2).

(3) If for the nonlinear goal program subprograms $1,2,\ldots,k-1$ are linear programs and subprograms $k, k+1, \ldots, K$ are nonlinear programs, then by the above algorithm, we can solve subprograms $1,2,\ldots,k-1$ by the simplex method directly. This saves effort when solving problems by hand. This aspect will be clarified in Section 5.9 and Appendix D.

5.9 Example 5.1

In order to demonstrate the procedures of the algorithm given in the previous section, we solve again the following example which was presented and solved by Ignizio [37, page 163].

Nonlinear goal program

Find $x = (x_1, x_2)$

so as to

$$\text{lexico-min } a = \{(d_3^+), (2d_1^- + d_2^+)\} \tag{5.125}$$

subject to

$$G_1 : x_1 x_2 + d_1^- - d_1^+ = 16 \tag{5.126}$$

$$G_2 : (x_1-3)^2 + x_2^2 + d_2^- - d_2^+ = 9 \tag{5.127}$$

$$G_3 : x_1 + x_2 + d_3^- - d_3^+ = 6 \tag{5.123}$$

(see Figure 5.1).

Solution

Step 1   From (5.125)-(5.128), the 1^{st} subprogram is:

$$\text{minimize} \quad a_1 = d_3^+ \tag{5.129}$$

subject to

$$x_1 + x_2 + d_3^- - d_3^+ = 6 \tag{5.130}$$

$$x_1,\ x_2,\ d_3^-,\ d_3^+ \geq 0 \tag{5.131}$$

The above program is a linear program. In turn, using the simplex method (see the third property of the algorithm in Section 5.8), the optimal value of objective function (5.129) is:

$$a_1^* = d_3^+ = 0 \tag{5.132}$$

Step 2

1. From (5.125)-(5.128) and (5.132), the 2^{nd} subprogram is:

$$\text{minimize} \quad a_2 = 2d_1^- + d_2^+ \tag{5.133}$$

subject to

$$x_1x_2 + d_1^- - d_1^+ = 16 \tag{5.134}$$

$$(x_1-3)^2 + x_2^2 + d_2^- - d_2^+ = 9 \tag{5.135}$$

$$x_1 + x_2 + d_3^- - d_3^+ = 6 \tag{5.136}$$

$$d_3^+ = 0 \tag{5.137}$$

$$x_j,\ d_i^-,\ d_i^+ \geq 0 \qquad j=1,2 \quad i=1,2,3 \tag{5.138}$$

2. From Section 5.7 and (5.113), program (5.133)-(5.138) is equivalent to:

$$\text{minimize} \quad a_2 = 2d_1^- + d_2^+ \tag{5.139}$$

subject to

$$x_1x_2 + d_1^- \geq 16 \tag{5.140}$$

$$x_1^2 + x_2^2 - 6x_1 - d_2^+ \le 0 \tag{5.141}$$

$$x_1 + x_2 - d_3^+ \le 6 \tag{5.142}$$

$$d_3^+ \le \epsilon \tag{5.143}$$

where $\epsilon \to 0$.

3. From (5.120)-(5.123), the program (5.139)-(5.143) is equivalent to the following generalized geometric program $(g\,g\,p)_2$, where:

<u>$(g\,g\,p)_2$</u> minimize d_o (5.143)

subject to

$$G_{12} = \frac{P_{12}(x,d)}{1+Q_{12}(x,d)} = \frac{2d_1^- + d_2^+}{d_o} \le 1 \tag{5.144}$$

$$G_{22} = \frac{P_{22}(x,d)}{1+Q_{22}(x,d)} = \frac{16}{x_1x_2 + d_1^-} \le 1 \tag{5.145}$$

$$G_{32} = \frac{P_{32}(x,d)}{1+Q_{32}(x,d)} = \frac{x_1^2 + x_2^2}{6x_1 + d_2^+} \le 1 \tag{5.146}$$

$$G_{42} = \frac{P_{42}(x,d)}{1+Q_{42}(x,d)} = \frac{x_1 + x_2}{6 + d_3^+} \le 1 \tag{5.147}$$

$$G_{52} = \frac{P_{52}(x,d)}{1+Q_{52}(x,d)} = \frac{d_3^+}{\epsilon} \le 1 \tag{5.148}$$

$$\left.\begin{array}{l} \epsilon \le d_o \le 42 \\ \epsilon \le x_1, x_2, d_3^+ \le 6 \\ \epsilon \le d_1^- \le 16 \\ \epsilon \le d_2^+ \le 10 \end{array}\right| \tag{5.149}$$

where the bounds on the variables in (5.149) are artificial bounds (see subsection 5.7.3).

Step 3 By guessing, we consider $(x,d)^{o,1}$ to be a feasible solution point to program (5.143)-(5.149)

where: $(x,d)^{o,1} = \{ d_o = 42,\ x_1 = 4.53,\ x_2 = 1.46,\ d_1^- = 16,$

$d_2^+ = 10,\ d_3^+ = \epsilon \}$

Step 4 Now, we solve $(g\,g\,p)_2$ using the phase 2 algorithm (see subsection 5.6.1) as follows:

1. We consider the condensation of posynomials [1] $1 + Q_{t2}$, $t = 2,3,4$ at the point $(\bar{x},\bar{d}) = (x,d)^{o,1}$

$$\overline{Q_{22}} = \left(\frac{x_1 x_2}{\delta_{221}(\bar{x},\bar{d})} \right)^{\delta_{221}(\bar{x},\bar{d})} \left(\frac{d_1^-}{\delta_{222}(\bar{x},\bar{d})} \right)^{\delta_{222}(\bar{x},\bar{d})}$$

$$= 1.8301\ x_1^{.2925}\ x_2^{.2925}\ (d_1^-)^{.7075} \tag{5.150}$$

$$\overline{Q_{32}} = \left(\frac{6x_1}{\delta_{321}(\bar{x},\bar{d})} \right)^{\delta_{321}(\bar{x},\bar{d})} \left(\frac{d_2^+}{\delta_{322}(\bar{x},\bar{d})} \right)^{\delta_{322}(\bar{x},\bar{d})}$$

$$= 4.6593\ x_1^{.73104}\ (d_2^+)^{.269} \tag{5.151}$$

$$\overline{Q_{42}} = \left(\frac{6}{\delta_{421}(\bar{x},\bar{d})} \right)^{\delta_{421}(\bar{x},\bar{d})} \left(\frac{d_3^+}{\delta_{422}(\bar{x},\bar{d})} \right)^{\delta_{422}(\bar{x},\bar{d})} = 6 \tag{5.152}$$

where the weights δ are computed at $(\bar{x},\bar{d}) = (x,d)^{o,1}$ according to (5.34), for example:

$$\delta_{221}(\bar{x},\bar{d}) = \frac{\bar{x}_1 \bar{x}_2}{1+Q_{22}(\bar{x},\bar{d})} = .2925 \text{ ; and}$$

---

[1] $1 + Q_{12}$ and $1 + Q_{52}$ are not condensed because they are single terms.

$$\delta_{222}(\bar{x},\bar{d}) = \frac{\bar{d}_1^-}{1+Q_{22}(\bar{x},\bar{d})} = .7075 \quad \text{etc.}$$

2. Thus $gp^{(o,1)}$ will be the following program:

minimize $\quad d_o$

subject to

$$g_{12}(x,d) = 2d_1^-(d_o)^{-1} + d_2^+(d_o)^{-1} \leq 1 \tag{5.153}$$

$$g_{22}(x,d) = 8.7427\ x_1^{-.2925}\ x_2^{-.2925}(d_1^-)^{-.7075} \leq 1 \tag{5.154}$$

$$g_{32} = .2146\ x_1^{1.269}(d_2^+)^{-.269} + .2146\ x_1^{-.7310}\ x_2^2(d_2^+)^{-.269} \leq 1 \tag{5.155}$$

$$g_{42} = 1.667\ x_1 + .1667\ x_2 \leq 1 \tag{5.156}$$

$$g_{52} = \epsilon^{-1}\ d_3^+ \leq 1 \tag{5.157}$$

where $g_{tk}(x,d) = \dfrac{P_{tk}(x,d)}{\bar{Q}_{tk}}$

3. Now, we condense g_{tk}[1], $t=1,3,4$ into single posynomial terms at the point $(\bar{x},\bar{d}) = (x,d)^{o,1}$:

$$\bar{g}_{12} = 2.9358\ (d_1^-)^{.76191}(d_o)^{-.9991} \leq 1 \tag{5.158}$$

$$\bar{g}_{22} = 8.7427\ x_1^{-.2925}x_2^{-.2925}(d_1^-)^{-.7075} \leq 1 \tag{5.159}$$

$$\bar{g}_{32} = 1.5419\ x_1^{1.1496}x_2^{.1882}(d_2^+)^{.269} \leq 1 \tag{5.160}$$

$$g_{42} = .2904\ x_1^{.7563}\ x_2^{.2437} \leq 1 \tag{5.161}$$

---

[1] g_{22} and g_{52} are single posynomial terms and do not need to be condensed.

$$\bar{g}_{52} = (10)^6 \, d_3^+ \leq 1 \qquad (5.162)$$

(We assume $\epsilon = (10)^{-6}$) , see subsection 5.7.3).

4. Using transformations (5.53) and (5.54)

$$z_o = \ln d_o - \ln d_o^{LB} \quad , \quad z_o^{UB} = \ln d_o^{UB} - \ln d_o^{LB} \qquad (5.163)$$

$$z_1 = \ln x_1 - \ln x_1^{LB} \quad , \quad z_1^{UB} = \ln x_1^{UB} - \ln x_1^{LB} \qquad (5.164)$$

$$z_2 = \ln x_2 - \ln x_2^{LB} \quad , \quad z_2^{UB} = \ln x_2^{UB} - \ln x_2^{LB} \qquad (5.165)$$

$$z_3 = \ln d_1^- - \ln (d_1^-)^{LB}, \quad z_3^{UB} = \ln (d_1^-)^{UB} - \ln (d_1^-)^{LB} \qquad (5.166)$$

$$z_4 = \ln d_2^+ - \ln (d_2^+)^{LB} \quad , \quad z_4^{UB} = \ln (d_2^+)^{UB} - \ln (d_2^+)^{LB} \qquad (5.167)$$

$$z_5 = \ln d_3^+ - \ln (d_3^+)^{LB} \quad ; \quad z_5^{UB} = \ln (d_3^+)^{UB} - \ln (d_3^+)^{LB} \qquad (5.168)$$

We obtain the program $LP^{(o,1)}$

minimize $\quad z_o$ (5.169)

subject to

$$\bar{g}_{12}(z,(\bar{x},\bar{d})) = .7619 \, z_3 - .9991 \, z_o \leq -4.3544 \qquad (5.170)$$

$$\bar{g}_{22}(z,(\bar{x},\bar{d})) = -.2925 \, z_1 - .2925 \, z_2 - .7075 \, z_3 \leq -20.0248 \qquad (5.171)$$

$$\bar{g}_{32}(z,(\bar{x},\bar{d})) = 1.1496 \, z_1 + .1882 \, z_2 + .269 \, z_4 \leq 21.7327 \qquad (5.172)$$

$$\bar{g}_{42}(z,(\bar{x},\bar{d})) = .7563 \, z_1 + .2437 \, z_2 \leq 15.0519 \qquad (5.173)$$

$$\bar{g}_{52}(z,(\bar{x},\bar{d})) = z_5 \leq 0 \qquad (5.174)$$

$$\left.\begin{array}{l} 0 \leq z_o \leq 17.55 \\ 0 \leq z_1, z_2, z_5 \leq 15.607 \\ 0 \leq z_3 \leq 16.588 \\ 0 \leq z_4 \leq 16.118 \end{array}\right\} \qquad (5.175)$$

5. The solution to the above program is $(x,d)^{1,1}$: $d_o = 2.867$, $x_1 = .3378$, $x_2 = 6$, $d_1^- = 16$, $d_2^+ = .00013$ and $d_3^+ = \epsilon$. The values of the $(g\,gp)_2$ constraints (5.144)-(5.148) at $(x,d)^{1,1}$ are:

$$G_{12} = 11.1615 > 1 \tag{5.176}$$

$$G_{22} = .8876 < 1 \tag{5.177}$$

$$G_{32} = 17.8174 > 1 \tag{5.178}$$

$$G_{42} = 1.0563 > 1 \tag{5.179}$$

$$G_{52} = 1 \tag{5.180}$$

6. Constraints G_{12}, G_{32}, G_{42} are violated at point $(x,d)^{1,1}$. So we linearize G_{32} (see (5.67)) at the present solution $(x,d)^{1,1}$

$$\bar{g}_{32}(z,(x,d)^{1,1}) = -.99303\, z_1 + 1.994\, z_2 - .00006\, z_4 \le 15.599 \tag{5.181}$$

Inequality (5.181) represents cut number 1 .

