DOCTOR OF PHILOSOPHY

Computer graphics studies of Islamic geometrical patterns and designs

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COMPUTER GRAPHICS STUDIES OF ISLAMIC GEOMETRICAL PATTERNS AND DESIGNS

BY

AMER SHAKER SALMAN

B.Sc. Department of Mathematics
University of Baghdad

M.Sc. School of Mathematics
University College of North Wales

MBCS, MEG.

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SUMMARY

This thesis results from the following work:

(1) We have carried out a comprehensive study of Islamic geometrical patterns. More then 300 patterns have been studied. We have given a critique of the work of previous authors on this subject and have discussed our own ideas on the evolution of Islamic geometrical designs.

(2) We have performed symmetry analysis on the patterns and classified them according to their symmetry groups.

(3) We have extracted numerical data for efficient generation of the patterns based on the analysis in (2). The data for more than 300 patterns is provided on the disk.

(4) We have developed a mathematical formalism based on group theory and constructed algorithms suitable for the generation of the patterns using computer graphics.

(5) the algorithms have been proved by writing an interactive computer graphic program called Islamic Geometrical Patterns 'IGP'. A library of geometric Islamic patterns has been constructed.

(6) At the end of this thesis, in an Appendix, we have provided suggestions for further extension of this work.

TABLE OF CONTENTS

ERRATA

Symmetry groups of Nets, Frieze patterns and Crystallographic patterns have their elements listed in chapter 3. Corrected versions of these lists are given below, where $F_{p,q}$ denotes reflection in the line through q in direction p and $G_{p,q}$ denotes the glide having translation p and reflection $F_{p,q}$. Also, parameters α, β, γ are arbitrary integers with $\gamma \neq 0$.

Frieze Patterns

$$\Xi_{p111} = \{ T_{\alpha h} \}$$

$$\Xi_{p112} = \Xi_{p111} \cup \{ R_{180, \alpha h/2} \}$$

$$\Xi_{pm11} = \Xi_{p111} \cup \{ F_{k, \alpha h/2} \} \qquad \text{where } k \perp h$$

$$\Xi_{p1m1} = \Xi_{p111} \cup \{F_{h,0}, G_{\gamma h, 0} \}$$

$$\Xi_{pmm2} = \Xi_{\mathbb{N}_F} = \Xi_{p112} \cup \Xi_{pm11} \cup \Xi_{p1m1}$$

$$\Xi_{p1a1} = \Xi_{p111} \cup \{ G_{(2\alpha+1)h/2, 0} \}$$

$$\Xi_{pma2} = \Xi_{p1a1} \cup \{ R_{180, \alpha h/2}, F_{k, (2\alpha+1)h/4} \}$$

Crystallographic Patterns

$$\Xi_{p1} = \{ T_{\alpha u+\beta v} \}$$

$$\Xi_{p211} = \Xi_{\mathbb{N}_P} = \{ T_{\alpha u+\beta v}, R_{180, \alpha u/2+\beta v/2} \}$$

$$\Xi_{p1m1} = \{ T_{\alpha u+\beta v}, F_{u, \beta v/2}, G_{\gamma u, \beta v/2} \} \qquad \text{or}$$

$$\bar{\Xi}_{p1m1} = \{ T_{\alpha u+\beta v}, F_{v, \beta u/2}, G_{\gamma v, \beta u/2} \}$$

$$\Xi_{p2mm} = \Xi_{\mathbb{N}_R} = \Xi_{p211} \cup \Xi_{p1m1} \cup \bar{\Xi}_{p1m1}$$

$$\Xi_{p1g1} = \{ T_{\alpha u+\beta v}, G_{(2\alpha+1)u/2, \beta v/2} \} \qquad \text{or}$$

$$\bar{\Xi}_{p1g1} = \{ T_{\alpha u+\beta v}, G_{(2\alpha+1)v/2, \beta u/2} \}$$

$$\Xi_{p2mg} = \Xi_{p1g1} \cup \Xi^{-}_{p1m1} \cup \{ R_{180,(2\alpha+1)u/4+\beta v/2} \}$$

$$\Xi_{p2gg} = \Xi_{p1g1} \cup \Xi^{-}_{p1g1} \cup \{ R_{180,(2\alpha+1)u/4+(2\beta+1)v/4} \}$$

$$\Xi_{c1m1} = \{ T_{\alpha u+\beta v}, F_{u+v,\beta(u-v)/2}, G_{\gamma(u+v),\beta(u-v)/2}, G_{(2\alpha+1)(u+v)/2,(2\beta+1)(u-v)/4} \}$$

$$\Xi_{c2mm} = \Xi_{\mathbb{N}_C} = \Xi_{c1m1} \cup \{ F_{u-v,\beta(u+v)/2}, G_{\gamma(u-v),\beta(u+v)/2}, G_{(2\alpha+1)(u-v)/2,(2\beta+1)(u+v)/4}, R_{180,\alpha u/2+\beta v/2} \}$$

$$\Xi_{p4} = \Xi_{p211} \cup \{ R_{\pm 90,\alpha u+\beta v}, R_{\pm 90,(2\alpha+1)u/2+(2\beta+1)v/2} \}$$

$$\Xi_{p4mm} = \Xi_{p4} \cup \{ F_{u+v,\beta(u-v)/2}, G_{\gamma(u+v),\beta(u-v)/2}, F_{(u-v),\beta(u+v)/2}, G_{\gamma(u-v),\beta(u+v)/2}, F_{u,\beta v/2}, G_{\gamma u,\beta v/2}, F_{v,\beta u/2}, G_{\gamma v,\beta u/2} \}$$

$$\Xi_{p4gm} = \Xi_{p4} \cup \{ F_{u+v,(2\beta+1)(u-v)/4}, G_{\gamma(u+v),(2\beta+1)(u-v)/4}, F_{(u-v),(2\beta+1)(u+v)/4}, G_{\gamma(u-v),(2\beta+1)(u+v)/4}, G_{(2\alpha+1)u/2,(2\beta+1)v/4}, G_{(2\alpha+1)v/2,(2\beta+1)u/4}, G_{(2\alpha+1)(u+v)/2,\beta(u-v)/2}, G_{(2\alpha+1)(u-v)/2,\beta(u+v)/2} \}$$

$$\Xi_{p3} = \{ T_{\alpha u+\beta v}, R_{\pm 120,\alpha u+\beta v}, R_{\pm 120,(3\alpha+1)u/3+(3\beta+1)v/3}, R_{\pm 120,(3\alpha-1)u/3+(3\beta-1)v/3} \}$$

$$\Xi_{p3m1} = \Xi_{p3} \cup \{ F_{u,\beta v}, G_{\gamma u,\beta v}, G_{(2\alpha+1)u/2,(2\beta+1)v/2}, F_{v,\beta u}, G_{\gamma v,\beta u}, G_{(2\alpha+1)v/2,(2\beta+1)u/2}, F_{u-v,\beta u}, G_{\gamma(u-v),\beta u}, G_{(2\alpha+1)(u-v)/2,(2\beta+1)u/2} \}$$

$$\Xi_{p31m} = \Xi_{p3} \cup \{ F_{u+v,\beta v}, G_{\gamma(u+v),\beta v}, G_{(2\alpha+1)(u+v)/2,(2\beta+1)v/2}, F_{2u-v,\beta u}, G_{\gamma(2u-v),\beta u}, G_{(2\alpha+1)(2u-v)/2,(2\beta+1)u/2}, F_{2v-u,\beta v}, G_{\gamma(2v-u),\beta v}, G_{(2\alpha+1)(2v-u)/2,(2\beta+1)v/2} \}$$

$$\Xi_{p6} = \Xi_{p3} \cup \{ R_{\pm 60,\alpha u+\beta v}, R_{\pm 180,\alpha u/2+\beta v/2} \}$$

$$\Xi_{p6mm} = \Xi_{\mathbb{N}_H} = \Xi_{p6} \cup \Xi_{p3m1} \cup \Xi_{p31m}$$

1- INTRODUCTION

1.1 INTRODUCTION

The work in this thesis sets out to achieve the following:

(1) To carry out a comprehensive study of Islamic geometrical patterns.

(2) To perform symmetry analysis on the patterns and to classify them according to their symmetry group.

(3) To extract numerical data for efficient generation of the patterns based on the analysis in (2).

(4) To develop a mathematical formalism for the construction of algorithms suitable for the generation of the patterns using computer graphics.

(5) To prove the algorithms developed in (4) by writing an interactive computer graphic program and by constructing a library of geometric Islamic patterns.

After the introduction in this opening section we discuss the importance of symmetry in general and then give examples of symmetry in Islamic art. This is intended to explain the motivation for this work.

The purpose of chapter 2 is to present our own views on how the Islamic patterns originated and developed and to compare our thoughts with the views put forward by previous authors who have published in this subject.

The main aim of chapter 3 is to apply group theoretic methods of analysis and generation to Islamic geometrical patterns. Following our review of the subject, we develop a set of algorithms suited to interactive generations of frieze and crystallographic patterns. These algorithms are used in the computer program which is described in chapter 5.

The first part of the work for this thesis involved an extensive study of more than 300 Islamic patterns. The majority of the patterns studied appear in the books by Bourgoin [9], Critchlow [13], El-Said & Parman [63] and Wade [73]. Also, about ten patterns which do not occur in these references where collected by the author on a study tour of Islamic architecture to be found in Spain.

The patterns were studied using the CAD package AutoCAD and data was extracted to make it possible to recreate these patterns using the insights provide by symmetry analysis. Chapter 4 reports the first part of our work.

Finally in chapter 5, we describe the interactive program Islamic Geometrical Pattern (IGP), which was written to utilize our data and our algorithms. Although the program was written specifically to recreate the Islamic geometrical patterns studied by us, it is in fact a general purpose program capable of generating the full set of plane crystallographic patterns from template motif data given in a file or created interactively. The program allows for interactive modification and coloring of related patterns obtained from library data. Example of output produced by the program are given at the end of the Chapter.

The program listing and the numerical data are attached in a floppy disk.

In the next section we discuss the importance of the subject of symmetry which will be followed by examples of symmetry in Islamic art.

2.1 IMPORTANCE OF SYMMETRY

Symmetry is a vast subject. The term 'symmetry' is meaningful in ordinary every day occurrences, in the arts, in literature and also in exact sciences. In the arts and in ordinary language the term is used rather vaguely in two different ways; to express exact correspondence of size, shape, color etc between opposite sides of an object; to express harmony of proportion, balance and regularity between parts. Mathematicians, on the other hand, define it precisely in terms of invariance of a set under a group of automorphic transformations, see for example Coxeter[12a,b].

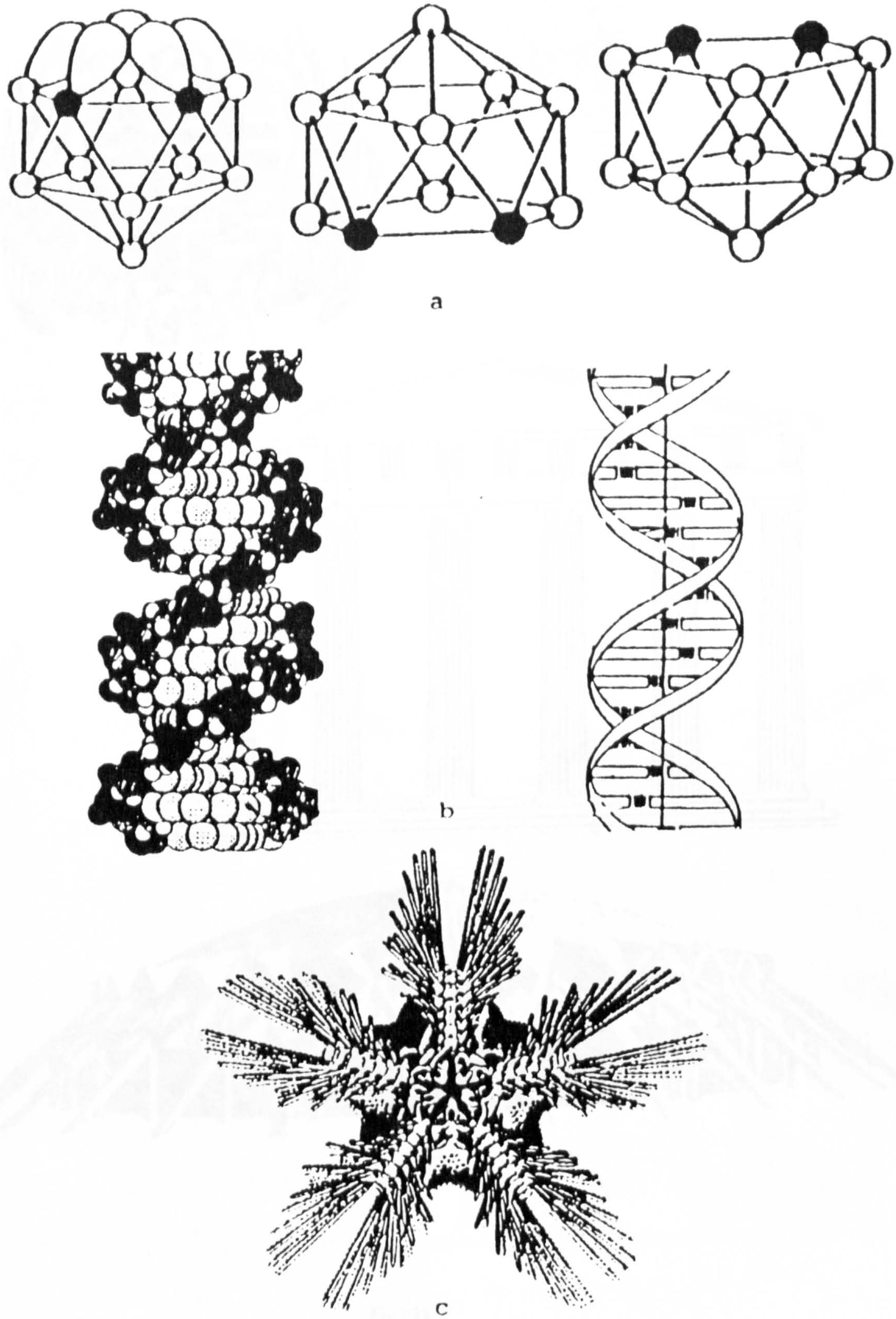
It is not our intention in this thesis to discuss the range of the subject of symmetry in great detail. This has been done elsewhere and the reader is referred to the classical monograph by Weyl[74] and the more recent collection of papers given in references (see authors, El-Said & Parman [63], Jones [37]) as sources for appreciating the range of application of symmetry. Our intention in this opening chapter is simply to indicate with the aid of figures the widespread occurrence of symmetric forms. This is done to justify the application of computer graphics to studies of symmetry which is the nature of the work carried out for this thesis.

We find symmetrical structures right at the micro-scale of living organisms and non living matter. Fig(1a) shows the chemical structures of certain compounds known as metallacarboranes. Fig(1b) shows the well known structure of the DNA molecule which is the basis of all living matter. Fig(1c) shows the five-fold symmetrical structure of a type of sea Urchin. This kind of symmetry is quite common in biological forms, particularly flowers.

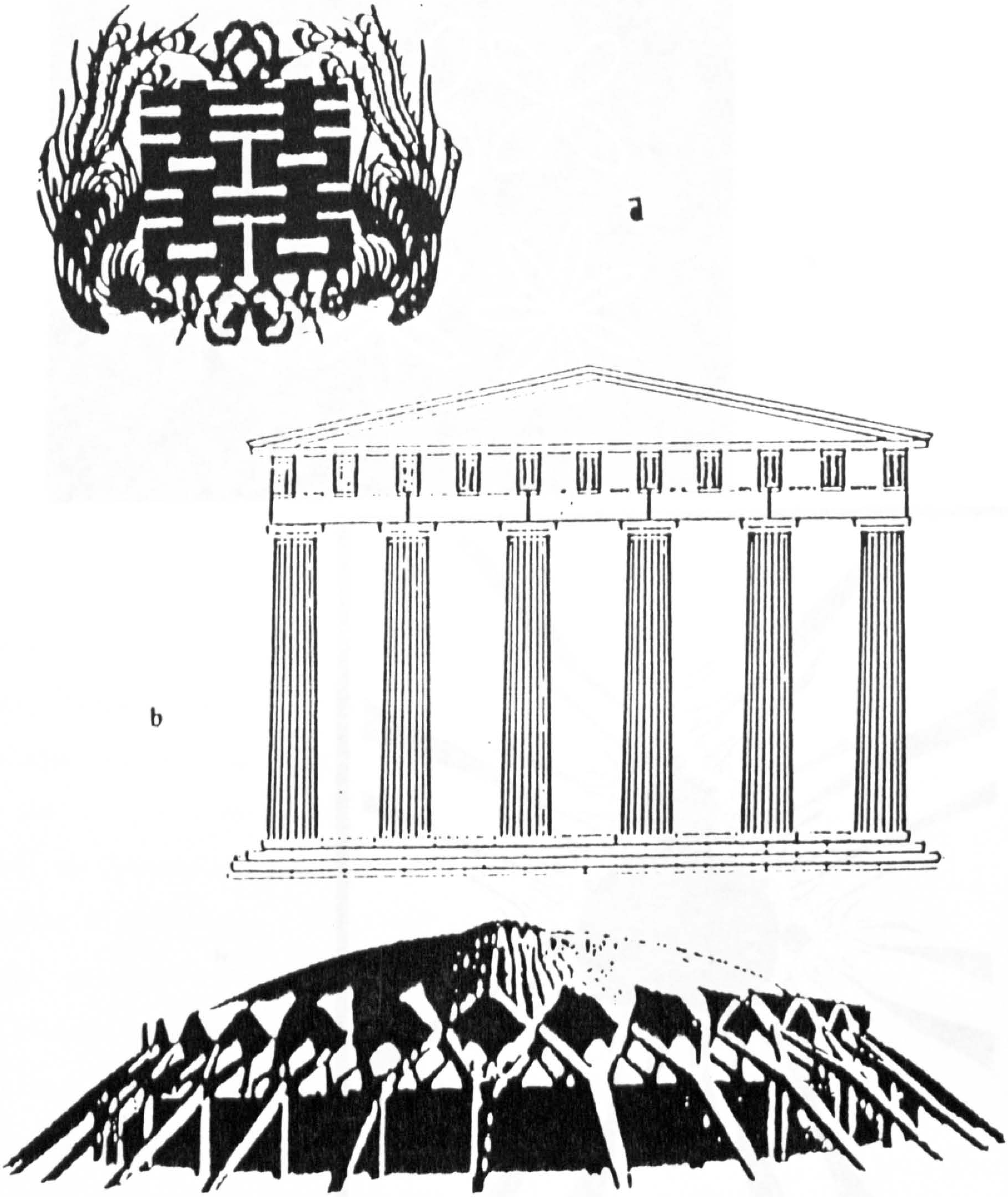
Of course the most common association of symmetry is with beauty and in their search for beauty human artists and designers have always explored symmetry. Fig(2a) shows the Chinese character which means double happiness. One cannot imagine a great building which did not have symmetry. Fig(2b) shows a typical structure associated with the classical Greek architecture and fig(2c) shows the design of a modern sports stadium by Pier Luigi Nervi.

The study of geometry has always produced and continues to produce many beautiful symmetric forms. Fig(3a) represents seven cardioids generated by the formula $r=a(1-\cos(\alpha\phi))$, a cardioid of order 1 is the basic shape, increasing α increases the lobes in the patterns as shown in the figure. The availability of the computer has made it possible to explore many complex symmetric structures which could not be explored before. One growth area has been in the field of Fractals invented by the French mathematician Benoit Mandelbort[46]. Fig(3b) shows an artistic rendering of the famous Mandelbrot set.

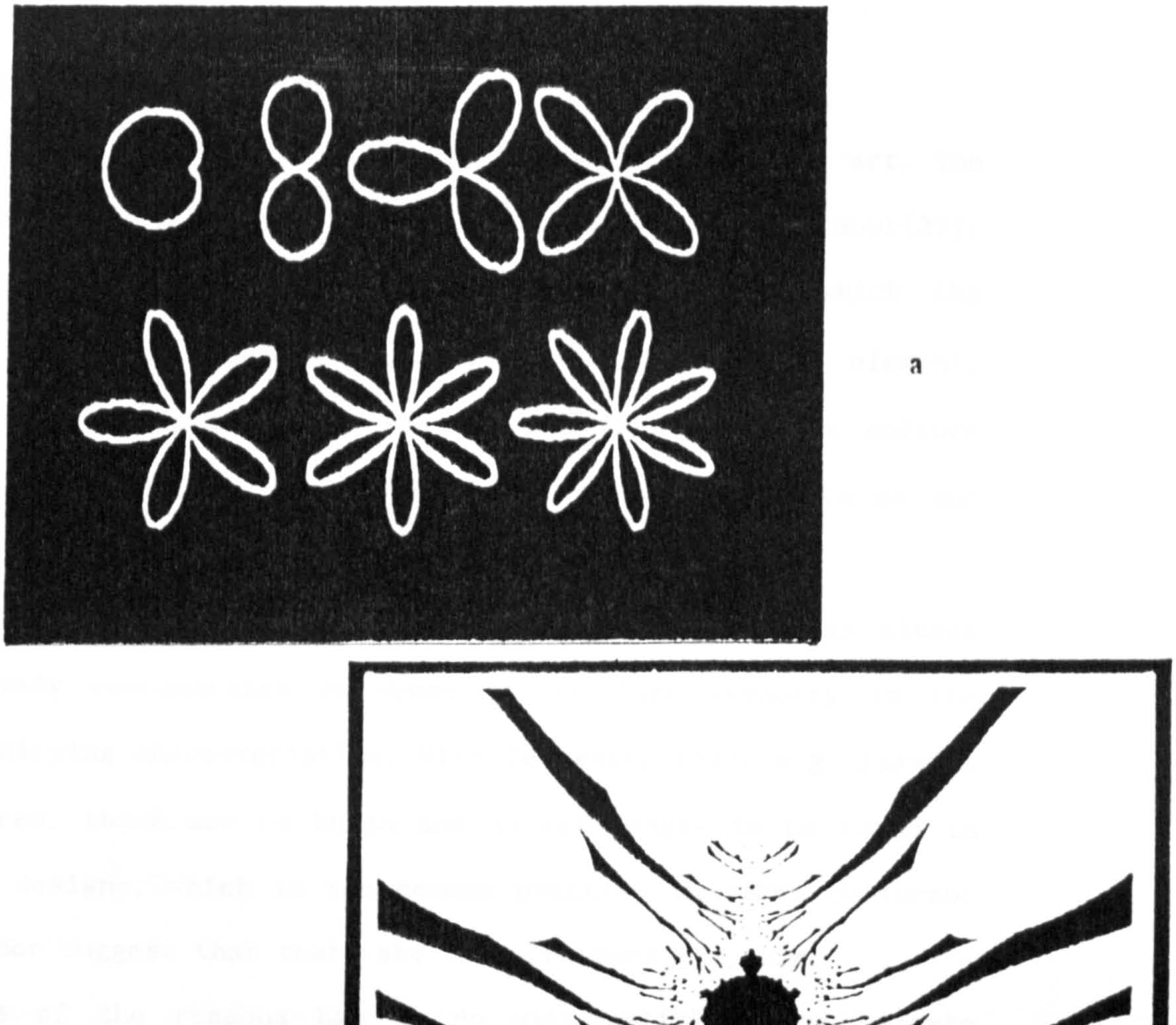
The above set of examples illustrate the range of symmetrical structures and We now move to our main interest in symmetry in this thesis which is the context of Islamic art.



fig(1)



fig(2)



fig(3)

1.3 SYMMETRY IN ISLAMIC ART

It is useful to know first, what we mean by islamic art. The most useful definition of 'Islamic' is suggested by Grabar[27]: " Islamic refers to a culture or civilization in which the majority of the population, or at least the ruling element, profess the faith of Islam, the art produced by such a culture could then be called Islamic art ". We will accept this as our definition.

Islamic art, unlike the arts of other nations, has almost exclusively concentrated on symmetry, so that symmetry is the major unifying characteristics. With few exceptions, e.g. Persian miniatures, there are no human and animal images to be found in Islamic designs, which is the common practice in other cultures. The author suggest that there are three reasons for this.

One of the reasons has to do with religion. The Islamic artist is prohibited from representing human or animal forms according to the Hadith (the tradition concerning the actions and sayings of prophet Mohammed, collected by his followers). This does contain the admonition that the punishment of the people who paint any living thing will be very hard on the day of judgment. Another reason is that the Quran (the holy book of Islam, contains the words of Allah in the chosen language of Arabic). Arabic, thus came to have a sanctified position in Islamic society, especially when it was used by artists in calligraphically to quote verses from the Quran. The final reasons comes from the value placed on education in Islamic culture. Since geometry was regarded with great respect in education, the artist naturally thought that this was the correct way to express their work.

Calligraphy is an integral part of design in Islamic culture which is its unique feature. Islamic artists have produced many calligraphic scripts and symmetrical calligraphic designs from the earliest times and continue to do so today.

Fig(4a) shows an example of a simple calligraphic design in the interior of Uiu Cami, Bursa, Turkey, from the Othoman period (1359-1420). Fig(4b) shows an example of a more complex design from a recent work by the Iraqi calligrapher Hashem Al-Khattat[39]. The calligraphic panel is structured in Jali Thuluth script. The text is a verse from the Surat Al-Omran in Quran enjoining Muslims to put their trust in God.

Of course, Islamic art is best known for the use of infinite repeat patterns in tiling. The Great Mosque in Baghdad, the palace of Alhambra in Spain and the Taj Mahal in India are well known examples of building which have been admired universally. In this area its achievements are greater than that of any previous culture and examples of all the seventeen Crystallographic groups are to be found (see Montesinos[50]) in Islamic tiling decoration. Bourgoin [9] was the first one to collect and publish a large collection of these designs. Since then many authors have written on this subject, and the recent work by Grunbaum and Shephard [28h] contains many examples from the work of Islamic artists.

Several photographic plates taken by the author, during a study tour of Spain are included here. Plate(1) shows a view of the court of the lions in the Alhambra palace at Granada (1354). This was one of the latest additions to the palace which served the local rulers of that part of Spain.

Plate(2) shows an example of an infinite uni-directional repeat pattern from the Great Mosque at Cordova, The mosque was started in eighth century but has frequently been added to.

Plate(3) shows another infinite uni-directional repeat pattern. The combined use of geometry and calligraphy is very common in Islamic designs.

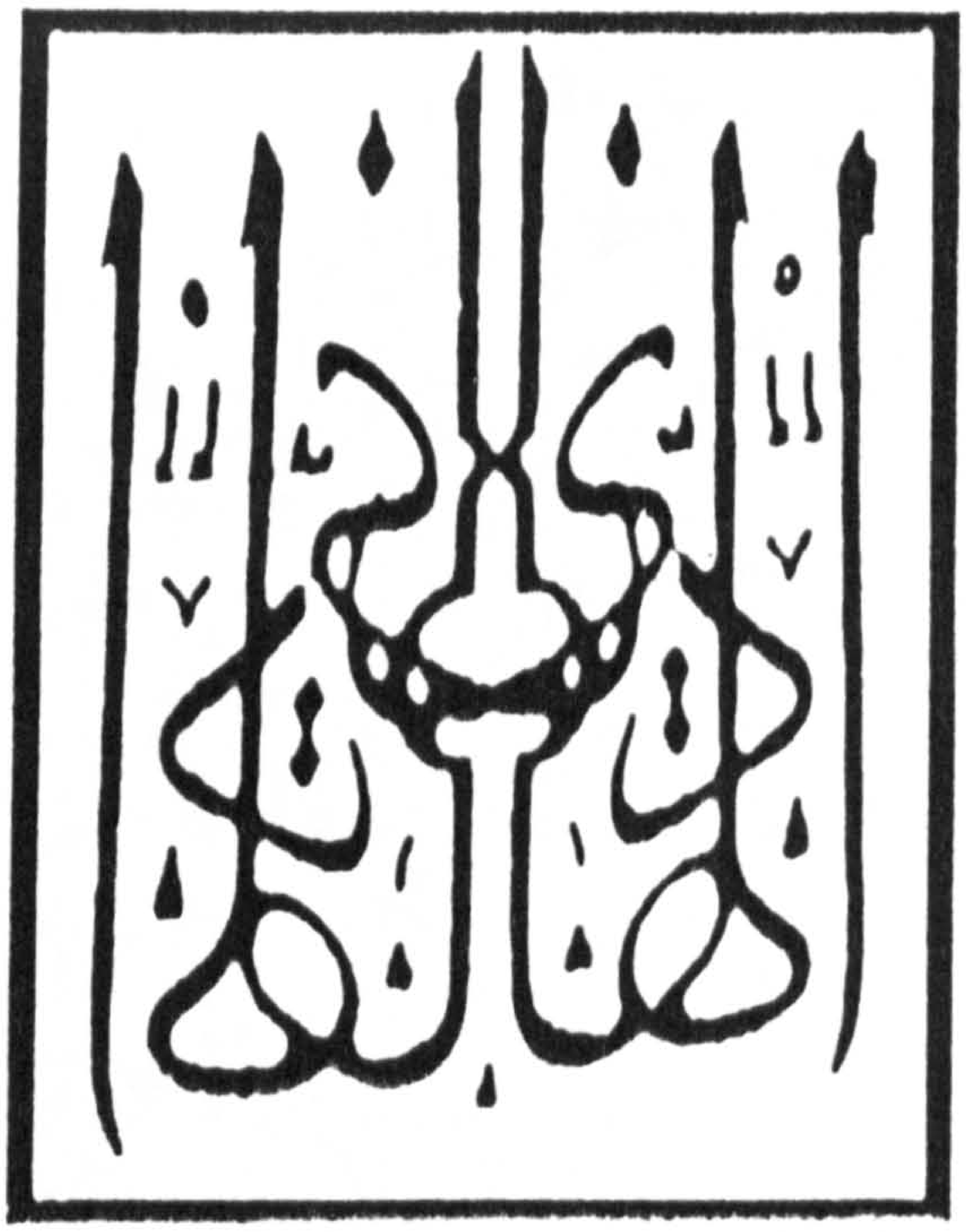
Another concern of Islamic art has been centered on the effective use of colour. It is surprising to discover that it was to record and display the colors of Islamic architecture that color lithography was first developed in Britain. Owen Jone's treatise on Alhambra in Spain, produced during (1836-1845), was the first color book to be produced in Britain.

Plate(4) shows a repeat pattern in two dimensions. Islamic artists used colors systematically to reflect different ideas and to create the Islamic feeling. This topic requires research to clarify the exact principles used, however, we can see that green, black and blue are very common colors used to produce a large set of designs. In calligraphy the verses from the Quran are written with white and blue in the background . White is associated with good and blue is intended to suggest the feeling of the sky. Plates(5) and (6) show two typical examples of repeat pattern decorations and surrounding borders. Plates(4),(5) and (6) where photographed by the author in Algiralda, Seville, Spain, which was built in 1248.

Apart from its aesthetic value, the importance of Islamic design in mathematics and other sciences, comes from the use of repeat patterns. The study of these can provide a pleasurable lead into group theory which is the basis of advanced pure mathematics and also basis of modern thinking in physics.

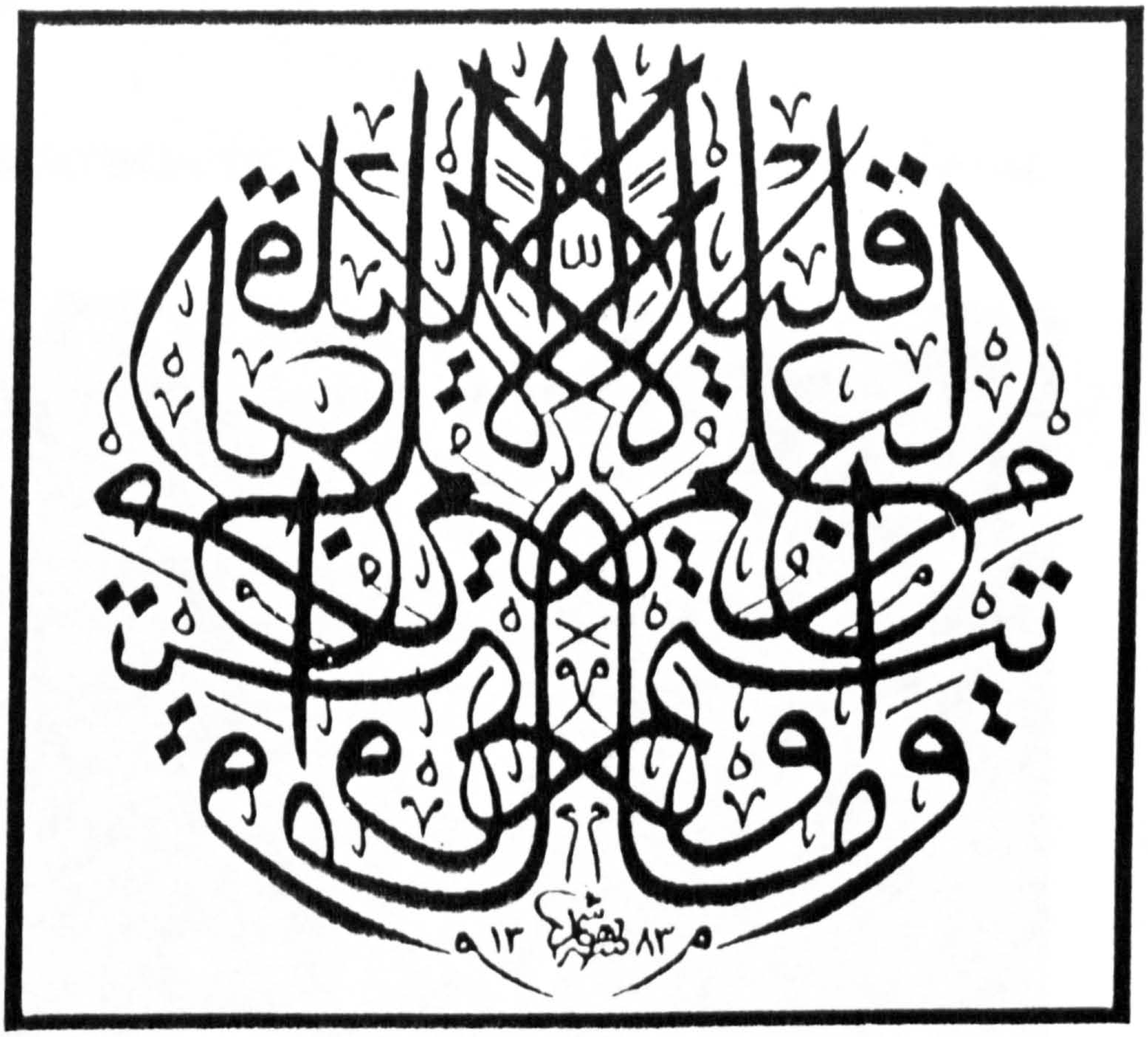
Many highly creative teachers of mathematics in the West have in recent years discovered Islamic art to be an ideal medium for the teaching of mathematical concepts (see authors Jones[37], Makovicky[43], Niman & Norman[54], Norman & Stahl[55]). This has sadly not been appreciated to any extent in Islamic culture where the study of these patterns could be used to teach something of historical importance and which could also lead to modern thinking.