7. (5.181) is added to program $LP^{(o,1)}$ to obtain $LP^{(1,1)}$. The solution to $LP^{(1,1)}$ is $(x,d)^{2,1}$:

$d_o = 8.371$, $x_1 = 1.3134$, $x_2 = 1.5432$, $d_1^- = 16$, $d_2^+ = .01143$ and $d_3^+ = (10)^{-6}$.

The values of the $(g\,gp)_2$ constraints (5.144)-(5.148) at $(x,d)^{2,1}$ are:

$$G_{12} = 3.8241 > 1 \tag{5.182}$$

$$G_{22} = .8876 < 1 \tag{5.183}$$

$$G_{32} = .5204 < 1 \tag{5.184}$$

$$G_{42} = .4761 < 1 \tag{5.185}$$

$$G_{52} = 1 \tag{5.186}$$

8. From (5.182) we note that the first constraint of $(ggp)_2$ is not satisfied at the point $(x,d)^{2,1}$. Therefore we continue with constructing the cuts and solving the linear programs. After adding the 8^{th} cut, we obtain the point $(x,d)^{9,1}$, $d_o = 18.2587$, $x_1 = 3.7177$, $x_2 = 2.2704$, $d_1^- = 8.8707$, $d_2^+ = .5280$, $d_3^+ = \epsilon$.

9. The point $(x,d)^{9,1}$ satisfies the $(ggp)_2$ constraints and may be used to form the program $gp^{(o,2)}$. For $(ggp)_2$, convergence to the local minimum is shown in Table 5.1 [22] .

Table 5.1

Program $(ggp)_2$: Convergence to the local minimum

Phase 2 Iteration	Number of Cuts	Next approximating point $(d_o, x_1, x_2, d_1^-, d_2^+, d_3^+)$	Comments
0	-	$(x,d)^{o,1}$:(42, 4.53,1.46,16 , 10, 0)	
1	8	$(x,d)^{9,1}$:(18.23,3.72,2.27,8.87,.53, 0)	
2	8	$(x,d)^{9,2}$:(14.40,3.17,2.82,7.2, .01, 0)	
3	3	$(x,d)^{4,3}$:(14.07,3.08,2.91,7.04, 0 , 0)	
4	3	$(x,d)^{4,4}$:(14, 3, 3, 0, 0)	Local optimum (global also)

From Table 5.1, point $(x,d)^{4,4}$ is a local minimum. It is also a global minimum, i.e., that is the best solution to $(ggp)_2$ (See Figure 5.1).

<u>Step 5</u> Although we cannot obtain a better solution to $(ggp)_2$ than $(x,d)^{4,4}$ (see Figure 5.1), we demonstrate the use of phase 1 below:

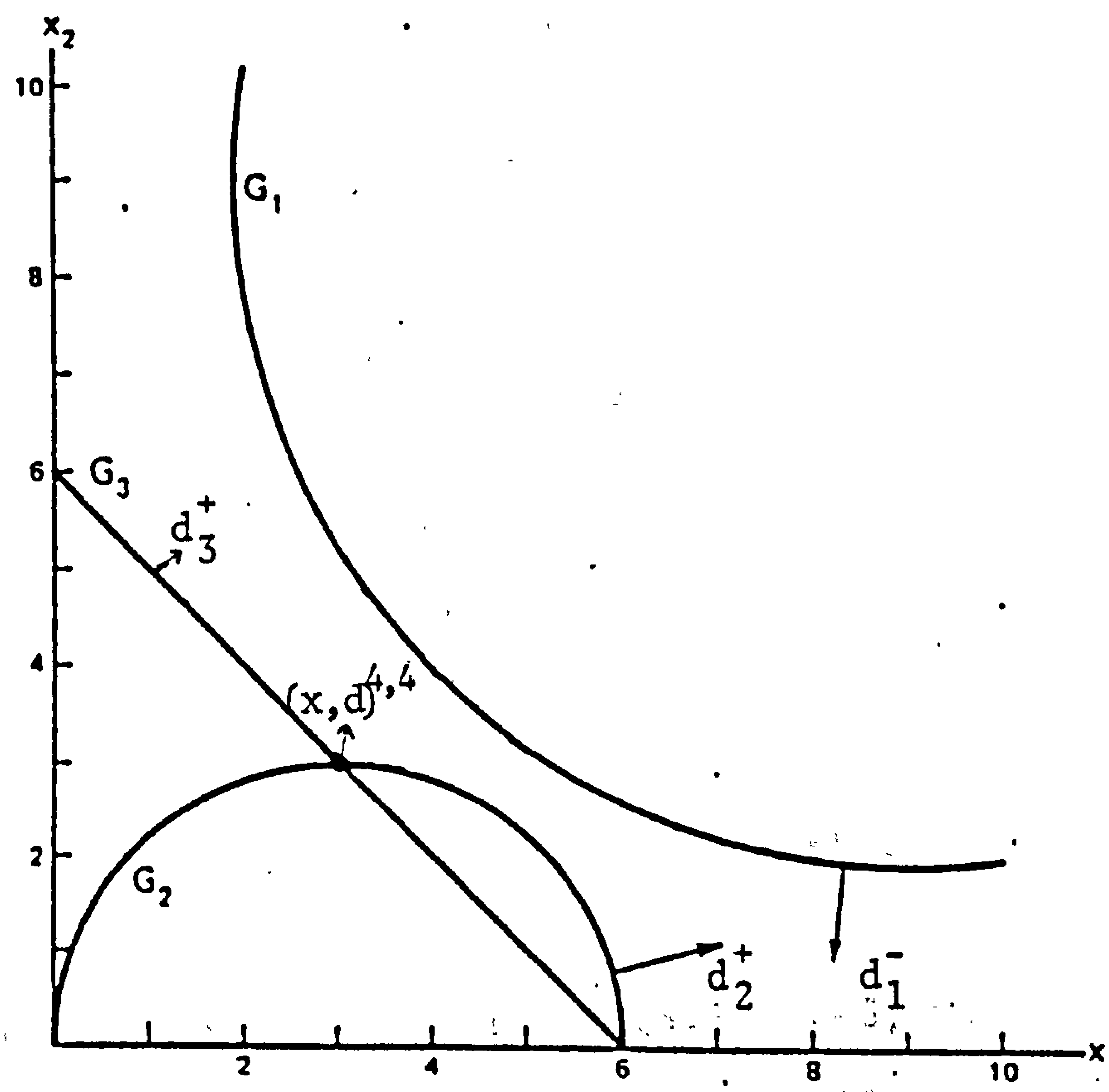

Figure 5.1. Solution to Example 5.1.

1. $(g\,g(w))_2$: minimize $\prod_{i=1}^{5} w_i$ (5.187)

subject to

$$\frac{2d_1^- + d_2^+}{d_o} \leq w_1 \tag{5.188}$$

$$\frac{16}{x_1 x_2 + d_1^-} \leq w_2 \tag{5.189}$$

$$\frac{x_1^2 + x_2^2}{6x_1 + d_2^+} \leq w_3 \tag{5.190}$$

$$\frac{x_1 + x_2}{6 + d_3^+} \leq w_4 \tag{5.191}$$

$$\frac{d_3^+}{\epsilon} \leq w_5 \tag{5.192}$$

$$\left.\begin{array}{ll} \epsilon \leq d_o \leq 13.5 & \\ \epsilon \leq x_1, x_2 . d_3^+ \leq 6 & \\ \epsilon \leq d_1^- \leq 16 & \\ \epsilon \leq d_2^+ \leq 10 & \\ 1 \leq w_i \leq 3 & i=1,2,\ldots,5 \end{array}\right\} \tag{5.193}$$

Starting at the point $d_o = (13.5,\ x_1=3,\ x_2=3,\ d_1^-=7,\ d_2^+=0,\ d_3^+=0)$. Note that d_o violates constraint (5.144). However, starting point $(x,d,w)^o$: $d_o = 13.5, x_1=3, x_2=3, d_1^-=7, d_2^+=d_3^+=0, w_i=1.5,\ i=1,2,\ldots,5$ satisfies constraints (5.188)-(5.193).

2. Solve $(g\,g\,p(w))_2$ by the phase 2 algorithm with initial feasible point $(x,d,w)^o$ as shown in Table 5.2.

Table 5.2

Program $(g\,g\,p(w))_2$: Convergence to a local minimum

Phase 1 iteration	No. of cuts	Next approximating point $(d_o,x_1,x_2,d_1^-,d_2^+,d_3^+,w_1,w_2,w_3,w_4,w_5)$	Comments
0	-	(13.5,3,3,7,0,0,2,2,2,2,2)	
1	2	(13.5,3.087,2.999,0,0,1,1,1,1.014,1)	
2	1	(13.5,3.085,2.999,6.75,0,0,1,1,1,1.014,1)	Local optimum

From Table 5.2 the local minimum point to $(g\,g\,(w))_2$ is: $d_o = 13.5$, $x_1 = 3.085$, $x_2 = 2.999$, $d_1^- = 6.75$, $d_2^+ = 0$, $d_3^+ = 0$, $w_1 = w_2 = w_3 = w_5 = 1$, $w_4 = 1.014$.

Since $w_4 = 1.04 > 1$, then the algorithm has failed to converge.

5.10 Conclusion

In this chapter, we have specified how to formulate a nonlinear goal program as a sequence of generalized geometric programs. We have also reformulated the "double condensed geometric programming" algorithm (phase 2) into one which is easier to apply. Additionally, we have presented "sequential double condensed geometric goal programming" algorithm for solving nonlinear goal programs generally and CCGP programs in particular.

Finally, the procedures of our algorithm have been illustrated by a numerical example.

CHAPTER 6

C C G P AND THE DISTRIBUTION OF EXPORTS & IMPORTS ON THE MARINE PORTS OF THE EMERGING COUNTRIES

6.1 Introduction

It is not uncommon for most of the marine ports of the emerging countries to be suffering from congestion [25,55] in some or all of the stages in the turnover [1] of the goods they handle despite the fact that other ports in the same countries do not use all their available capacities. [2]

It is generally agreed that the most important factor leading to congestion is a misdistribution of exports and imports on the ports [25,34] .

The problem of optimizing distribution of exports and imports differs in the following ways from traditional distribution problems:

(1) because there are competitive and conflicting goals, as shown in the next section;

(2) often the amounts exported and imported and the transport prices are non-negative random variables [28,35] where the random variations depend on many factors such as weather, demand and supply, ... etc.

---

[1] The stages in the turnover of exports are (i) transporting the exports from the exporting centers to the ports, (ii) storage at the ports, and (iii) loading on the quays or wharfs. The stages in the turnover of imports are (i) discharging the imports on the quays or wharfs, (ii) storage at the ports, and (iii) transporting from the ports to the importing centers.