To conclude, in this chapter we have given a brief and general introduction to the subject of symmetry, Islamic art and the importance of repeat patterns. This was intended to show that an extensive study of symmetric Islamic repeat patterns using modern mathematics and computer graphics is a worthwhile task to attempt. We shall now proceed to carry out this task.



a

b

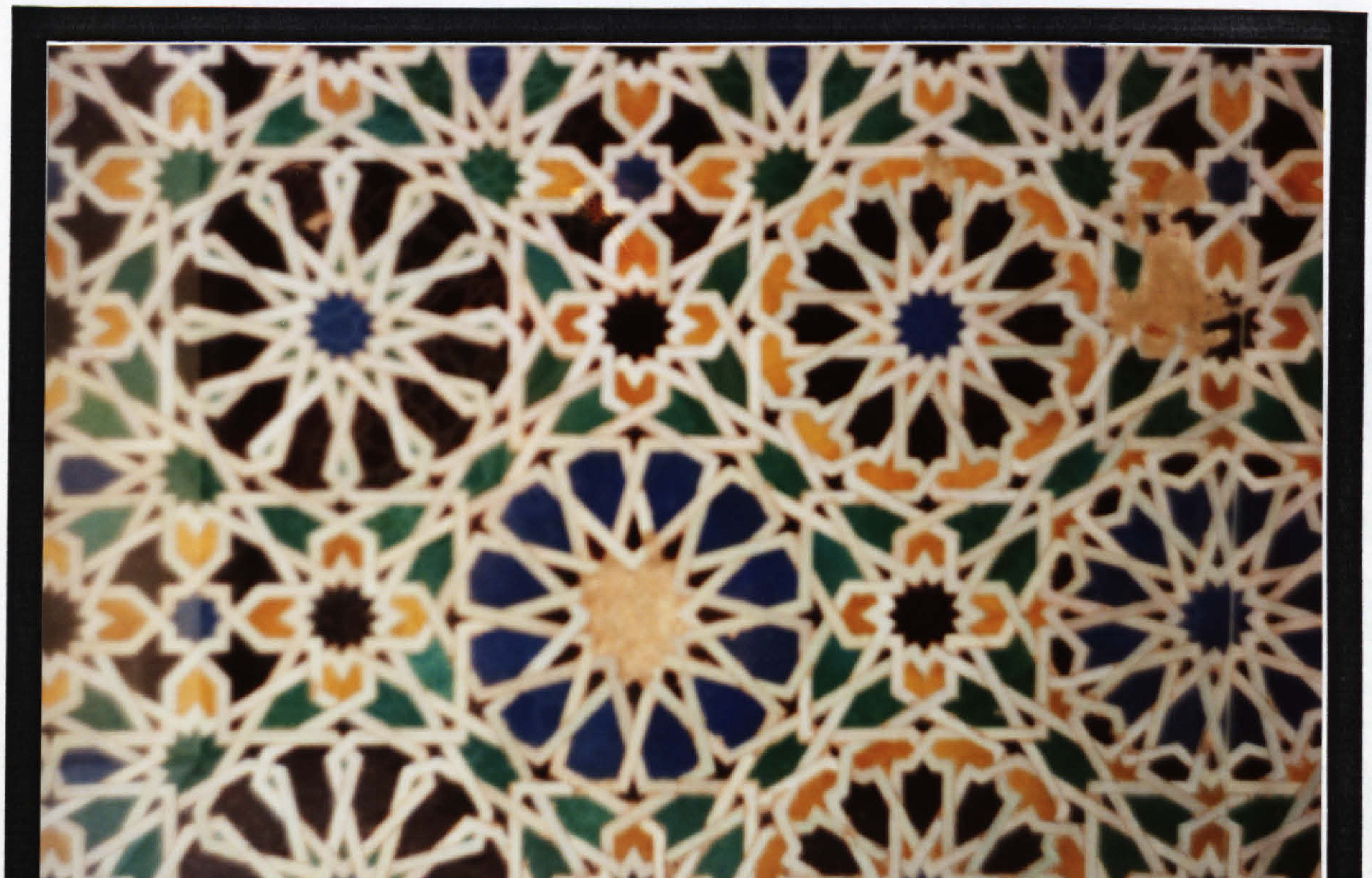


fig(4)

plate(1)

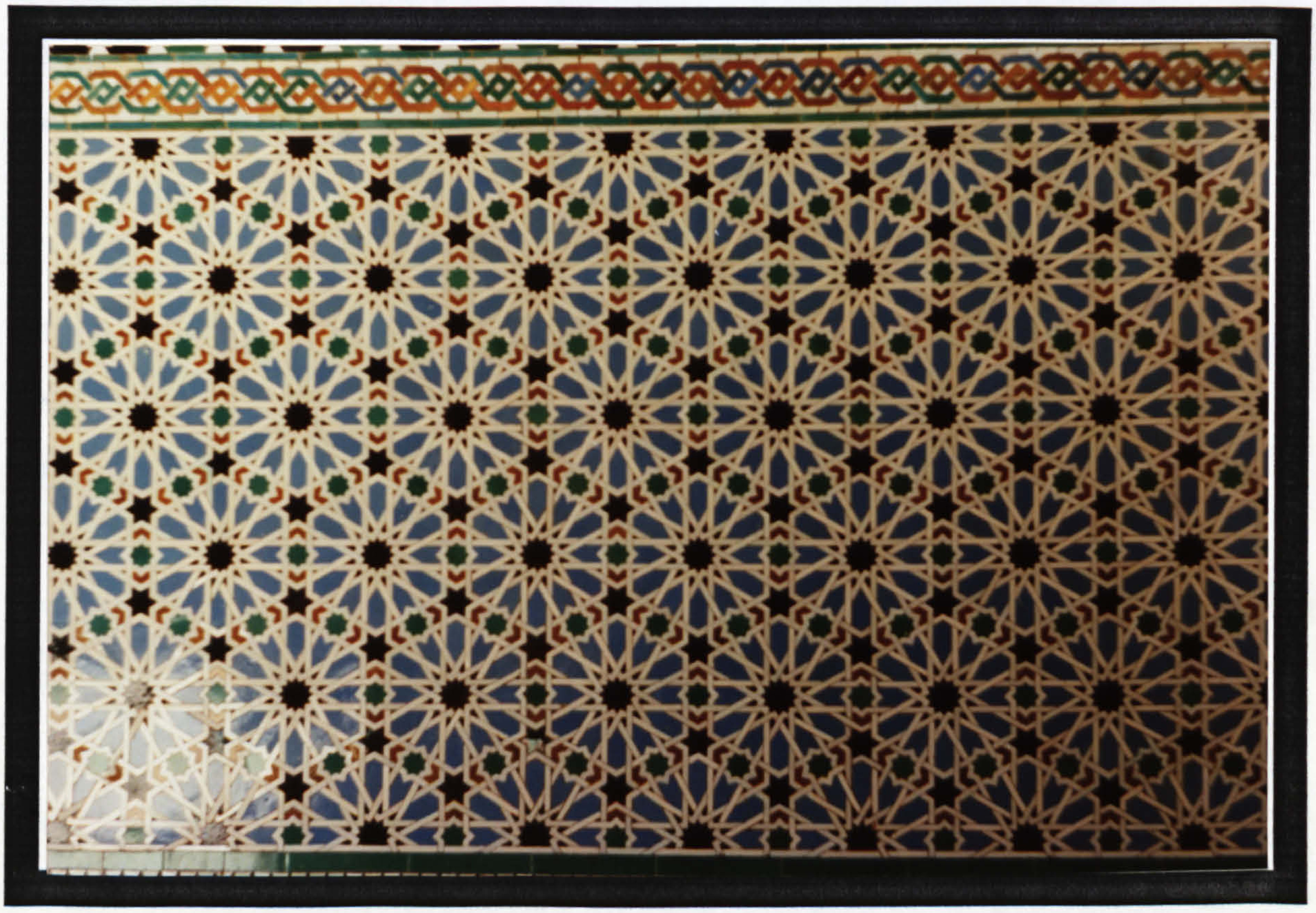
plate(2)

plate(3)



plate(4)

plate(5)



plate(6)

2- CLASSICAL METHOD OF ISLAMIC PATTERN DESIGN

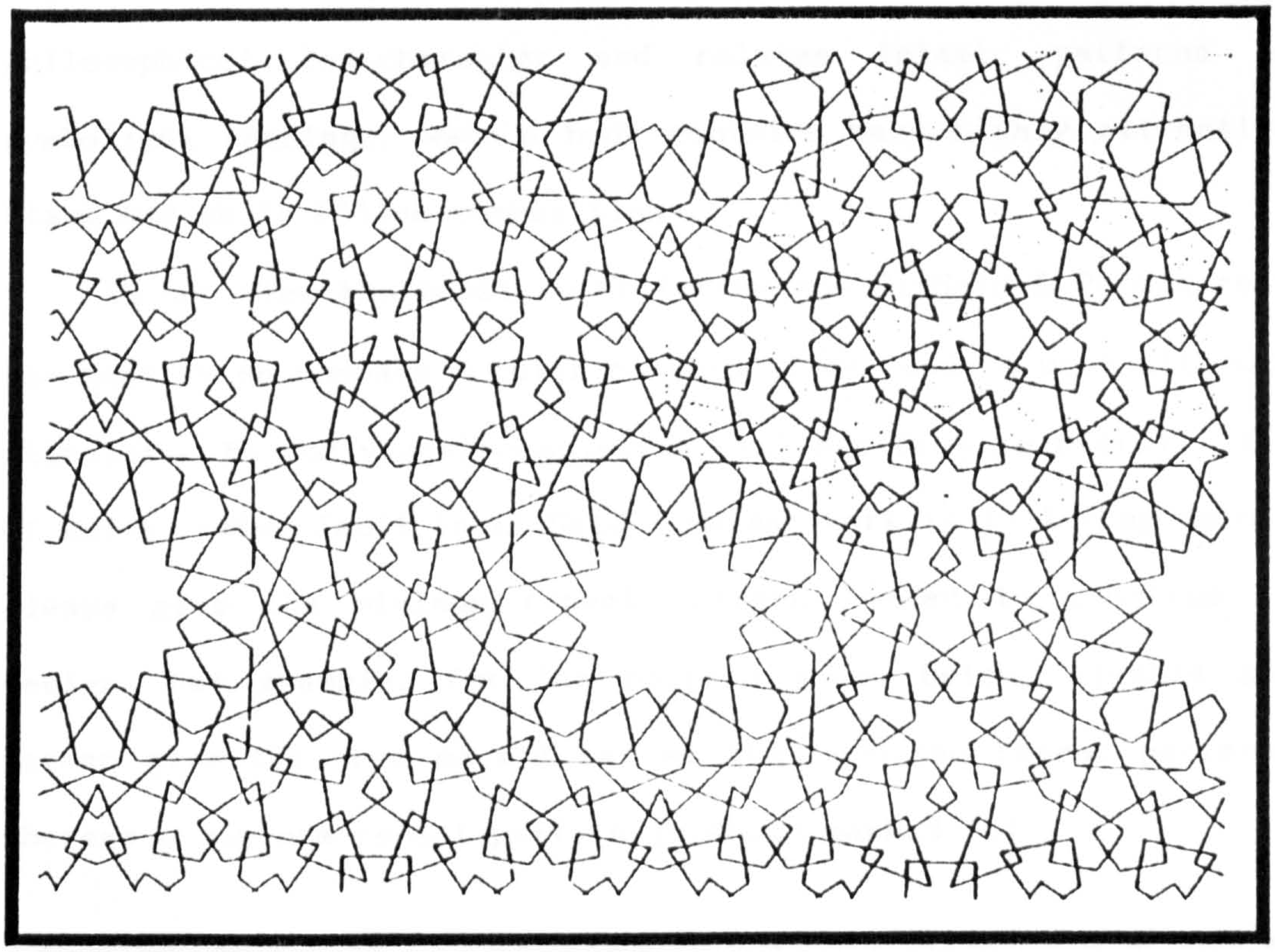
The reader who is unfamiliar with the subject will not know that the original artists who designed Islamic patterns were secretive and did not disclose their methods. Although some limited documents exist in a few libraries and museums (see authors, Christie [10,a,b], Chorbachi [11]), no comprehensive treatise on the subject has come down from the past. The methods proposed are therefore speculative in nature.

Several authors (see, Bourgoin [9], Critchlow [13], El-Said & Parman [63], Wade [73]) have published large collections of algorithms for Islamic patterns but the methods proposed are speculative in nature. In our view they rely unnecessarily on compass/ ruler and net based constructions.

From our extensive study of Islamic geometrical patterns, We have formulated our own view as to how the Islamic patterns originated. The purpose of this chapter is to present this view. We shall propose the concept of a 'tile' as being much simpler to explain the origin of the Islamic geometrical patterns. Both from the point of view of historical as well as mathematical development, this seems to be a much more realistic and useful explanation.

The first major collection of Islamic patterns was published in 1856 in a book containing patterns from many cultures by Owen Jones [38]. Soon after, from 1869 through 1877, the French art historian Prisse D'Avennes [16] published L'art arabe, a sumptuous set of plates (of wood engravings, helio-gravures and color lithographs) illustrating a wide range of art treasures located in and around the city of Cairo (along with a few comparison pieces from European collections). Neither the book by Owen Jones nor the book by Prisse D'Aennes included any algorithms.

The pioneering work which contained a large collection of Islamic patterns together with suggested methods of construction was by Bourgoin [9], published in 1879. Whereas this work provides a rich source of patterns, it suffers from the fact that it is not always possible to work out the method of construction being proposed. For example, fig(1) below appears on page 152 of Bourgoin's book, the dotted construction lines are barely visible.



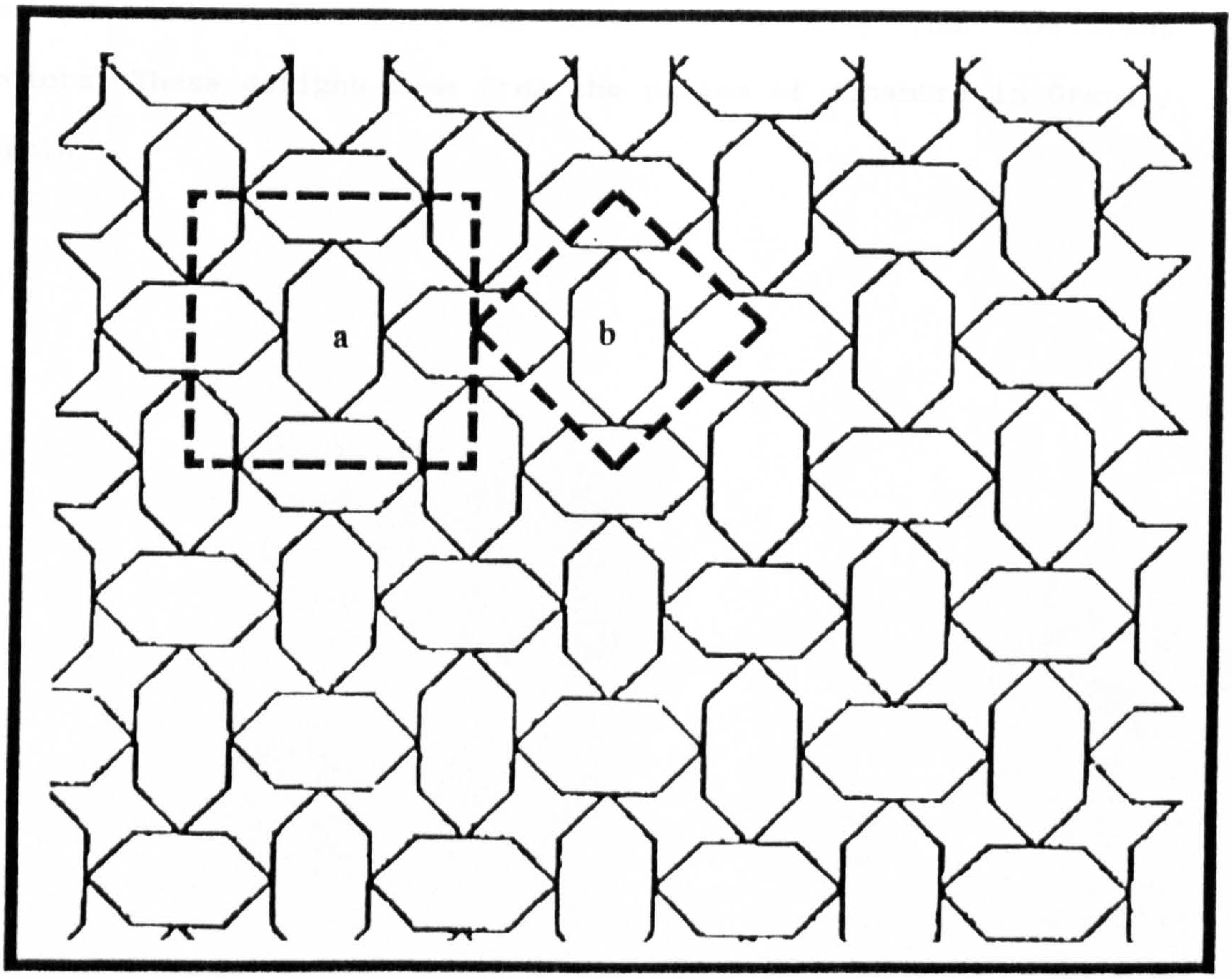
fig(1)

Christie's book [10b] which appeared in 1929 is another multi-cultural book and not exclusively concerned with Islamic patterns, but nevertheless it contains many islamic patterns. It does not give detailed algorithms but does give some general methods.

Three books on Islamic patterns appeared in 1976. They are by Critchlow [13], El-Said & Parman [63], and Wade [73]. All these books have something individual and interesting to offer.

El-Said & parman's book gives very clear geometrical constructions and also contains sections on calligraphy and architecture. There are many photographs of actual building together with the patterns discussed. Critchlow's book is highly philosophical in character and relates Islamic patterns to symbolical meanings. Wade's book contains many highly attractive black and white filled compositions.

In our view the books by Critchlow and El-Said & Parman rely too heavily on compass / ruler constructions. Wade's work although attractive in individual composition suffers from an overall lack of unity. other fault in El-Said & Parman work is that they do not always give the minimum repeat pattern needed to construct a design. For example, for the pattern shown below, El-said and Parman give the area marked (a) as the required repeat pattern, whereas a minimum repeat pattern is shown marked (b).

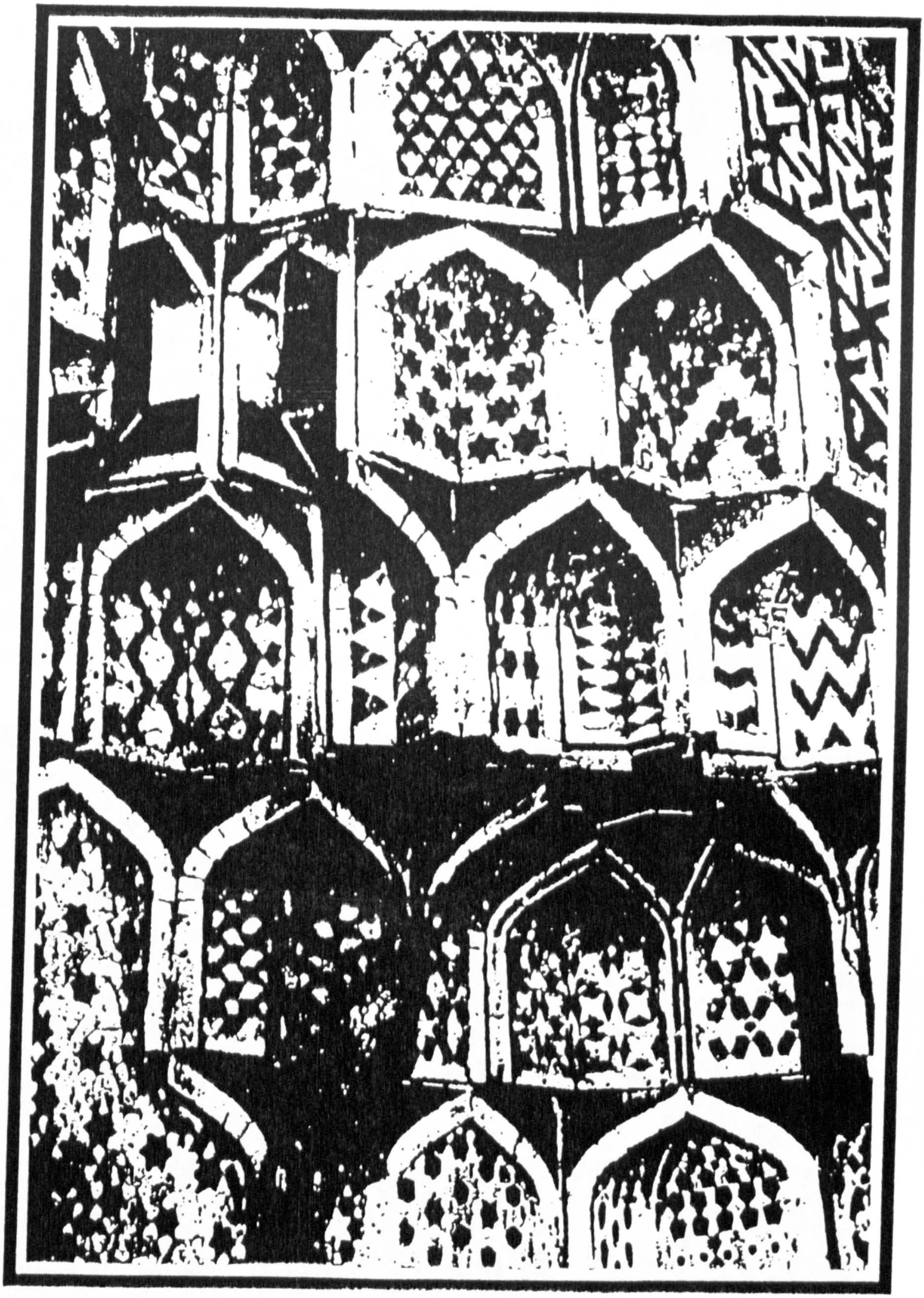


fig(2)

2.1 THE ORIGINS OF ISLAMIC PATTERNS

If one asks the question as to how the Islamic patterns originated, then it would seem most logical to start with the practical experience of tiling with simple shapes e.g. triangles, rectangles, squares, and hexagons. These shapes would have been decorated with simple colors and patterns.

From this beginning it would be natural to experiment with multiple shaped tiles, new shapes produced by overlapping tiles and to invent better colorings and patterns. Fig(3) shows an inside view of Sircali Madrassa, Konya in Turkey, and displays typically how various shaped tiles were used to experiment with patterns. Plates(1), (2) show examples of the use of simple shaped colored tiles. In these the mosaic work uses four different colors. These designs come from the palace of Alhambra in Granda, Spain.

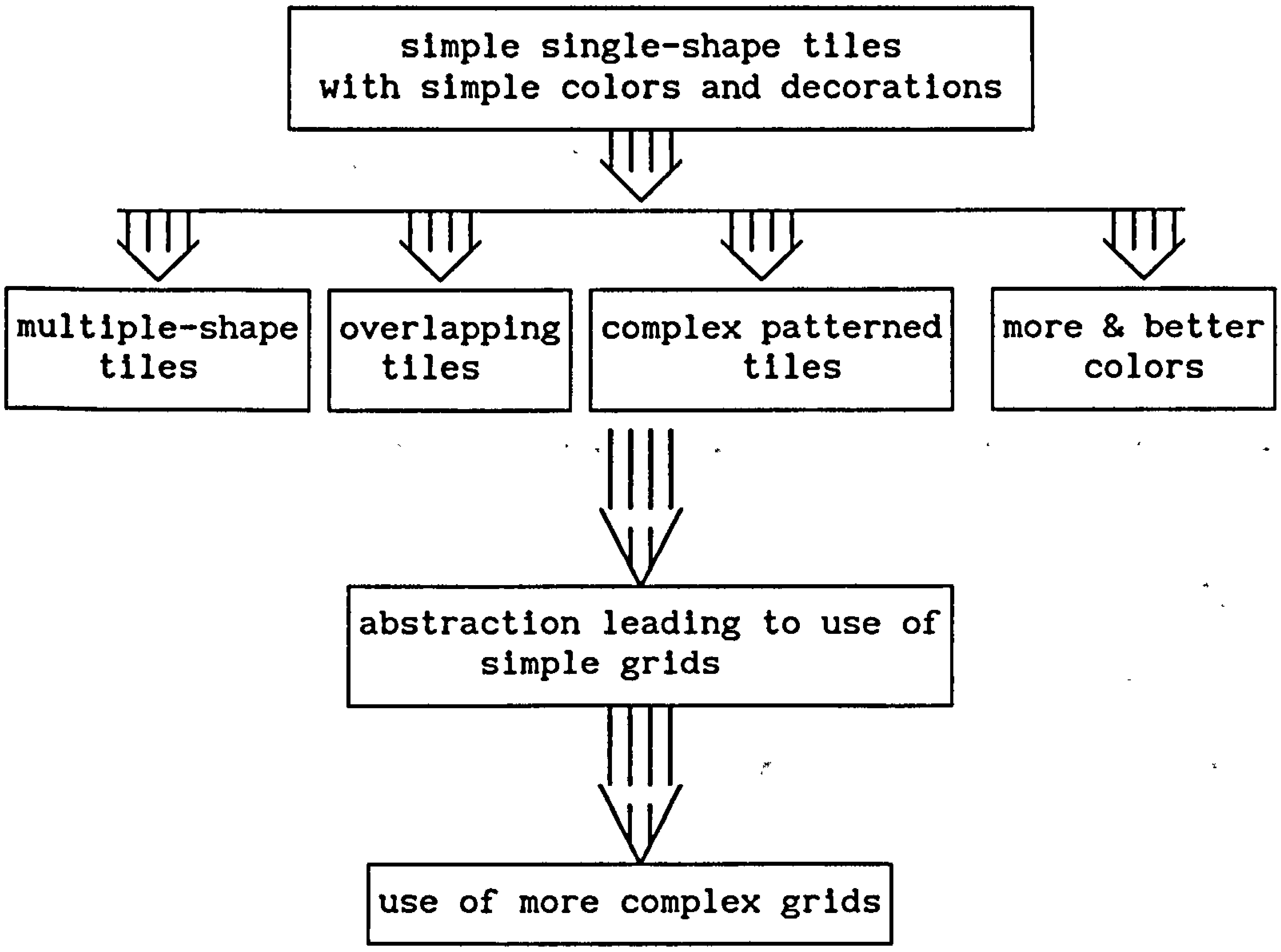


fig(3)

plate(1)

plate(2)

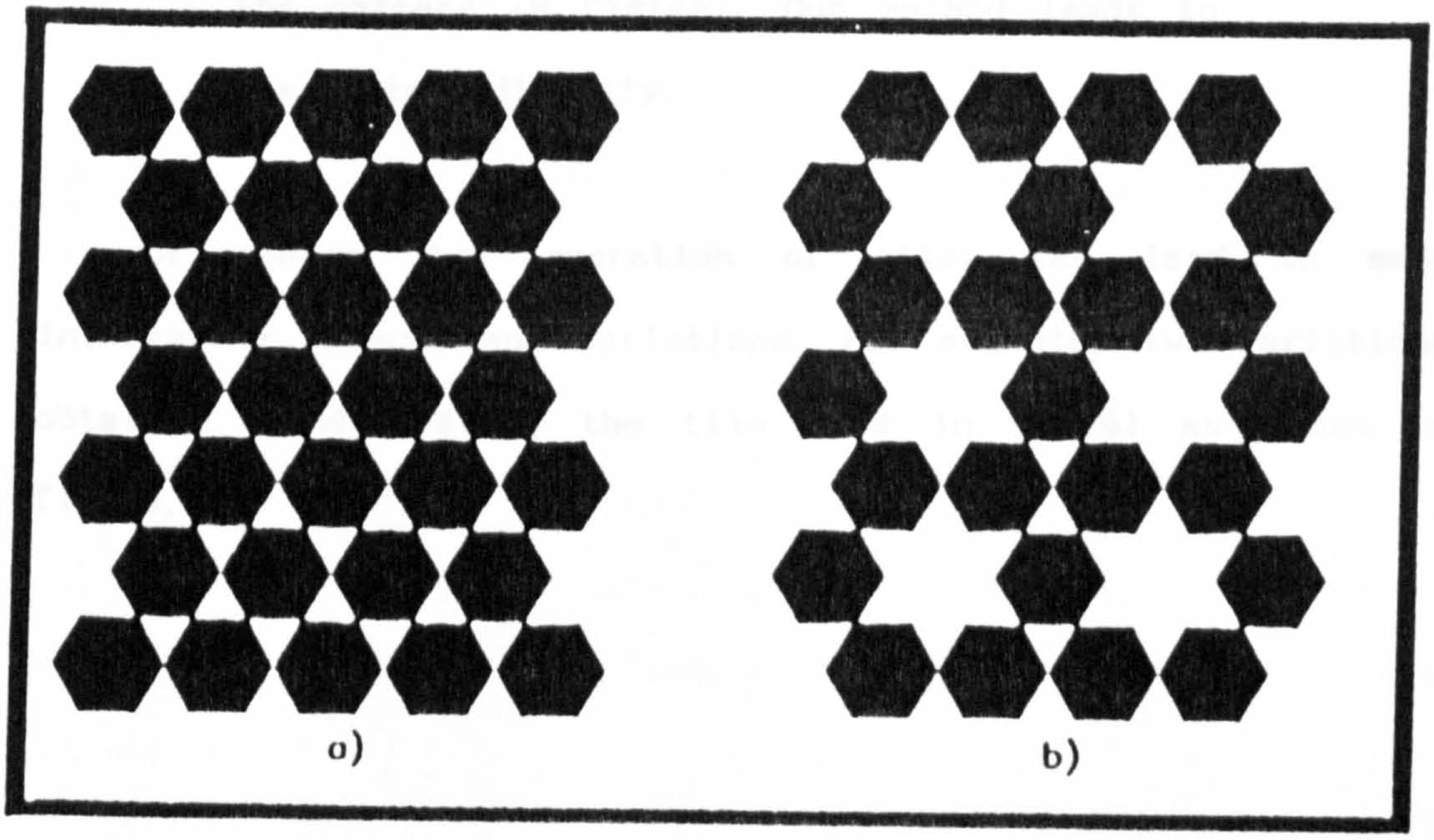
The experience with simple colored tiles would then lead to abstraction in the design of patterns and give rise to the use of simple and complex nets. This development is summarized in the chart below and we shall evolve our description on this structure.



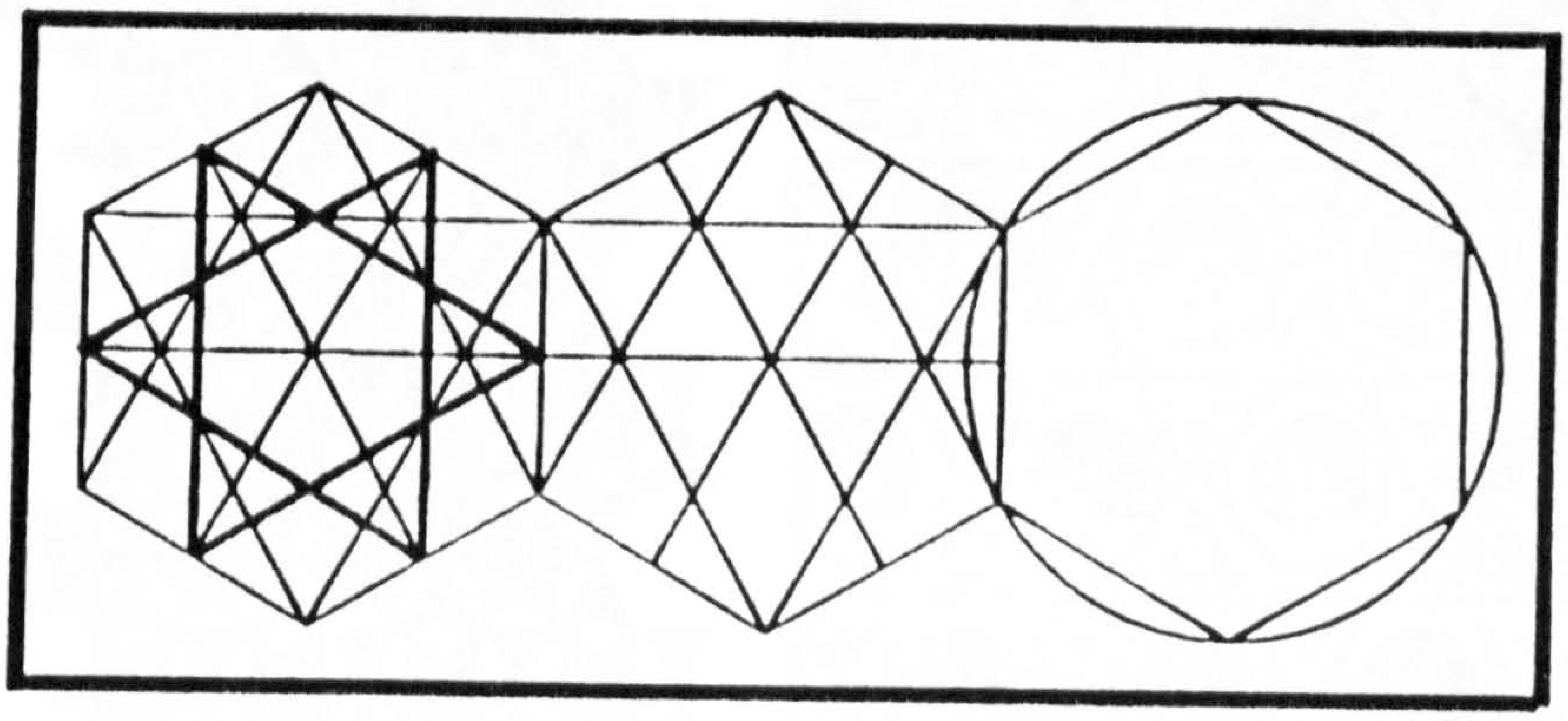
2.2 <u>SOME ISLAMIC PATTERNS ARISING FROM SIMPLE SINGLE-SHAPED TILES</u>:

Many Islamic patterns can be made quite easily from simple single shaped tiles. The hexagonal tile is the most familiar tile used in Islamic patterns and in fig(4a,b), we show two patterns which arise from the use of a single tile of this shape. We simply arrange the hexagons touching each other in rows. Of course, this produces triangular holes which could be filled with another simple tile or we could think of the geometrical pattern by itself without any reference to tiles. If we were to remove every third hexagonal tile from such a row then star-shape holes are produced. This could be thought of as a tiling with a star-shape tile and a hexagonal-shape tile or as a pattern, as shown in fig(4b).

To make clear the point we made earlier regarding excessive use of compass/ruler by El-Said & Parman [63], the reader may like to compare the method suggested by us here with that given by El-Said & Parman which is shown in fig(5).



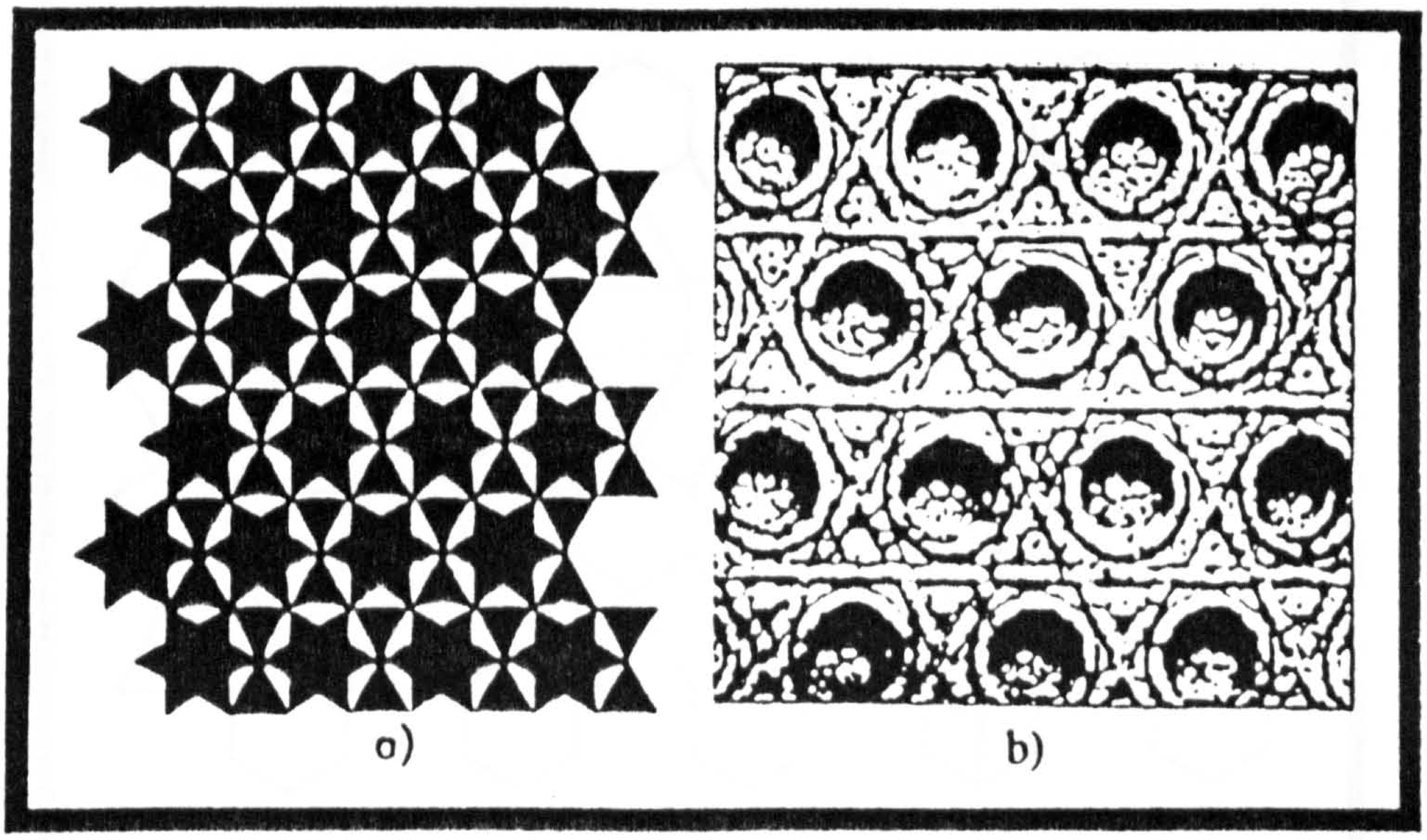
fig(4)



fig(5)

El-said & parman start with circle and proceed as show in this figure to obtain a decreased hexagonal tile from which they suggest making the pattern in fig(4c). Our method leads to the pattern directly.

Of course the decoration of tiles can lead to many interesting effects and variations. For example, two variations obtained by decorating the tile used in fig(4) as shown in fig(6a,b).

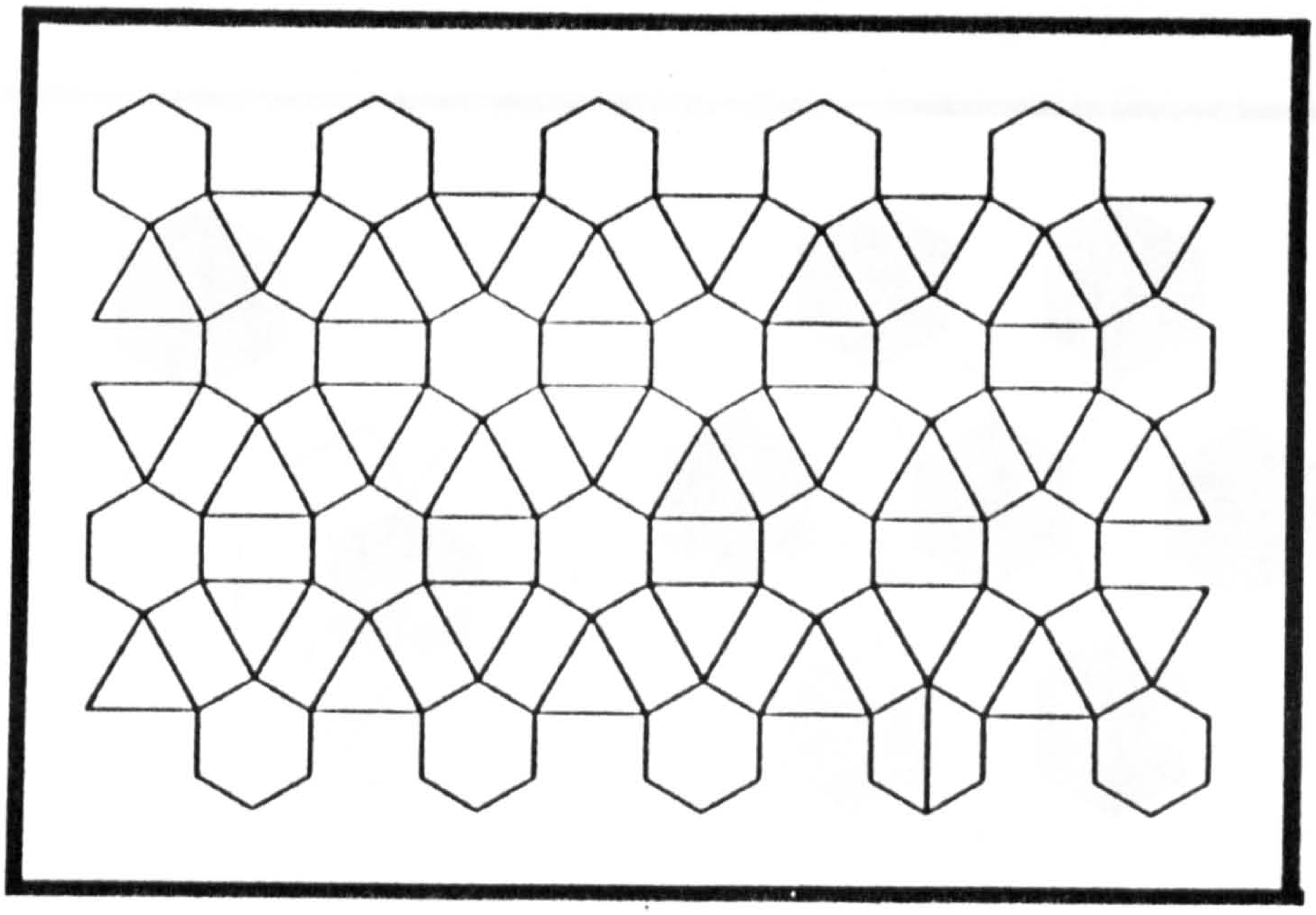


fig(6)

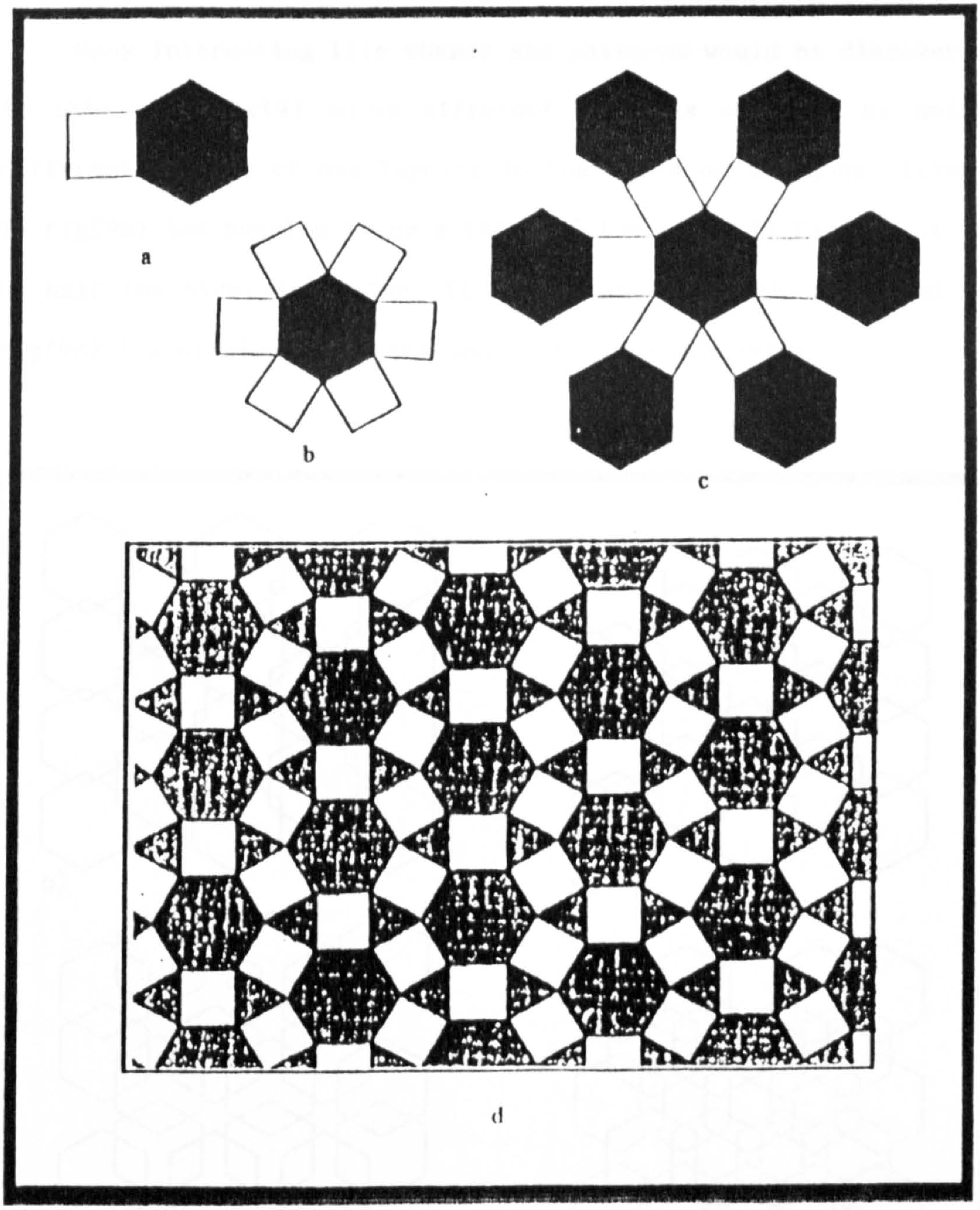
2.3 EXAMPLES OF PATTERNS PRODUCED FROM USING MULTIPLE-SHAPED TILES:

Some other well known patterns of Islamic art can be derived equally easily if we use multiple-shaped tiles. For example, if we introduce rectangular tiles with hexagonal tiles, which are very common, then we obtain the pattern shown in fig(7).

Fig(8) shows another example of a pattern produced by making use of square tiles and hexagonal tiles. Again the triangular holes that appear can be filled by using simple tiles in a tiling or we can ignore the difference between a tile and hole, and see it as a pattern.



fig(7)

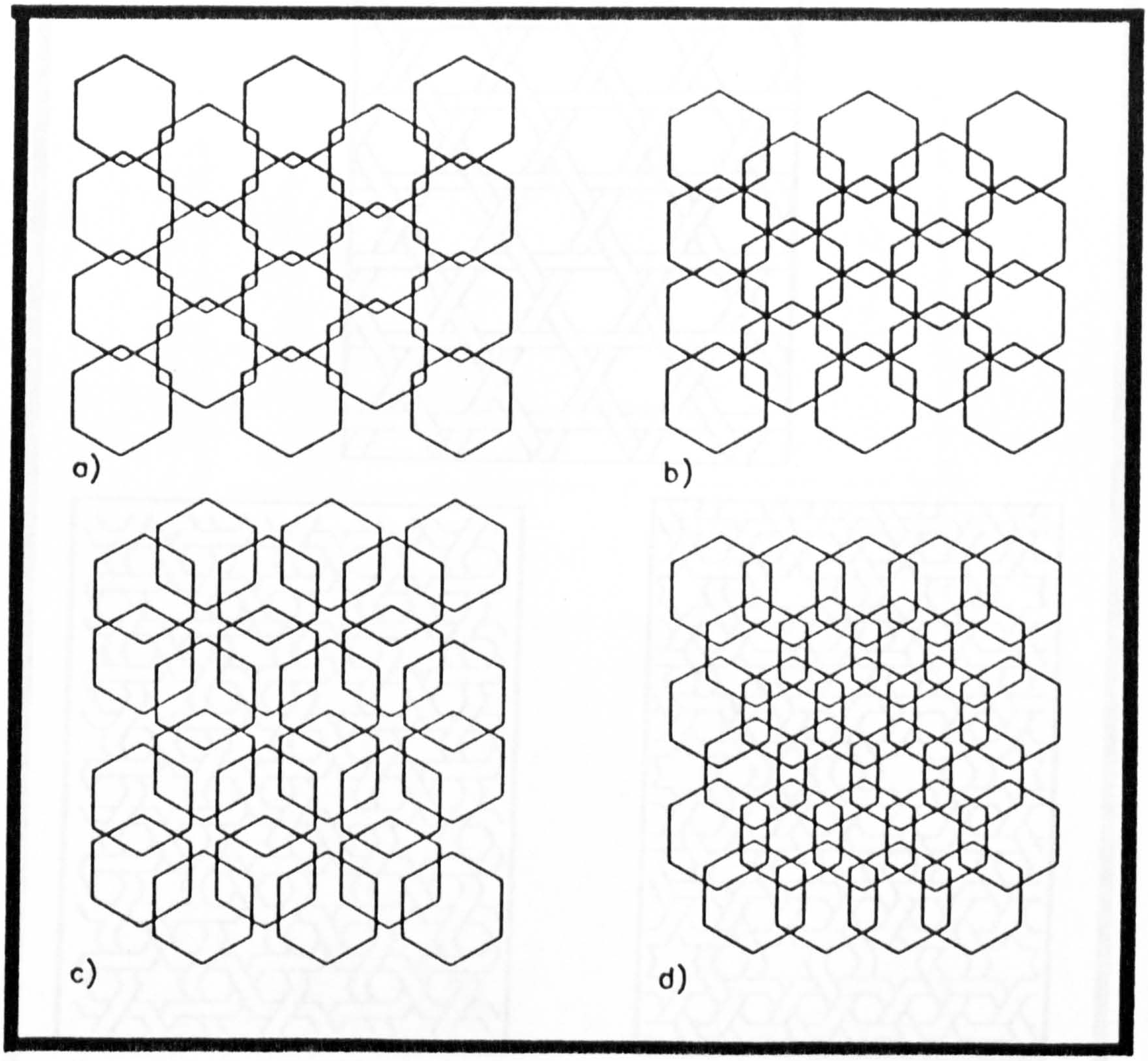


Fig(8)

2.4 PATTERNS BASED ON OVERLAPPING TILES

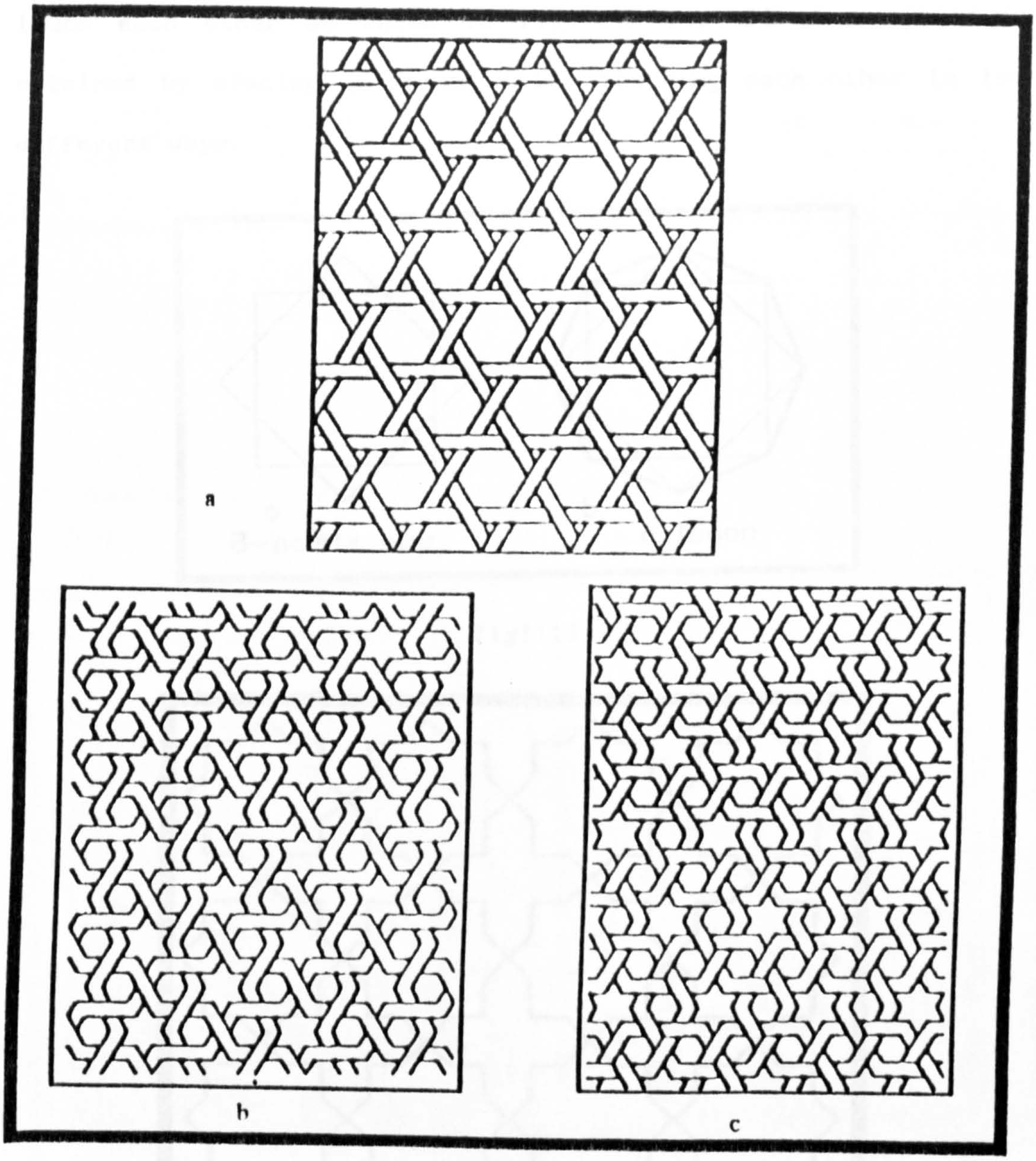
One would expect that the availability of simple tiles would naturally lead to some experimentation with overlapping tiles.

Many interesting tile shapes and patterns would be discovered in this way. Fig(9) shows different patterns obtained by using different amounts of overlapping in the sides of hexagonal tiles. In fig(9a) the overlap is by a third of the side. In fig(9b) it is by half the side. In fig(9c) it is by two thirds the side and in fig(9d) the overlap is by the whole side and a quarter.



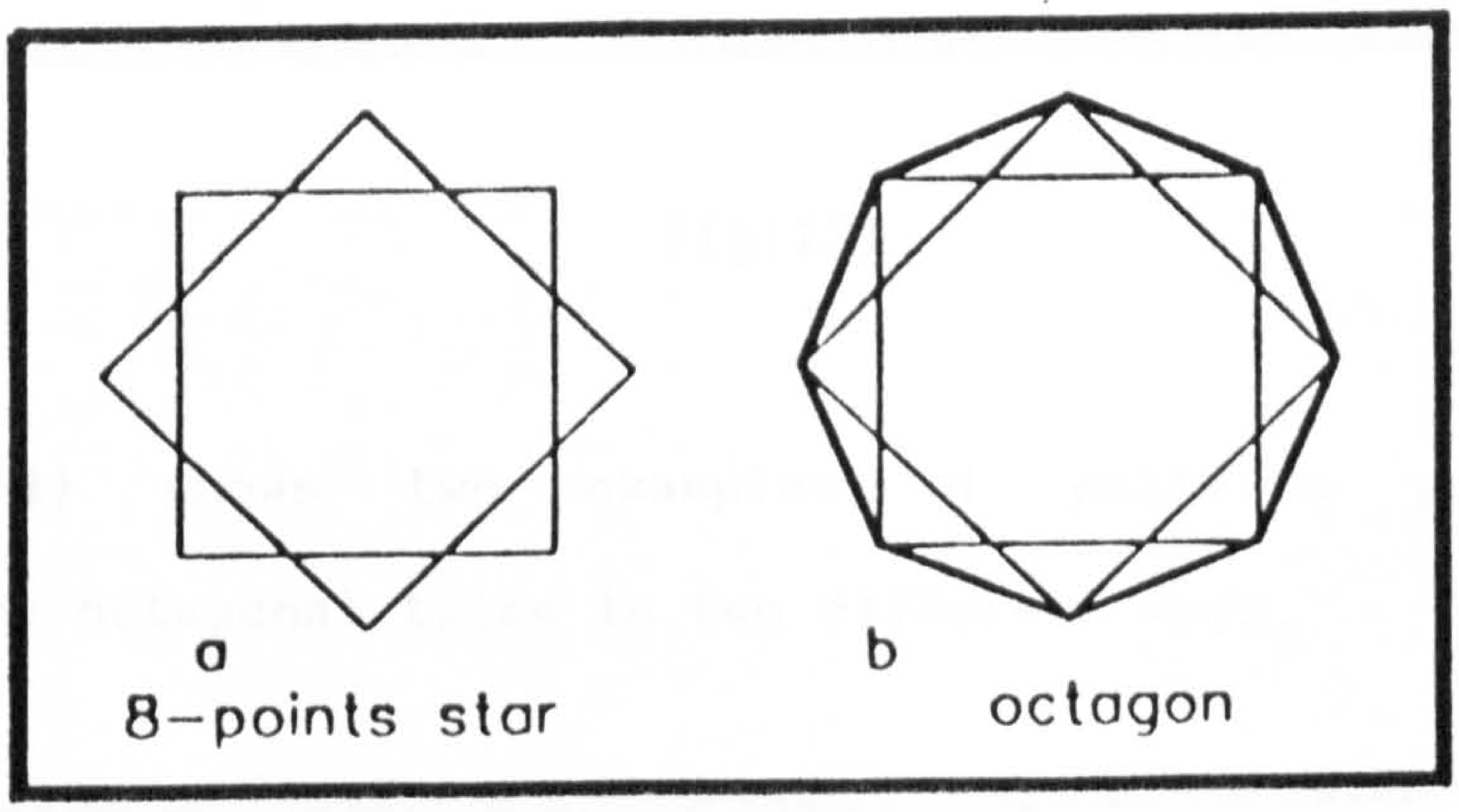
fig(9)

One very common enhancement used in Islamic patterns is the interlacing of lines. Fig10(a,b,c) below show the interlacing patterns obtained by replacing the lines in fig4(a,b) and fig(9c) with interlaced lines.

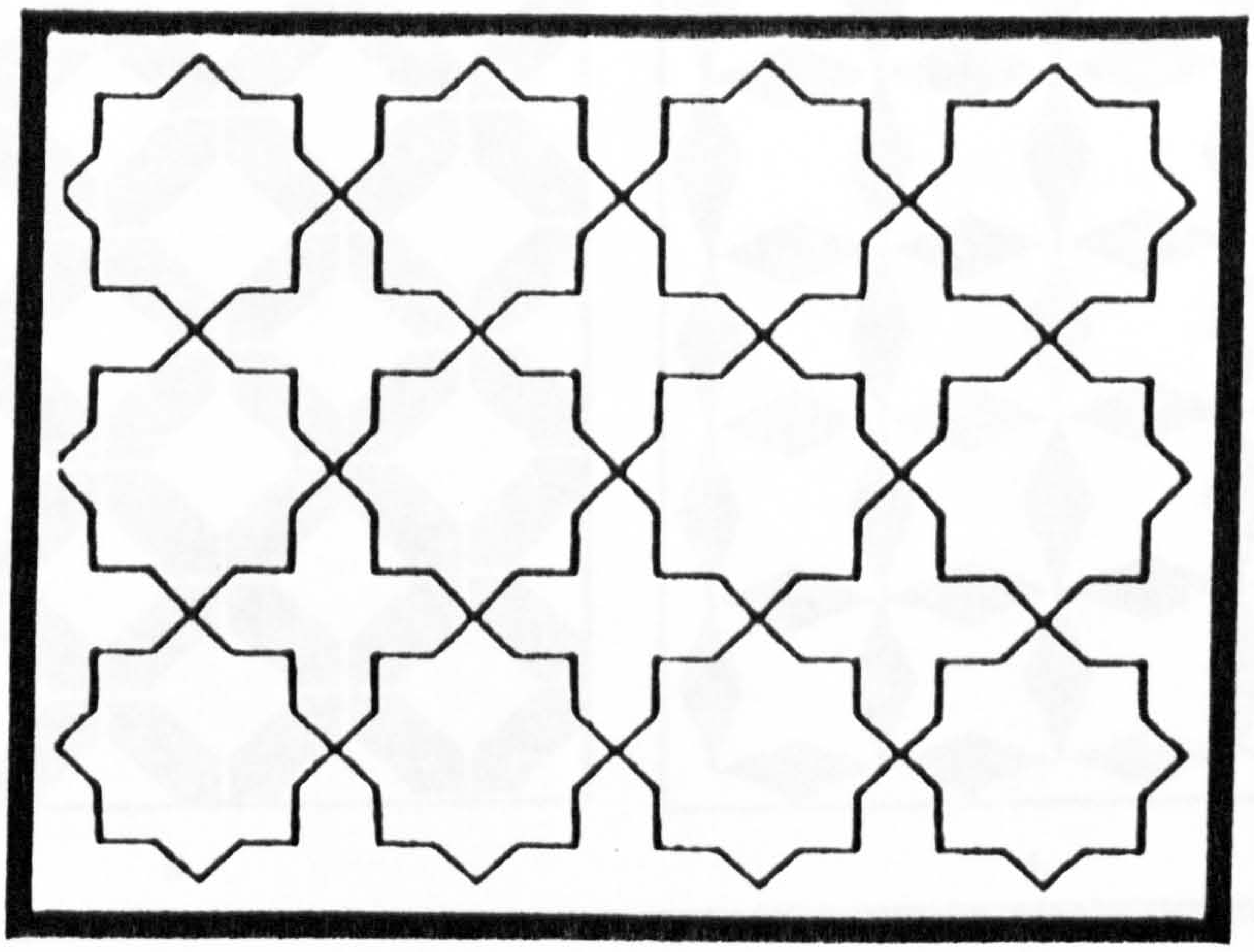


fig(10)

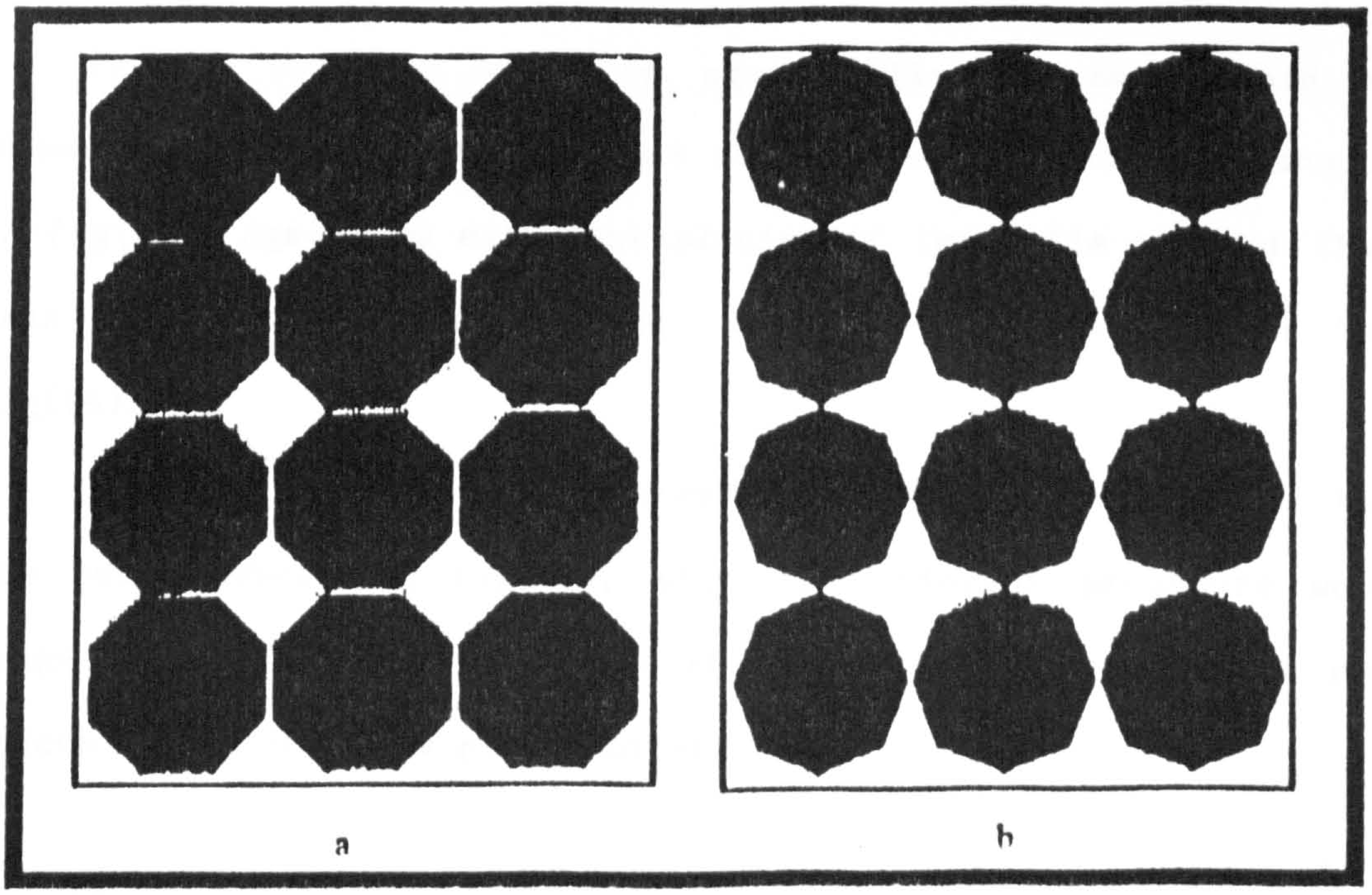
The most common tile shape in Islamic world is the one obtained by superimposing two square tiles to obtain an octagonal star shape, shown in fig(11a). Related to it is the simple octagonal tile shown in fig(11b). Many patterns arise from the use of these tiles. The most familiar pattern in Islamic would is obtained by using 8-pointed star shapes, placing them so that they touch each other as shown in fig(12). Fig(13) shows patterns obtained by placing octagonal tiles touching each other in two different ways.



fig(11)

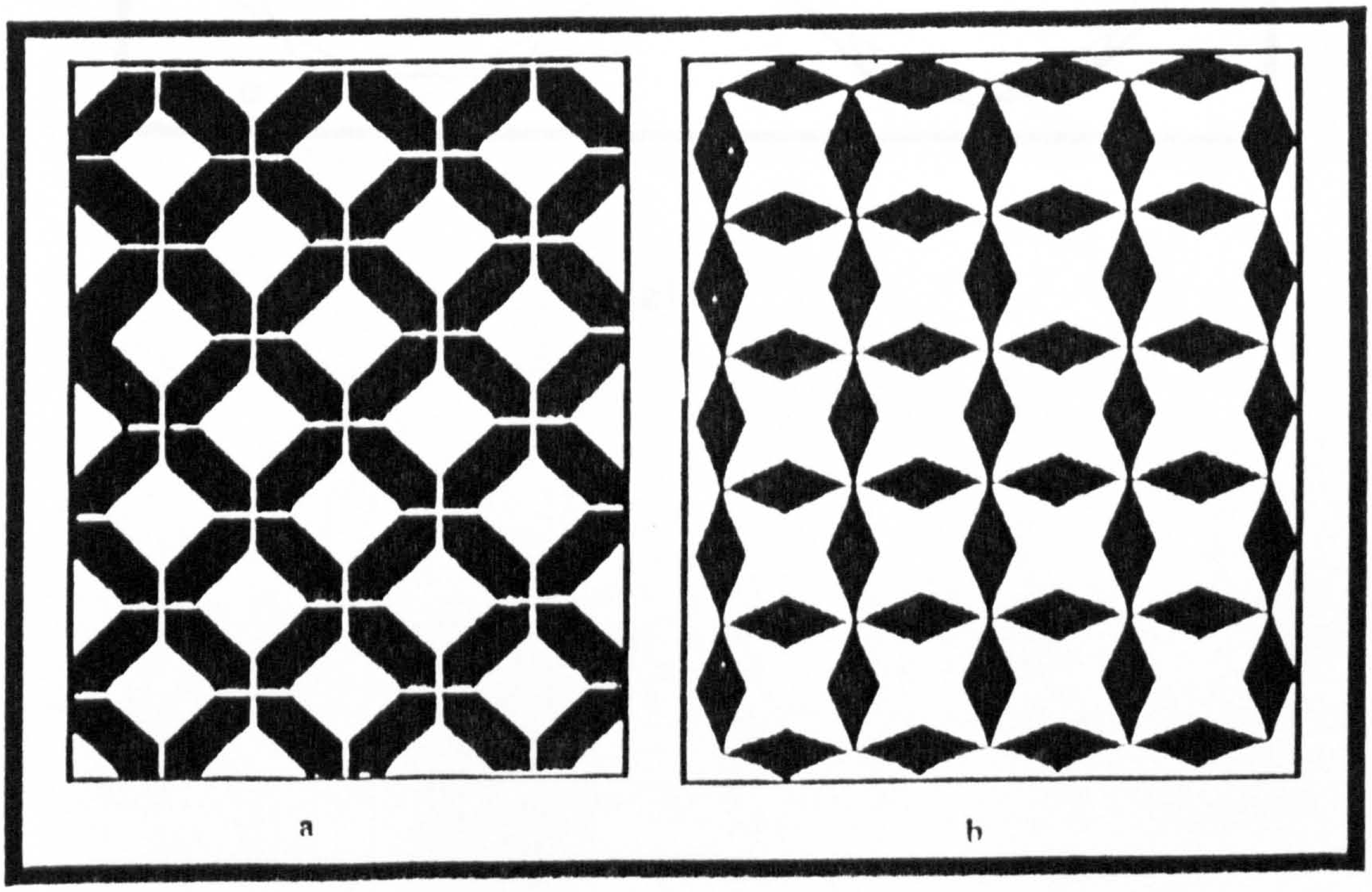


fig(12)



fig(13)

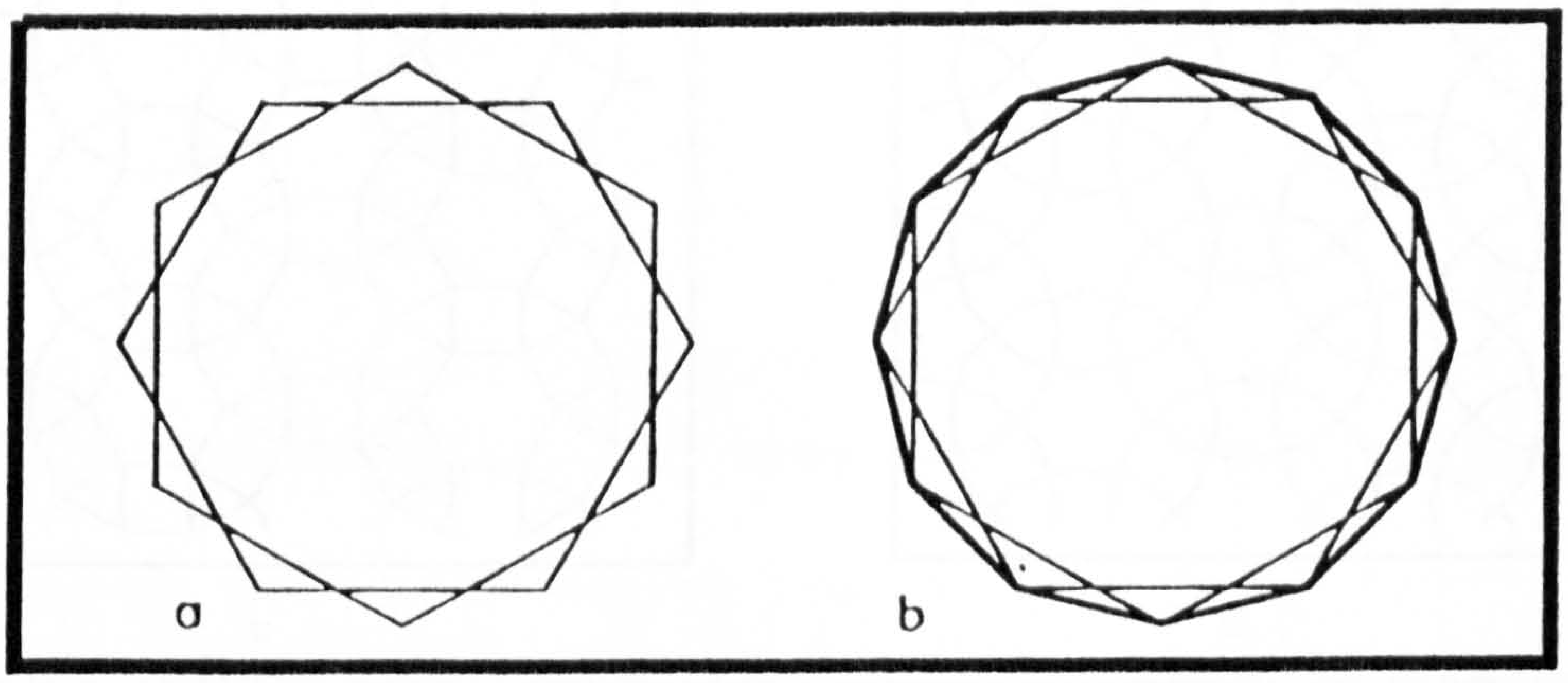
Fig(14) shows two examples of patterns produced by overlapping octagonal tiles in two different ways.



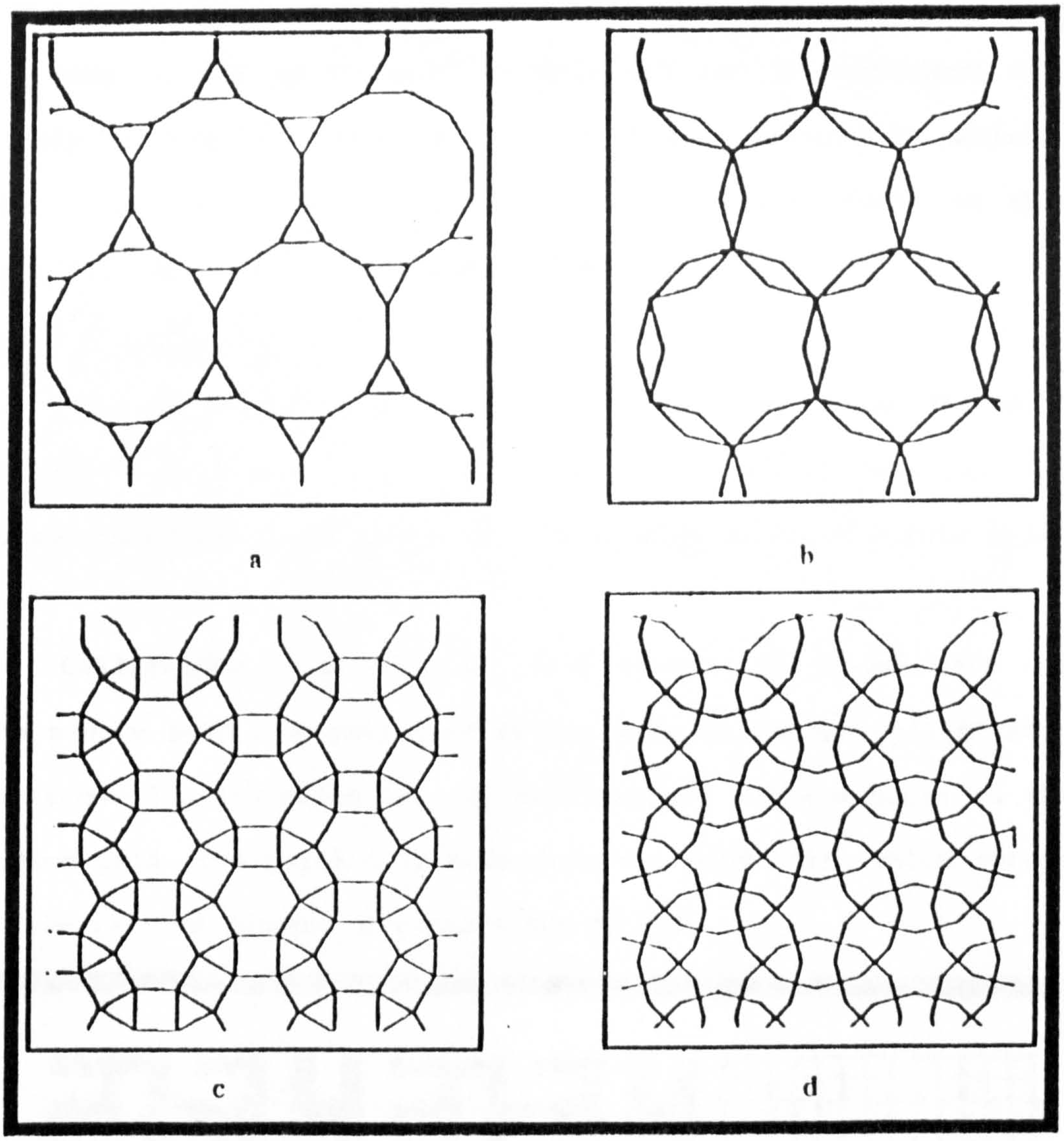
fig(14)

Exactly as done with the square tile, we can obtain a dodecagon tile from two hexagonal tiles by superimposing as shown in fig(15). Again, by different placing of this tile many of the Islamic patterns are generated. Some of these are shown in fig(16).