[2] e.g. in Egypt, the port of Alexandria is usually congested although the ports at Matroh and El-ghardaka have unused capacities [73] .

In Section 6.2, we present a CCGP model to optimize distribution of exports and imports on the marine ports. In Section 6.3 the formulation and solution to the model is illustrated by a numerical example.

6.2 A CCGP model for the distribution of exports and imports

In general, any country is divided into exporting and imports centers. We consider that there are M centres. The goods exported and imported are classificated into groups according to their kinds (e.g. general goods, food-stuffs, wood, ..., etc.). In addition, the kinds of goods that are handled determine the kinds of quays, wharfs, storages that are required and the means of transport (rivers, roads, and railways) to be used [25]. We consider that there are T groups of goods. Further we assume that transport prices and the amount of exports and imports of some of the groups have exponential and χ^2 distributions respectively.

We now define the decision variables and parameters used in the model.

x_{ijt} : The amount of goods belonging to group t, $t=1,2,...,T$ Which can be exported from the i^{th} exporting center trough the j^{th} port, $i=1,2,...,M$; $j=1,2,...,N$.

A_t : The amount of goods belonging to group t, $t=1,2,...,T$ which are required to be exported. We assume that the quantities A_t for $t=1,2,...,t'$ are $\chi^2(S_t)$ random variables and for $t=t'+1,t'+2,...,T$, are constants.

y_{ijt} : The amount of goods belonging to group t which can be imported through the j^{th} port, to the i^{th} importing center, $j=1,2,...,N$; $i=1,2,...,M$.

B_t : The amount of goods belonging to group t, $t=1,2,...,T$ which are required to be imported, we assume that the quantities B_t, $t=1,2,...,t''$ are $\chi^2(S_t')$ random variables and for $t=t''+1,t''+2,...,T$, are constants.

γ_t : Is the probability that the amount of goods belonging to group t, $t=1,2,...,t'$ $(t'<T)$ which are to be exported is less than or equal to the amount which can be exported.

λ_t : Is the probability that the amount of goods belonging to group t, $t=1,2,...,t''$ $(t''<T)$ which are to be imported is less than or equal to the amount which can be imported.

L_{jt} : The loading and discharging capacity of the j^{th} port for the t^{th} group of goods. $J=1,2,...,N$ and $t=1,2,...,T$.

d_t : The transport capacity to transport goods of group t, $t=1,2,...,T$ either from the ports to the importing centers or from the exporting centers to the ports.

c_{ijt} : The price of transporting one unit of the goods belonging to group t, $t=1,2,...,T$ either from the i^{th} exporting and importing center to the j^{th} port or from the j^{th} port to the i^{th} exporting and importing center. We assume that the c_{ijt} for $i=1,2,...,m$ $(m<M)$ and $j=1,2,...,n$ $(n<N)$ are exponentially distributed random variables with parameters $(\alpha_{ijt},\sigma_{ijt})$ and that the c_{ijt} for $i=m+1,m+2,...,M$; $j=n+1,n+2,...,N$ are constants.

c_t : The total cost of transporting the goods belonging to group t.

ψ_t : The probability that the transport cost of goods belonging to group t is less than or equal to c_t, $t=1,2,\ldots,T$.

<u>Goals related to the amount exported and imported</u>

If the decision maker wants to export amount A_t and import amount B_t of goods belonging to group t, $t=1,2,\ldots,T$ such that the probabilities of exporting amount A_t, $t=1,2,\ldots,t'$ and of importing amount B_t, $t=1,2,\ldots,t''$ are γ_t and λ_t respectively., While at the same time minimizing the occurrence of congestion, these goals can be written as follows:

$$P_r\left(\sum_{i=1}^{M}\sum_{j=1}^{N} x_{ijt} \geq A_t\right) = \gamma_t \qquad t=1,2,\ldots,t' \qquad (6.1)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} x_{ijt} \geq A_t \qquad t=t'+1,t'+2,\ldots,T \qquad (6.2)$$

$$P_r\left(\sum_{i=1}^{M}\sum_{j=1}^{N} y_{ijt} \geq B_t\right) = \lambda_t \qquad t=1,2,\ldots,t'' \qquad (6.3)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} y_{ijy} \geq B_t \qquad t=t''+1,t''+2,\ldots,T \qquad (6.4)$$

Since A_t for $t=1,2,\ldots,t'$ and B_t for $t=1,2,\ldots,t''$ are $\chi^2(S_t)$, $\chi^2(S_t')$ random variables respectively, then the following transformed deterministic goals in standard from are equivalent to goals (6.1)-(6.4).

$$\sum_{i=1}^{M}\sum_{j=1}^{N} x_{ijt} + x_t^- - x_t^+ = F^{-1}(\gamma_t) \qquad t=1,2,\ldots,t' \qquad (6.5)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} x_{ijt} + a_t^- - a_t^+ = A_t \qquad t=t'+1,t'+2,\ldots,T \qquad (6.6)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} y_{ijt} + y_t^- - y_t^+ = F^{-1}(\lambda_t) \qquad t=1,2,\ldots,t'' \qquad (6.7)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} y_{ijt} + b_t^- - b_t^+ = B_t \qquad t=t''+1,t''+2,\ldots,T \qquad (6.8)$$

(For goals (6.5),(6.7), see Section 4.3; for goals (6.6), (6.8), Section 1.2).

Where

$F^{-1}(\gamma_t)$ and $F^{-1}(\lambda_t)$: are the inverse functions of the cumulative functions of the variables $\chi^2(S_t)$ and $\chi^2(S_t')$ respectively.

x_t^- and y_t^- : are the lower levels of the amounts of goods belonging to group t that cannot be exported (i.e. that cannot be arrived to the ports from the exporting centers or arrived to ports and cannot be loaded) or arrived at the importing centers, with probabilities $(1-\gamma_t)$ and $(1-\lambda_t)$ respectively. They represent the blocking in the ports or in the means of transport.

x_t^+ and y_t^+ : are the lower levels of the additional amounts of goods belonging to group t that can be exported or imported, with probabilities γ_t and λ_t respectively.
(See third section 3.3).

a_t^- and b_t^- : are the amounts of goods belonging to group t that cannot be exported or arrived at the importing centers. They represent the blocking in the ports or in the means of transport.

a_t^+ and b_t^+ : are the additional amounts of goods belonging to group t that can be exported or imported respectively.

The loading and discharging goals

The purpose of these goals is to minimize the occurrence of congestion (in any port, of any group of goods) arising from the loading and discharging processes. These goals can be formulated as follows:

$$\sum_{i=1}^{M}(x_{ijt} + y_{ijt}) + L_{jt}^- - L_{jt}^+ = L_{jt} \qquad j=1,2,\ldots,N \quad t=1,2,\ldots,T \qquad (6.9)$$

L_{jt}^- and L_{jt}^+ are respectively the under-achievement and the over-achievement of the loading and discharging capacity of goods of group t in the j^{th} port.

The transport goals

The purpose of these goals is to minimize the occurrence of congestion arising from the transport capacities. They are:

$$\sum_{i=1}^{M}\sum_{j=1}^{N}(x_{ijt} + y_{ijt}) + d_t^- - d_t^+ = d_t \qquad t=1,2,\ldots,T \qquad (6.10)$$

Where

d_t^- and d_t^+ are respectively the under-achievement and the over-achievement of the transport capacities in transporting goods of group t.

The transport cost goals

The purpose of these goals is to minimize the total transport cost between the ports and the exporting and importing centers given that the probability that the total transport cost of the t^{th} group of goods is less than or equal to c_t, is greater than or equal to ψ_t, $t=1,2,\ldots,T$. These goals can be written as:

$$P_r\left(\sum_{i=1}^{M}\sum_{j=1}^{N} c_{ijt}\,(x_{ijt}+y_{ijt}) \le c_t\right) = \psi_t \qquad t=1,2,\ldots,T \tag{6.11}$$

Since each c_{ijt}, $i=1,2,\ldots,m$; $j=1,2,\ldots,n$ has an exponential distribution with parameters $(\alpha_{ijt},\sigma_{ijt})$ then, from (3.69), the following transformed deterministic goals in standard form are equivalent to goals (6.11):

$$1 - \left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\prod_{dd'\ne ij}\left[1-\frac{\sigma_{dd't}(x_{dd't}+y_{dd't})}{\sigma_{ijt}(x_{ijt}+y_{ijt})}\right]^{-1} \exp - \left[c_t - \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ijt}(x_{ijt}+y_{ijt}) - \sum_{i=m+1}^{M}\sum_{j=n+1}^{N} c_{ijt}(x_{ijt}+y_{ijt})\right] \Big/ \sigma_{ijt}(x_{ijt}+y_{ijt})\right\} + \psi_t^- - \psi_t^+ = \psi_t \qquad t=1,2,\ldots,T \tag{6.12}$$

where

$$0 \le \psi_t^-, \psi_t^+ \le 1, \quad \psi_t^- \cdot \psi_t^+ = 0 \qquad t=1,2,\ldots,T \tag{6.13}$$

See results 3.1 and 3.2.

The achievement function

Since the decision-maker's objective is to decrease the occurrence of congestion and to minimize the total transport cost then one possible priority structure is:

first priority: to minimize the exporting and importing amounts or their lower levels that cannot be exported or arrived to importing centers. These amounts represent a blocking in the ports or in the means of transport.

The quantity to be minimized is:

$$\sum_{t}^{T} [(x_t^- + a_t^-) + (y_t^- + b_t^-)]$$

second priority: to minimize the over-achievement of the loading and discharing capacities and the over-achievement of the transport capacities. The quantity to be minimized is:

$$(\sum_{j}^{N} \sum_{t}^{T} L_{jt}^+) + (\sum_{t}^{T} d_t^+)$$

third priority: to minimize the probabilities that the transport cost goals are not satisfied.

The quantity to be minimized is:

$$(\sum_{t}^{T} \psi_t^-)$$

This priority structure will yield the following goal program.

Find x_{ijt}, y_{ijt} for $i=1,2,\ldots,M$; $j=1,2,\ldots,N$; $t=1,2,\ldots,T$

So as to

$$\text{lexico-min} \quad a = \left\{ (\sum_{t=1}^{t'} x_t^- + \sum_{t=t'+1}^{T} a_t^- + \sum_{t=1}^{t''} y_t^- + \sum_{t=t''+1}^{T} b_t^-), \right.$$

$$\left. (\sum_{t=1}^{T} \sum_{j=1}^{N} L_{jt}^+ + d_t^+), (\sum_{t=1}^{T} \psi_t^-) \right\} \qquad (6.13)$$

subject to

$$\sum_{i=1}^{M} \sum_{j=1}^{N} x_{ijt} + x_t^- - x_t^+ = F^{-1}(\gamma_t) \qquad t=1,2,\ldots,t' \qquad (6.14)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} x_{ijt} + a_t^- - a_t^+ = A_t \qquad t=t'+1,t'+2,\ldots,T \qquad (6.15)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} y_{ijt} + y_t^- - y_t^+ = F^{-1}(\lambda_t) \qquad t=1,2,\ldots,t'' \qquad (6.16)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} y_{ijt} + b_t^- - b_t^+ = B_t \qquad t=t''+1,t''+2,\ldots,T \qquad (6.17)$$

$$\sum_{i=1}^{M} (x_{ijt} + y_{ijt}) + L_{jt}^- - L_{jt}^+ = L_{jt} \qquad j=1,2,\ldots,N \quad t=1,2,\ldots,T \qquad (6.18)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} (x_{ijt} + y_{ijt}) + d_t^- - d_t^+ = d_t \qquad t=1,2,\ldots,T \qquad (6.19)$$

$$1 - \left\{ \sum_{i=1}^{m}\sum_{j=1}^{n} \prod_{dd'\neq ij} \left[1 - \frac{\sigma_{dd't}(x_{dd't} + y_{dd't})}{\sigma_{ijt}(x_{ijt}+y_{ijt})}\right]^{-1} \exp - \left[c_t - \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ijt}(x_{ijt} \pm y_{ijt}) - \sum_{i=m+1}^{M}\sum_{j=n+1}^{N} c_{ijt}(x_{ijt} + y_{ijt}) \right] / \sigma_{ijt}(x_{ijt}+y_{ijt}) \right\} + \psi_t^- - \psi_t^+ = \psi_t \qquad t=1,2,\ldots,T \qquad (6.20)$$

The equivalent signomial program:

The above program is equivalent to the following signomial program (see subsection 3.4.2).