Note that the pattern produced in fig(16c) is identical to the one produced in fig(8d), where a different procedure was suggested. This emphasizes, the obvious point that there is no unique way to make any given pattern.



fig(15)



fig(16)

Patterns obtained by placing dodecagons

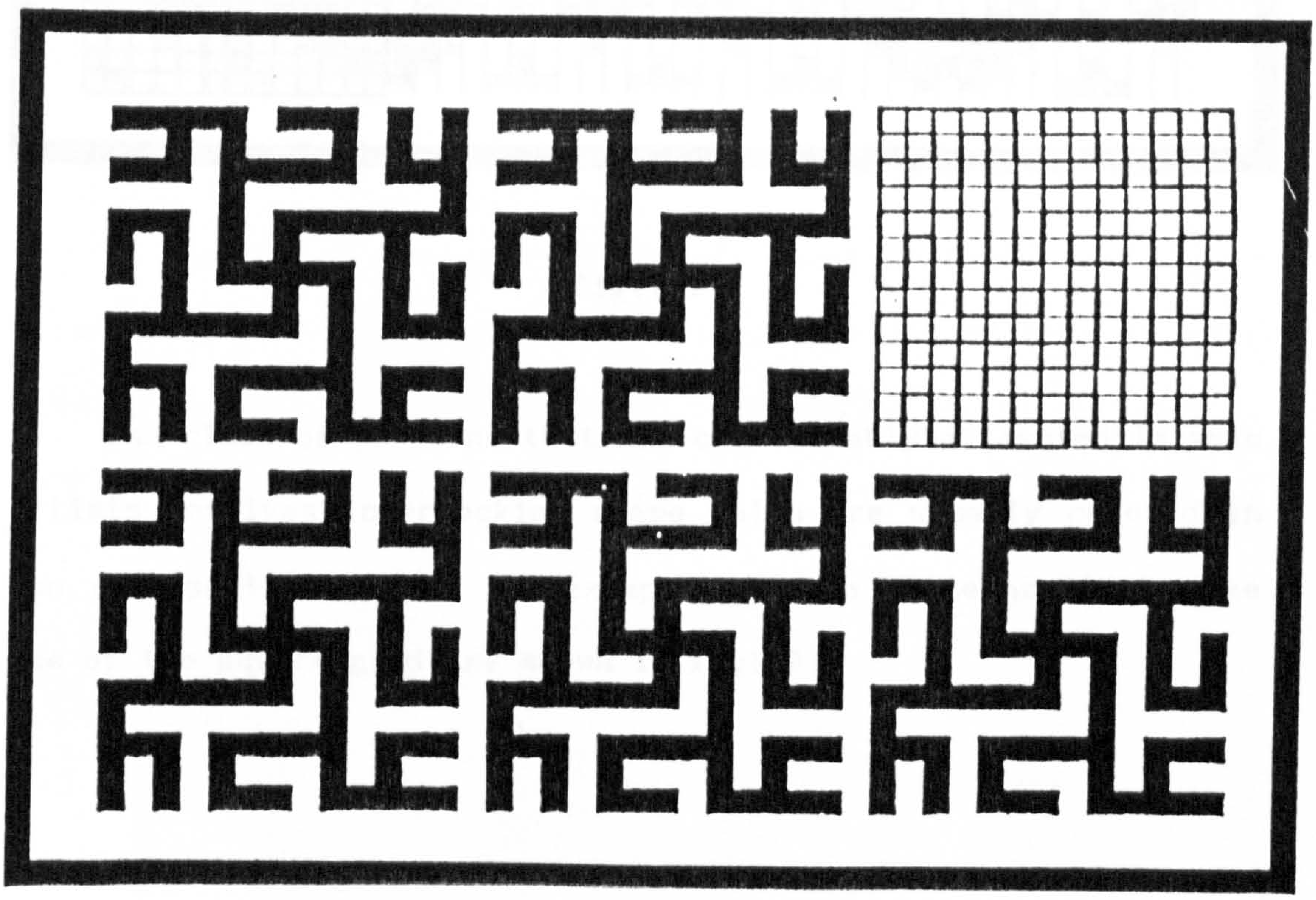
2.5 USE OF GRIDS

The above discussion was intended to show how simple practical experience with tiles can give rise to a large class of patterns. This experience will undoubtedly lead to abstraction and the next stage would involve geometrical construction without their being necessarily any connection with tiles. Having shown how many of the patterns of Islamic art can be explained very simply in terms of tiles, we will now look at example of patterns which can be derived making use of simple grids. First We show examples based on two of the most common grids.

2.5.1 SQUARE GRID

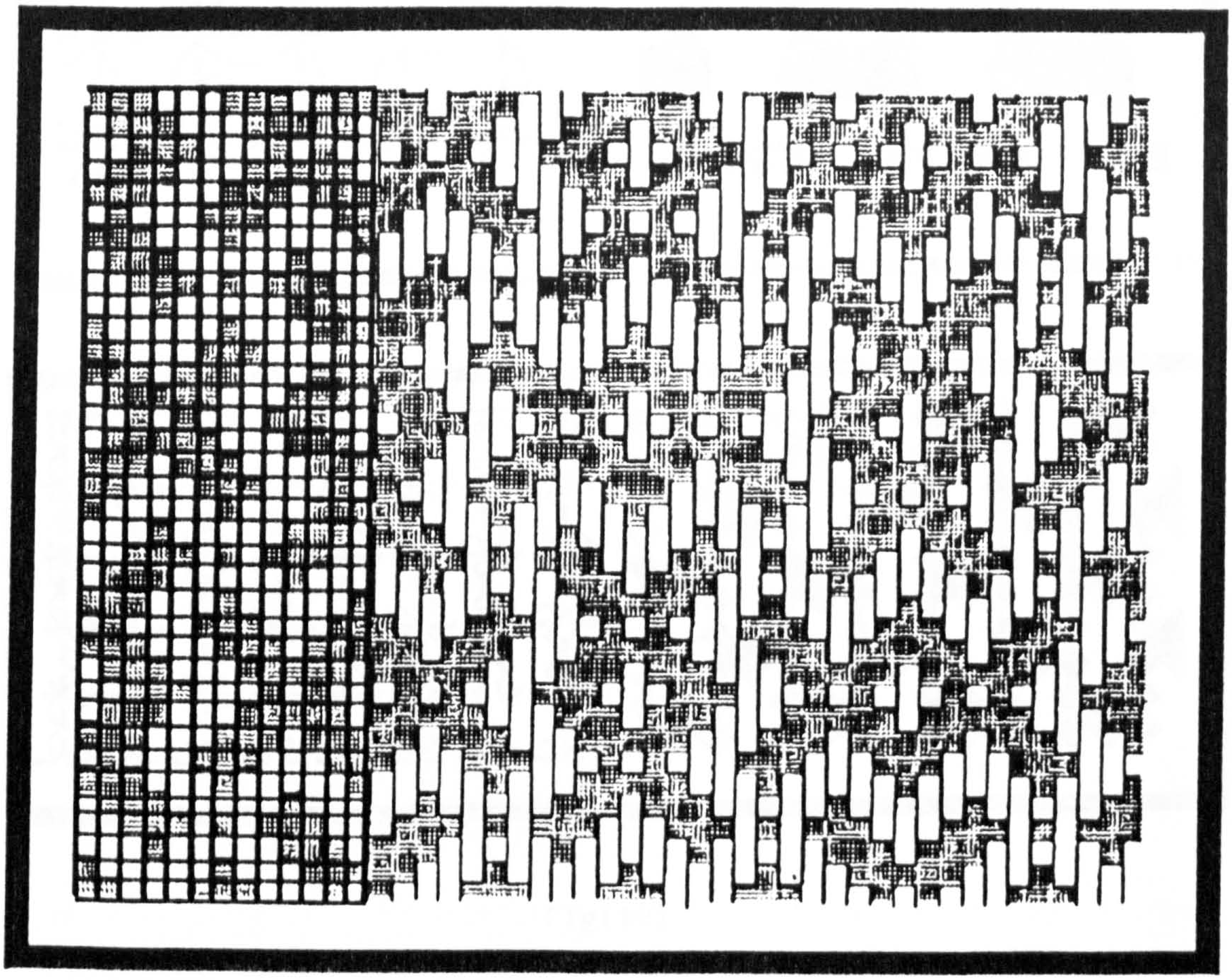
The simplest grid is the square grid. It has a high degree of symmetry and is also very useful from the practical point of view because designs based on it can be translated easily into brick work.

Calligraphy is a very important feature of Islamic art and the square grid has been used extensively to design calligraphic patterns. Fig(17) shows typical calligraphic pattern based on the square grid. The pattern is made from the word 'Ali' which refers to the name of prophet Mhammad's son-in-law.



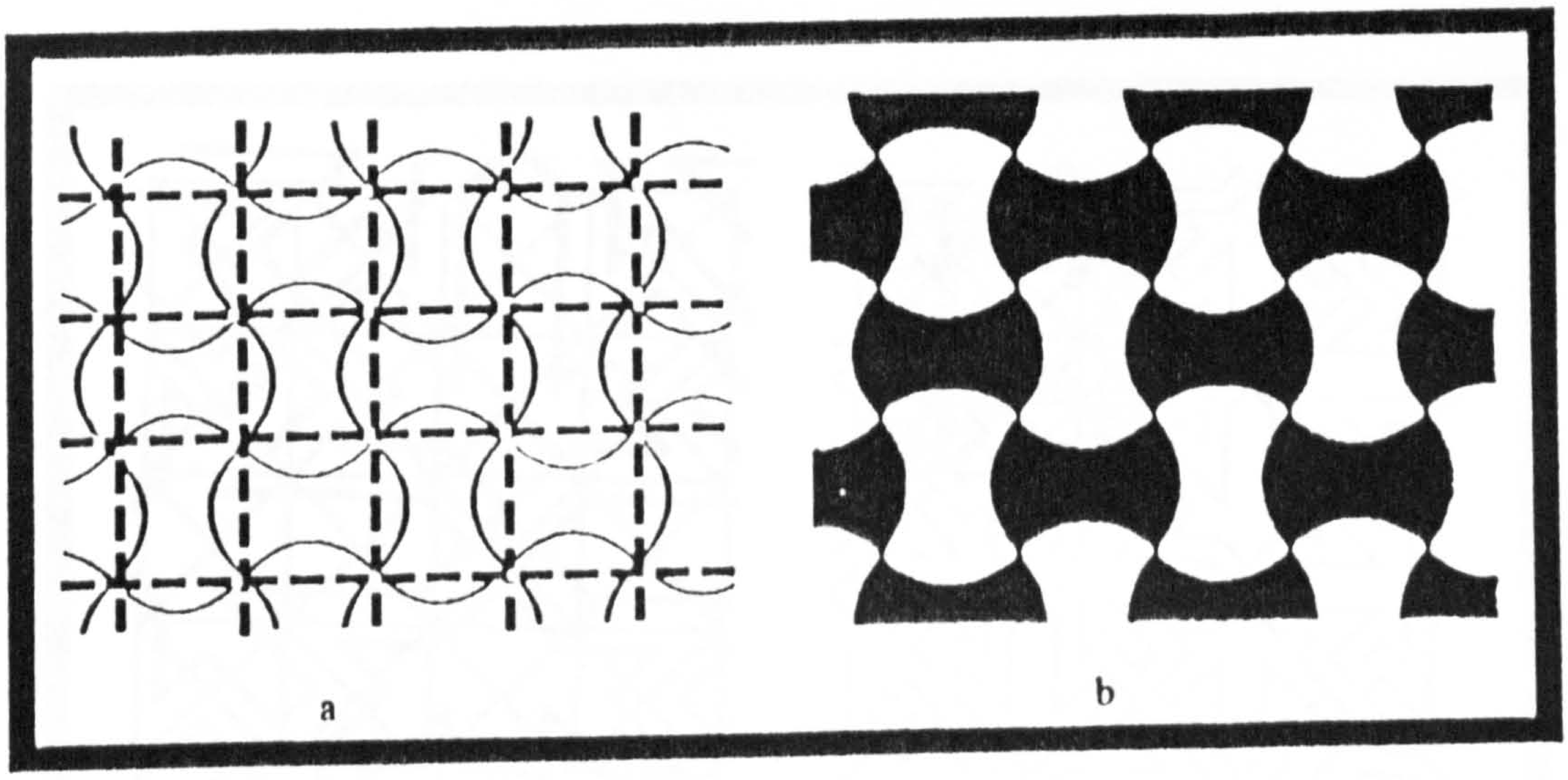
fig(17)

Fig(18) shows an example of a design which is often found in brick work. Its method of construction is shown in the right of the figure.



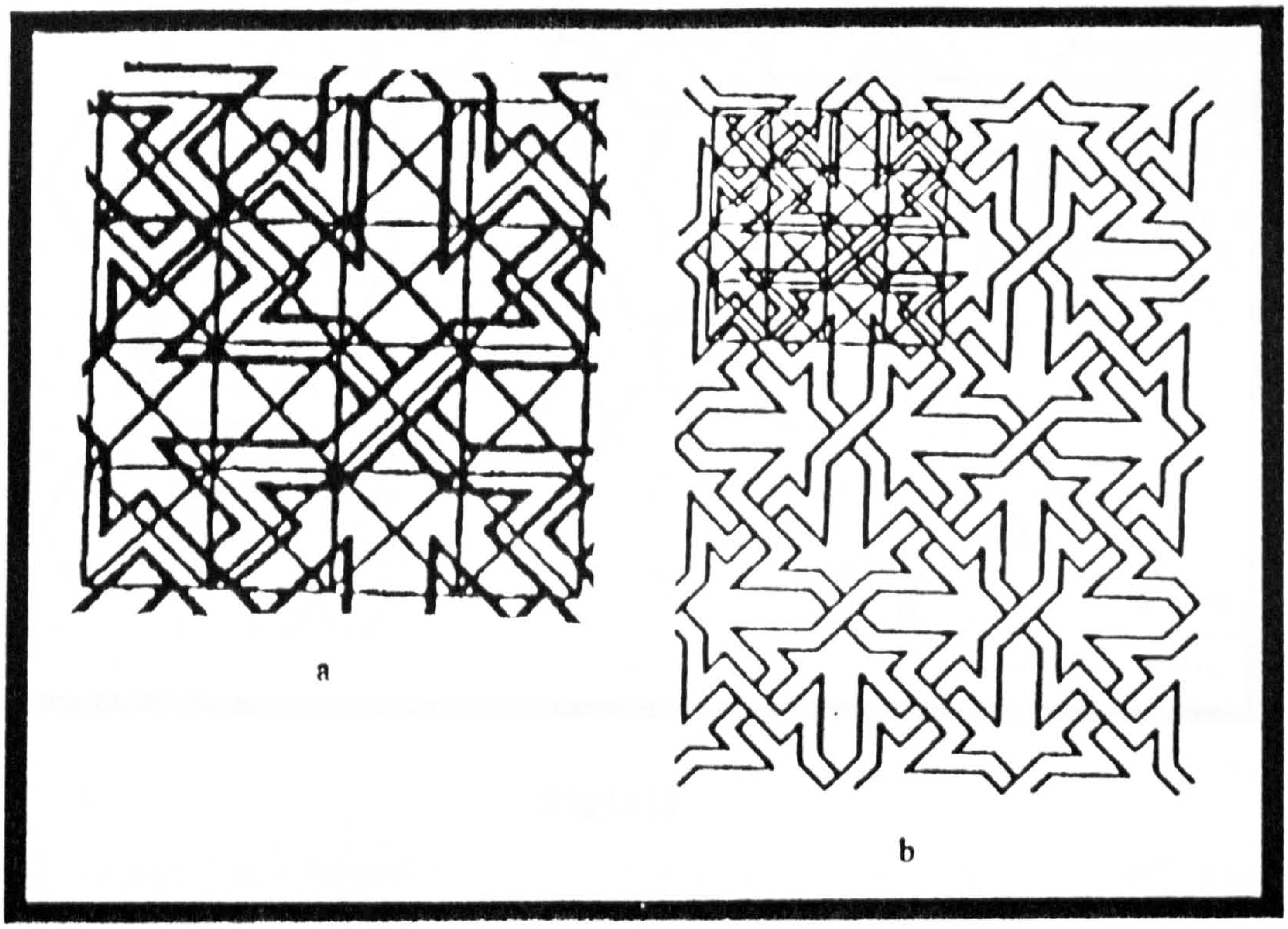
fig(18)

One class of patterns that has considerably attracted Islamic artists involves interlocking shape which are usually colored in two contrasting colors. Two example of such patterns which make use of the square grid are shown in fig(19).



fig(19)

An example of a very pleasing interlaced pattern based on the square grid is shown in fig(20b) and its method of construction, are shown in the fig(20a). The pattern is obtain by replacing the line in fig(20a) by thick interlaced lines.

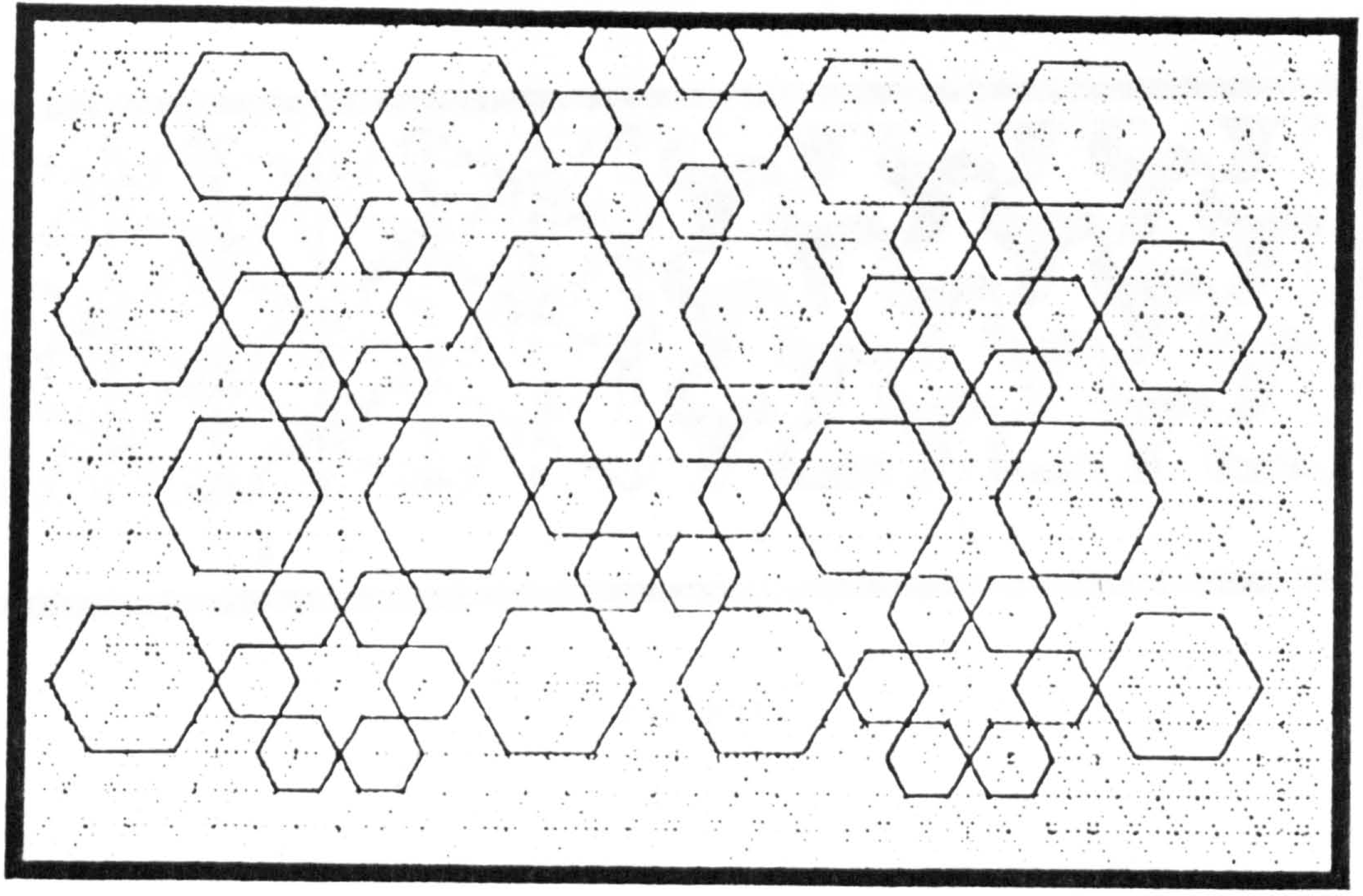


fig(20)

2.5.2 ISOMETRIC TRIANGULAR GRID

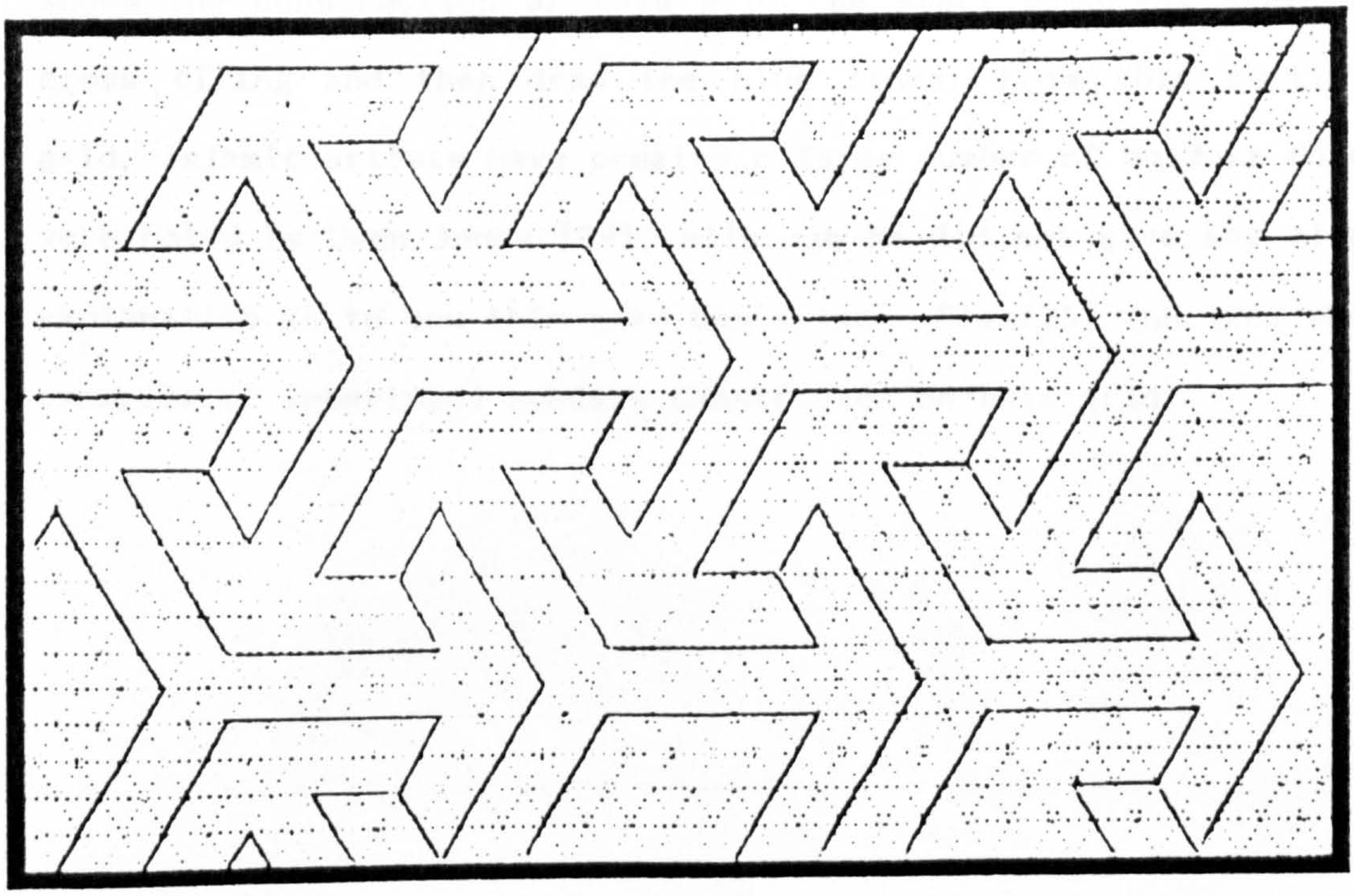
This is anther very popular grid and also has a high degree of symmetry. This give arise to a massive number of star patterns which occur very commonly in islamic art.

Fig(21) shows an example based on this grid. This pattern has been used to great effect in the stone work in the famous Jomah Masjed (Friday mosque) of Isfahan.

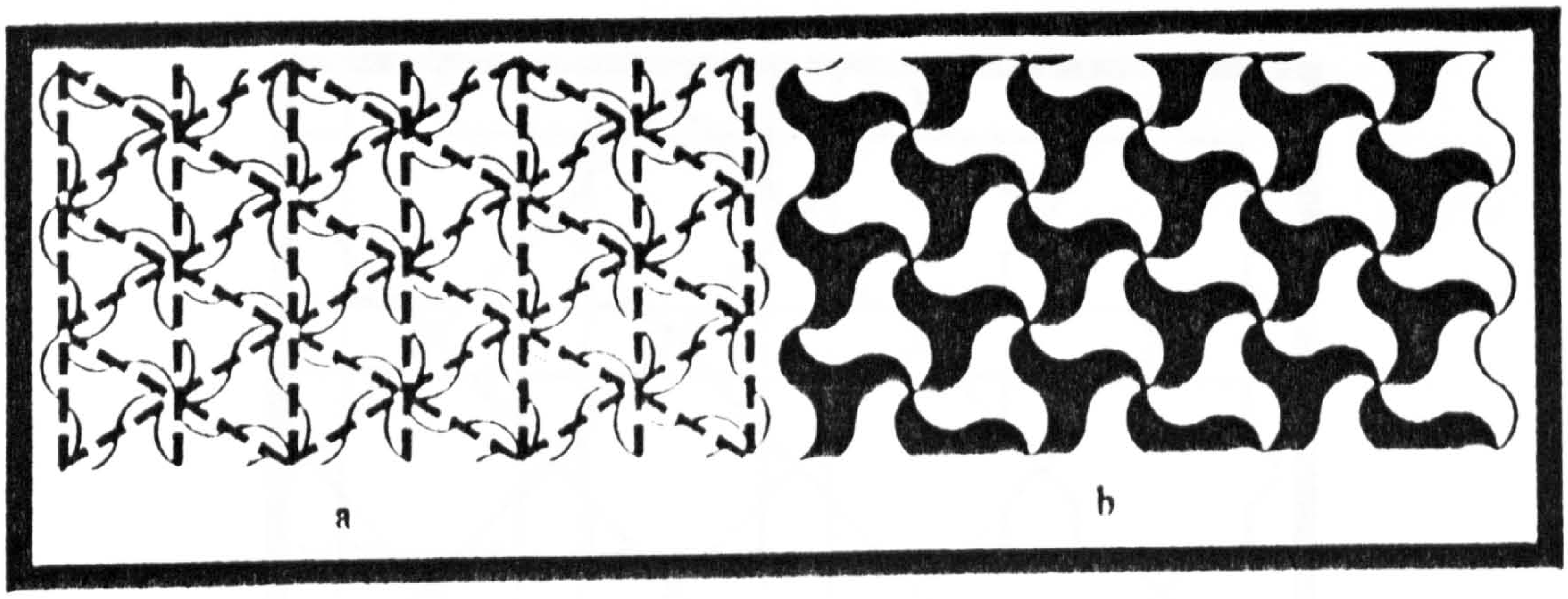


fig(21)

Figs(22) and (23) show examples of interlocking patterns based on the triangular grid. The pattern in fig(23) is very popular all over the Islamic world and is executed in the widest range of materials.



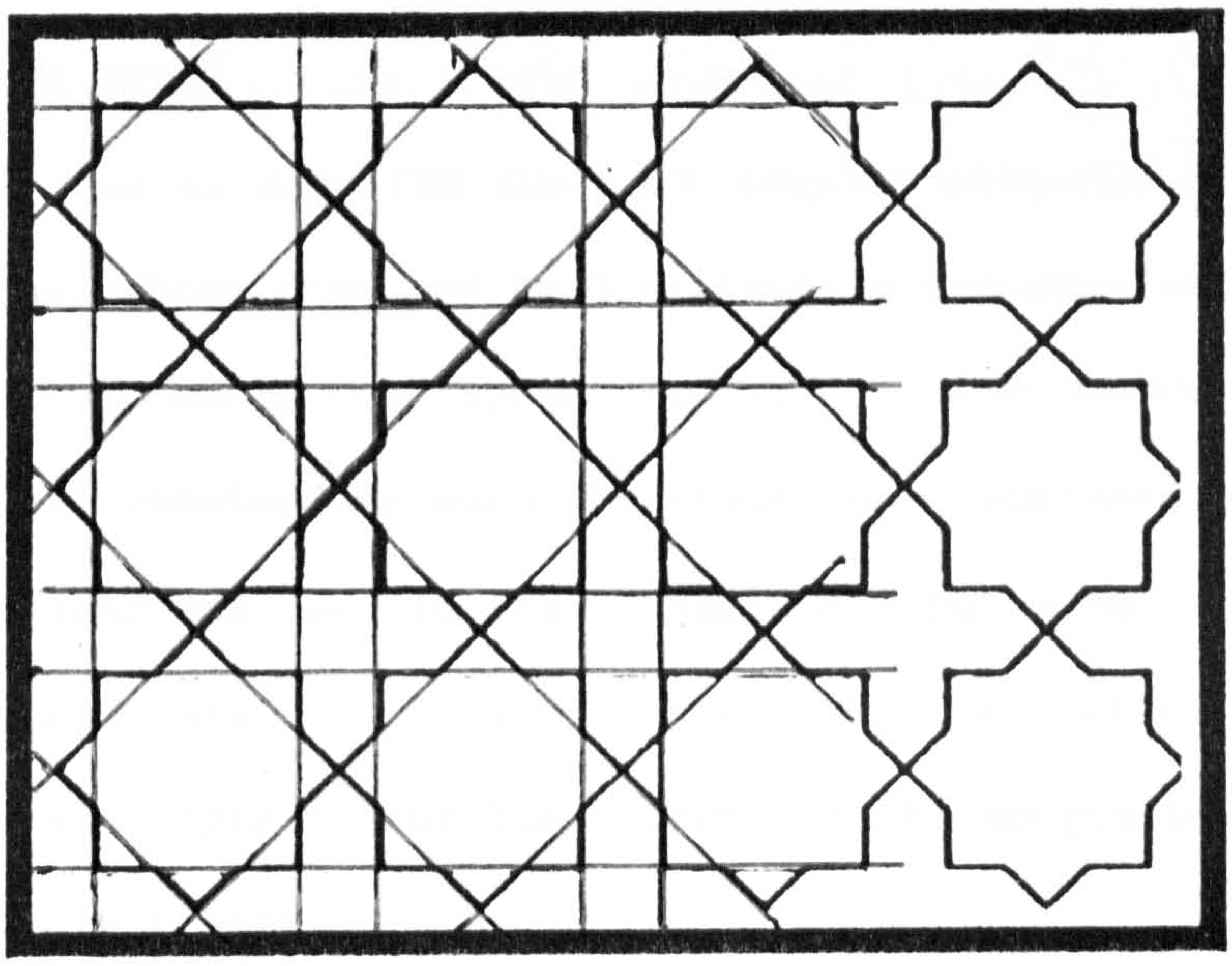
fig(22)



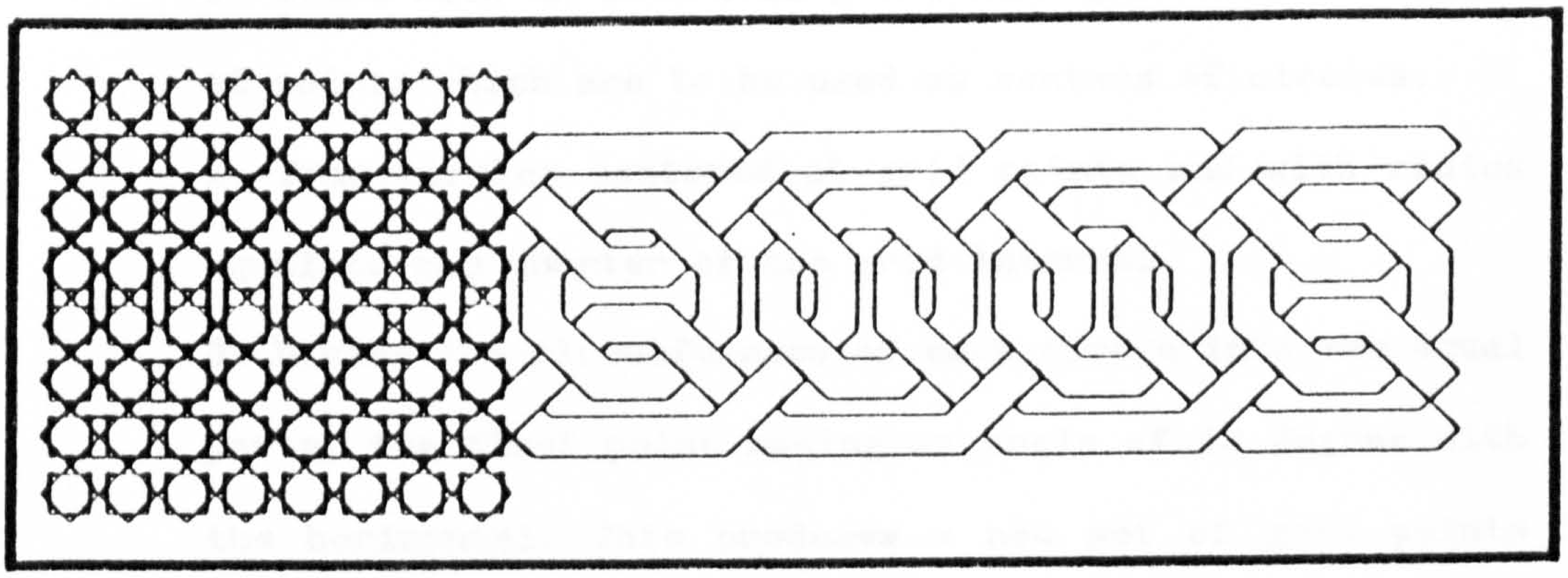
fig(23)

2.5.3 EXAMPLES OF PATTERNS PRODUCED FROM USING MULTIPLE GRIDS

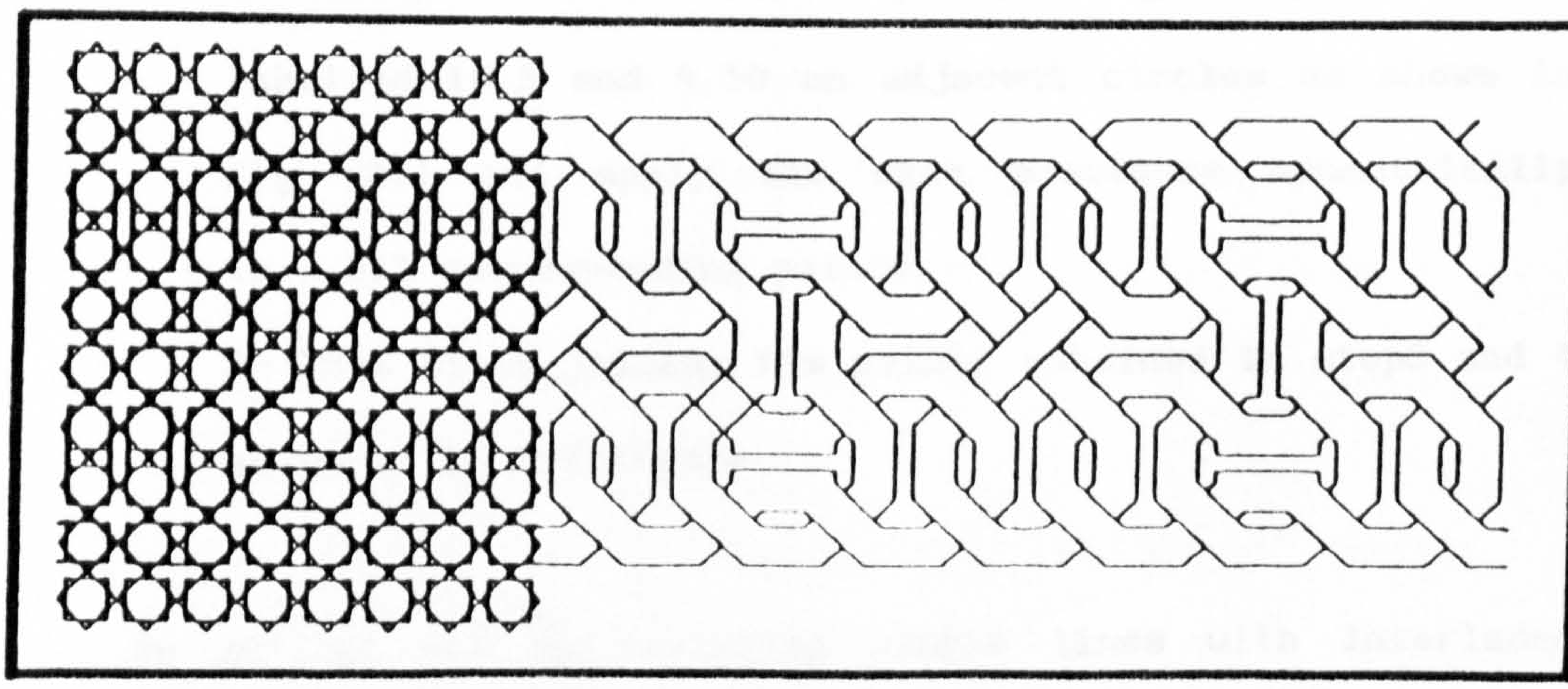
After experience with tiles and simple grids, the next stage of development would naturally lead to some experimentation with multiple grids. One very interesting example of the use of multiple grids is star and cross grids which is derived easily from the best known tiling used in the Islamic world. Fig(24) shows the construction of this grid. We start with the star and cross tiling and then draw the blue lines. From this multiple grid, Islamic artists have created a large number of borders which were noted by Owen Jones [38], although he did not give any clear explanation as to how this grid has arisen. Figs(25) and (26) show examples of interlaced borders constructed on this grid.



fig(24)



fig(25)



fig(26)

2.5.4 COMPLEX GRIDS

We now come to describe the most complex patterns of Islamic art. These have been produced by distributing polygons and circles on grids and dividing them symmetrically. In some cases the grid used is itself obtained by such distributions. Imaginative joining of the divisions has lead to truly remarkable patterns.

To demonstrate the typical approach, we will give an algorithm (see fig(27)) for the pattern which emerges in the final stage shown in fig(28).

1- Start with an isometric grid, this gives rise to a set of points which are to be used as centers of circles.

2- Draw circles centered at grid points and with radius equal to one quarter of the grid interval.

3- Divide the circumference of each circle into ten equal parts, the first point making an angle of 18 degree with the horizontal. This produces a new set of grid points which will be used in the construction.

4- Obtain a further set of points by joining the points labelled 10,1 and 9,10 on adjacent circles as shown in fig (27) and apply the same procedure symmetrically to all corresponding points.

5- Draw lines joining the points obtained in step3 and 4 as shown in the figure.

By filling and by replacing single lines with interlaced lines, many variations can now be produced.

Fig(29) to (33) show more examples of patterns produced by following similar procedures to the one we have just described.

Finally, We will describe a procedure to produce the pattern in fig(34). This procedure was devised by the author to demonstrate the method which obtains auxiliary grid points by making use of initial distribution of polygons. This is intended to give an example which does not start with circles, the shape which occurs most frequently.

1. First distribute dodecagons and equilateral triangles constructed on their sides as shown in the first stage of the figure, (the squares appear automatically).

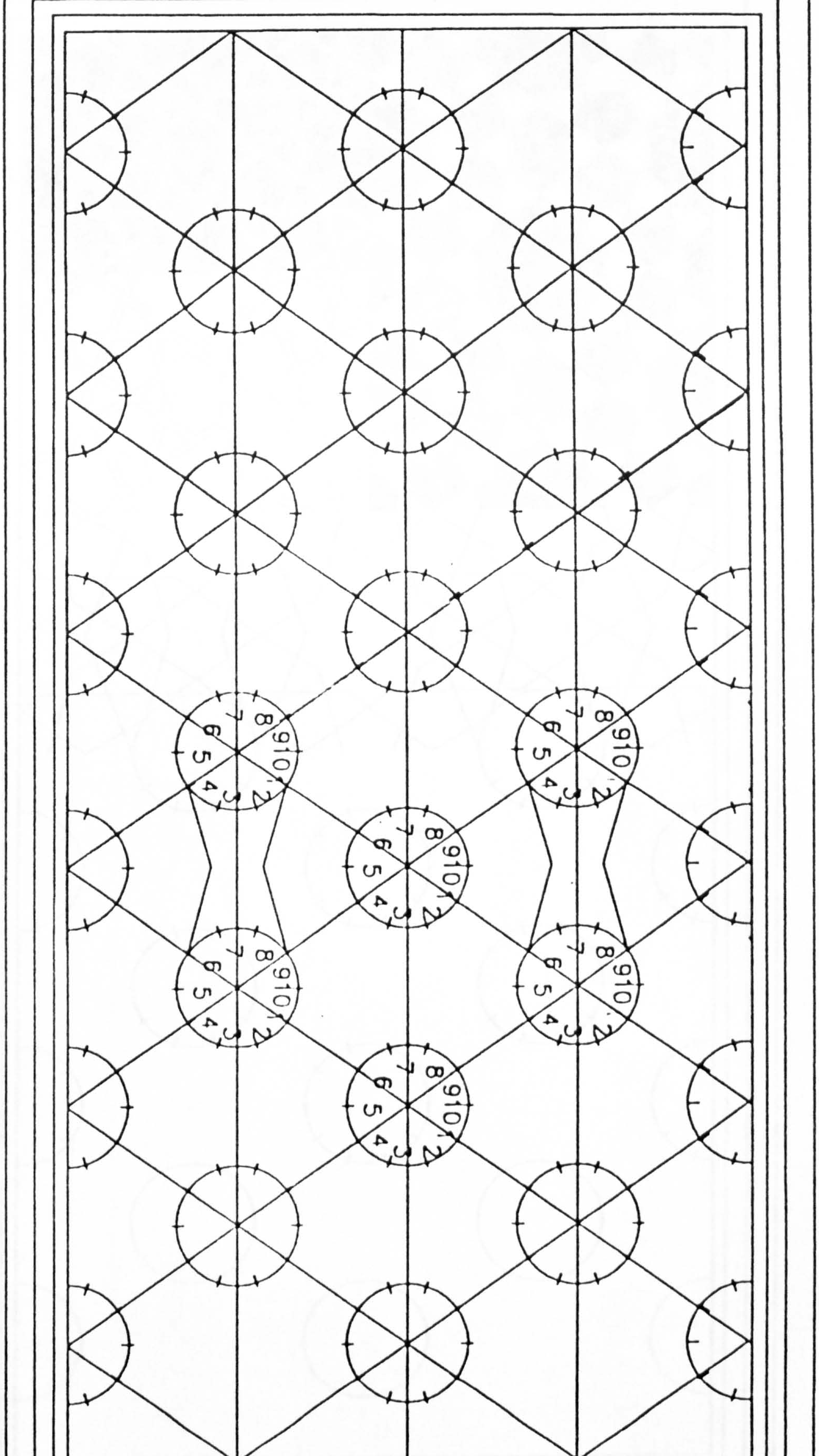
2. Select points in the middle of each sides of the dodecagons and the triangles as shown in the second stage of the figure. This gives a grid with one set of auxiliary points.

3. Draw L1 and L2 as shown to find the point A at their intersection. Similarly, find sets of points in region Q1, Q2 and Q3. Add these points to the auxiliary grid.

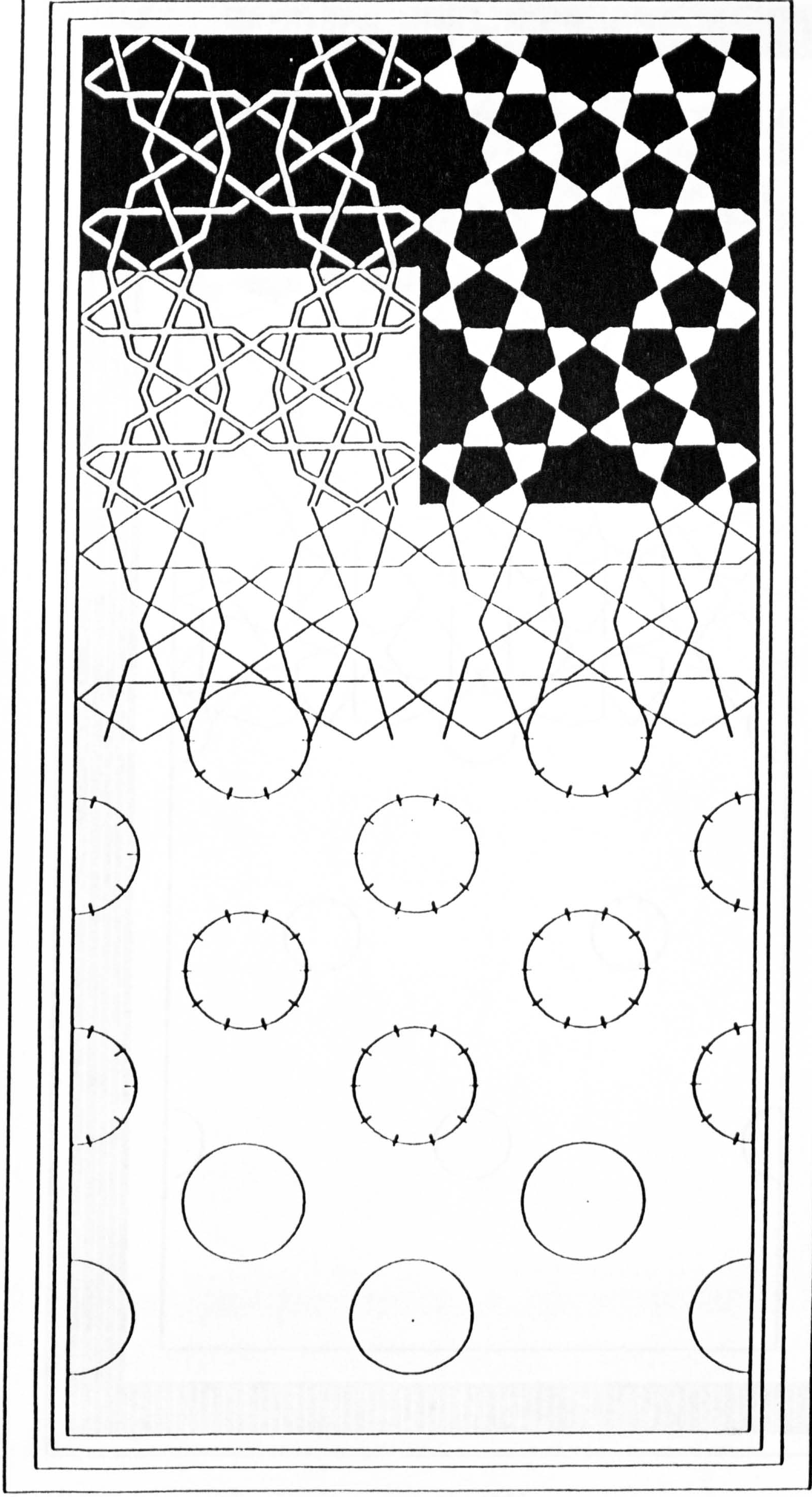
4. Draw lines join the grid points as shown in the fourth stage of the figure. This produces the simple line version of the pattern.

Again, fillings and interlacings lead to a variety of enhancement.

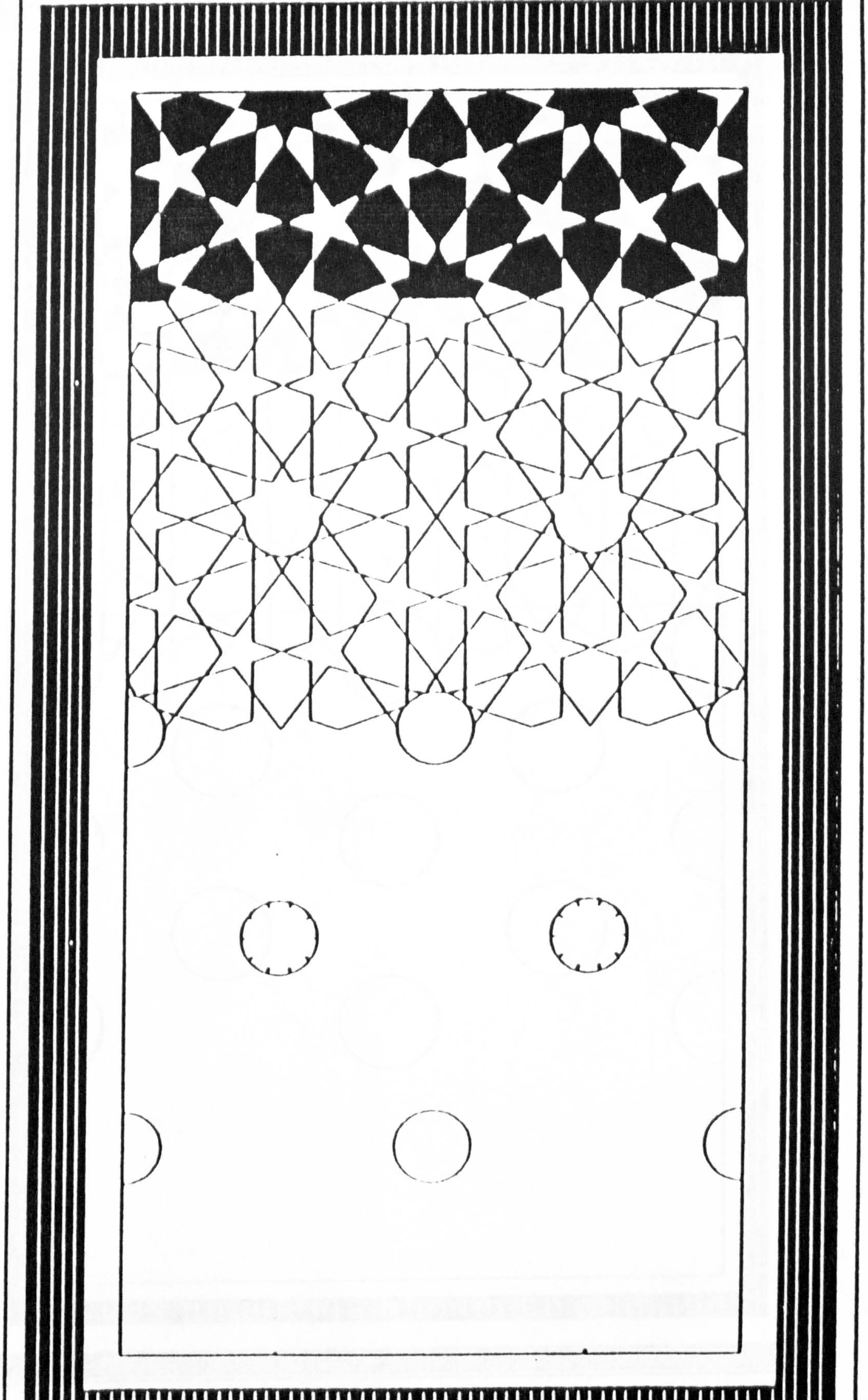
In this chapter we have given our explanation as to how starting with tiles and using only simple geometry the patterns of Islamic art have arisen. In chapter three we shall approach the method from group theoretic point of view and of modern computer graphics.



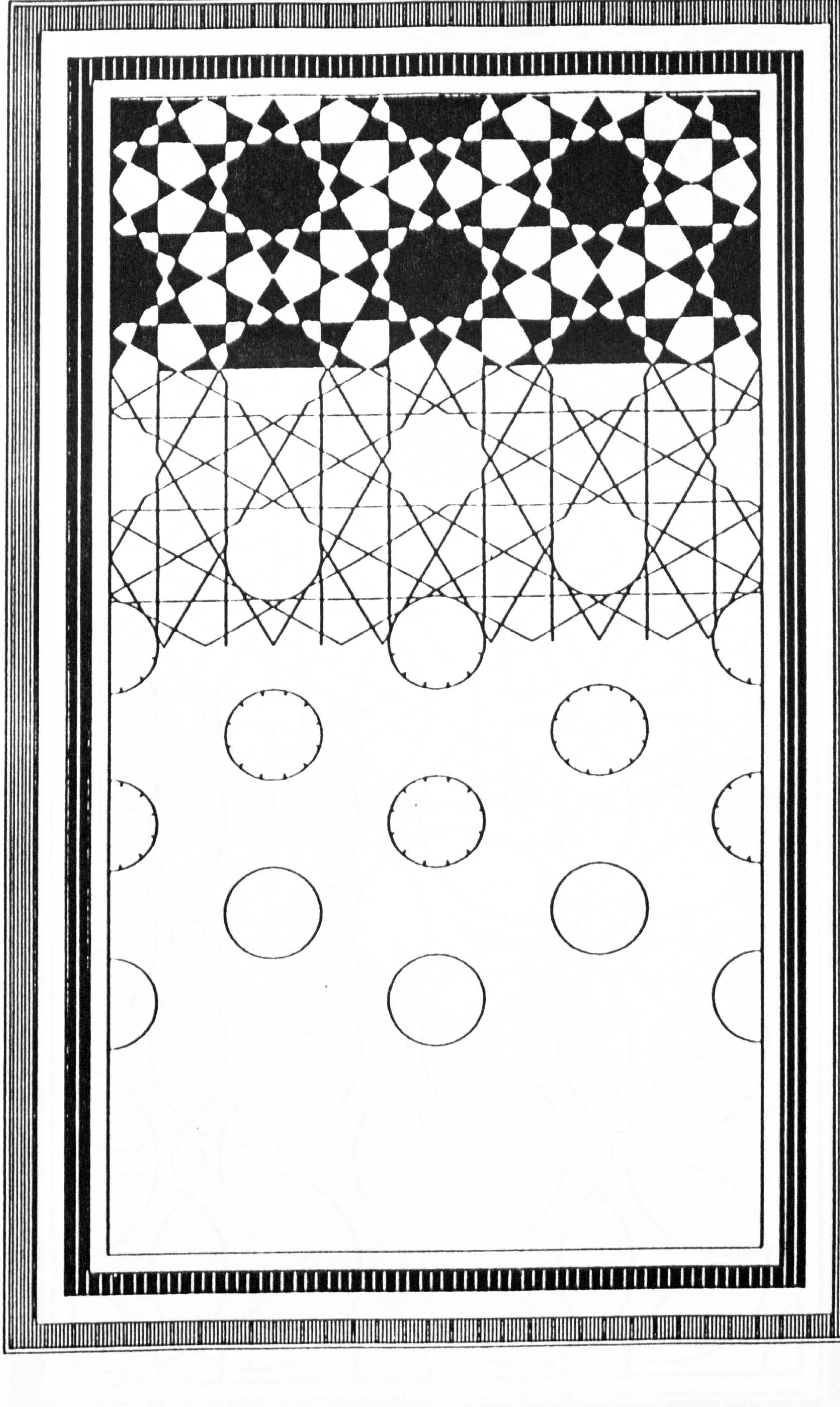
fig(27)



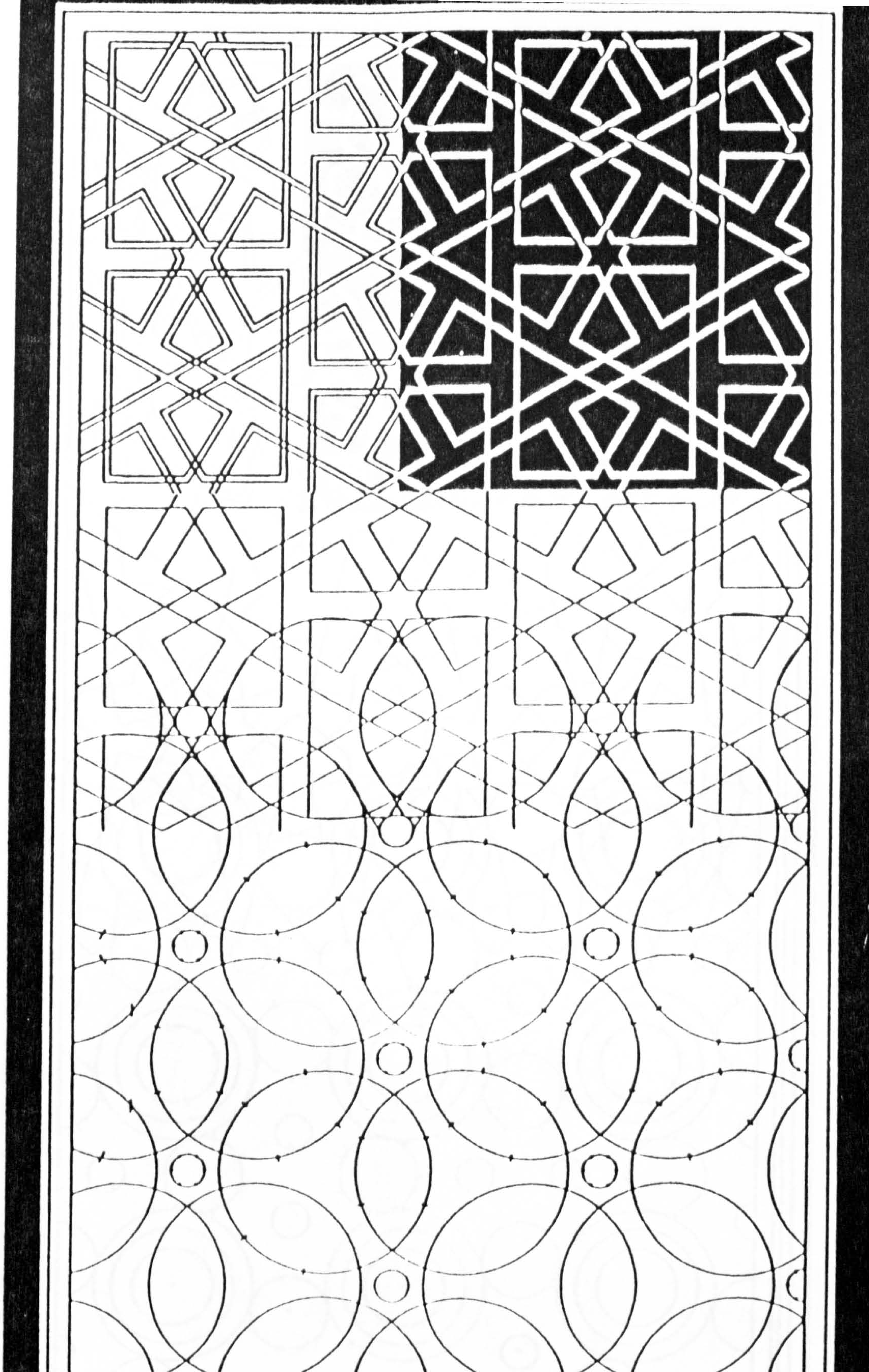
fig(28)



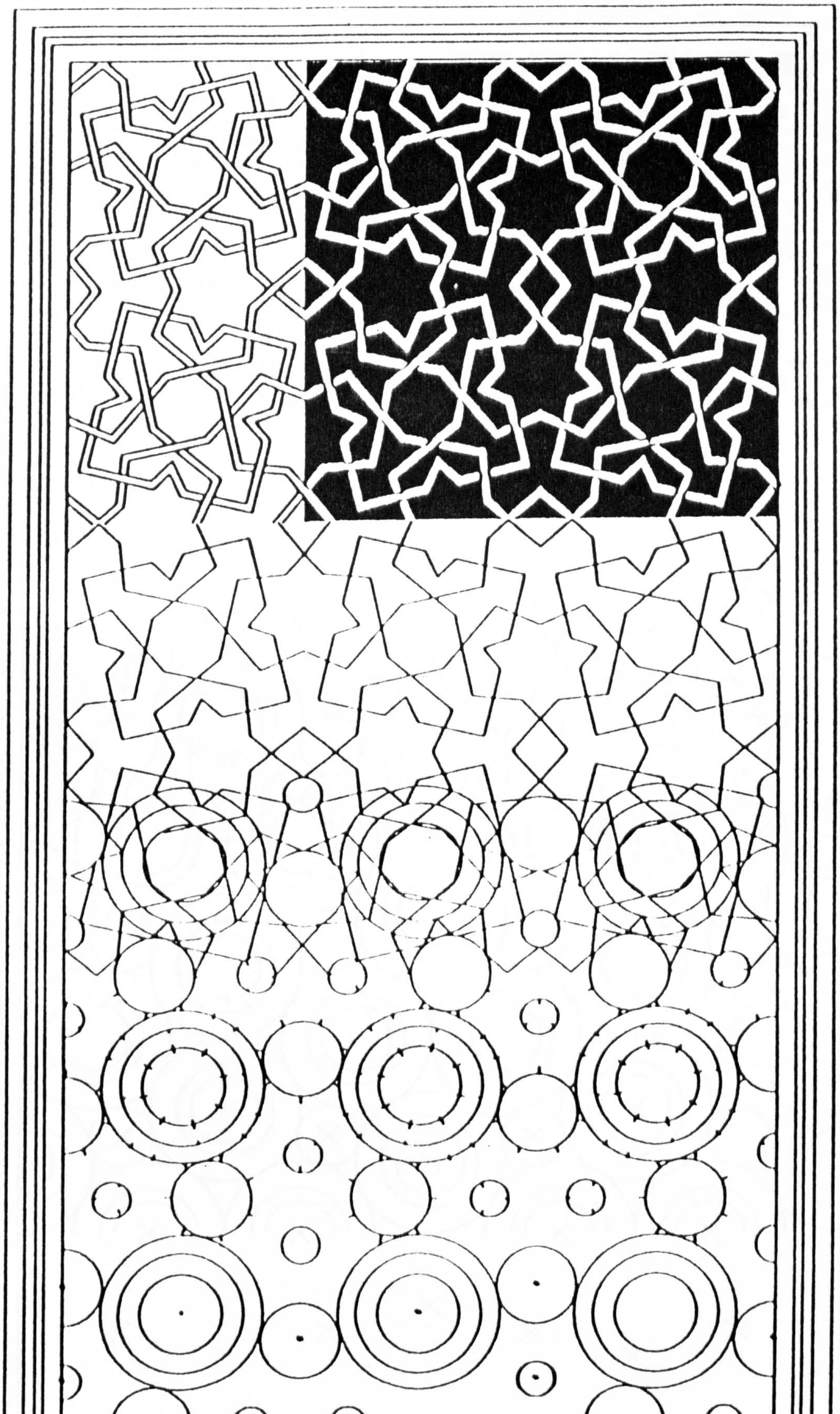
fig(29)



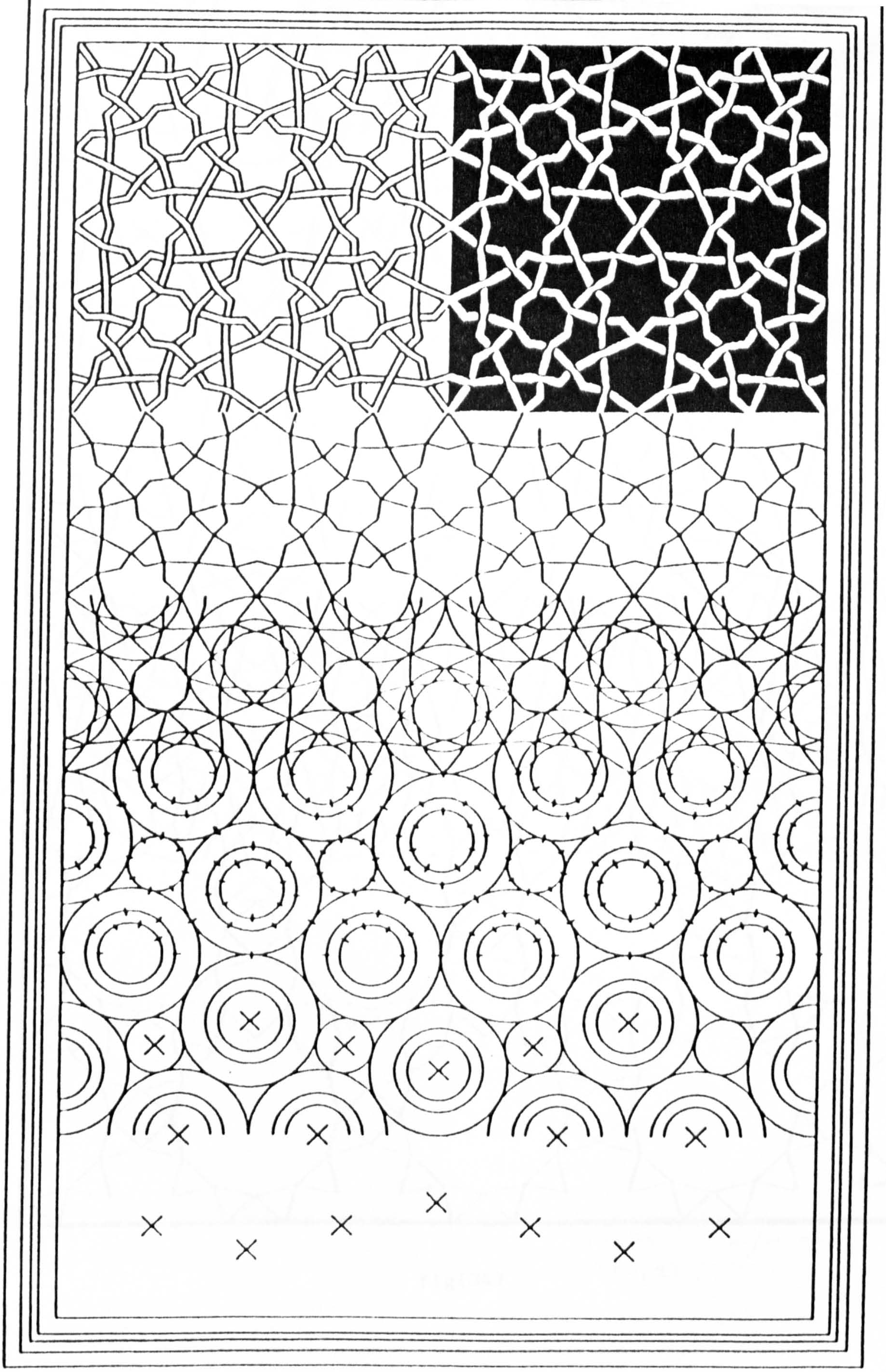
fig(30)



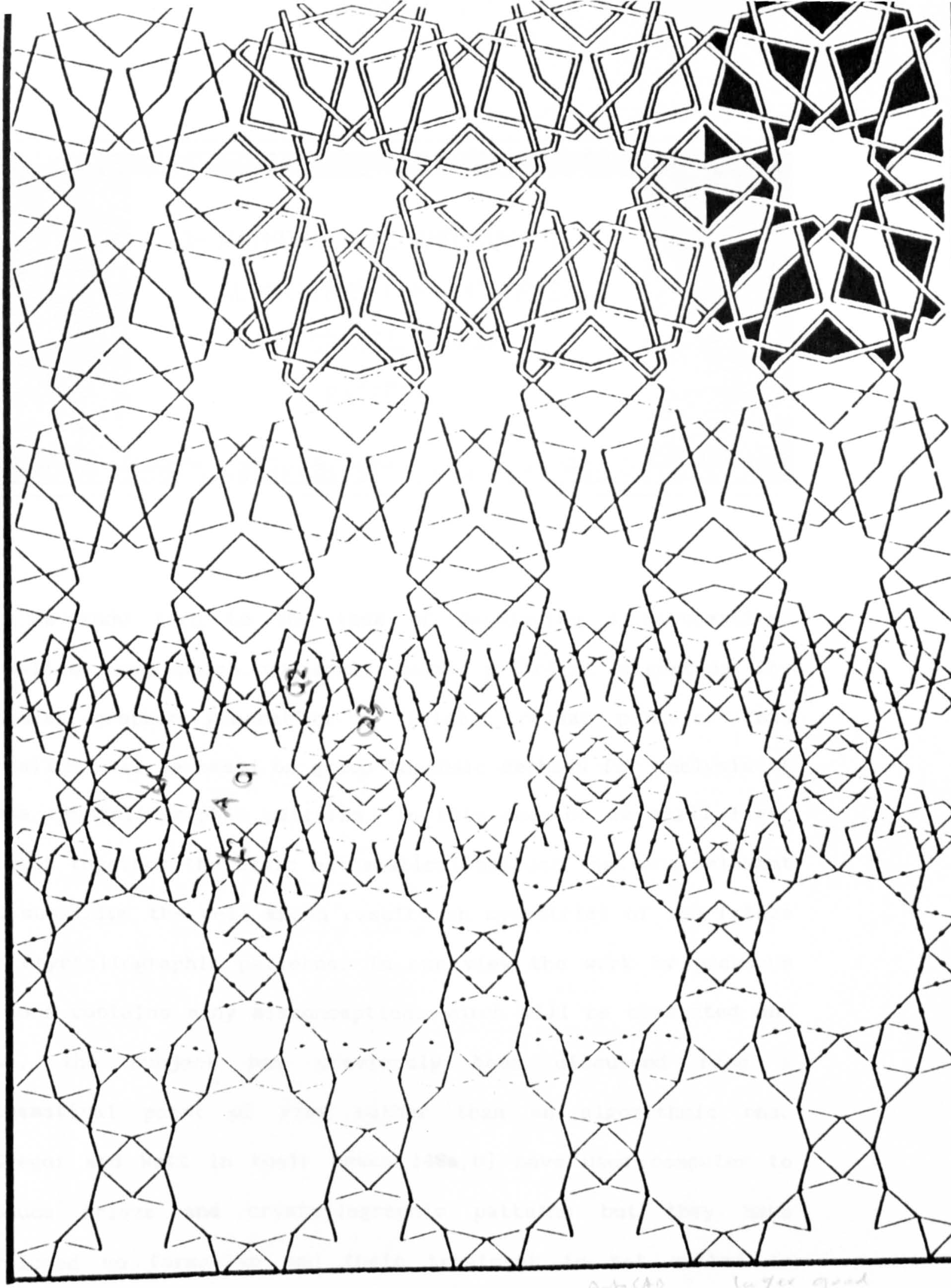
fig(31)



fig(32)



fig(33)



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chap21

fig(34)

3- MATHEMATICAL AND COMPUTER ALGORITHMS FOR THE FRIEZE AND CRYSTALLOGRAPHIC PATTERNS

We know turn to the task of developing a mathematical formalism from which we shall derive efficient algorithms for computer graphic generation of Islamic repeat pattern. This formalism will be based on group theoretic methods for analysis of plane crystallographic patterns. In this chapter we shall first collect together the basic mathematical notions that are relevant and summarise the well known results on symmetries of the frieze and crystallographic patterns. In our view the work by previous authors contains many misconceptions which will be commented on. Also, the subject has previously been discussed from a mathematical point of view rather than an algorithmic one. McGregor and Watt in their books [48a,b] have used a computer to produce frieze and crystallographic patterns but they have developed no formalism and their treatment is not suited to generalization to other types of symmetry such as color symmetry.

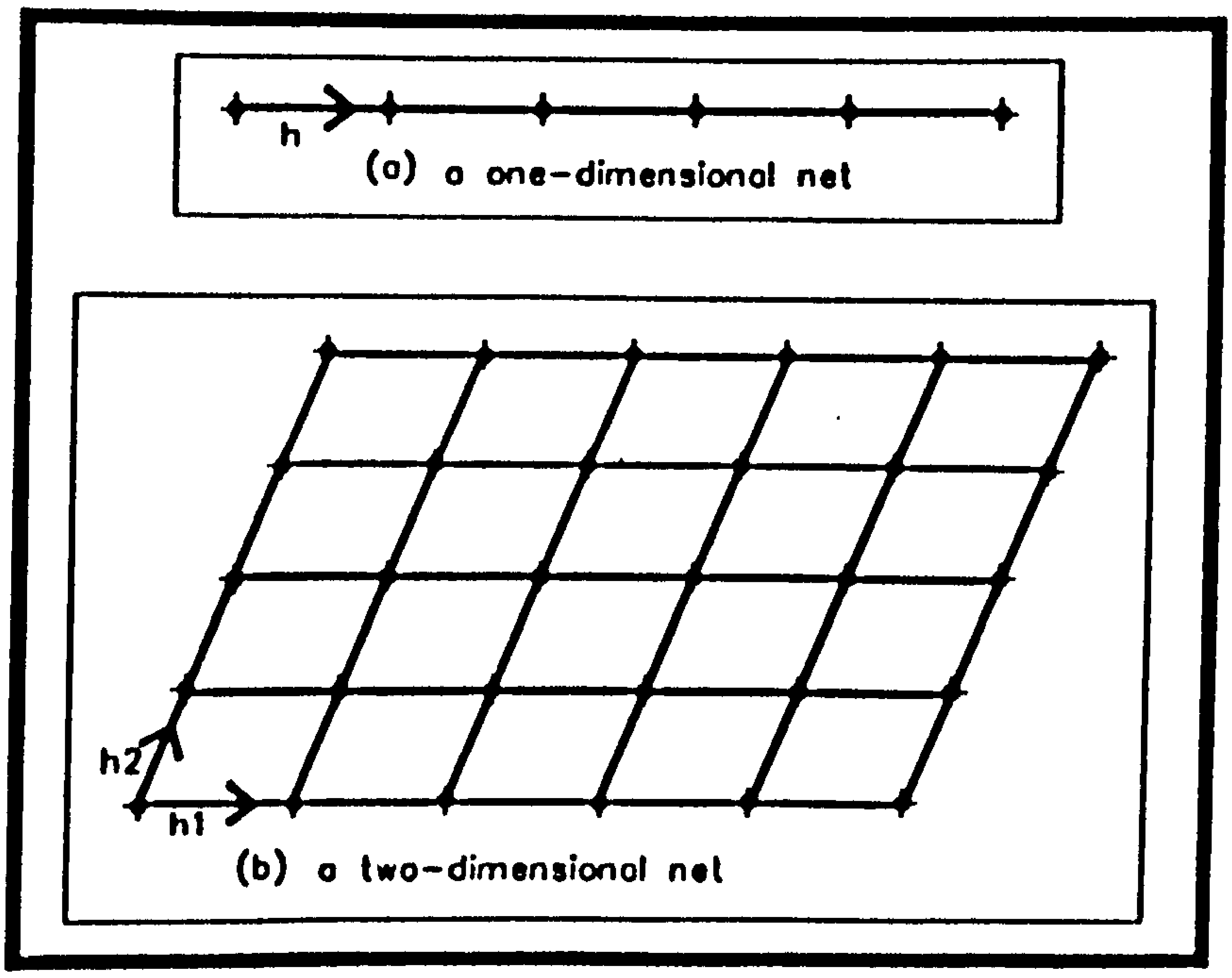
Following our review, we shall first develop a set of simple algorithms which are suited to interactive generations of these patterns and will produce illustrative examples. Finally, we shall develop a general purpose algorithms and again will give examples produced to illustrate. These algorithms are the ones that we have used to develop our computer program which will be discussed in next chapter.

3.1 BASIC MATHEMATICAL CONCEPTS IN SYMMETRY

3.1.1 NET

Given a vector $h \in \mathbb{R}$, a one-dimensional net is the set of points $N(h)=\{ \alpha h \mid \alpha \in \mathbb{Z} \}$. We say that h generates the net.

Given two non-parallel vectors $h_1, h_2 \in \mathbb{R}^2$, a two-dimensional net is the set of points $N(h_1,h_2)=\{ \alpha h_1+\beta h_2 \mid \alpha,\beta \in \mathbb{Z} \}$. In this case h_1 and h_2 generate the net. We will refer to any point of N as a node.



fig(1)

3.1.2 TRANSFORMATION

A **transformation** T on a set σ is an action which changes the **initial state** of σ to an **image state** $\bar{\sigma}$. We shall be interested in sets σ whose elements are geometrical entities, e.g. points, lines, polygons etc. and a state of σ will be defined by specifying the positions and orientations. In general other attributes could also be included, e.g. colors, styles, fill patterns etc. of the elements. The other types of sets that will be of interest are those whose elements are transformations.

We shall denote the action of the transformation T on σ by writing $T\sigma = \bar{\sigma}$. If U is another transformation then by $UT\sigma$ we shall mean $U(T\sigma)$. The composite transformation UT will be referred to as the product of T and U.

3.1.3 ISOMETRY

An **isometry** A is a transformation which preserves distances, i.e. If p_1, p_2 are any two points, then the distance between p_1, p_2 is equal to the distance between their images Ap_1 and Ap_2. This implies that the corresponding angles between any two lines are also preserved, although the image of the angle may be in the opposite sense, in which case the isometry is called **indirect** otherwise it is call **direct**.

The **identity** isometry denoted by I is an isometry which transforms every point onto itself.

An **invariant** point of an isometry is one which remains unchanged after the isometry is performed.

It may be shown by Martin[47] and Coxeter[12a] that any isometry is one of four kinds:

3.1.3.1 TRANSLATION

A **translation** T_r, is an isometry in which each point is moved by the vector **r**, see fig(2a). This isometry is direct and has no invariant points.

3.1.3.2 ROTATION

A **rotation** $R_{\varphi,c}$, is an isometry which rotates a points p_1 by φ degrees in an anti-clockwise sense around the point **c**, which is called the **center of rotation.** The isometry $R_{\varphi,c}$ is direct as shown in fig(2b). The rotation $R_{\varphi,c}$ always has the point **c** as the only invariant point.