Find x_{ijt}, y_{ijt} for $i=1,2,\ldots,M$, $j=1,2,\ldots,N$, $t=1,2,\ldots,T$

So as to

$$\text{lexico-min} \quad a = \left\{ \left(\sum_{t=1}^{t'} x_t^- + \sum_{t=t'+1}^{T} a_t^- + \sum_{t=1}^{t''} y_t^- + \sum_{t=t''+1}^{T} b_t^-\right), \left(\sum_{t=1}^{T}\sum_{j=1}^{N} L_{jt}^+ + \sum_{t=1}^{T} d_t^+\right), \left(\sum_{t=1}^{T} \psi_t^-\right) \right\} \qquad (6.21)$$

subject to

$$\sum_{i=1}^{M}\sum_{j=1}^{N} x_{ijt} + x_t^- - x_t^+ = F^{-1}(\gamma_t) \qquad t=1,2,\ldots,t' \qquad (6.22)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} x_{ijt} + a_t^- - a_t^+ = A_t \qquad t=t'+1,t'+2,\ldots,T \qquad (6.23)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} y_{ijt} + y_y^- - y_t^+ = F^{-1}(\lambda_t) \qquad t=1,2,\ldots,t'' \qquad (6.24)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} y_{ijt} + b_t^- - b_t^+ = B_t \qquad t=t''+1,t''+2,\ldots,T \qquad (6.25)$$

$$\sum_{i=1}^{M} (x_{ijt} + y_{ijt}) + L_{jt}^- - L_{jt}^+ = L_{jt} \qquad j=1,2,\ldots,N \quad t=1,2,\ldots,T \qquad (6.26)$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} (x_{ijt} + y_{ijt}) + d_t^- - d_t^+ = d_t \qquad t=1,2,\ldots,T \qquad (6.27)$$

$$1 - \left\{ \sum_{i=1}^{m}\sum_{j=1}^{n} \prod_{dd'\neq ij} \left[1 - \frac{\sigma_{dd't}(x_{dd't}+y_{dd't})}{\sigma_{ijt}(x_{ijt}+y_{ijt})}\right]^{-1} \beta_{ijt}^{\phi} \right\} +$$

$$\psi_t^- - \psi_t^+ = \psi_t \qquad t=1,2,\ldots,T \qquad (6.28)$$

$$\sigma_{ijt} z_{ijt}(x_{ijt}+y_{ijt}) + \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ijt}(x_{ijt}+y_{ijt}) +$$

$$\sum_{i=m+1}^{M}\sum_{j=n+1}^{N} c_{ijt}(x_{ijt}+y_{ijt}) = c_t \qquad i=1,2,\ldots,m \quad j=1,2,\ldots,n \quad t=1,2,\ldots,T \qquad (6.29)$$

$$\beta_{ijt} + \phi^{-1} z_{ijt} = 1 \qquad i=1,2,\ldots,m \quad j=1,2,\ldots,n \quad t=1,2,\ldots,T \qquad (6.30)$$

where

$$z_{ijt} = \Big[c_t - \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ijt}(x_{ijt}+y_{ijt}) - \sum_{i=m+1}^{M}\sum_{j=n+1}^{N}c_{ijt}(x_{ijt}+y_{ijt})\Big] \Big/ \sigma_{ijt}(x_{ijt}+y_{ijt})$$

$$z_{ijt} \geq 0 \qquad (6.31)$$

$$\beta_{ijt} = 1 - \phi^{-1} z_{ijt} \qquad \beta_{ijt} \geq 0 \qquad (6.32)$$

and $\phi \to \infty$

This program can be solved using the algorithm presented in Section 5.8.

6.3 A Numerical Example

We consider program (6.21)-(6.30) in Section 6.2 and assume two exporting and importing centers $i=1,2$; two ports $j=1,2$ and two groups of goods $t=1,2$. Such that:

$$\begin{array}{lll}
A_1 \sim \chi^2(70), & \gamma_1 = .90, & A_2 = 100 \\
B_1 = 50, & B_2 \sim \chi^2(50), & \lambda_2 = .90 \\
L_{11} = 80, & L_{21} = 80 & \\
L_{12} = 100, & L_{22} = 50 & \\
d_1 = 150, & d_2 = 90 & \\
c_{211} = 10, & c_{221} = 8 & \\
c_{212} = 5, & c_{222} = 12 & \\
c_1 = 600, & c_2 = 900 & \\
\psi_1 = .70, & \psi_2 = .60 &
\end{array}$$

and c_{111}, c_{121}, c_{112}, c_{122} have exponential distributions with parameters:

$(\alpha_{111} = 5 \quad , \quad \sigma_{111} = 2\ )$

$(\alpha_{121} = 4 \quad , \quad \sigma_{121} = .6)$

$(\alpha_{112} = 3 \quad , \quad \sigma_{112} = 1\ )$

$(\alpha_{122} = 6 \quad , \quad \sigma_{122} = .5)$

respectively.

Substituting the above values of the parameters in program (6.21)-(6.30), we have the following program:

Find x_{ijt}, y_{ijt} for $i=1,2$; $j=1,2$ and $t=1,2$

so as to

$$\text{lexico-min} \quad a = \left\{(x_1^- - a_2^- + y_2^- + b_1^-),\ (L_{11}^+ + L_{12}^+ + L_{21}^+ + L_{22}^+ + d_1^+ + d_2^+)\ ,\ (\psi_1^- + \psi_2^-)\right\} \tag{6.33}$$

subject to

$$x_{111} + x_{121} + x_{211} + x_{221} + x_1^- - x_1^+ = F^{-1}(.90) = 100.4 \tag{6.34}$$

$$x_{112} + x_{122} + x_{212} + x_{222} + a_2^- - a_2^+ = 100 \tag{6.35}$$

$$y_{111} + y_{121} + y_{211} + y_{221} + b_1^- - b_1^+ = 50 \tag{6.36}$$

$$y_{112} + y_{122} + y_{212} + y_{222} + y_2^- - y_2^+ = F^{-1}(.90) = 76.2 \tag{6.37}$$

$$x_{111} + y_{111} + x_{211} + y_{211} + L_{11}^- - L_{11}^+ = 80 \tag{6.38}$$

$$x_{121} + y_{121} + x_{221} + y_{221} + L_{21}^- - L_{21}^+ = 80 \tag{6.39}$$

$$x_{112} + y_{112} + x_{212} + y_{212} + L_{12}^- - L_{12}^+ = 100 \tag{6.40}$$

$$x_{122} + y_{122} + x_{222} + y_{222} + L_{22}^- - L_{22}^+ = 50 \tag{6.41}$$

$$x_{111} + y_{111} + x_{121} + y_{121} + x_{211} + y_{211} + x_{221} + y_{221} + d_1^- - d_1^+ = 150 \tag{6.42}$$

$$x_{112} + y_{112} + x_{122} + y_{122} + x_{212} + y_{212} + x_{222} + y_{222} + d_2^- - d_2^+ = 90 \tag{6.43}$$

$$1 - \left\{ \frac{2(x_{111}+y_{111})\beta_{111}^{\phi} - .6(x_{121}+y_{121})\beta_{121}^{\phi}}{2(x_{111}+y_{111}) - .6(x_{121}+y_{121})} \right\} + \psi_1^- - \psi_1^+ = .70 \qquad (6.44)$$

$$1 - \left\{ \frac{1(x_{112}+y_{112})\beta_{112}^{\phi} - .5(x_{122}+y_{122})\beta_{122}^{\phi}}{1(x_{112}+y_{112}) - .5(x_{122}+y_{122})} \right\} + \psi_2^- - \psi_2^+ = .60 \qquad (6.45)$$

$$2z_{111}(x_{111}+y_{111}) + 5(x_{111}+y_{111}) + 4(x_{121}+y_{121}) + 10(x_{211}+y_{211}) + 8(x_{221}+y_{221}) = 600 \qquad (6.46)$$

$$.6z_{121}(x_{121}+y_{121}) + 5(x_{111}+y_{111}) + 4(x_{121}+y_{121}) + 10(x_{211}+y_{211}) + 8(x_{221}+y_{221}) = 600 \qquad (6.47)$$

$$z_{112}(x_{112}+y_{112}) + 3(x_{112}+y_{112}) + 6(x_{122}+y_{122}) + 5(x_{212}+y_{212}) + 12(x_{222}+y_{222}) = 900 \qquad (6.48)$$

$$.5z_{122}(x_{122}+y_{122}) + 8(x_{112}+y_{112}) + 6(x_{122}+y_{122}) + 5(x_{212}+y_{212}) + 12(x_{222}+y_{222}) = 900 \qquad (6.49)$$

$$\begin{aligned} \beta_{111} + \phi^{-1}z_{111} &= 1 \\ \beta_{121} + \phi^{-1}z_{121} &= 1 \\ \beta_{112} + \phi^{-1}z_{112} &= 1 \\ \beta_{122} + \phi^{-1}z_{122} &= 1 \end{aligned}$$

The solution to this program using the algorithm presented in Section 5.8 (see the solution to example 3.1, Appendix D) is:

$$a^* = \{0,\ 112.8,\ 0\}$$

$$\begin{aligned} x_{111} &= 20.4, & y_{111} &= 50 \\ x_{121} &= 80, & y_{121} &= 0 \\ x_{211} &= 0, & y_{211} &= 0 \end{aligned}$$

$x_{221} = 0$,	$y_{221} = 0$
$x_{222} = 0$,	$y_{222} = 0$
$x_{212} = 0$,	$y_{212} = 0$
$x_{112} = 50$,	$y_{112} = 76.2$
$x_{122} = 50$,	$y_{122} = 0$
$x_1^- = 0$,	$x_1^+ = 0$
$a_2^- = 0$,	$a_2^+ = 0$
$b_1^- = 0$,	$b_1^+ = 0$
$y_2^- = 0$,	$y_2^+ = 0$
$L_{11}^- = 9.6$,	$L_{11}^+ = 0$
$L_{21}^- = 0$,	$L_{21}^+ = 0$
$L_{12}^- = 0$,	$L_{12}^+ = 26.2$
$L_{22}^- = 0$,	$L_{22}^+ = 0$
$d_1^- = 0$,	$d_1^+ = .4$
$d_2^- = 0$,	$d_2^+ = 86.2$
$\psi_1^- = 0$,	$\psi_1^+ = .078$
$\psi_2^- = 0$,	$\psi_2^+ = .184$

6.4 Conclusion

In this chapter, we present a CCGP model to optimise the distribution of the amounts exported and imported by the marine ports. A numerical example is presented to illustrate the use of the model and its solution.

The model allows a decision-maker:

1. To determine the optimum method of distributing exports and imports, taking into account the priorities of the goals and the probabilities that the goals are not satisfied and hence to estimate the risk involved.
2. To construct schedules to determine the amounts of goods to be exported and imported by each port; either to avoid congestion in any stage of the turnover of goods or to minimize its cost.
3. To determine whether congestion is caused solely by a misdistribution of the goods to be exported and imported by the ports or rather by such a misdistribution together with some or all of the other factors mentioned in Section 6.1.
4. To estimate the amount and the kind of new investments to put into the existing ports and/or to determine where to construct new ports and what their specifications should be.