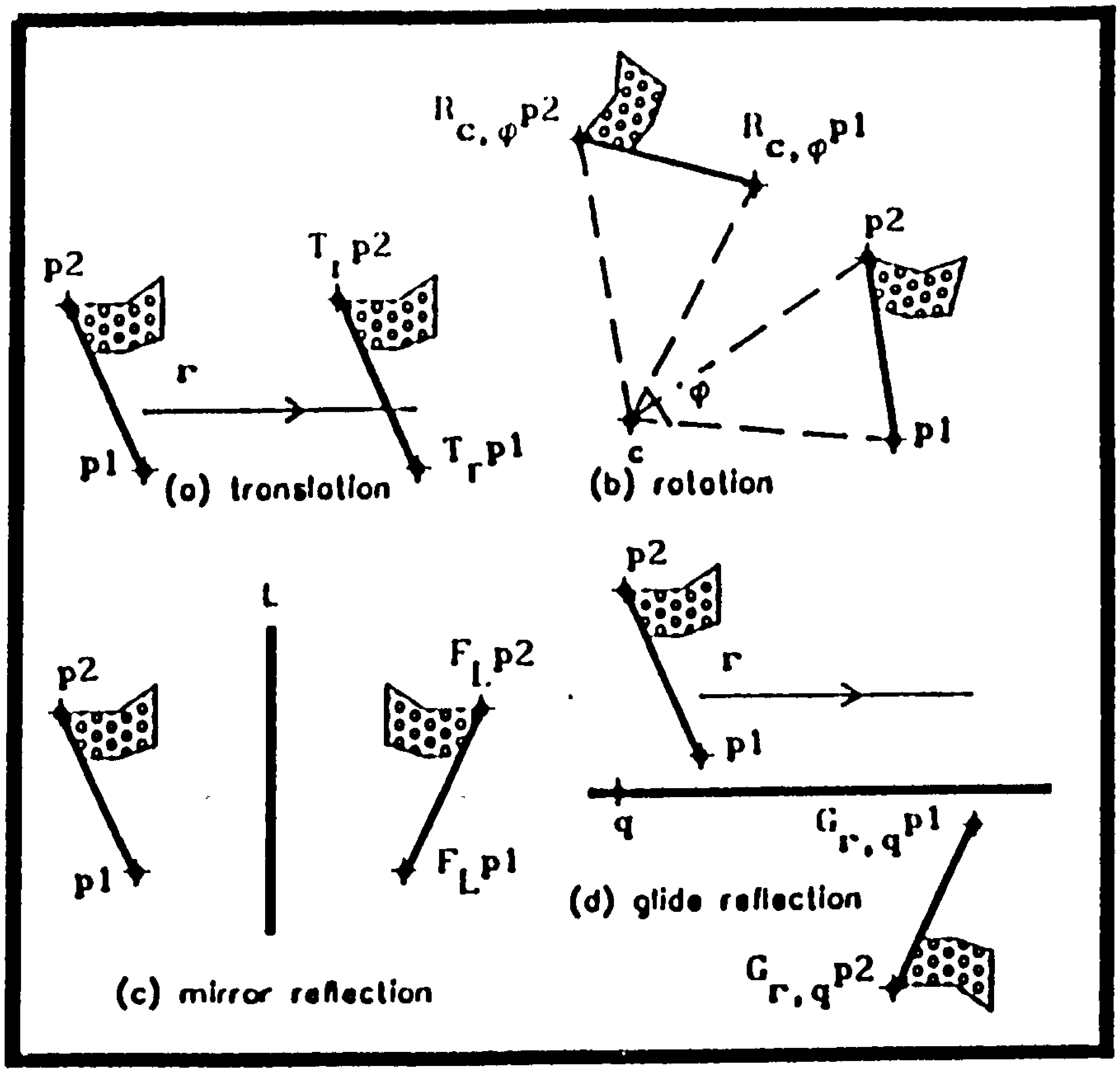
When the angle of the rotation is $360^\circ/n$ the rotation is called an **N-fold rotation.** When the angle φ is 180 degrees, the rotation is called a **half-turn.**

3.1.3.3 MIRROR REFLECTION

A **mirror reflection** F_L, of a point **p** in the line L sends **p** to its mirror image $F_L p$. If **p** lies on L then it is left fixed, see fig(2c). We shall also use $F_{p,q}$ to represent a reflection in the line passing through the points **p** and **q**.

3.1.3.4 GLIDE REFLECTION

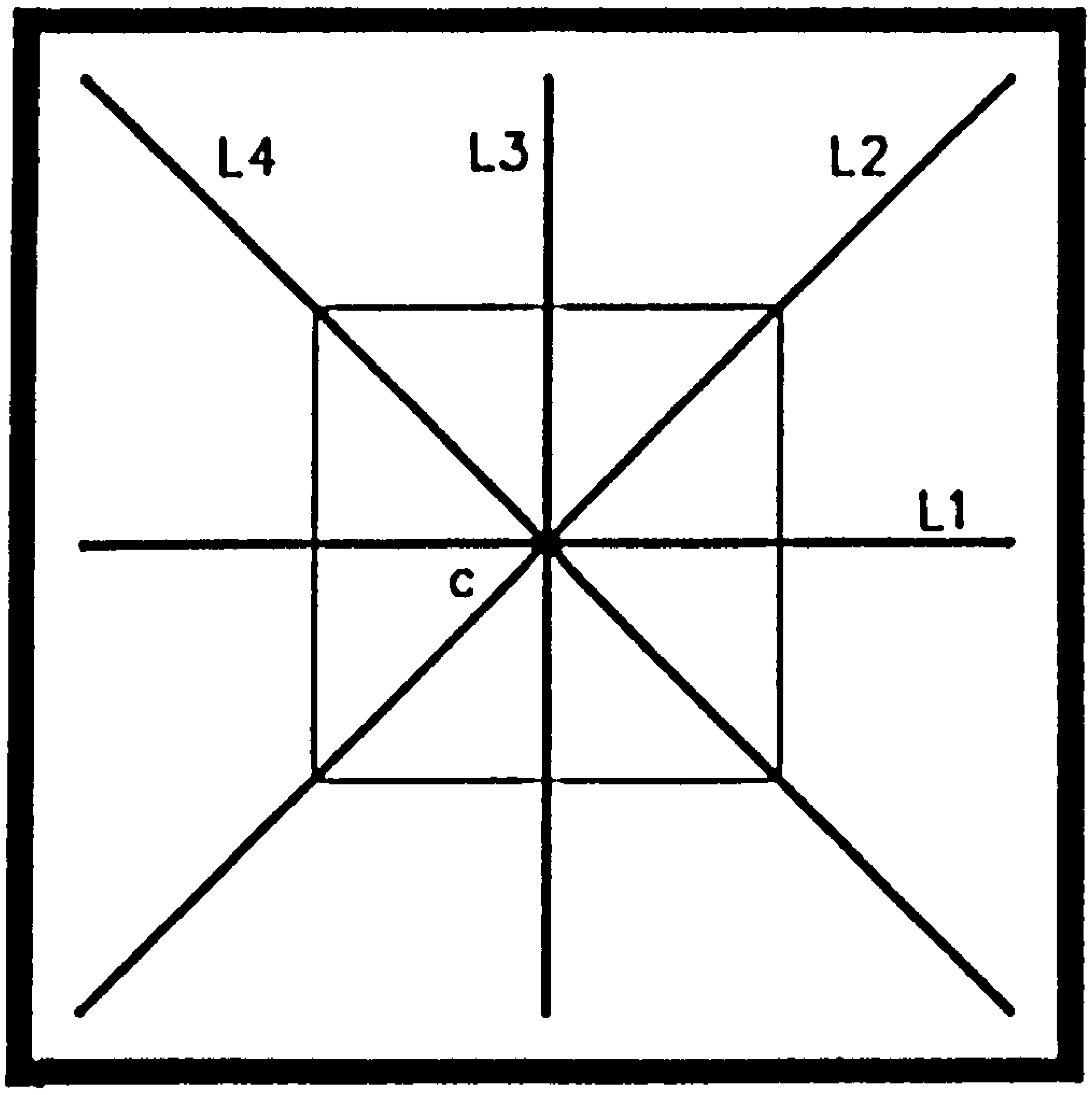
A **glide reflection** $G_{r,q}$, is the combination of a translation by the vector **r** and a mirror reflection in a line parallel to **r** and passing through the point **q**, see fig(2d). We shall also use, $G_{p,q}$ to represent a glide reflection which involve a translation by distance **pq** followed by reflection in the line joining the points **p** and **q**. The isometry has no invariant points.



fig(2) four types of isometry

3.1.4 <u>SYMMETRY</u>

A **symmetry** is an isometry transformation which produces an image state which is indistinguishable from the initial state. If A is a symmetry of σ then $A\sigma=\sigma$. For example, any rotation about the center of a circular disk is a symmetry of the disk, and so also is a reflection in any line through the center of the disk. In the case of a square, the reflections in the lines L_1, L_2, L_3 and L_4 are symmetries, see fig(3), as are rotations through angles $\pi/2$, π and $3\pi/2$ in a counterclockwise direction about its center c, which is a center of 4-fold rotational symmetry.



fig(3)

Reflections in the four lines L_1, L_2, L_3 and L_4 are symmetries of the square. The other symmetries of the square are the identity isometry, and counterclockwise rotations through angles $\pi/2$, π and $3\pi/2$ about the center c.

3.1.4.1 SYMMETRY GROUP

The symmetry group Ξ_σ of a set σ is the set that consists of all the symmetries of σ. The elements of Ξ_σ form a group, i.e. they satisfy the following:

(i) Given any two elements A, B in Ξ_σ, their product AB is in Ξ_σ.

(ii) Give any three elements A, B, C in Ξ_σ, A(BC)=(AB)C.

(iii) There is a special element I in Ξ_σ, called the identity element, such that IA=A for every element A in Ξ_σ.

(iv) Given any element A in Ξ_σ there exists an element A^{-1} in

Ξ_σ, called the **inverse** of A, such that $AA^{-1} = A^{-1}A = I$.

We say that two elements A, B **commute** if AB = BA. Ξ_σ is a **commutative** or **abelian** group if all the elements of Ξ_σ commute.

The **order** of the symmetry group Ξ_σ, denoted by $|\Xi_\sigma|$ is the number of elements in Ξ_σ.

Ξ_σ has symmetry, or is symmetric , if $|\Xi_\sigma| \geq 2$. It is asymmetric if $|\Xi_\sigma|=1$, i.e if the symmetry group contains only the identity element I. $\Xi_{\sigma 1}$ has a greater degree of symmetry than $\Xi_{\sigma 2}$ if $|\Xi_{\sigma 1}| \geq |\Xi_{\sigma 2}|$.

Ξ_σ is said to be **finite** order if it has a finite number of element otherwise Ξ_σ has **infinite** order.

3.2 SYMMETRIES OF FRIEZE AND CRYSTALLOGRAPHIC PATTERNS

3.2.1 FRIEZE PATTERNS

Consider a set σ in $\mathbb{R}^2$ with an arbitrary reference point $\underline{r}_0$. IF σ is copied by repeated translations onto a one-dimensional net to make $\underline{r}_0$ coincide with the nodes then we obtain a **frieze pattern,** also called a **band** or a **strip pattern.**

3.2.2 FRIEZE GROUPS

A **frieze group** is the symmetry group of a frieze pattern.

Theorem: There are seven different types of frieze groups.

(See for example, Martin[47])

3.2.3 CRYSTALLOGRAPHIC PATTERNS

Consider a set σ in $\mathbb{R}^2$ with an arbitrary reference point $\underline{r}_0$. IF σ is copied by repeated translations onto a two-dimensional net to make $\underline{r}_0$ coincide with the nodes then we obtain a **crystallographic**, also called a **wallpaper pattern.**

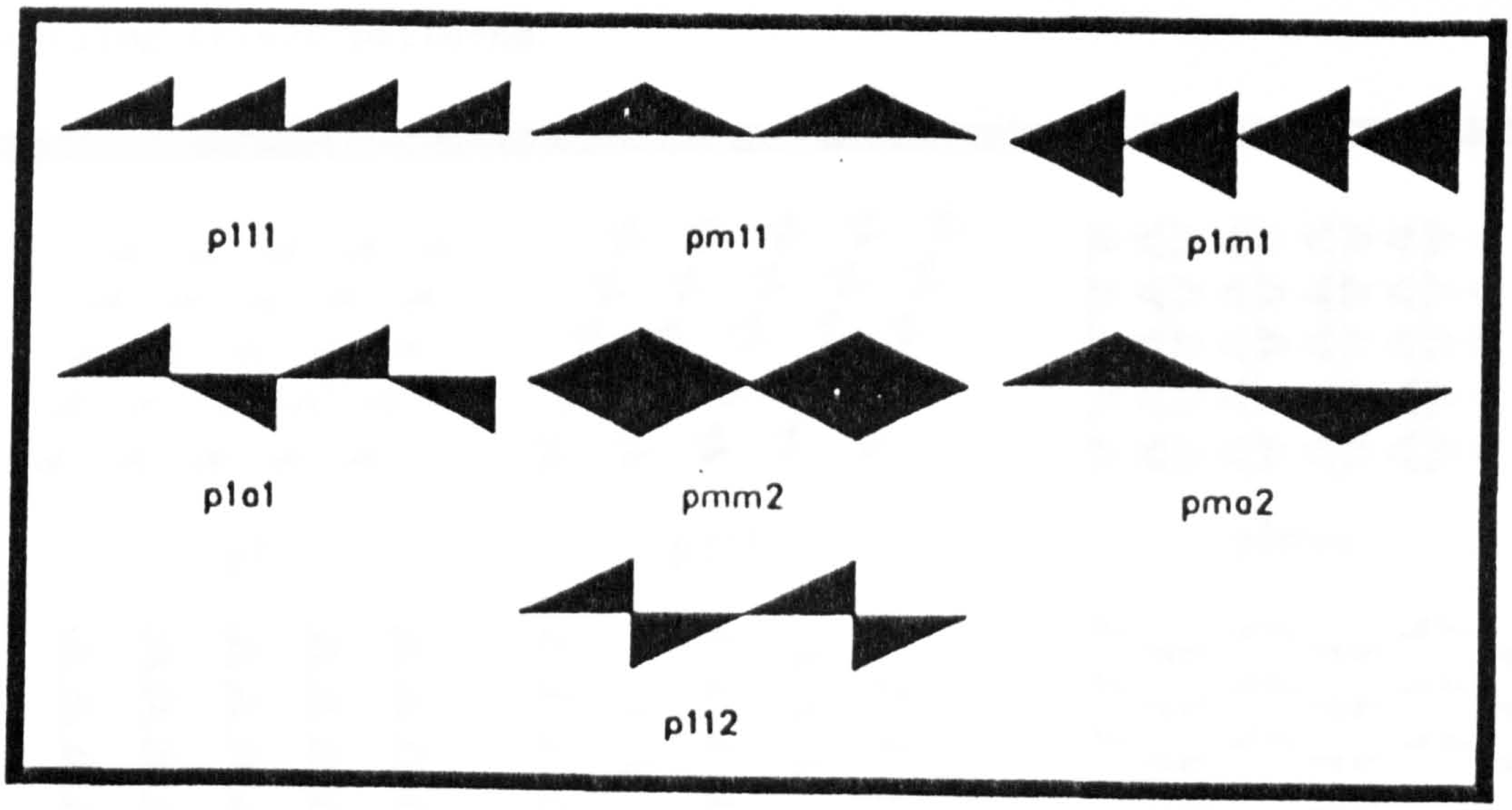
3.2.4 CRYSTALLOGRAPHIC GROUPS

A **crystallographic group** is the symmetry group of a crystallographic pattern.

Theorem: There are seventeen different types of crystallographic groups. (See for example, Martin [47]).

3.2.1 INTERNATIONAL CRYSTALLOGRAPHIC NOTATIONS

Several notations have been used to classify frieze and crystallogrphic patterns, see for example Doris Schattschneider [65] and Crowe & Washburn [15]. In this work, we shall use the notation adopted by Henry & Consdale [31]. The notation is made up of four symbols which will be explained below. While reading the next two sections the reader will find it helpful to refer to fig(4) and fig(5).

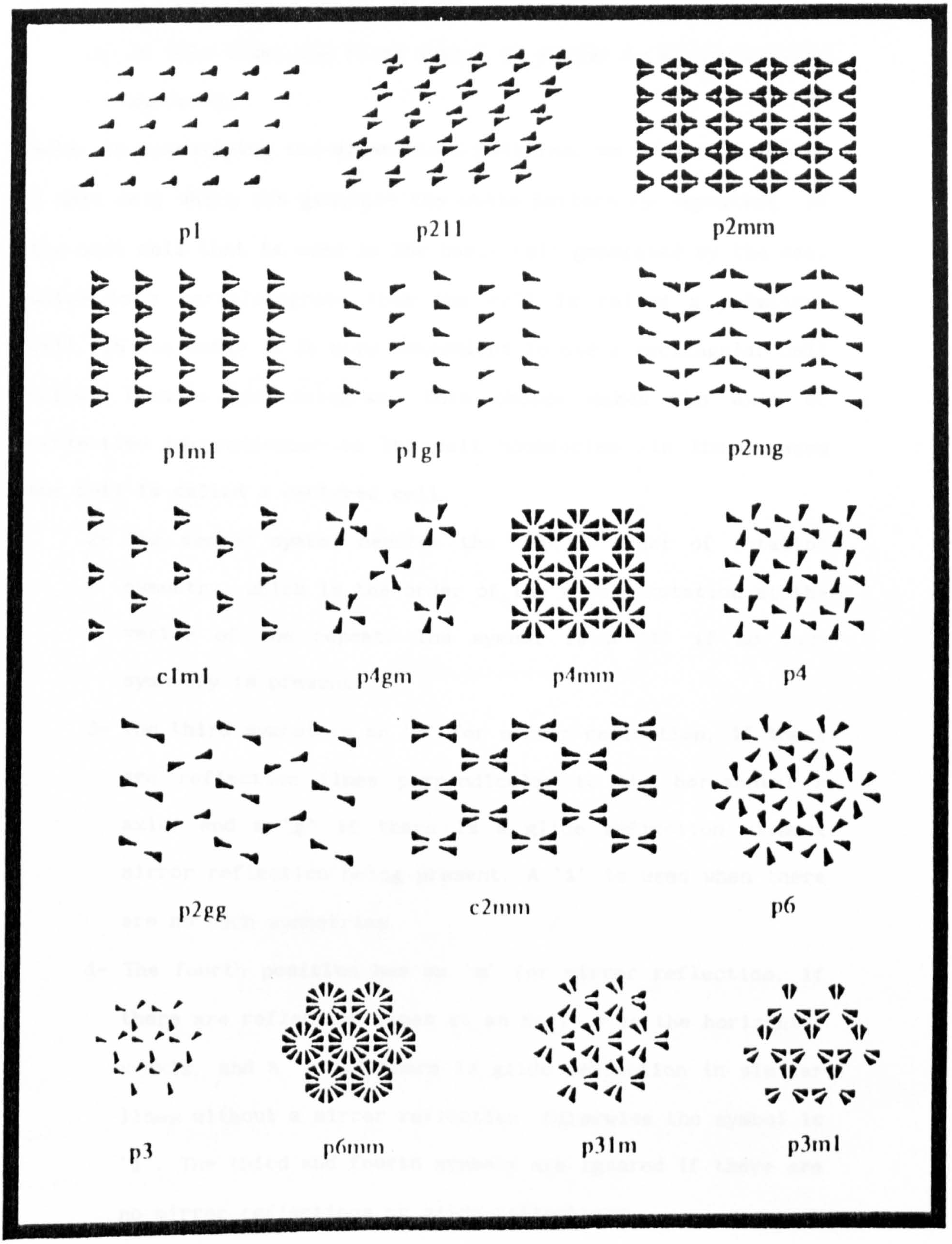


fig(4) The seven distinct types of frieze patterns.

3.2.1.1 NOTATION FOR FRIEZE PATTERNS

1. **The first symbol is always denoted by 'p', for primitive. (The meaning of this term will be explained later when we** come to the notation for crystallogrphic patterns).
2. The second symbol is an '**m**', for mirror reflection, if the pattern has vertical reflection lines. A '**1**' in this position indicates that there are no reflection lines.
3. **The third symbol is an 'm', if the central axis along the length of the pattern is a mirror reflection line, and an 'a' if** a glide reflection takes place without mirror reflection being present. Again, a '**1**' indicates that the pattern has no such symmetries.
4- **The fourth symbol is a '2', if the pattern had two-fold rotations as symmetries, otherwise the symbol is a '1'.**

Crowe [14a] and Zuslow [76] give useful flowcharts for classifying frieze patterns.



fig(5) The seventeen distinct types of crystallographic pattern

3.2.1.2 NOTATION FOR CRYSTALLOGRAPHIC PATTERNS

1- In this case, the first symbol is either a '**p**' or '**c**' (for centered).

Note: In classifying two-dimensional patterns, we need to identify a unit cell which can generate the whole pattern by repeating. If the unit cell that is used is the basic cell generated by the net, which is a parallelogram, then the cell is called a **primitive cell.** In two cases it is more convenient to use a rectangular cell rather than a parallelogram. This choice makes the axis of reflection perpendicular to the cell boundaries. In these cases the cell is called a **centered cell.**

2- The second symbol denotes the highest order of rotation symmetry, which is the order of the n-fold rotation at the vertex of the repeat. The symbol is a '**1**' if no such symmetry is present.

3- The third symbol is an '**m**' for mirror reflection, if there are reflection lines perpendicular to the horizontal x axis, and a '**g**' if there is a glide reflection without mirror reflection being present. A '**1**' is used when there are no such symmetries.

4- The fourth position has an '**m**' for mirror reflection, if there are reflection lines at an angle φ to the horizontal x-axis, and a '**g**' if there is glide reflection in similar lines without a mirror reflection. Otherwise the symbol is '**1**'. The third and fourth symbols are ignored if there are no mirror reflections or glide reflections.

3.2.2 SYMMETRY GROUPS OF NETS

The symmetry groups of frieze and crystallographic patterns are constrained by the symmetries of the nets on which the patterns are constructed. We shall describe the symmetry groups of the various nets that are of interest. We write down below the notation that will be used in diagrams to depicte various types of symmetries.

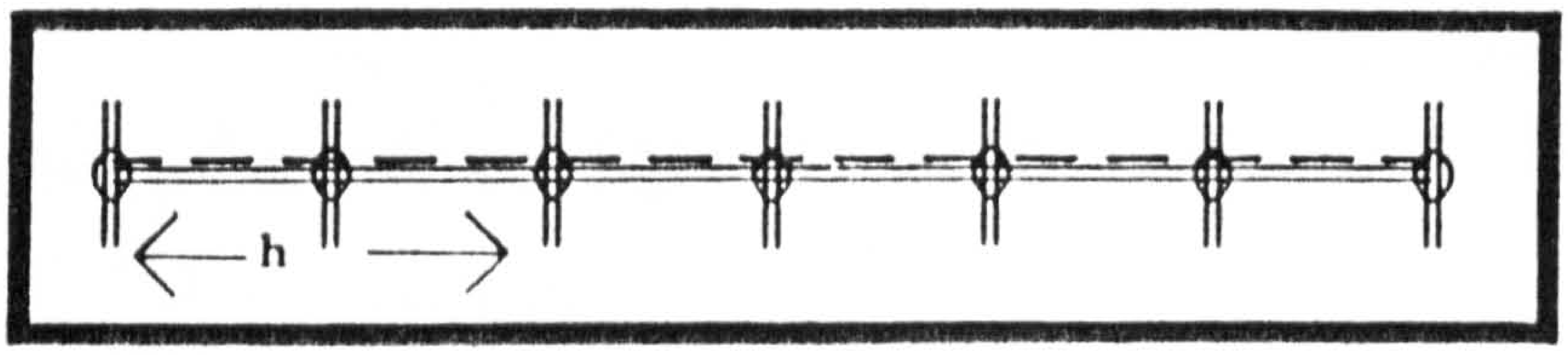
symbol	meaning
═══	Line of mirror reflection
— —	Line of glide reflection
o	Center of 2-fold rotation
△	Center of 3-fold rotation
□	Center of 4-fold rotation
O	Center of 6-fold rotation

3.2.2.1 SYMMETRY GROUP OF A ONE-DIMENSIONAL NET

Let N_F be a one dimensional net. The symmetry group of this net is:

$$\Xi_{N_F} = \{ T_{\alpha h}, R_{180,\alpha h/2}, F_L, G_{r,p} \mid \alpha \in Z \}$$

i.e it contains the identity, the translations αh, 180 degree rotations about the points $\alpha h/2$, mirror reflections in line L, where L is of the form $y=\alpha|h|/2$, $\alpha \in Z$, or the x-axis, and glide reflections. The glide vector r is of the form αh and passes through the axis of the frieze, see fig(6).



fig(6)

3.2.2.2 SYMMETRY GROUPS OF FRIEZE PATTERNS

p111

The symmetry group of a **p111** pattern is

$$\Xi_{\mathbf{p111}} = \{ T_{\alpha h} \quad | \ \alpha \in \mathbb{Z} \}$$

i.e it admits only translations.

p112

The symmetry group of a **p112** pattern is

$$\Xi_{\mathbf{p2}} = \Xi_{\mathbf{p111}} \cup \{ R_{180,\alpha h/2} \ | \ \alpha \in \mathbb{Z} \}$$

i.e apart from translations it contains half turns about the nodes and about mid points between the nodes.

pm11

The symmetry group of a **pm11** pattern is

$$\Xi_{\mathbf{pm11}} = \Xi_{\mathbf{p111}} \cup \{ F_L \}$$

i.e apart from translations it admits mirror reflections in the lines L which are of the form $y=\alpha|\mathbf{h}|/2$, $\alpha \in \mathbb{Z}$.

p1m1

The symmetry group of a **p1m1** pattern is

$$\Xi_{\mathbf{p1m1}} = \Xi_{\mathbf{p111}} \cup \{ F_L \}$$

i.e apart from translations it contains mirror reflections in the line L which is the x-axis.

pmm2

The symmetry group of a **pmm2** pattern is

$$\Xi_{\mathbf{pmm2}} = \Xi_{\mathbf{pm11}} \cup \Xi_{\mathbf{p1m1}} \cup \{ R_{180,\alpha h/2} | \alpha \in \mathbb{Z} \}$$

it contains translations, half turns about nodes and about mid points between the nodes, and mirror reflections in the line L, which is of the form $y=\alpha|h|/2$, $\alpha \in \mathbb{Z}$, or the x-axis.

p1a1

The symmetry group of a **p1a1** pattern is

$$\Xi_{\mathbf{p1a1}} = \Xi_{\mathbf{p111}} \cup \{ G_{\mathbf{r},\mathbf{p}} \}$$

i.e apart of translations, it contains glide refections. The glide vector $\mathbf{r}$ is of the form $\alpha\mathbf{h}$ and passes through the axis of the frieze.

pma2

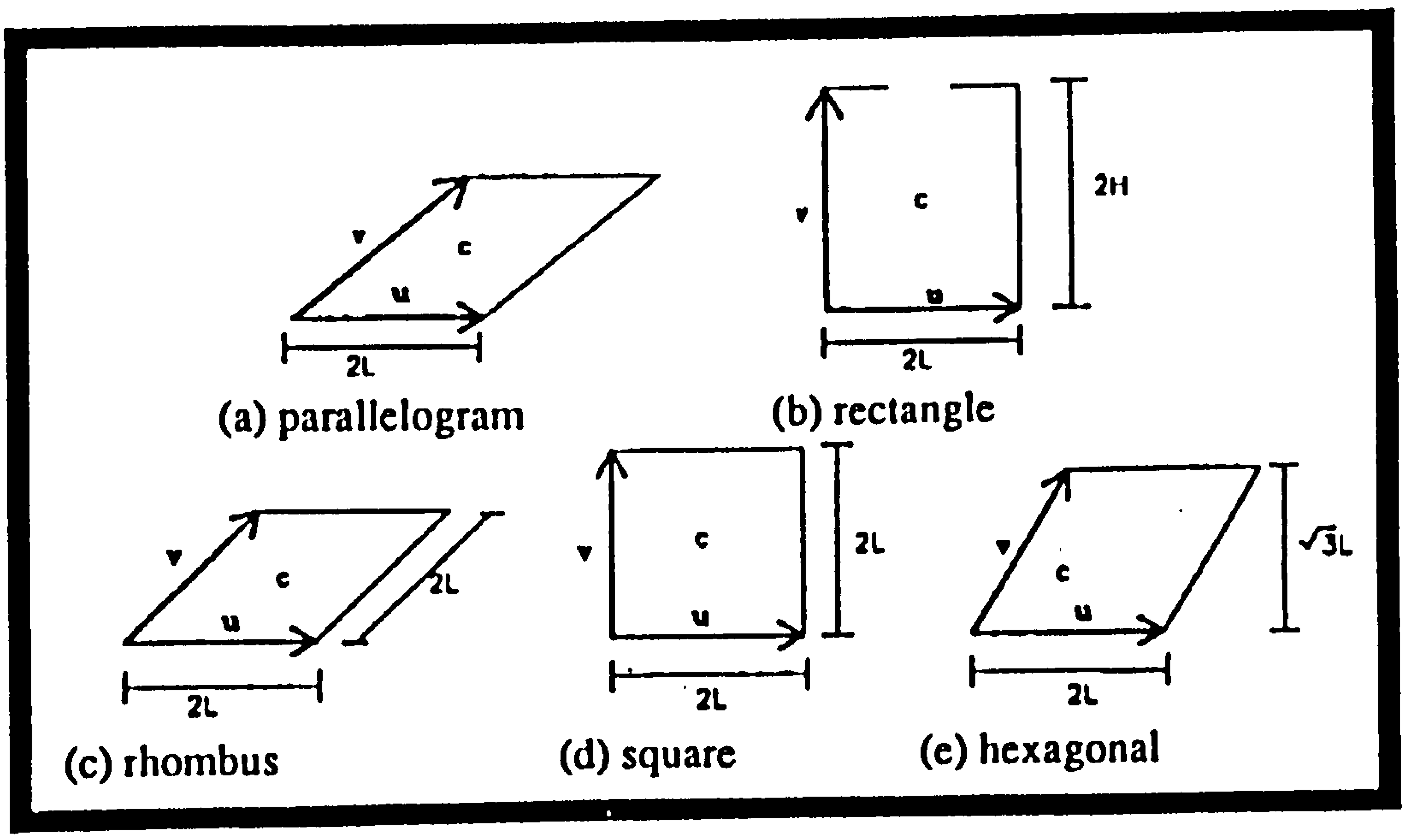
The symmetry group of **pma2** pattern is

$$\Xi_{\mathbf{pma2}} = \Xi_{\mathbf{p1a1}} \cup \{ R_{180,\alpha h}, F_L | \alpha \in \mathbb{Z} \}$$

it contains translation, half turns about the nodes and about the mid points between nodes, mirror reflections where line L is of the form $y=(\alpha|\mathbf{h}|+1)/2$, $\alpha\in\mathbb{Z}$, and glide reflections. The glide vector r is of the form αh and passes through the axis of the frieze.

3.2.2.3 SYMMETRY GROUP OF TWO-DIMENSIONAL NETS

There are five different types of nets categorised by their symmetries as shown in fig(7). These are parallelogram, rectangle, rhombus, square and hexagon , we shall refer to them as N_P, N_R, N_C, N_S and N_H respectively. The points marked c will be used later when we construct algorithms for generating crystallographic patterns.



fig(7) show five types of nets

We give below the symmetry group of five different types of nets in two-dimension.

Vectors u,v generate five different types of nets which are categorized according to their symmetry groups. These are:

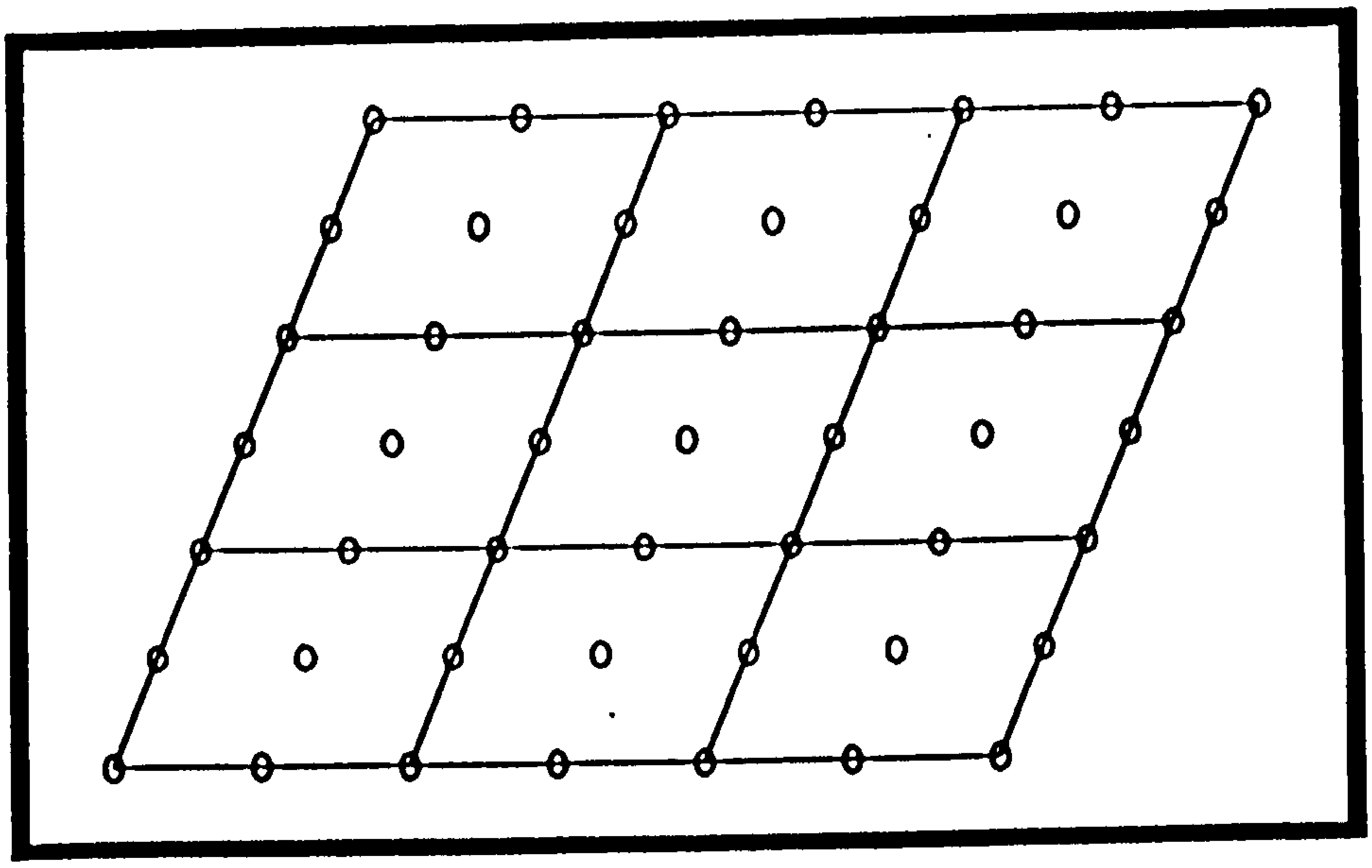
(i) A parallelogram net N_P which arises when $|u|\neq|v|$ and

$u \cdot v \neq 0$,

The symmetry group of N_P is:

$$\Xi_{N_P} = \{ T_{\alpha u+\beta v}, R_{180,\alpha u/2+\beta v/2} \mid \alpha,\beta \in Z \}$$

i.e it contains the identity, the translations $\alpha u+\beta v$ and 180 degree rotations (half turns) about the vertices, the centers and the mid-points of the parallelogram cells of the net, see fig(8).