CHAPTER 7

SUMMARY AND SUGGESTIONS FOR FURTHER RESEARCH

In this chapter, we give a summary of the research work presented in this thesis and offer some suggestions for further research.

7.1 The Contributions and Summary of the Thesis

The general objective behind this research was to develop the approach of chance-constrained linear goal programming when the parameters in the goal set are random variables having non-negative distributions. Two possible distributions were considered for those parameters: the exponential and the chi-square distributions.

The main contributions presented in this thesis are:

First, we have developed a method for transforming probabilistic linear goal programs into equivalent deterministic linear goal programs when the right hand side coefficients of the goals have exponential or chi-square distributions. Also, the probabilistic interpretation of the deviational random variables and their levels *is* presented.

Second, we have also developed a method for transforming probabilistic linear goal programs into equivalent deterministic nonlinear goal programs when the input coefficients in the goal set have exponential or chi-square distributions. We have further transformed the equivalent deterministic nonlinear goal programs into equivalent signomial goal programs.

In both cases, probabilistic deviational variables were introduced.

In addition, we have proved that Sengupta's transformation to obtain an approximate distribution for $\sum_j a_{ij}x_j$ when a_{ij}'s have chi-square distributions does not lead to a solvable program.

Third, we have presented a set of propositions which make it possible to formulate a nonlinear goal program as a sequence of generalized geometric programs and developed an algorithm "the sequential double condensed geometric goal programming algorithm" to solve nonlinear goal programs generally, and the signomial goal programs equivalent to the transformed deterministic nonlinear goal programs in particular.

Fourth, we have formulated the problem of optimizing the distribution of exports and imports on marine ports and solved it using methods presented in the thesis and the sequential double condensed geometric goal programming algorithm.

We now summarize the contents of each chapter.

Chapter 1: The fundamental concepts of goal programming and the standard form of a goal program are presented through an account of the historical development of goal programming. In addition, the sequential goal programming algorithm due to Dauer & Krueger is presented because any optimization algorithm appropriate to the problem under consideration can be incorporated in it for solving linear or nonlinear goal programs as is shown.

Chapter 2: A brief account is given of the main works presenting the study and applications of probabilistic linear goal programming. The most important drawbacks of these studies are determined and we indicate the points about which more research is needed. Further, the effective factors which lead us to use a chance-constrained programming approach to study probabilistic linear goal programming are given.

Chapter 3: The chance-constrained goal programming approach with linear goals having exponentially distributed parameters is presented. The probabilistic interpretation of the deviational random variables and their levels is given.

Chapter 4: This chapter deals with the approach of chance-constrained goal programming when the linear goals have chi-square distributed parameters. In addition, it contains the proof that Sengupta's transformation to obtain equivalent deterministic goal programs when the input coefficients of the goals have chi-square distributions, does not lead to a solvable program.

Chapter 3 and 4 show that the study of chance-constrained programming when the input coefficients have exponential or chi-square distributions, is closely related to the methods for solving nonlinear goal programs.

Chapter 5: Here, a condensed geometric programming technique is employed to solve nonlinear goal programs, this is the first time for this to be done.

The formulation of subprograms of a goal program as generalized geometric programs and a "sequential double condensed geometric goal programming" algorithm are presented. This algorithm is constructed by combining a "sequential goal programming" algorithm with a "double condensed geometric programming" algorithm. Therefore, the fundamental concepts of the geometric programming technique, and the algorithms for solving condensed geometric programs which are necessary for a "double condensed geometric programming" algorithm are presented.

Chapter 6: The formulation of the "exports and imports distribution" problem in the emerging countries using a chance-constrained goal programming model has been presented. The model is transformed into a deterministic nonlinear goal program using the method presented previously.

Finally, a simple numerical example is given to illustrate the formulation and the solution to the model.

7.2 Suggestions For Further Research

The research work described in this thesis can be developed in several directions.

First, more research is needed about the chance-constrained goal programming approach when some right hand side coefficients b_i for $i = 1,2,\ldots, M$ or some single goal input coefficients a_{ij}, $j = 1,2,\ldots,N$, are dependent random variables and have exponential or chi-square distributions. We think that the use of a multivariate exponential distribution is important in these cases.

Second, the "sequential double condensed geometric goal programming" algorithm requires a more efficient algorithm than the phase 1 algorithm to obtain the starting points.

Third, it was shown that the study of chance-constrained goal programming when some of the parameters are non-negative random variables is closely related to nonlinear goal programming. As yet, three only, nonlinear programming methods have been employed to solve nonlinear goal problems. Hence, more research is needed about methods for solving nonlinear goal problems, especially since, most real world problems are formulated as nonlinear goal programming models.

Fourth, combining the chance-constrained goal programming approach and the interactive sequential goal programming approach is important for solving probabilistic multiple-objective decision problems. These problems involve trade-off decisions. This combining will provide the decision maker with a learning process about the system.

Fifth, in most real life situations, the solution is only part of the information that is really needed. Often, more important than obtaining a solution to the problem is to obtain information that will enable us to improve the system itself. We can obtain such information using sensitivity analysis. However, it appears to us that, for chance-constrained goal programming, the study of the use of sensitivity analysis for the tolerance measures or the parameters of the probability distributions has not been touched upon.

APPENDIX A

Logarithmic and Exponential Terms In Signomial Form

In many mathematical models related to the real-world logarithmic or exponential terms often appear in the formulation.

We can transform these terms into signomialform (see definition 5.3) by using limiting approximations as follows [3].

First: logarithmic terms

From elementary calculus, the logarithm of an arbitrary real number x is defined by

$$\ln(x) = \int_1^x \frac{1}{y}\, dy = \int_1^x y^{-1}\, dy \tag{A.1}$$

Suppose that we define an arbitrary small positive quantity ϵ, and restructure the above equation in the following manner:

$$\ln(x) \simeq \int_1^x y^{\epsilon-1}\, dy \tag{A.2}$$

Hence,

$$\ln(x) \simeq \frac{x^\epsilon}{\epsilon} - \frac{1}{\epsilon} = \epsilon^{-1}x^\epsilon - \epsilon^{-1} \tag{A.3}$$

and

$$\lim_{\epsilon \to 0} [\epsilon^{-1}x^\epsilon - \epsilon^{-1}] = \ln x \tag{A.4}$$

This procedure is valid numerically, since it is easily seen that as ϵ approaches 0 then $\epsilon^{-1}x^\epsilon - \epsilon^{-1}$ is very close to $\ln x$ as shown in Table A.1.

Table A.1

$x \backslash \epsilon$	$(10)^{-1}$	$(10)^{-2}$	$(10)^{-3}$	$(10)^{-4}$	$(10)^{-5}$	$(10)^{-6}$	ln x
1	0	0	0	0	0	0	0
2	.718	.696	.6934	.6932	.693	.693	.693
3	1.1612	1.1047	1.0992	1.0987	1.0986	1.0986	1.0986
4	1.487	1.3959	1.3876	1.3864	1.3863	1.3862	1.3863
5	1.7462	1.6225	1.6107	1.6096	1.6095	1.6094	1.6094

(the value in row x and column ϵ represents the value $\epsilon^{-1}x^{\epsilon} - \epsilon^{-1}$).

Second: exponential terms

From the calculus also, it is well knwon that:

$$e^{x} = \lim_{\phi \to \infty} (1 + \frac{x}{\phi})^{\phi} \quad (A.5)$$

Hence, $e^{x} \simeq (1 + x\,\phi^{-1})^{\phi}$ (A.6)

where $\phi \to \infty$

APPENDIX B

The Integration Of A Product Of Exponential And Rational Functions

If y is a random variable and n , a are constants such that n is a non-negative integer number then [1]:

$$\int y^n e^{ay}\,dy = e^{ay}\left(\frac{y^n}{a} + \sum_{k=1}^{n} (-1)^k \frac{n(n-1)(n-2)\ \dots\ (n-k+1)}{a^{k+1}}\, y^{n-k}\right) \tag{B.1}$$

Hence

$$P_r(\chi^2(2(g_{ij}-h)) > b_i/x_j)$$

$$= \int_{b_i/x_j}^{\infty} \frac{1}{\Gamma(g_{ij}-h)} \left(\frac{1}{2}\right)^{(g_{ij}-h)} y^{(g_{ij}-h)-1}\ e^{-\frac{1}{2}y}\,dy$$

$$= \frac{2^{-(g_{ij}-h)}}{(g_{ij}-h-1)!} \int_{b_i/x_j}^{\infty} y^{(g_{ij}-h)-1}\ e^{-\frac{1}{2}y}\,dy \tag{B.2}$$

Substituting (B.1) in (B.2)

$$P_r(\chi^2(2(g_{ij}-h)) > b_i/x_j)$$

$$= \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!} \Big\{ e^{-\frac{1}{2}b_i/x_j} \Big[(b_i x_j^{-1})^{g_{ij}-h-1} + \sum_{t=1}^{g_{ij}-h-1} 2^t\, (g_{ij}-h-1)(g_{ij}-h-2)\ \dots\ (g_{ij}-h-t)\,(b_i x_j^{-1})^{g_{ij}-h-t-1} \Big] \Big\}$$

$$g_{ij}-h \geq 1 \tag{B.3}$$

---

[1] I.S. Goradshteyn and I.M. Ryzlik (1965): "Table of integrals series and products" Academic Press, New York and London.

Also

$$P_r\left(\chi^2(2(g_{ij}-h) > \frac{b_i - \sum_{j=n+1}^{N} a_{ij}x_j}{x_j}\right)$$

$$= \frac{2^{-(g_{ij}-h-1)}}{(g_{ij}-h-1)!}\left\{e^{-\frac{1}{2}(x_j^{-1}b_i - x_j^{-1}\sum_{j=n+1}^{N} a_{ij}x_j)}\left[(x_j^{-1}b_i - x_j^{-1} * \sum_{j=n+1}^{N} a_{ij}x_j)^{g_{ij}-h-1} + \sum_{t=1}^{g_{ij}-h-1} 2^t(g_{ij}-h-1) * (g_{ij}-h-2)\ldots(g_{ij}-h-t)(x_j^{-1}b_i - x_j^{-1} * \sum_{j=n+1}^{N} a_{ij}x_j)^{g_{ij}-h-t-1}\right]\right\}$$

$$g_{ij} - h \geq 1 \qquad \text{(B.4)}$$

APPENDIX C

The Mean And Variance of n_{kj}

This Appendix presents the values of $E(n_{kj})$ and $Var(n_{kj})$ used in section 4.5.

We calculate them by Taylor's Theorem [59] as follows:

If y is written for χ^2 and y_o is the mean of y we have:

$$y^{\frac{1}{2}} = y_o^{\frac{1}{2}} + \frac{1}{2}(y-y_o)y_o^{-\frac{1}{2}} - \frac{1}{8}(y-y_o)^2 y_o^{-\frac{3}{2}} + \frac{1}{16}(y-y_o)^3 y_o^{-\frac{5}{2}}$$

$$- \frac{15}{384}(y-y_o)^4 y_o^{-\frac{7}{2}} + \ldots \quad (C.1)$$

Also, since $n_{kj}^2 \sim \chi^2(s_{kj})$,

$$E(n_{kj}^2) = s_{kj} \quad , \quad var(n_{kj}^2) = 2s_{kj}$$

then

$$n_{kj} = [E(n_{kj}^2)]^{\frac{1}{2}} + \frac{1}{2}[n_{kj}^2 - E(n_{kj}^2)][E(n_{kj}^2)]^{-\frac{1}{2}}$$

$$- \frac{1}{8}[n_{kj}^2 - E(n_{kj}^2)]^2[E(n_{kj}^2)]^{-\frac{3}{2}}$$

$$+ \frac{1}{16}[n_{kj}^2 - E(n_{kj}^2)]^3[E(n_{kj}^2)]^{-\frac{5}{2}}$$

$$- \frac{15}{385}[n_{kj}^2 - E(n_{kj}^2)]^4[E(n_{kj}^2)]^{-\frac{7}{2}} + \ldots \quad (C.2)$$

By taking expectations on both sides of (C.2) we have

$$E(n_{kj}) = s_{kj}^{\frac{1}{2}} + \frac{1}{2} E(n_{kj}^2 - s_{kj}) s_{kj}^{-\frac{1}{2}} - \frac{1}{8} E(n_{kj}^2 - s_{kj})^2 s_{kj}^{-\frac{3}{2}}$$

$$+ \frac{1}{16} E(n_{kj}^2 - s_{kj})^3 s_{kj}^{-\frac{5}{2}} - \frac{5}{128} E(n_{kj}^2 - s_{kj})^4 s_{kj}^{-\frac{7}{2}}$$

$$+ \frac{7}{256} E(n_{kj}^2 - s_{kj})^5 s_{kj}^{-\frac{9}{2}} \quad \ldots\ldots$$

$$\simeq (8s_{kj}^4 - 2s_{kj}^3 - 56s_{kj}^2 + 20s_{kj} + 84)/8s_{kj}^3 \sqrt{s_{kj}}$$

$$= (4s_{kj}^4 - s_{kj}^3 - 28s_{kj}^2 + 10s_{kj} + 42)/4s_{kj}^3 \sqrt{s_{kj}}$$

$$= A_{kj} \tag{C.3}$$

where A_{kj} is constant

Also

$$\mathrm{Var}(n_{kj}) = E(n_{kj}^2) - [E(n_{kj})]^2$$

$$\simeq s_{kj} - [(2s_{kj}^4 - s_{kj}^3 - 28s_{kj}^2 + 10s_{kj} + 42) /$$

$$4s_{kj}^3 \sqrt{s_{kj}} \,] = B_{kj} \tag{C.4}$$

where B_{kj} is constant.

APPENDIX D

The Solution to Example 3.1

1. From Section 3.5, the subprogram associated with the first level priority of program (3.145)-(3.154) is:

minimize $\quad a_1 = d_2^+ + d_3^- \qquad$ (D.1)

subject to

$$2x_1 + x_2 + x_3 + d_2^- - d_2^+ = 10.07 \qquad (D.2)$$

$$x_1 + x_2 + d_3^- - d_3^+ = 6.408 \qquad (D.3)$$

$$x, d^-, d^+ \geq 0 \qquad (D.4)$$

The above program is a linear program, the solution by the simplex method is:

$$a_1^* = d_2^+ + d_3^- = 0 \qquad (D.5)$$

2. From (3.145)-(3.154) and (D.5) the subprogram associated with the second level priority of program (3.145)-(3.154) is:

minimize $\quad a_2 = d_1^- \qquad$ (D.6)

subject to

$$2x_1 + x_2 + x_3 + d_2^- - d_2^+ = 10.07 \qquad (D.7)$$

$$x_1 + x_2 + d_3^- - d_3^+ = 6.408 \qquad (D.8)$$

$$1 - \left\{\left(1 - \frac{x_2}{x_1}\right)^{-1} \beta_{11}^{\phi} + \left(1 - \frac{x_1}{x_2}\right)^{-1} \beta_{12}^{\phi}\right\} + d_1^- - d_1^+ = .55 \qquad (D.9)$$

$$.04x_1z_{11} + .12x_1 + .16x_2 + .12x_2 = 1 \qquad (D.10)$$

$$.04x_2z_{12} + .12x_1 + .16x_2 + .12x_3 = 1 \qquad (D.11)$$

$$\beta_{11} + z_{11}\phi^{-1} = 1 \qquad (D.12)$$

$$\beta_{12} + z_{12}\phi^{-1} = 1 \qquad (D.13)$$

$$a_1^* = d_2^+ + d_3^- = 0 \tag{D.14}$$

$$x_1, x_2, x_3, d_2^-, d_2^+, d_3^-, d_3^+, z_{11}, z_{12}, \beta_{11}, \beta_{12} \geq 0 \tag{D.15}$$

$$0 \leq d_1^- \leq .55 \quad , \quad 0 \leq d_1^+ \leq .45 \tag{D.16}$$

and

$$\phi \to \infty$$

3. From Section 5.7 and inequality (5.113) the above program is equivalent to:

minimize $a_2 = d_1^-$ (D.17)

subject to

$$2x_1 + x_2 + x_3 - d_2^+ \leq 10.07 \tag{D.18}$$

$$x_1 + x_2 + d_3^- \geq 6.408 \tag{D.19}$$

$$.55 + x_1^{-1}x_2 + \beta_{11}^{\phi} + x_1^{-1}x_2 d_1^- - .55x_1^{-1}x_2 - x_1^{-1}x_2\beta_{12}^{\phi} - d_1^- \leq 1 \tag{D.20}$$

$$.04x_1z_{11} + .12x_1 + .16x_2 + .12x_3 \leq 1 \tag{D.21}$$

$$.04x_2z_{12} + .12x_1 + .16x_2 + .12x_3 \leq 1 \tag{D.22}$$

$$\beta_{11} + z_{11}\,\phi^{-1} \geq 1 \tag{D.23}$$

$$\beta_{12} + z_{12}\,\phi^{-1} \geq 1 \tag{D.24}$$

$$d_2^+ + d_3^- \leq \epsilon \tag{D.25}$$

$$x_1, x_2, x_3, d_2^-, d_2^+, d_3^-, d_3^+, z_{11}, z_{12}, \beta_{11}, \beta_{12} \geq 0 \tag{D.26}$$

$$0 \leq d_1^- \leq .55 \quad , \quad 0 \leq d_1^+ \leq .45 \tag{D.27}$$

and

$$\phi \to \infty \; , \quad \epsilon \to 0 \tag{D.28}$$

Note that equalities (D.10)-(D.13) have been replaced by inequalities (D.21)-(D.24), where inequalities (D.21)-(D.24) are tight in the optimal solution (see subsection 5.7.2).

4. From (5.120)-(5.123), the above program is equivalent to the generalized geometric program $(ggp)_2$:

$(ggp)_2$ minimize $a_2 = d_1^-$ (D.29)

subject to

$$\frac{2x_1^- + x_2 + x_3}{10.07 + d_2^+} \le 1 \tag{D.30}$$

$$\frac{6.408}{x_1 + x_2 + d_3^-} \le 1 \tag{D.31}$$

$$\frac{.55x_1 + x_2 + \beta_{11}^{\phi} x_1 + x_2 d_1^-}{x_1 + .55x_2 + x_2 \beta_{12}^{\phi} + x_1 d_1^-} \le 1 \tag{D.32}$$

$$.04x_1 z_{11} + .12x_1 + .16x_2 + .12x_3 \le 1 \tag{D.33}$$

$$.04x_2 z_{12} + .12x_1 + .16x_2 + .12x_3 \le 1 \tag{D.34}$$

$$\frac{1}{\beta_{11} + z_{11} \phi^{-1}} \le 1 \tag{D.35}$$

$$\frac{1}{\beta_{12} + z_{12} \phi^{-1}} \le 1 \tag{D.36}$$

$$\frac{d_2^+ + d_3^-}{\epsilon} \le 1 \tag{D.37}$$

$$x_1, x_2, x_3, d_2^-, d_2^+, d_3^-, d_3^+, z_{11}, z_{12}, \beta_{11}, \beta_{12} \ge \epsilon \tag{D.38}$$

$$\epsilon \le d_1^- \le .55 \quad , \quad \epsilon \le d_1^+ \le .45 \tag{D.39}$$

and

$$\phi \to \infty \, , \quad \epsilon \to 0 \tag{D.40}$$

5. Consider the initial point:

$$\{d_1^- = .55,\ x_1 = 3.69,\ x_2 = 2.73,\ x_3 = 0,\ d_2^+ = 0\ ,\ d_3^- = 0\ ,\ z_{11} = .85\ ,\ z_{12} = .37\ ,\ \beta_{11} = .999,\ \beta_{12} = 1\} \tag{D.41}$$

The point (D.41) does not satisfy constraints (D.30)-(D.37).

6. Construct $(g\,gp(W))_2$ to obtain an initial feasible point:

$\underline{(g\,gp(w))_2}$ minimize $\prod_{i=1}^{8} W_i$ (D.43)

subject to

$$\frac{2x_1+x_2+x_3}{10.07+d_2^+} \le W_1 \qquad (D.44)$$

$$\frac{6.408}{x_1+x_2+d_3^-} \le W_2 \qquad (D.45)$$

$$\frac{.55x_1+x_2+\beta_{11}^{\phi}x_1+x_2d_1^-}{x_1+.55x_2+x_2\beta_{11}^{\phi}+x_1d_1^-} \le W_3 \qquad (D.46)$$

$$.04x_1z_{11} + .12x_1 + .16x_2 + .12x_3 \le W_4 \qquad (D.47)$$

$$.04x_2z_{12} + .12x_1 + .16x_2 + .12x_3 \le W_5 \qquad (D.48)$$

$$\frac{1}{\beta_{11}+z_{11}\phi^{-1}} \le W_6 \qquad (D.49)$$

$$\frac{1}{\beta_{11}+z_{12}\phi^{-1}} \le W_7 \qquad (D.50)$$

$$\frac{d_2^+ + d_3^-}{\epsilon} \le W_8 \qquad (D.51)$$

$$W_i \ge 1 \qquad i=1,2,\ldots,8 \qquad (D.52)$$

7. The solution to $(g\,g(W))_2$ by the phase 2 algorithm (see example 5.1) is shown in Table D.1.

Table D.1

Phase 1 iteration	No. of Cuts	Next approximating point $(d_1^-,x_1,x_2,x_3,d_2^+,d_3^-,z_{11},z_{12},\beta_{11},\beta_{12})$	Comments
0	-	(.55,3.69,2.73,0,0,0,.85,.37,.99,1)	not feasible
1	3	(.55,3.642,2.772,0,0,0,.82,.037, .999,1)	feasible

8. We consider the feasible point as an initial point. Using the Phase 2 algorithm the optimal solution to $(ggp)_2$ is computed. The result is shown in Table D.2 [22] .

Table D.2

Phase 3 iterations	No. of Cuts	Next approximating point $(d_1^-,x_1,x_2,x_3,d_2^+,d_3^-,z_{11},z_{12},\beta_{11},\beta_{12})$	Comments
0	-	(.55,3.642,2.772,0,0,0,.82,.037, .999,1)	
1	3	(0,2.77,3.96,0,0,0,.302,.014,1,1)	
2	13	(0,3.204,3.204,0,0,0,.6773,.6882, .9998,.9999)	local and global solution

Hence, the global solution to example 3.1 is:

$$a_2^* = 0$$

$$x_1 = 3.204 \quad , \quad x_2 = 3.204 \quad , \quad x_3 = 0$$

$$d_1^- = 0 \quad , \quad d_1^+ = .45$$

$$d_2^- = .458 \quad , \quad d_2^+ = 0$$

$$d_3^- = 0 \quad , \quad d_3^+ = 0$$

<u>Note</u> : in this example $\phi = (10)^3$.