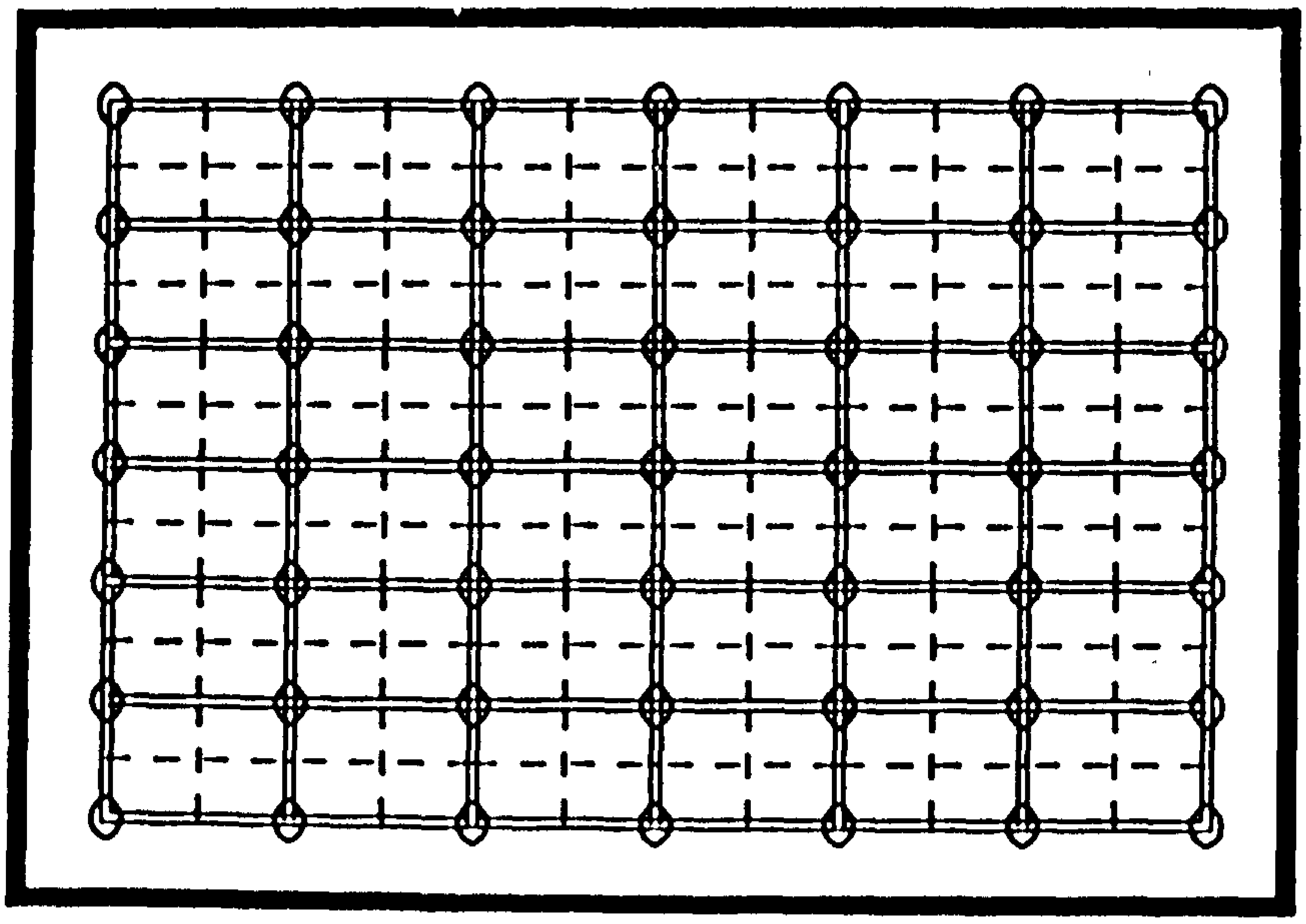


fig(8)

(ii) A rectangular net N_R which arises when $|u| \neq |v|$ and $u \cdot v=0$, The symmetry group of N_R is:

$$\Xi_{N_R} = \Xi_{N_P} \cup \{ F_L, G_{r,p} \}$$

where the line L is of the form $x=\alpha|u|/2$ or $y=\beta|v|/2$, $\alpha,\beta \in Z$, the glide vector r is of the form αu or βv with $\alpha,\beta \in Z$, $\alpha,\beta \neq 0$ and p is any node point of the net, see fig(9).



fig(9)

(iii) A centered rectangular net $\mathbb{N}_C$ which arises when

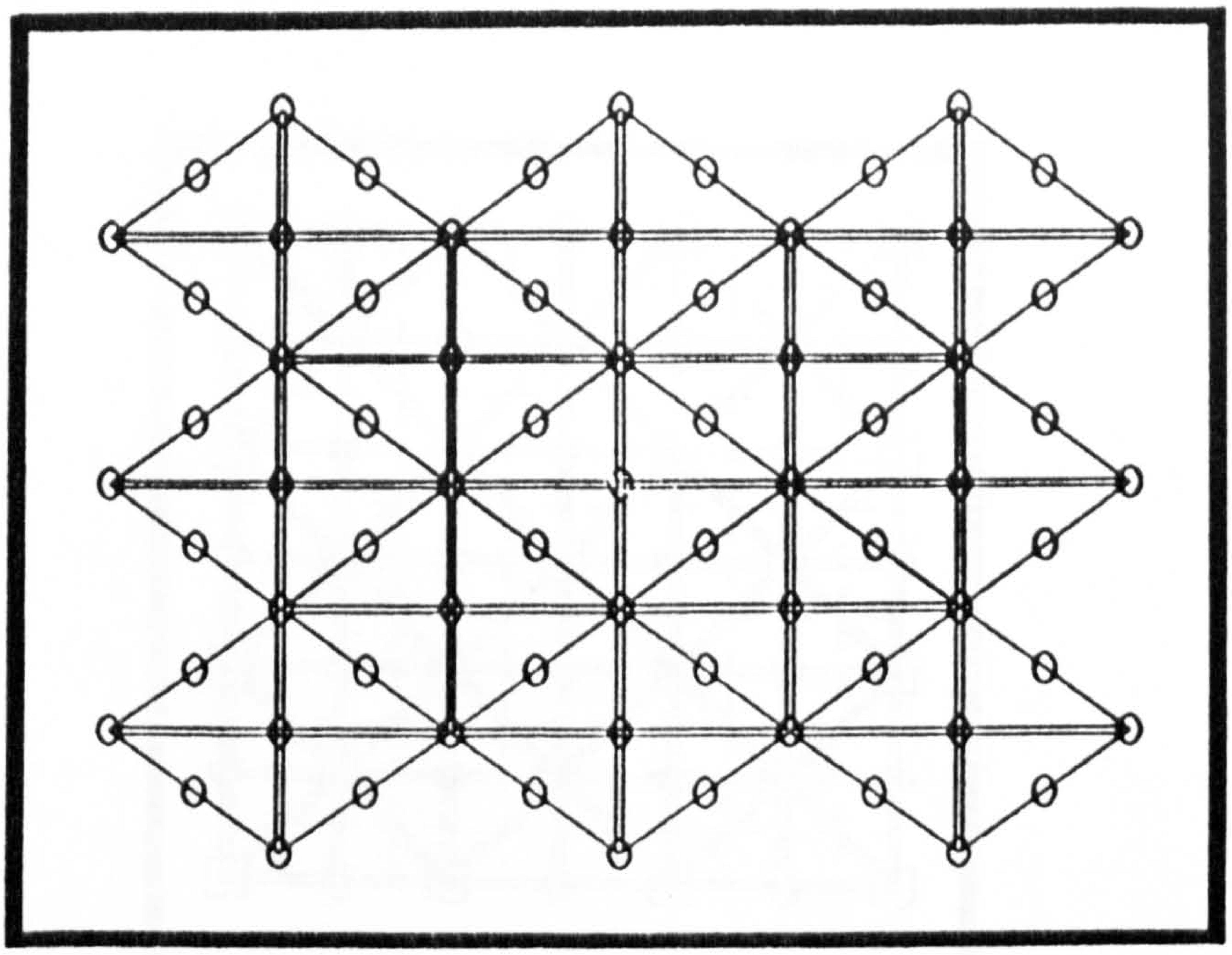
$\theta = 90$ $\neq \theta = 60$

$|u|=|v|$, $u \cdot v \neq 0$ and $u \cdot v/|u||v| \neq \pi/3$,

The symmetry group of $\mathbb{N}_C$ is:

$$\Xi_{\mathbb{N}_C} = \Xi_{\mathbb{N}_P} \cup \{ \ F_L \ \}$$

where the L refers to two families of lines parallel to the vectors u+v and u − v and passing through the nodes of the net, see fig(10).



fig(10)

(iv) A square net which arises when $|\mathbf{u}|=|\mathbf{v}|$ and $\mathbf{u}\cdot\mathbf{v}=0$.

The symmetry group of $\mathbb{N}_S$ is:

$$\Xi_{\mathbb{N}_S} = \{ T_{\alpha\mathbf{u}+\beta\mathbf{v}}, R_{90,\alpha\mathbf{u}+\beta\mathbf{v}}, R_{90,((2\alpha+1)/2)\mathbf{u}+((2\beta+1)/2)\mathbf{v}},$$
$$R_{180,(1/2+\alpha)\mathbf{u}+\beta\mathbf{v}}, R_{180,\alpha\mathbf{u}+(1/2+\beta)\mathbf{v}}, F_L, G_{\mathbf{r},\mathbf{p}} \}$$

where the symbols have the following meaning:

α, β are integers i.e. $\alpha, \beta \in \mathbb{Z}$

L refers to four families of lines whose equations are:

$x=\alpha|\mathbf{u}|/2$

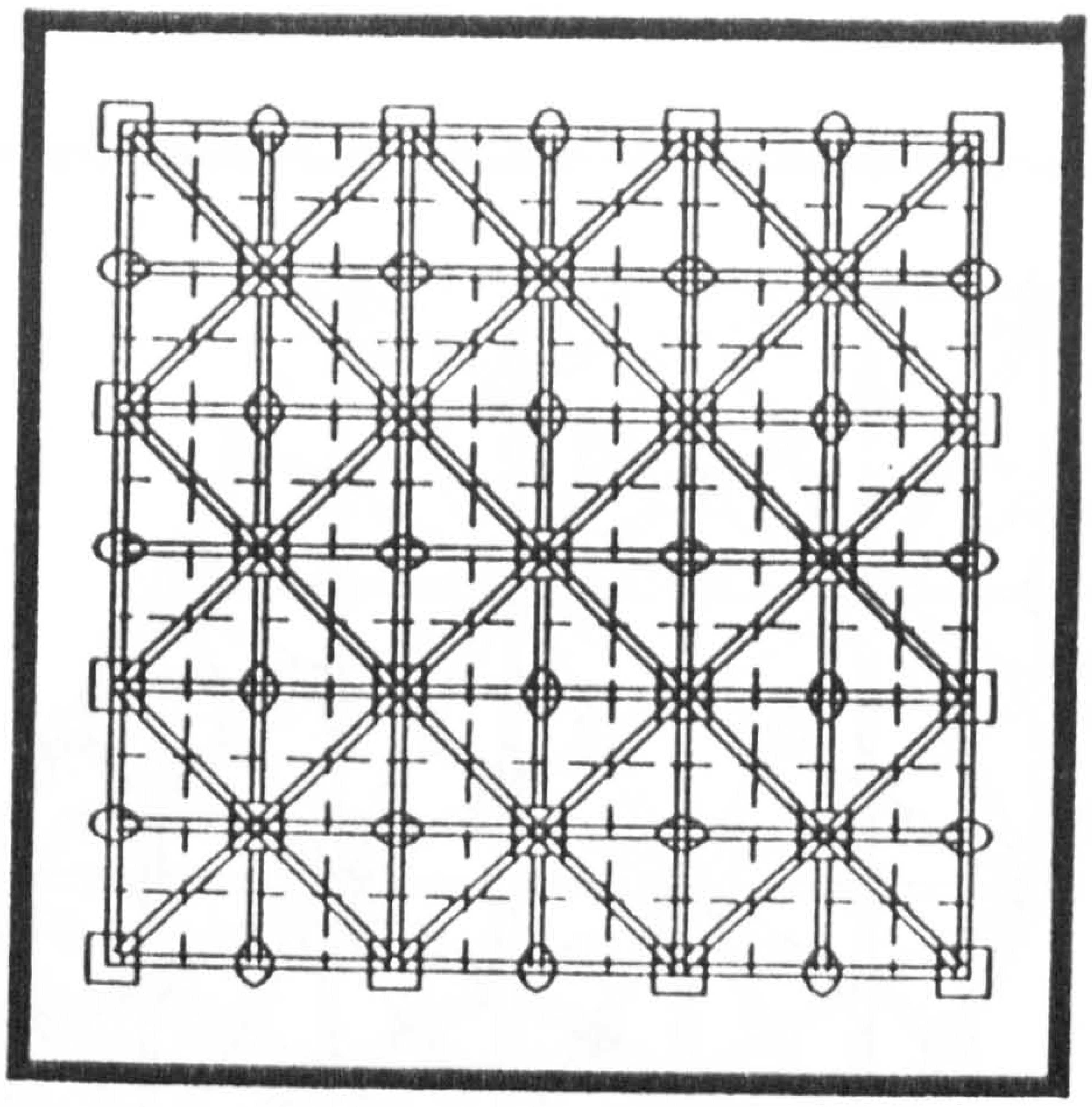
$y=\beta|\mathbf{v}|/2$

$y=x+\alpha \quad y=-x+\alpha$

The glide vector $\mathbf{r}$ can have the forms

$$\mathbf{r}=\alpha\mathbf{u} \text{ or } \mathbf{r}=\beta\mathbf{v} \quad \text{with } \alpha,\beta \neq 0$$

and finally $\mathbf{p}$ is any node i.e $\mathbf{p} \in \alpha\mathbf{u}+\beta\mathbf{v}$, see fig(11).



fig(11)

(v) A hexagonal net $\mathbb{N}_H$ which arises when $|\mathbf{u}|=|\mathbf{v}|$ and $\mathbf{u}\cdot\mathbf{v}/|\mathbf{u}||\mathbf{v}|=\pi/3$. $\rightarrow \theta=60$

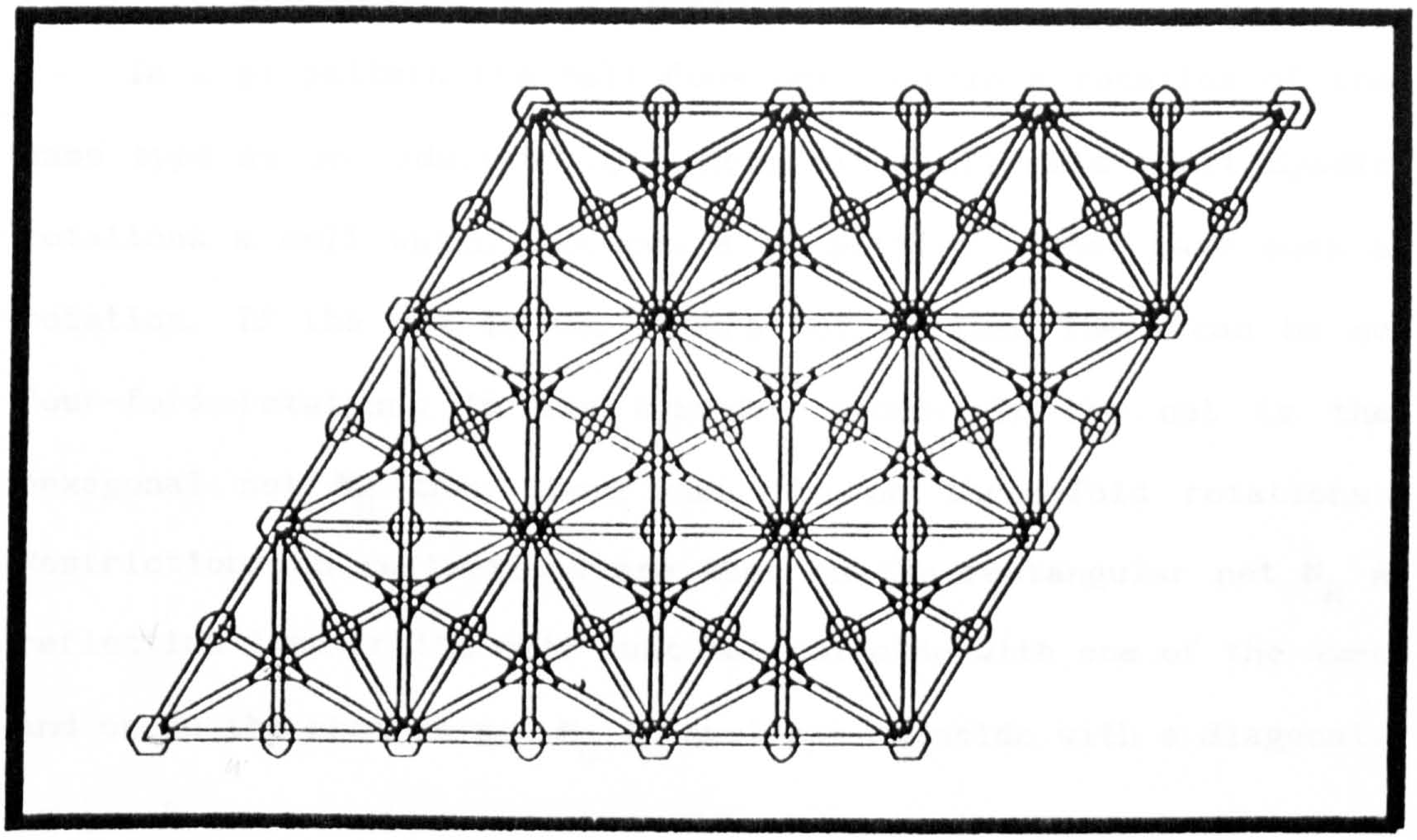
The symmetry group of $\mathbb{N}_H$ is:

$$\Xi_{\mathbb{N}_H} = \{ T_{\alpha\mathbf{u}+\beta\mathbf{v}}, R_{60,\alpha\mathbf{u}+\beta\mathbf{v}}, R_{180,((2\alpha+1)/2)\mathbf{u}+((2\beta+1)/2)\mathbf{v}}, R_{180,(1/2+\alpha)\mathbf{u}+\beta\mathbf{v}}, R_{180,\alpha\mathbf{u}+(1/2+\beta)\mathbf{v}}, R_{120,\mathbf{p}}, F_L \}$$

where $\alpha,\beta \in \mathbb{Z}$; $\mathbf{p}$ refers to the set of points lying on lines parallel to the vectors $\mathbf{u}+\mathbf{v}$ or $\mathbf{u}-\mathbf{v}$ and passing through the nodes. The distances of the points $\mathbf{p}$ measured from a node through which the line passes are $(2\gamma\pm1)|u|/\sqrt{3}+2\gamma|u|/(2\sqrt{3})$, where $\gamma \in \mathbb{Z}$. The lines l comprise 6 families of lines passing through the nodes and parallel to the lines whose equations are:

$y = \tan(\phi)x, \quad \phi \in \{ 0, \pi/6, \pi/3, \pi/2, 2\pi/6, 5\pi/6 \}$

see fig(12).



fig(12)

3.2.2.4 SYMMETRY GROUPS OF CRYSTALLOGRAPHIC PATTERNS

The kind of symmetries that can arise in a periodic pattern depend on the symmetry group of the cell and the symmetry group of the net on which the cell is copied. This is discussed below.

P1

The symmetry group of a **p1** pattern is

$$\Xi_{\mathbf{p1}} = \{ T_{\alpha u+\beta v} | \alpha, \beta \in \mathbb{Z} \}$$

i.e it admits only translations.

If the cell has no symmetries then a **p1** pattern is produced no matter what type of net is employed.

In a **p1** pattern the cell does not contain a rotation of the same type as one admitted by a net. Since all nets admit dyadic rotations a cell which produces a **p1** pattern cannot have such a rotation. If the net is the square net $\mathbb{N}_S$ then there can be no four-fold rotations in the symmetry group. If the net is the hexagonal net $\mathbb{N}_H$ then there can be no three-fold rotations. Restrictions on reflections are that on the rectangular net $\mathbb{N}_R$ a reflection line in the cell must not coincide with one of the axes and on an the rhombic net $\mathbb{N}_C$ it must not coincide with a diagonal.

p211

The symmetry group of a **p211** pattern is the same as the symmetry group of the net $\mathbb{N}_P$, i.e.

$$\Xi_{\mathbf{p211}} = \{ T_{\alpha u+\beta v}, R_{180,\alpha u/2+\beta v/2} | \alpha, \beta \in \mathbb{Z} \}$$

Apart from translations it admits dyadic rotations (half turns) about the vertices, the centers and the mid-points of the parallelogram cells of the net.

For a **p211** type pattern to arise, it is necessary that the cell has a dyadic rotation. On the net $\mathbb{N}_P$ there is no further restriction on rotations and reflection which may occur in the cell.

On an $\mathbb{N}_R$ net the cell must not have a line of reflection which coincides with one of the axes to produce a **p211** pattern.

On an $\mathbb{N}_S$ net the cell must not possess any four-fold centers of rotation.

On an $\mathbb{N}_C$ net the cell must not have any lines of reflections which coincide with the diagonals of the cells of the net.

Finally, on an N_H net the cell must not possess any three-fold centers of rotation if a **p211** type pattern is to be produced.

P1m1

A **p1m1** type pattern can be generated on the nets $\mathbb{N}_R$ and $\mathbb{N}_S$. The symmetry group of the pattern is

$$\Xi_{\mathbf{p1m1}} = \{ T_{\alpha u+\beta v}, F_L | \ \alpha, \beta \ \varepsilon \ \mathbb{Z} \}$$

where the lines L are a single family of lines of the form $x=\alpha|u|/2$ or $y=\beta|v|/2$ (but not both). The cell of such a pattern must have a line of reflection which can be made to coincide with the x or the y axis and must not have any two-fold centers of rotation.

P2mm

A **p2mm** type pattern, like the **p1m1** pattern arises on the nets $\mathbb{N}_R$ and $\mathbb{N}_S$. The symmetry group of the pattern is

$$\Xi_{\mathbf{p2mm}} = \{ T_{\alpha u+\beta v}, F_L, R_{180, \alpha u/2+\beta v/2} | \ \alpha, \beta \ \varepsilon \ \mathbb{Z} \}$$

where the lines L comprise two family of lines of the form $x=\alpha|u|/2$ and $y=\beta|v|/2$. The cell of such a pattern must have two lines of reflection at right angles which can be made to coincide with the x and the y axis.

p1g1

A **p1g1** type pattern can be generated on nets $\mathbb{N}_R$ and $\mathbb{N}_S$. The symmetry group of the pattern is

$$\Xi_{\mathbf{p1g1}} = \{ T_{\alpha u+\beta v}, G_{\mathbf{r},\mathbf{p}} \mid \alpha,\beta \in \mathbb{Z} \}$$

where the glide vector **r** of the form **r**=**u**/2, and **p**=(x=α, y=(β/2)**v**).

p2mg

A **p2mg** type pattern, like the **p1g1** pattern arises on the nets $\mathbb{N}_R$ and $\mathbb{N}_S$. The symmetry group of the pattern is

$$\Xi_{\mathbf{p2mg}} = \{ T_{\alpha u+\beta v}, F_L, R_{180,\alpha u/2+\beta v/2}, G_{\mathbf{r},\mathbf{p}} \mid \alpha,\beta \in \mathbb{Z} \}$$

where the lines L are a single family of lines of the form

$x=(2\alpha+1)|\mathbf{u}|/4$ or $y=(2\beta+1)|\mathbf{v}|/4$,

The glide reflection is

r=**u**/2, or **r**=**v**/2, and

p=((2α+1)**u**/4, (2β+1)**v**/4).

p2gg

A **p2gg** type pattern, like the **p1g1** pattern arises on the nets $\mathbb{N}_R$ and $\mathbb{N}_S$. The symmetry group of the pattern is

$$\Xi_{\mathbf{p2gg}} = \{ T_{\alpha u+\beta v}, R_{180,\alpha u/2+\beta v/2}, G_{\mathbf{r},\mathbf{p}} \mid \alpha,\beta \in \mathbb{Z} \}$$

The glide reflection is

r=**u**/2, **r**=**v**/2, and

$p=((2\alpha+1)u/4,\ (2\beta+1)v/4)$.

c1m1

A **c1m1** type pattern can be generated on nets $\mathbb{N}_R$ and $\mathbb{N}_S$. The symmetry group of the pattern is

$$\Xi_{\mathbf{c1m1}} = \{\ T_{\alpha u+\beta v}, F_L, G_{\mathbf{r},\mathbf{p}}\ |\ \alpha,\beta \in \mathbb{Z}\ \}$$

where the lines L are a single family of lines which are parallel to **u+v** and passes through any nodes. The glide vector **r** of the form $\mathbf{r}=(\mathbf{u}+\mathbf{v})/2$, and $\mathbf{p}=((2\alpha+1)\mathbf{u}/2+\beta\mathbf{v})$.

c2mm

A **c2mm** type pattern, like the **c1m1** pattern arises on nets $\mathbb{N}_R$ and $\mathbb{N}_S$. The symmetry group of the pattern is

$$\Xi_{\mathbf{c2mm}} = \{\ T_{\alpha u+\beta v}, F_L, G_{\mathbf{r},\mathbf{p}}\ |\ \alpha,\beta \in \mathbb{Z}\ \}$$

where the lines L comprise two families of lines. The first family are parallel to **u+v**, the second family are parallel to **u-v** and passes through any nodes.

The glide vector **r** of the form

$\mathbf{r}=(\mathbf{u}+\mathbf{v})/2$, $\mathbf{p}=((2\alpha+1)\mathbf{u}/2+\beta\mathbf{v})$

and $\mathbf{r}=(\mathbf{u}-\mathbf{v})/2$, $\mathbf{p}=(\alpha\mathbf{u}+(2\beta+1)\mathbf{v}/2)$.

p4

A **p4** type pattern can be generated on nets $\mathbb{N}_S$ only. The symmetry group of the pattern is

$\Xi_{\mathbf{p4}} =\{ T_{\alpha u+\beta v}, R_{90,\alpha u+\beta v}, R_{90,((2\alpha+1)/2)u+((2\beta+1)/2)v},$
$R_{180,(1/2+\alpha)u+\beta v}, R_{180,\alpha u+(1/2+\beta)v} \mid \alpha,\beta \in \mathbb{Z} \}$

p4mm

A **p4mm** type pattern, like the **p4** pattern arises on net $\mathbb{N}_S$ only. The symmetry group of the pattern is

$$\Xi_{\mathbf{p4mm}} = \Xi_{\mathbf{p4}} \cup \{ F_L, G_{\mathbf{r},\mathbf{p}} \}$$

where the reflection lines L refer to four families of lines whose equations are:

$x=\alpha|\mathbf{u}|/2$

$y=\beta|\mathbf{v}|/2$

$y=x+\alpha \quad y=-x+\alpha$

The glide vector **r** of the form

$\mathbf{r}=(\mathbf{u}+\mathbf{v})/2$, $\mathbf{p}=((2\alpha+1)\mathbf{u}/2+\beta\mathbf{v})$,

and $\mathbf{r}=(\mathbf{u}-\mathbf{v})/2$, $\mathbf{p}=(\alpha\mathbf{u}+(2\beta+1)\mathbf{v}/2)$.

p4gm

A **p4gm** type pattern, like the **p4** pattern arises on net $\mathbb{N}_S$ only. The symmetry group of the pattern is

$$\Xi_{\mathbf{p4gm}} = \Xi_{\mathbf{p4}} \cup \{ F_L, G_{\mathbf{r},\mathbf{p}} \}$$

where the reflection lines L refer to two families of lines whose equations are:

$y=x+\alpha/2 \quad y=-x+\alpha/2$

The glide vector **r** of the form

$\mathbf{r}=\mathbf{u}/2$, $\mathbf{r}=\mathbf{v}/2$, $\mathbf{p}=((2\alpha+1)\mathbf{u}/4,(2\beta+1)\mathbf{v}/4)$.

p3

A **p3** type pattern can be generated on nets $\mathbb{N}_H$ only. The symmetry group of the pattern is

$$\Xi_{\mathbf{p3}} = \{ T_{\alpha u+\beta v}, R_{120,\alpha u+\beta v}, R_{120,p} \mid \alpha,\beta \in \mathbb{Z} \}$$

where **p** refers to the set of points lying on lines parallel to the vectors **u+v** or **u−v** and passing through the nodes. The distances of the points **p** measured from a node through which the line passes are $(2\gamma\pm1)|\mathbf{u}|/\sqrt{3}+2\gamma|\mathbf{u}|/(2\sqrt{3})$, where $\gamma \in \mathbb{Z}$.

p3m1

A **p3m1** type pattern, like the **p3** pattern arises on net $\mathbb{N}_H$ only. The symmetry group of the pattern is

$$\Xi_{\mathbf{p3m1}} = \Xi_{\mathbf{p3}} \cup \{ F_L, G_{\mathbf{r},\mathbf{p}} \}$$

where The lines L comprise three families of lines passing through the nodes and parallel to the lines whose equations are:

$y = \tan(\emptyset)x, \quad \emptyset \in \{ \pi/6, \pi/2, 5\pi/6 \}$

The glide vector **r** of the form

$\mathbf{r}=(\mathbf{u}+\mathbf{v})/2, \ \mathbf{p}=((2\alpha+1)\mathbf{u}/2, \ (2\beta+1)\mathbf{v}/2).$

$\mathbf{r}=(\mathbf{u}-\mathbf{v})/2, \ \mathbf{P}=(\alpha\mathbf{u}+(1/2+\beta)\mathbf{v})$, and

$\mathbf{r}=\mathbf{v}/2, \ \mathbf{P}=((1/2+\alpha)\mathbf{u}+\beta\mathbf{v}).$

p31m

A **p31m** type pattern, like the **p3** pattern arises on net $\mathbb{N}_H$ only. The symmetry group of the pattern is

$$\Xi_{\mathbf{p31m}} = \Xi_{\mathbf{p3}} \cup \{ F_L, G_{\mathbf{r},\mathbf{p}} \}$$

where The lines L comprise three families of lines passing through the nodes and parallel to the lines whose equations are:

$y = \tan(\varnothing)x, \quad \varnothing \in \{ 0, \pi/3, 2\pi/6 \}$

The glide vector **r** of the form

$r=(u)/4, \; p=((2\alpha+1)u/4),$

$r=(v)/4, \; p=((2\beta+1)v/4),$

$r=(u+v)/4, \; r=(u+v)/2, \; p=((2\alpha+1)u/2+ \beta v, \alpha u+ (2\beta+1)v/2).$

p6

A p6 type pattern can be generated on nets $\mathbb{N}_H$ only. The symmetry group of the pattern is

$$\Xi_{p6} =\{ S_{\alpha u+\beta v}, R_{60,\alpha u+\beta v}, R_{180,((2\alpha+1)/2)u+((2\beta+1)/2)v}, R_{180,(1/2+\alpha)u+\beta v}, R_{180,\alpha u+(1/2+\beta)v}, R_{120,p} \mid \alpha,\beta \in Z \}$$

where **p** refers to the set of points lying on lines parallel to the vectors u+v or u-v and passing through the nodes. The distances of the points p measured from a node through which the line passes are $(2\gamma\pm1)|u|/\surd3+2\gamma|u|/(2\surd3)$, where $\gamma \in \mathbb{Z}$.

p6mm

A **p6mm** type pattern, like the **p6** pattern arises on net $\mathbb{N}$ only. The symmetry group of the pattern is

$$\Xi_{p6mm} = \Xi_{p6} \cup \{ F_L, G_{r,p} \}$$

where **p** refers to the set of points lying on lines parallel to the vectors u+v or u-v and passing through the nodes. The distances of the points p measured from a node through which the line passes are $(2\gamma\pm1)|u|/\surd3+2\gamma|u|/(2\surd3)$, where $\gamma \in \mathbb{Z}$. The lines L comprise 6 families of lines passing through the nodes and parallel to the lines whose equations are:

$y = \tan(\varnothing)x, \quad \varnothing \in \{ 0, \pi/6, \pi/3, \pi/2, 2\pi/6, 5\pi/6 \}$

The glide refelction is of the same form of **p3m1** and **p3m1**.

3.3 COMPUTER ALGORITHMS FOR FRIEZE AND CAYSTALLOGRAPHIC PATTERNS:

By **action set** Ω we shall mean the isometries (I, T_r, $R_{\varphi,c}$, F_L, $F_{p,q}$, $G_{r,q}$, $G_{p,q}$) which define previously.

We can combine the elements of Ω to form expressions. These expressions are to be interpreted in the following way:

Let $A,B,C,D \in \Omega$ and $p \in \mathbb{R}^2$. We have already defined Ap as the result of applying the transformation A to p and $(AB)p$ to mean $A(Bp)$. By $(A+B)p$ we shall mean $Ap \cup Bp$ In expressions involving additions and products of the elements of Ω the distributive and associative laws apply. Thus $A(B+C)p=ABp+ACp$ and we can write $A(B+C)=AB+AC$. Similarly $(A+B)(C+D)=AC+AD+BC+BD$ and $A(B+C)D=ABD+ACD$.

A **motif element** is an ordered pair (E,m) where E is an expression involving elements from the action set Ω and m is a geometrical entity, e.g. a polyline.

A **motif set** is a set of motif elements $\{(E_i,m_i) \mid i=1,2..n\}$.

A motif is the set of geometrical elements $\{ m_i \mid i=1,2..n\}$ contained in a motif set.

Given a motif set, a **template motif** M (or more simply a template)is the set created from the actions of each E_i on m_i i.e.

$$M=\sum_{i=1}^{n} E_i m_i$$

where Σ is being used here to denote a union of sets.

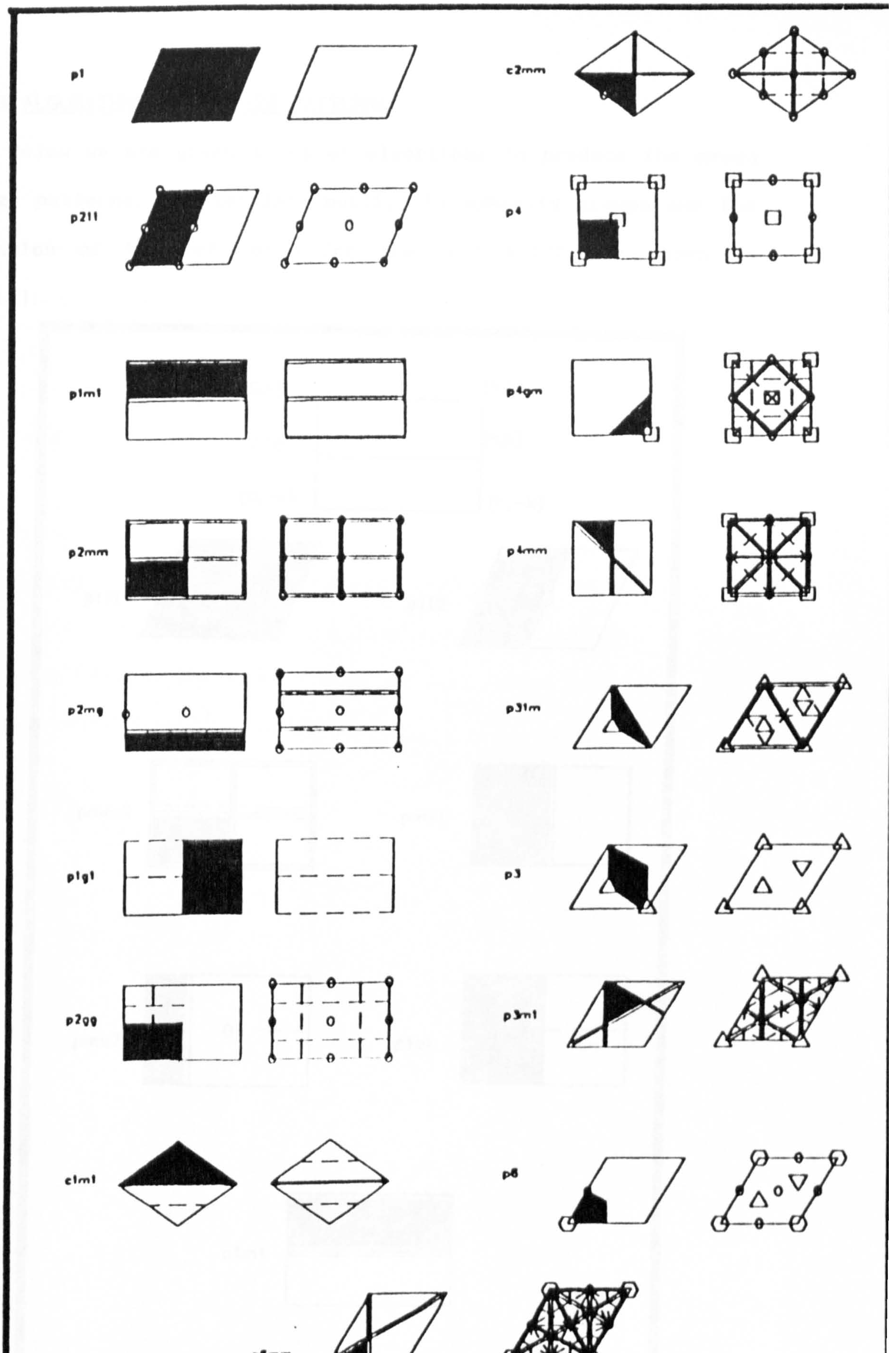
Given a template motif $\mathbb{M}$, a **unit motif** $\bar{\mathbb{M}}$ is the set created from the action of an expression $\bar{E}$ on $\mathbb{M}$, i.e. $\bar{\mathbb{M}} = \bar{E}\mathbb{M}$.

A **periodic pattern** $\mathbb{P}=(\mathbb{N},\bar{\mathbb{M}})$ is created when a **unit motif** $\bar{\mathbb{M}}$ is copied on all the nodes of the net $\mathbb{N}$, i.e.

$$\mathbb{P} = \sum_{w \in \mathbb{N}} T_w \bar{\mathbb{M}}$$

$$\text{or} \qquad \mathbb{P} = \sum_{w \in \mathbb{N}} T_w \bar{E} \sum_{i=1}^{n} E_i m_i$$

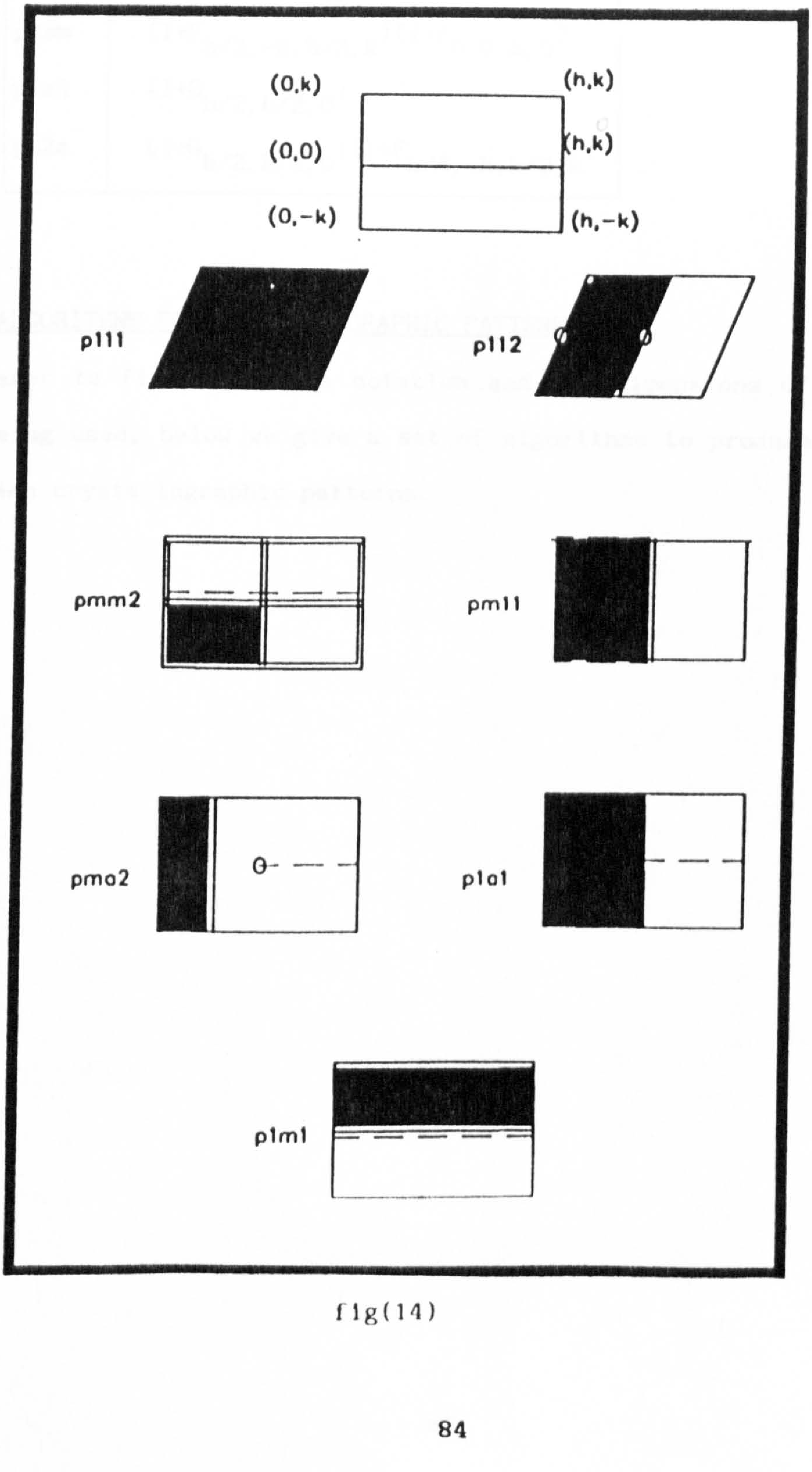
The expression $\bar{E}$ depends on the group type of the pattern and is supplied in the table below for frieze and crystallographic patterns. As we said earlier, $\bar{E}$ is not unique and in general it is possible to write down several equivalent forms. The reader may refer to the symmetry diagrams as shown in fig(13) for elucidation. In this diagram the left figure show a suitable region for template motif and the right diagram shows all the isometries that are symmetries of each group.


p1
c2mm
p211
p4
p1m1
p4gm
p2mm
p4mm
p2mg
p31m
p1g1
p3
p2gg
p3m1
c1m1
p6
p6mm

fig(13)

3.3.1 ALGORITHMS FOR FRIEZE PATTERNS:

Below we are given a set of algorithms to produce the seven frieze patterns. The template motif, the symmetry groups and the dimension of the cell used for the unit motif are shown in fig(14).



fig(14)

GROUP	EXPRESSION $\bar{E}$
p111	I
p112	$I+R_{180,h/2,0}$
p1m1	$I+F_{0,0,h,0}$
pm11	$I+F_{h/2,-k,h/2,k}$
p2mm	$(I+F_{h/2,-k,h/2,k})(I+F_{0,0,h,0})$
p1a1	$(I+G_{h/2,h/2,0})$
pm2a	$(I+G_{h/2,h/2,0})(I+F_{h/4,-k,h/2,k})$

3.3.2 ALGORITHMS FOR CRYSTALLOGRAPHIC PATTERNS:

Refer to fig(7) for the notation and the dimensions of the nets being used, below we give a set of algorithms to produce the seventeen crystallographic patterns.