APPENDIX E

The Solution to Example 4.1

1. From Section 4.6, the subprogram associated with the first level priority of program (4.98)-(4.103) is:

minimize $\quad a_1 = d_1^-$ (E.1)

subject to

$$1 - \left\{(1 - \frac{x_2}{x_1})^{-1} \beta_{11}^{\phi} + (\frac{1}{2} - \frac{x_2}{2x_1})^{-1} (\frac{x_1}{x_2} - 1)^{-2} \beta_{12}^{\phi} + (\frac{x_1}{x_2} - 1)^{-2}(1 + 10x_2^{-1})\beta_{12}^{\phi}\right\} + d_1^- - d_1^+ = .75 \qquad \text{(E.2)}$$

$$\beta_{11} + 10\,\phi^{-1}\,x_1^{-1} = 1 \qquad \text{(E.3)}$$

$$\beta_{12} + 10\,\phi^{-1}\,x_2^{-1} = 1 \qquad \text{(E.4)}$$

$$x_1, x_2, \beta_{11}, \beta_{12} \geq 0 \qquad \text{(E.5)}$$

$$0 \leq d_1^-\,,\ d_1^+ \leq 1 \qquad \text{(E.6)}$$

and

$$\phi \rightarrow \infty$$

2. From Section 5.7 the above program is equivalent to the following program:

minimize $\quad a_1 = d_1^-$ (E.7)

subject to

$$1 - \left\{(1 - \frac{x_2}{x_1})^{-1} \beta_{11}^{\phi} + (\frac{1}{2} - \frac{x_2}{2x_1})^{-1} (\frac{x_1}{x_2} - 1)^{-2} \beta_{12}^{\phi} + (\frac{x_1}{x_2} - 1)^{-2}(1 + 10x_2^{-1})\beta_{12}^{\phi}\right\} + d_1^- \geq .75 \qquad \text{(E.8)}$$

$$\beta_{11} + 10\,\phi^{-1}\,x_1^{-1} \geq 1 \qquad \text{(E.9)}$$

$$\beta_{12} + 10\,\phi^{-1}\,x_2^{-1} \geq 1 \qquad \text{(E.10)}$$

$$x_1, x_2, \beta_{11}, \beta_{12} \geq 0 \qquad \text{(E.11)}$$

Note that equalities (E.3), (E.4) have been replaced by inequalities (E.9), (E.10), where the inequalities (E.9), (E.10) are tight in the optimal solution (see subsection 5.7.2.)

3. From (5.120)-(5.123), the program (E.7)-(E.11) is equivalent to the generalized geometric program $(g\,g\,p)_1$, where:

$\underline{(g\,g\,p)_1}$ minimize $\quad a_1 = d_1^- \qquad$ (E.12)

subject to

$$\left\{ [\, 4\beta_{11}^{\phi} + 4x_1^{-2}x_2^2\beta_{11}^{\phi} + 12x_1^{-2}x_2^2\beta_{12}^{\phi} + 40x_1^{-2}x_2\beta_{12}^{\phi} + 12x_1^{-1}x_2d_1^- + 4x_1^{-3}x_2^3d_1^- + 3x_1^{-1}x_2 + x_1^{-3}x_2^3 \,] \,/\, [1 + 8x_1^{-1}x_2\beta_{11}^{\phi} + 4x_1^{-3}x_2^3\beta_{12}^{\phi} + 40x_1^{-3}x_2^2\beta_{12}^{\phi} + 4d_1^- + 12x_1^{-2}x_2^2d_1^- + 3x_1^{-2}x_2^2 \,] \right\} \le 1 \qquad \text{(E.13)}$$

$$\frac{1}{\beta_{11} + 10\,\phi^{-1}x_1^{-1}} \le 1 \qquad \text{(E.14)}$$

$$\frac{1}{\beta_{12} + 10\,\phi^{-1}x_2^{-1}} \le 1 \qquad \text{(E.15)}$$

$$x_1,\ x_2,\ \beta_{11},\ \beta_{12} \ge \epsilon \qquad \text{(E.16)}$$

$$\epsilon \le d_1^- \le .75 \qquad \text{(E.17)}$$

and

$\phi \to \infty$, $\quad \epsilon \to 0$.

4. Consider the initial point:

$$\left\{ d_1^- = 0,\ x_1 = .0001\ ,\ x_2 = .0001\ ,\ \beta_{11} = 0\ ,\ \beta_{12} = 0 \right\} \qquad \text{(E.18)}$$

The point (E.18) satisfies constraints (E.13)-(E.17) .

5. The optimal solution to $(g\,g\,p)_1$, obtained using the phase 2 algorithm (see example 5.1), is shown in Table E.1.

Table E.1

Phase 2 iterations	No. of cuts	Next approximating point $(d_1^-, x_1, x_2, \beta_{11}, \beta_{12})$	Comments
0	-	(0, .0001, .0001, 0, 0)	
1	0	(0, .0001, .0001, 0, 0)	local and global solution

From Table E.1 , $a_1^* = d_1^- = 0$. (E.19)

6. From (4.98) - (4.103) and (E.19) the subprogram associated with the second level priority of the program (4.98)-(4.103) is:

minimize $\quad a_2 = d_2^-$ (E.20)

subject to

$$1 - \left\{ \left(1 - \frac{x_2}{x_1}\right)^{-1} \beta_{11}^{\phi} + \left(\frac{1}{2} - \frac{x_2}{2x_1}\right)^{-1} \left(\frac{x_1}{x_2} - 1\right)^{-2} \beta_{12}^{\phi} + \left(\frac{x_1}{x_2} - 1\right)^{-2} \left(1 + 10x_2^{-1}\right) \beta_{12}^{\phi} \right\} + d_1^- \geq .75 \qquad (E.21)$$

$$\beta_{11} + 10\,\phi^{-1} x_1^{-1} \geq 1 \qquad (E.22)$$

$$\beta_{12} + 10\,\phi^{-1} x_2^{-1} \geq 1 \qquad (E.23)$$

$$x_1 + x_2 + d_2^- \geq 9.34 \qquad (E.24)$$

$$d_1^- \leq 0 \qquad (E.25)$$

$$x_1,\ x_2,\ \beta_{11},\ \beta_{12},\ d_1^-,\ d_2^- \geq 0 \qquad (E.26)$$

and

$$\phi \to \infty .$$

In turn, the above program is equivalent to the generalized geometric program $(ggp)_2$.

$\underline{(g\,g\,p)_2}$ minimize d_2^-

subject to

$$\{[4\beta_{11}^{\phi} + 4x_1^{-2}x_2^2\beta_{11}^{\phi} + 12x_1^{-2}x_2^2\beta_{12}^{\phi} + 40x_1^{-2}x_2\beta_{12}^{\phi} + 12x_1^{-1}x_2d_1^-$$

$$+ 4x_1^{-3}x_2^3d_1^- + 3x_1^{-1}x_2 + x_1^{-3}x_2^3] / [1 + 8x_1^{-1}x_2\beta_{11}^{\phi} + 4x_1^{-3}x_2^3\beta_{12}^{\phi}$$

$$+ 40x_1^{-3}x_2^2\beta_{12}^{\phi} + 4d_1^- + 12x_1^{-2}x_2^2d_1^- + 3x_1^{-2}x_2^2\ ]\} \le 1 \qquad (E.27)$$

$$\frac{1}{\beta_{11} + 10\ \phi^{-1}\ x_1^{-1}} \le 1 \qquad (E.28)$$

$$\frac{1}{\beta_{12} + 10\ \phi^{-1}x_2^{-1}} \le 1 \qquad (E.29)$$

$$\frac{9.34}{x_1 + x_2 + d_2^-} \le 1 \qquad (E.30)$$

$$\epsilon^{-1}\ d_1^- \le 1 \qquad (E.31)$$

$$x_1,\ x_2,\ \beta_{11},\ \beta_{12},\ d_1^-,\ d_2^- \ge \epsilon$$

7. Consider the initial point:

$$\{d_2^- = 2,\ x_1 = 2.5,\ x_2 = 4,\ d_1^- = 0,\ \beta_{11} = .9999,\ \beta_{12} = .9999\} \qquad (E.32)$$

The point (E.32) does not satisfy the constraints (E.27)-(E.31). Using the phase 1 algorithm we obtain a feasible point as shown in Table E.2.

Table E.2

Phase 1 iterations	No. of cuts	Next approximating point $(d_2^-, x_1, x_2, d_1^-, \beta_{11}, \beta_{12})$	Comments
0	-	(2, 2.5, 4, 0, .9999, .9999)	not feasible point
1	4	(9.34,1.648,2.934,0,.9999, .9999)	feasible point

8. The optimal solution to $(ggp)_2$', obtained using the phase 2 algorithm (see example 5.1), is shown in Table E.3 [22] .

Table E.3

Phase 2 iterations	No. of cuts	Next approximating point $(d_2^-, x_1, x_2, d_1^-, \beta_{11}, \beta_{12})$	Comments
0	-	(9.34,1.648,2.934,0,.9999,.9999)	
1	3	(4.935,2.462,4.551,0,.9999, .99997)	
2	4	(1,854,3.181,6,0,.9999,.99997)	
3	5	(.659,3.261,6,0,.9999,.99997)	
4	5	(.2588,3.279,6,0,.9999,.99997)	
5	5	(.1115,3.297,6,0,.9999,.99997)	
6	5	(.0528,3.312,6,0,.9999,.99997)	
7	5	(.0265,3.324,6,0,.999,.99997)	
8	5	(.0126,3.333,6,0,.9999,.99997)	
9	5	(.00415,3.341,6,0,.9999,.99997)	
10	3	(.00205,3.348,6,0,.9999,.99997)	
11	3	(0,3.341,6,0,.9999,.99997)	
12	2	(0,3.34,6,0,.9999,.99997)	local and global solution

Hence, the global solution to example 4.1 is:

$a_2^* = 0$

$x_1 = 3.34$, $x_2 = 6$

$d_1^- = 0$, $d_1^+ = .19$

$d_2^- = 0$, $d_2^+ = 0$

<u>Note</u> : In this example $\phi = (10)^5$.

REFERENCES

[1] Avriel, M. and Williams, A.C. (1970):
"Complementary Geometric Programming"
SIAM J. Appl. Math. Vol. 19, No. 1.

[2] Avriel, M., Dembo, R. and Passy, U. (1975):
"Solution of Generalized Geometric Programs"
International Journal for Numerical Methods
in Engineering, Vol. 9.

[3] Beightler, C. and Phillips, D.T. (1976):
"Applied Geometric Programming", John Wiley
& Sons, Inc., New York, London.

[4] Beightler, C.S., Phillips, D.T. and Wilde, D.T. (1979):
"Foundations of optimization" Prentice-Hall Inc.,
Englewood Cliffs, N.J. 07632.

[5] Blau, G.E. and Wilde, J. (1969):
"Generalized polynomial programming". The
Canadian Journal of Chemical Engineering,
Vol. 47, No. 3.

[6] Box, G.E.P. (1954):
"Some Theorems on Quadratic Forms Applied in the
Study of Analysis of Variance Problems,
I. Effect of Inequality of Variance in the
One-way Classification". Annals of Mathematical
Statistics, Vo. 25, No. 2.

[7] Charnes, A., Cooper, W.W. and Ferguson, R. (1955):
"Optimal estimation of executive compensation
by linear programming". Manag. S.C., Vol. 1,
No. 2.

[8] Charnes, A. and Cooper, W.W. (1961):
"Management models and industrial applications
of linear programming". Vol. 1, John Wiley &
Sons, Inc., New York.

[9] Charnes, A. and Cooper, W.W. (1959):
"Change-constrained programming", Mang. Sci.
Vol. 6, No. 1.

[10] Charnes, A. and Cooper W.W. (1969):
"Deterministic equivalent for optimizing and
satisficing under chance constraints in economic
models, estimation and risk programming"
Essays in Honor of G. Tintner, Springer Verlage,
Berlin-Heidelberg, New York.

[11] Charnes, A., Cooper, W.W. and Symonds, G. (1958):
"Cost Horizons and Certainty Equivalent:
An Approach to Stochastic Programming of Heating
Oil" Manag.Sci., Vol. 4, No. 3.

[12] Charnes, A. and Cooper, W.W. (1963):
"Deterministic equivalent for optimizing and statisticing under chance - constraints". O.R., Vol. 11, No. 1.

[13] Charnes, A. and Kirby, M. (1967):
"Some special P-models in chance - constrained programming". Manag. Sci., Vol. 14, No. 3.

[14] Charnes, A. and Kirby, M (1966):
"Optimal decision rules for E model of chance - constrained programming" Cahiers due Centre d'Etudes de Recherche, Operationnelle, Vol. 8, No. 1.

[15] Charnes, A., Cooper W.W., and Nilhaus, R.J. (1968):
"A goal programming model for manpower planning" Manag.Sci. Research report No. 115, Carnegie - Mellon University (August).

[16] Contini, B. (1968):
"Stochastic approach to goal programming" O.R. (May-June) pp. 576-86.

[17] Dantzing, G.B. (1955):
"Linear programming under uncertainty". Manag. Sci., Vol. 1, No. 3 and 4.

[18] Dantzing, G.B. (1966):
"Linear programming and extensions" Princeton University Press, Princeton.

[19] Dauer, J.P. and Krueger, R.J. (1977):
"An iterative approach to goal programming" Opl. Res. Q., Vol. 28, No. 3.

[20] Dauer, J.P. and Krueger, R. (1976):
"An iterative approach to goal programming" Department of Mathematics, University of Nebraska.

[21] David, F.N. and Kendall, M.G. (1949):
"Tables of symmetric functions - Part I." Biometrika, Vol. 36.

[22] Dembo, R.S. (1972):
"GGP - A program for solving generalized geometric programs, users manual", Department of Chemical Engineering, Technion, Haifa.

[23] Dembo, R.S. (1972):
"Solution of complementary geometric programming problems". M.Sc. thesis, Technion, Haifa.

[24] Duffin, R.J., Peterson, E.L. and Zener, C. (1967):
"Geometric programming theory and application" John Wiley & Sons Inc., New York.

[25] El-dash, A.A. (1979):
"Series service systems and planning in maritime transport sector". M.Sc. thesis, Cairo University, Egypt (English summary).

[26] El-maghraby, S.E. (1958):
"An approach to linear programming under uncertainty", O.R. Vol. 7, pp. 208-216.

[27] El-maghraby, S.E. (1960):
"Allocation under uncertainty with the demand HAS continuous D.F." Manag. Sci., Vol. 3, pp. 270-294.

[28] Esmail, N. (1970):
"The available-range to increase loading and discharging efficient" The National Planning Institute, Cairo, Egypt (Arabic).

[29] Edited by Avriel, M. (1980):
"Advances in geometric programming" Plenum Press, New York.

[30] Edited by Pearson and Hartley (1976):
"Biometrik tables for statisticians" Vol. II, Biometrika Trust, University Printing House, Cambridge.

[31] Gamkrelidze, R.V. (1967):
"Extremal problems in finite - dimensional spaces". J. of Optimization Theory and Applications, Vol. 1, No. 3.

[32] Griffith, R.E. and Stewart, R.A. (1961):
"A nonlinear programming technique for the optimization of continuous processing systems" Manag. Sci., Vol. 7, No. 4.

[33] Hansotia, B.J. (1980):
"Stochastic linear programs with simple recourse: the equivalent deterministic convex program for the normal, exponential, and erlang cases" Naval Research Logistics Quarterly, Vol. 27.

[34] Halawa, M.A. (1978):
"The Alexandria port's congestion problem and its effects on the national economy", the Marine Transport Economics and Administration Institute, Alexandria, Egypt.

[35] Harrison, A.J. (1974):
"The economics of transport appraisal". Croom Helm, London.

[36] Hooke, R. and Jeeves, T.A. (1961): "Direct search solution of numerical and statistical problems". J. Assoc. Comp. Mach., Vol. 9, No. 2.

[37] Ignizio, J.P. (1976): "Goal programming and extensions" Lexington Books, Massachusetts, Toronto, London.

[38] Ignizio, J.P. (1982): "Linear programming in single and multiple objective systems", Prentice Hall Inc., Englewood Cliffs.

[39] Ijiri, Y. (1965): "Management goals and accounting for control" North-Holland Publishing Company, Amsterdam.

[40] Kelley, J.E. (1960): "The cutting plane method for solving convex programs". J. Soc. Indust. Appl. Math., Vol. 8, No. 4.

[41] Kendall, M.G. and Stuart, A. (1979): "The advanced theory of statistics, Vol. 2, Inference and relationship ". Charles Griffin and Company Limited, London.

[42] Kendall, M.G. and Stuart, A. (1977): "The advanced theory of statistics, Vol. 1" Charles Griffen & Company Limited, London.

[43] Keown , A.J. and Martin, J.D. (1977): "A chance-constrained goal programming model for working capital management". The Engineering Economist, Vol. 22, No. 3.

[44] Keown , A.J. (1978): "A chance-constrained goal programming model for bank liquidity management", Decision Sciences, January, pp. 93-106.

[45] Keown , A.J. and Taylor, I.I.I. (1980): "A chance-constrained integer goal programming model for capital budgeting in the production area" J. Apl. Res. Soc., Vol. 31, pp. 579.

[46] Keown , A.J. and Duncan, C.P. (1976): "An integer goal programming model for advertising media selection" Proceedings of AIDS Meeting, Francisco , November.

[47] Kornbluth, J. (1973): "A survey of goal programming". Omega, Vol. 1, No. 3.

[48] Kuhn, H.W. and Tucker, A.W. (1951):
"Nonlinear programming" Second Berkley Symposium on Mathematical Statistics and Probability, Edited by Jerzy Neyman, University of California Press.

[49] Lancaster (1969):
"The chi-squared distribution". John Wiley Inc. London.

[50] Lee, S.M. (1972):
"Goal programming for decision analysis", Averbach Publishers Inc., New York.

[51] Lee, S.M. (1979):
"Linear optimization for management" Petrocelli/Charter, New York.

[52] Lin, W.T. (1980):
"A survey of goal programming applications" Omega, Vol. 8, No. 1.

[53] Markowski, C.A. (1980):
"Duality in linear goal programming". The Pennsylvania State University, Ph.D. thesis in Industrial Engineering and Operations Research.

[54] Masud, A.S. and Hwang, C.L. (1981):
"Interactive sequential goal programming". J. Opl. Res. Soc., Vol. 32, pp. 391-400.

[55] Mahmoud, M.W. (1974):
"Demurrages problem". Marine-Transport Research Journal, Vol. 2., The Arab-Marine Transport Academy, Alexandria, Egypt.

[56] Mood, A.M., Graybill, F.A. and Boes, D.C. (1974):
"Introduction to the theory of statistics" Third Edition, McGraw-Hill Inc.

[57] Näslund, B. (1967):
"Decisions under risk: economic applications of chance-constrained programming" The Economic Research Institute, Stockholm School of Economics.

[58] Owen, D.B. (1962):
"Handbook of statistical tables". Addison-Wesley Publishing Company, London.

[59] Patnaik, P.B. (1949):
"The non-central χ^2- and F distributions and their applications". Biometrika, 36, pp. 202-231.

[60] Passy, U. and Wilde, J. (1967):
"Generalized polynomial optimization" SIAM J. Appl. Math., Vol. 15, No. 5.

[61] Pearson, E.S. (1959):
"Note on an approximation to the distribution of non-central χ^2". Biometrika, 46, pp. 364.

[62] Sengupta, J.K. (1972):
"Stochastic programming methods and applications" North Holland Publishing Company, Amsterdam.

[63] Sengupta, J.K., Tintner, G. and Millhan, C. (1963):
"On some theorems of stochastic linear programming with applications". Manag. Sci., Vol. 10, No. 1.

[64] Sengupta, J.K. and Fox, K.A. (1969):
"Economic analysis and operations research: optimization techniques in quantitative economic models", North-Holland Publishing Company, Amsterdam, London.

[65] Sengupta, J.K. and Tintner, G. (1971):
"A review of stochastic linear programming" Rev. of the Inter. Stat. Inst., Vol. 39.

[66] Sengupta, J.K. (1973):
"Application of system reliability measure in chance-constrained programming" Cahiers du Centre D'Etudes de Recherche Operationnelle, Vol. 15, No. 4.

[67] Sengupta, J.K. (1972):
"Chance-constrained linear programming with chi-square type deviates" Manag. Sci., Vol. 14, No. 3.

[68] Sengupta, J.K. (1971):
"A system reliability approach to linear programming" Unternehmensforschung, 15, pp. 112-129.

[69] Sengupta, J.K. (1970):
"On the active approach of stochastic linear programming" Matrika, Vol. 15, pp. 59-70.

[70] Sengupta, J.K. (1969):
"Distribution problems in stochastic and chance-constrained programming" in Economic Models, Estimation in Honor of Tintmer - Springer Verlag, New York.

[71] Sengupta, J.K. and Gruver, G. (1969):
"A linear reliability analysis in programming with chance-constraints". The Swedish Journal of Economics, Vol. LXXI, No. 4.

[72] Sengupta, J.K. (1970):
"A generalization of some distribution aspects of chance-constrained linear programming" International Economic Review, Vol. 11, No. 2.

[73] Some Essays about (1980, 1981, 1982, 1983):
"Congestion problems in Egyptian marine ports"
El-Ahram, Economist Journal, Cairo, Egypt.

[74] Taha, H.A. (1976):
"Operations research - an introduction"
Second Edition, Macmillan Publishing Co.,
New York.

[75] Tintner, G. (1941):
"The theory of choice under subjective risk and uncertainty". Econometric, IX.

[76] Tintner, G., and Sengupta, J.K. (1972):
"Stochastic Economics: stochastic processes, control and programming". Academic Press,
New York.

[77] Vajda, S. (1972):
"Probabilistic programming". Academic Press,
New York and London.

[78] Walkup, D.W. and Wets, J.B. (1967):
"Stochastic programs with recourse".
SIAM J. Appl. Math., Vol. 15, No. 5.

[79] Walkup, D.W. and Wets, J.B. (1970):
"Stochastic programs with recourse: special forms", in Proceedings of Princeton Symposium on Mathematical Programming, Princeton University Press, Princeton, New Jersey.

[80] Wagner, H.M. (1975):
"Principles of operations research with applications to managerial decisions" Prentice Hall International Inc., London.

[81] Wagner, H.M. (1958):
"The dual simplex algorithm for bounded variables"
Naval Research Logistics Quarterly, 5, pp. 257-261.

[82] Wets, J.B. (1965):
"Programming under uncertainty". University of California, Berkeley, Ph.D. thesis, Mathematics.

[83] Wets, J.B. (1966):
"Programming under uncertainty: the equivalent convex program", J. SIAM Appl. Math. Vol. 14,
No. 1.

[84] Wets, J.B. (1966):
"Programming under uncertainty: the complete problem", Zeitschrift für Wahrscheinlichkeitsheorie und verwandte Gebiete, 4, 316-339.

[85] Wets, J.B. (1972):
"Characterization theorems for stochastic programs" Mathematical Programming, 2, p.166-175.

[86] Widhelm, W.B. (1981):
"Extensions of goal programming models". Omega, Vol. 9, No. 2.

[87] Wilde, D.J., and Beightler, C.S. (1967):
"Foundations of optimization" Prentice Hall Inc., Englewood Cliffs, N.J.

[88] Wolfe, P. (1970):
"Integer and nonlinear programming". Abadil, J. Ed. North Holland Publishing Co.