GROUP	NET TYPE	EXPRESSION $\bar{E}$
p1	$\mathbb{N}_p$	I
p211	$\mathbb{N}_p$	$I+R_{180,c}$
p1m1	$\mathbb{N}_R$	$I+F_{0,H,2L,H}$
p1g1	$\mathbb{N}_R$	$I+G_{L,H,L,2H}$
p2mm	$\mathbb{N}_R$	$(I+F_{L,0,L,2H})(I+F_{0,H,2L,H})$
p2mg	$\mathbb{N}_R$	$(I+R_{180,c})(I+F_{0,H/2,2L,H/2})$
p2gg	$\mathbb{N}_R$	$(I+R_{180,c})(I+G_{L/2,H,L/2,2H})$
c1m1	$\mathbb{N}_C$	$I+F_{0,0,c}$
c2mm	$\mathbb{N}_C$	$(I+F_{u,v})(I+F_{0,0,c})$
p4	$\mathbb{N}_S$	$I+R_{90,c}+R_{180,c}+R_{270,c}$
p4mm	$\mathbb{N}_S$	$(I+R_{90,c}+R_{180,c}+R_{270,c})(I+F_{0,0,L,L})$
p4gm	$\mathbb{N}_S$	$(I+R_{90,c}+R_{180,c}+R_{270,c})(I+F_{0,L,L,0})$
p3	$\mathbb{N}_H$	$I+R_{120,c}+R_{240,c}$
p3m1	$\mathbb{N}_H$	$(I+R_{120,c}+R_{240,c})(I+F_{0,0,c})$
p31m	$\mathbb{N}_H$	$(I+F_{u,v})(I+R_{120,c}+R_{240,c})$
p6	$\mathbb{N}_H$	$(I+R_{180,(u,v)/2})(I+R_{120,c}+R_{240,c})$
p6mm	$\mathbb{N}_H$	$(I+F_{u,v})(I+R_{120,c}+R_{240,c})(I+F_{0,0,c})$

4- ANALYSIS OF ISLAMIC PATTERNS

The first part of our study for this thesis involved an extensive study of a very large number of Islamic geometrical patterns. The majority of the patterns studied appear in the books by Bourgoin [9], Critchlow [13], El-Said & Parman[63] and Wade [73]. Also, about ten patterns which do not occur in these references were collected by the author on a study tour of Islamic architecture to be found in Spain.

The patterns were studied using the CAD package AutoCAD and data was extracted to make it possible to recreate these patterns using group theoretical methods which were described in the last chapter. The purpose of this chapter is to describe this first part of our work and to draw same conclusions from it.

We begin first by giving a brief history of the group theoretic studies of Islamic patterns. Most of these studies have considered only the patterns to be found in the Palace of Alhambra in Granada, Spain and until recently the conclusion drawn have been quite controversial.

Muller [51a] was the first one to carry out a study of the patterns to be found in Alhambra and she came to the conclusion that 11 types of pattern are to be found there. She was unable to find the crystallographic patterns (P1g1, P211, P2gg, P3m1, P1m1, P4gm).

In contrast with Muller's findings, Coxeter [12b] claimed that 13 types of patterns occur in Alhambra whilst Belov [2], Toth [72] and Martin [47] claimed that all 17 types of patterns are to be found there. B. Grunbaum ,Z. Grunbaum & Shephard [29] carried out anther study in 1982 and came to the conclusion that 13 crystallograpic patterns exist in the Palace of Alhambra. They were unable to find the crystallographic patterns (P1g1, P211, P2gg, P3m1).

It has been now established that all the 17 crystallographic patterns do exist in the Alhambra, see Montesinos [50]. The last crystallographic pattern P3m1 to be eliminated from the list of missing patterns was discovered by Gomez and Pareja [26].

The controversy described above is concerned with Moorish architecture in Alhambra and does not relate to patterns to be found elsewhere in the Islamic world. Lahza [41] and Bixler [5] have carried out studies of crystallographic patterns of Islamic art and Bixler has given examples of all the 17 types of pattern. Other work which have analysed Islamic patterns from a group theoretic point of view are by E. Makovicky [44], E. Makovicky & M. Makovicky [43] and Chorbachi [11].

Comment: The point to be clearly understood in any discussions of analysis of crystallographic patterns to be found in art work is that the classification will depend on whether or not colors, decoration ... etc, are taken into account, most of the analysis does not concern itself with colors, decorations and so on.

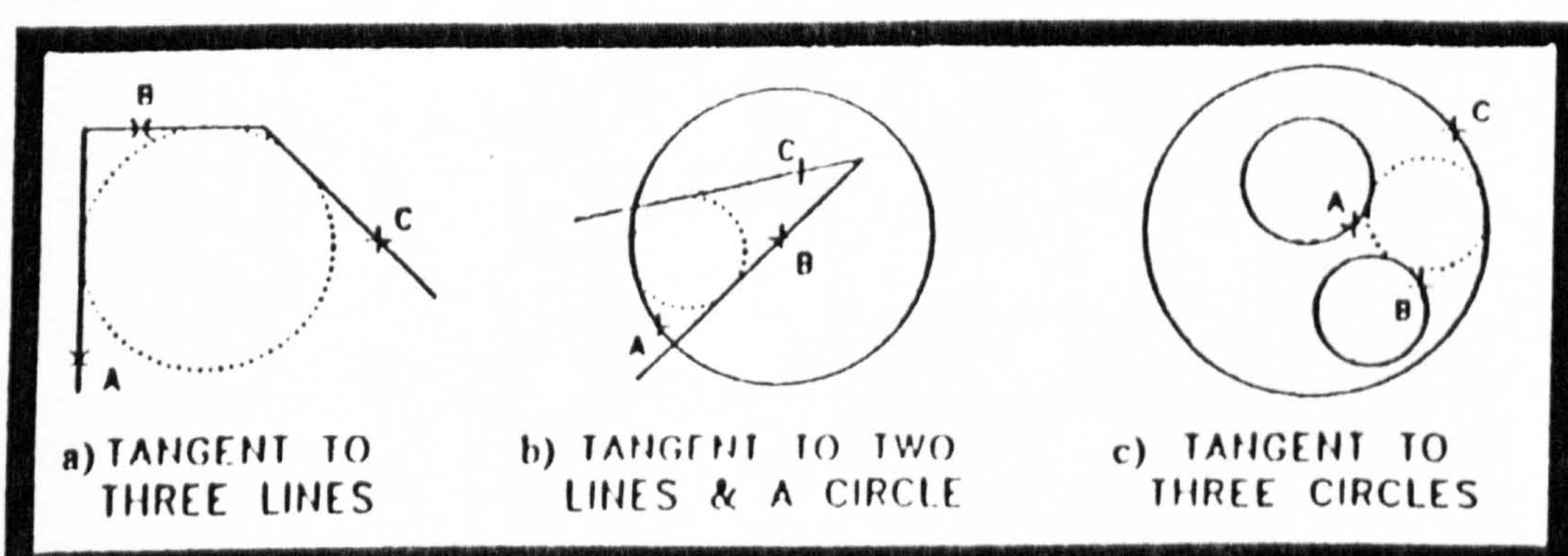
4.1 THE USE OF AUTOCAD

AutoCAD is a well known Computer Aided Drafting package. It would be inappropriate to discuss in any detail as to how this package works. Here we will comment on same features that were found to be particularly useful in our work.

We recall that the classical methods of constructing Islamic patterns involve the use of various shaped tiles, grids and the facility to construct and position certain shapes such as polygons and circles on these grids.

A CAD package is an ideal tool for these operations. The facility to construct shapes accurately, to manipulate them and do such operation as ERASE, MOVE, ROTATE, MIRROR, TRIM, EXTEND etc, are fairly commonly available in all CAD packages and are very suited in the context of the classical method.

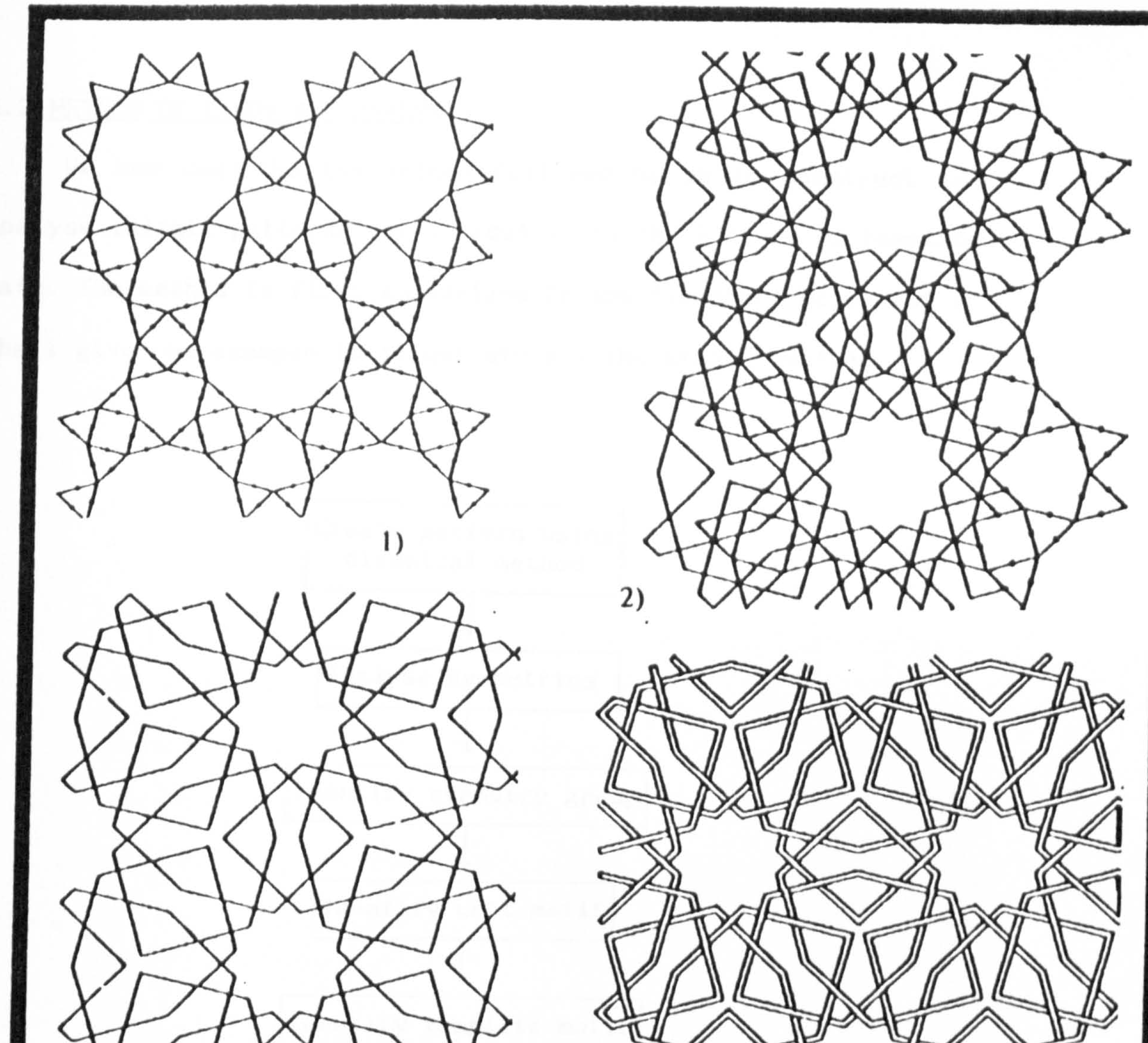
One feature of AutoCAD which was found to the very useful in our work is the facility to draw tangential lines to circles and touching circles. Typical examples are shown below which were utilised in the work described in chapter 2.



fig(1)

The facility to work in LAYERs is another feature which is highly useful in this kind of work. This allows for grids and intermediate constructions to be placed in separate layers which can be finally switched off when the pattern has emerged. Fig(2) shows an example of a construction which utilizes 5 layers to extract the final pattern.

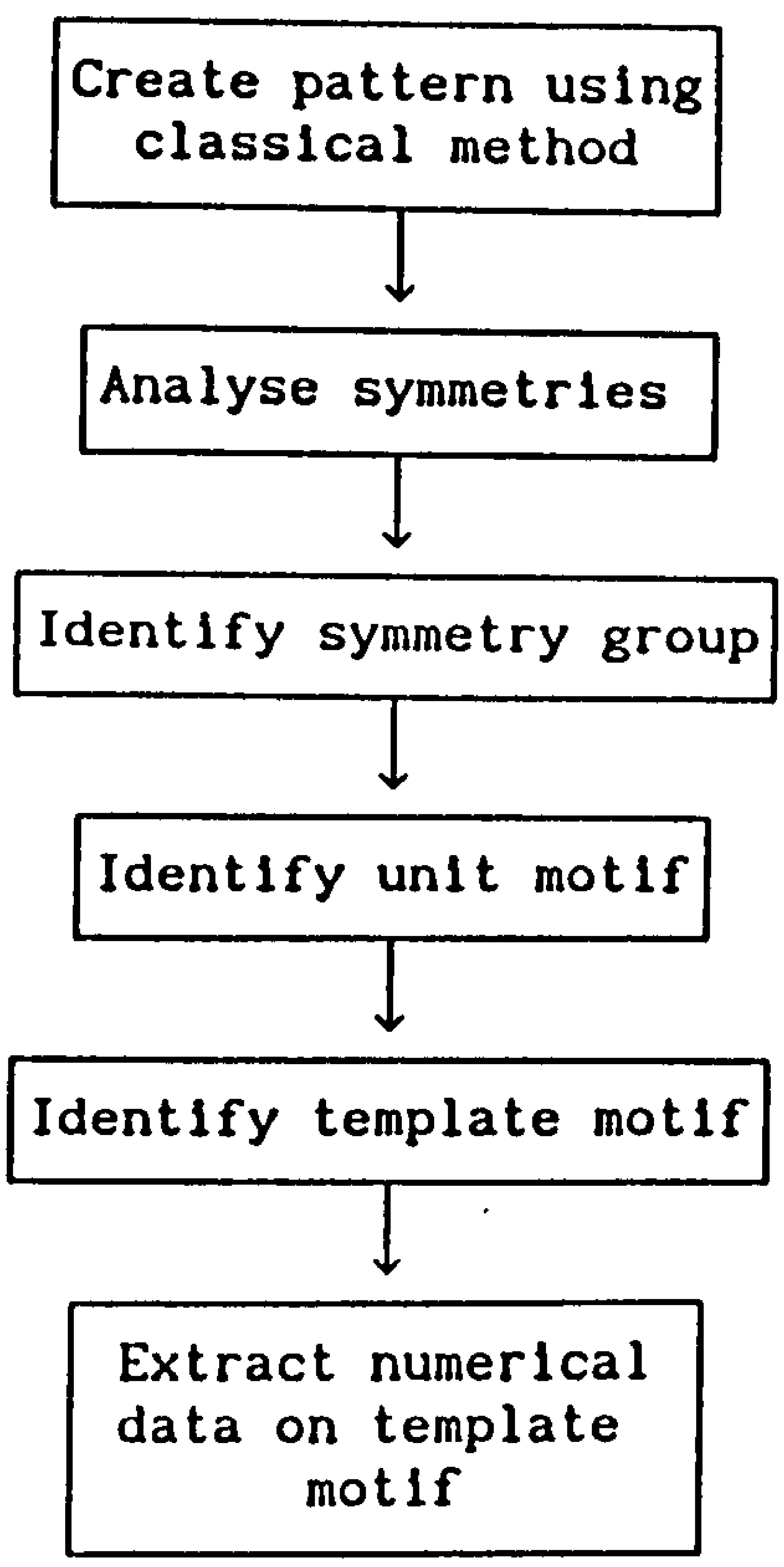
Finally, perhaps the BLOCK facility is the one that also needs to mentioned as being highly useful. This allows for various geometrical entities to be grouped together into a single unit which can be manipulated as a whole. In particular multiple copies can be made on a grid and the object can be scaled and rotated during the process of copying. The reader may refer to fig(9) in chapter 2 where the BLOCK command was used to construct a variety of patterns by scaling a single shape.



fig(2)

4.2 METHOD OF STUDY AND ANALYSIS

We now describe the method followed by us to construct and analyse Islamic patterns which resulted in the library of template data. The method is first summarized in the flowchart below and we shall give one example to illustrate all the steps involved.



The reader should refer to chapter 3 for the notation and the symbols used in this section.

4.2.1 ANALYSIS OF A P4MM CRYSTALLOGRAPHIC PATTERN

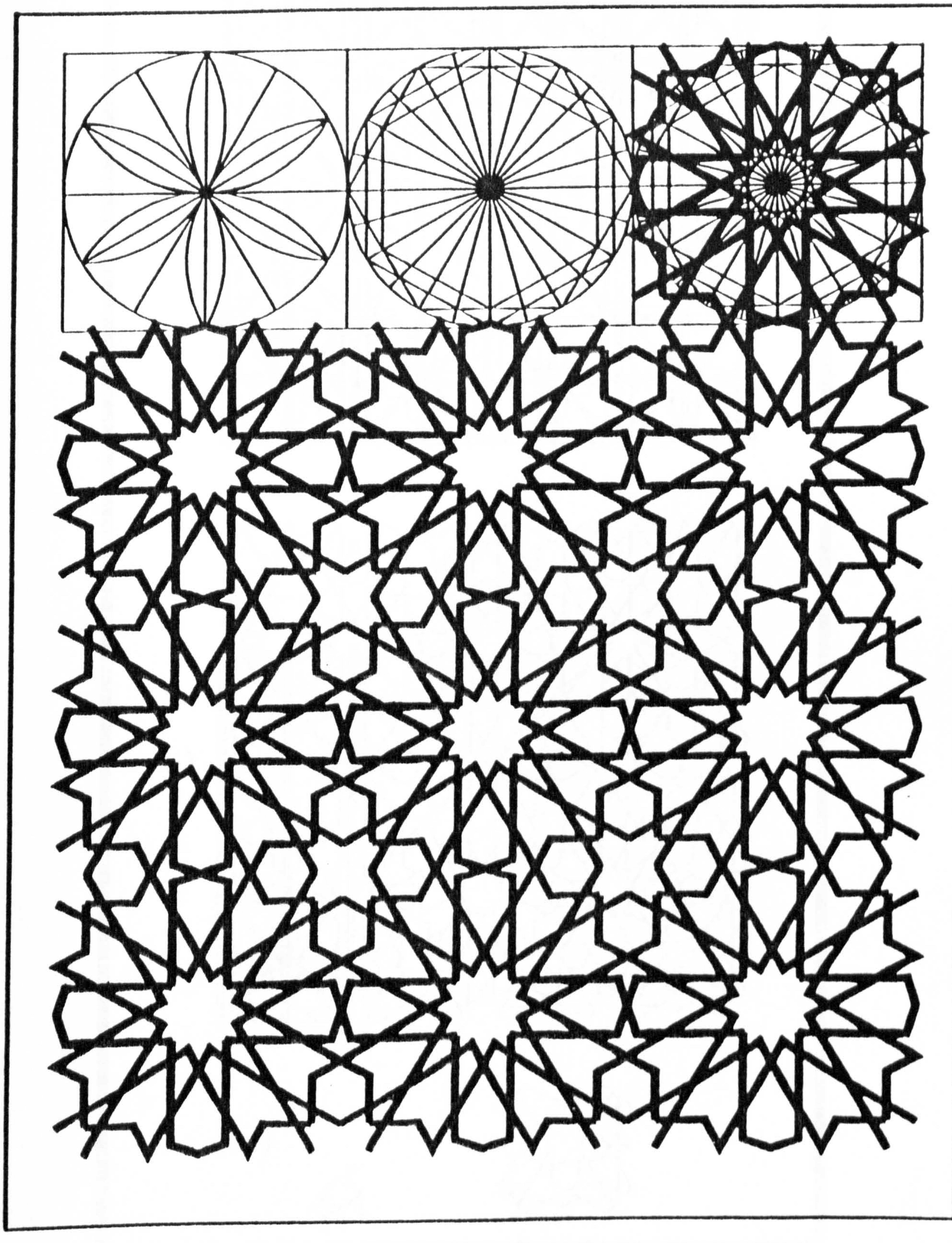
Fig(3) shows the construction of the pattern as suggested by El-Said [63]. The method can be followed fairly easily in AutoCAD.

Now, we examine the symmetries of the pattern for mirror reflections, glide reflections and centers of rotations. These are shown in fig(4a). Once again, AutoCAD is very useful in that we are able, by performing the operations, to verify that we are in fact correct. This allows us to identify the unit motif and the symmetry group of this pattern, see fig(4b) (the repeat motif produced by a suggested classical method doesn't necessarily give the minimum unit motif. For example, the repeat motif produced by El-Said [63] for the pattern on page 15 is not a unit motif). Having identified the symmetry group and the unit motif, we can identify the region of the template motif and the template motif itself. This is shown in fig(4c).

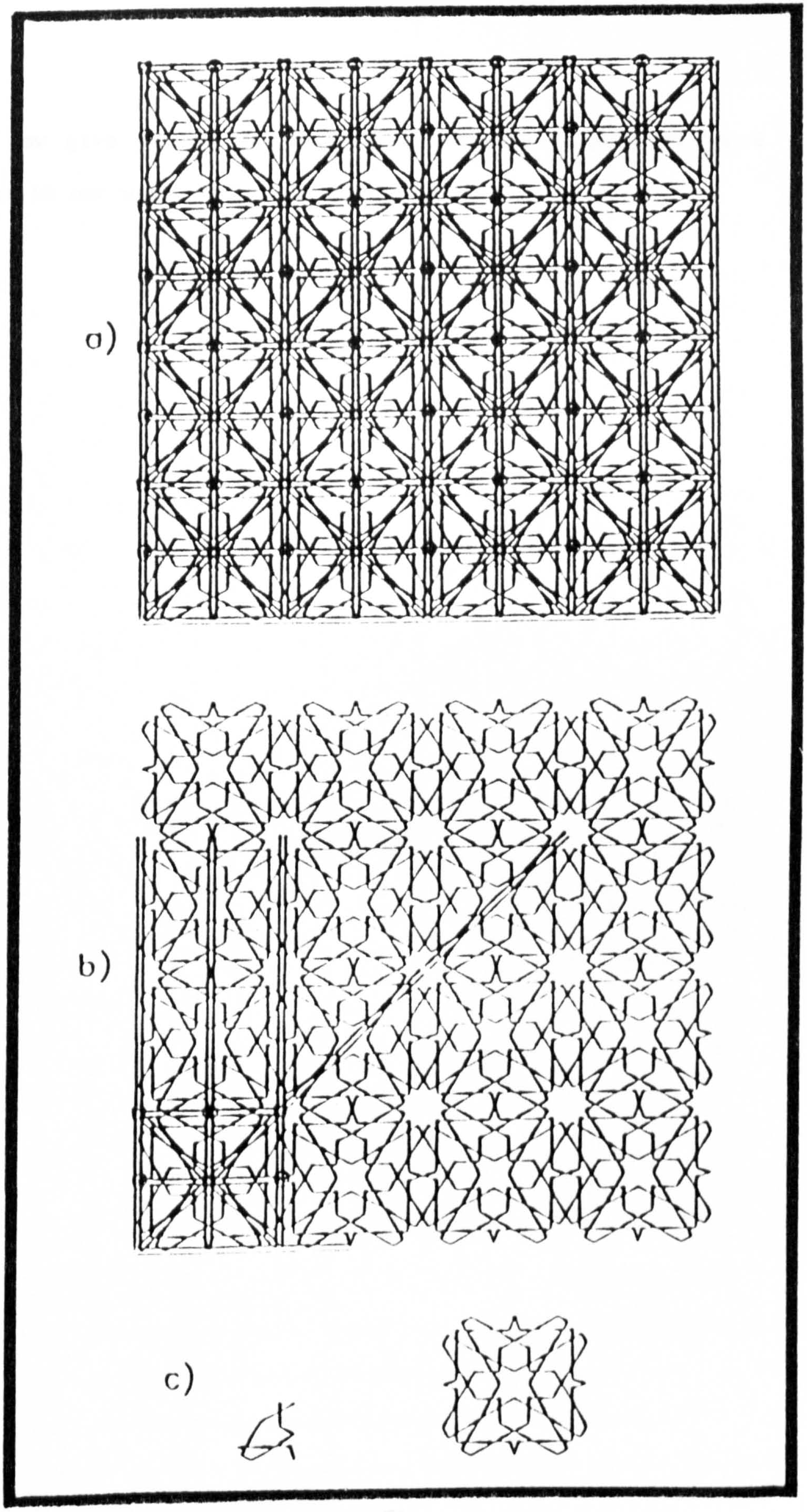
The enquire function LIST in AutoCAD allows one to extract the coordinates of each of these point from which we can construct the polylines. The data for this template motif, which comprises two polylines, shown below.

PolyLine1: (.5,.4),(.289,269),(.129,.041),(.199,0),
(.5,.175),(.4,.229),(.4,.4);

PolyLine2: (.5,0),(.468,.098),(.098,.098);

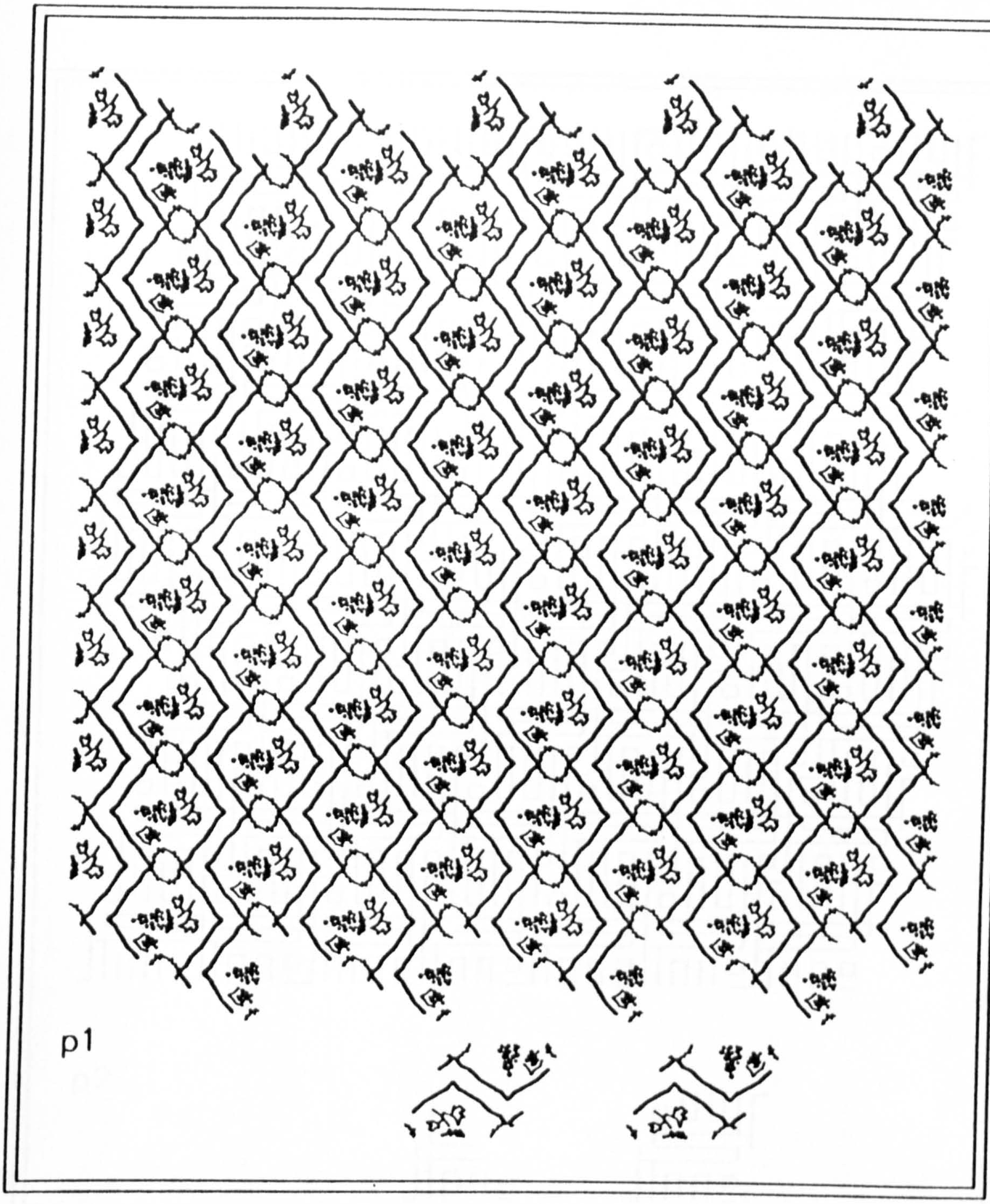


fig(3)



fig(4) chapT43

We now give example of each of 17 crystallographic patterns analysing in our work.



p1

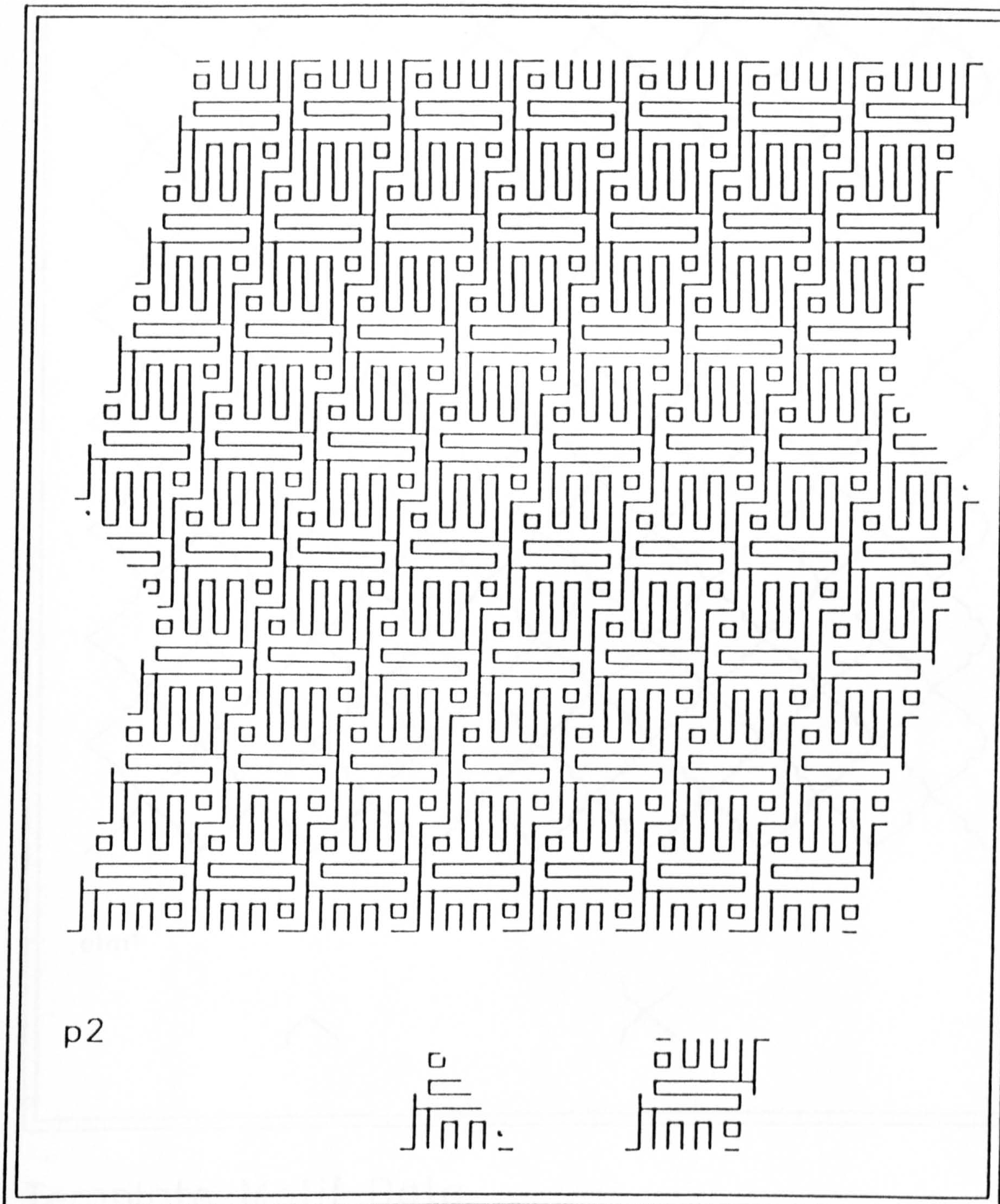
Template Motif Data

PolyLine1: (.442,.164),(.375,.131),(.319,.150),(.281,.141),(.310,.112),(.295,.075),(.329,.066),.352,.089)
(.389,.084),(.375,.131). PolyLine2: (.319,.263),(.399,.206),(.479,.117).

PolyLine3: (1.131,.681),(1.178,.634),(1.202,.582),(1.249,.634),(1.286,.657),(1.296,.718).

PolyLine4: (.413,.197),(.432,.234),(.526,.188),(.554,.112),(.582,.183),(.62,.183),(.577,.23),(.620,.253)
(.563,.296),(.54,.277),(.512,.286),(.498,.206).

Include the data in p211.1 in IDL



Generator Data

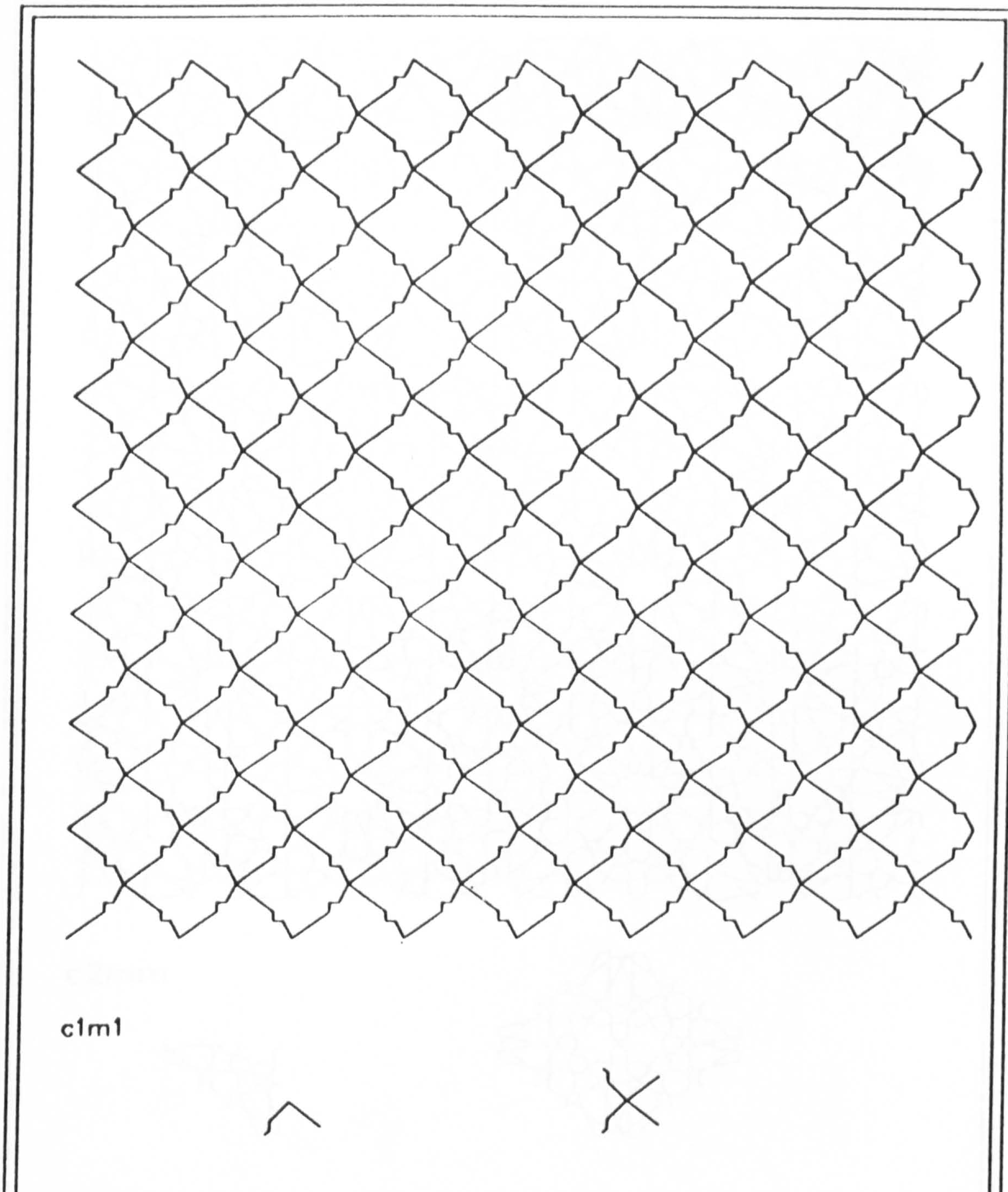
PolyLine1: (1,0),(.875,0). PolyLine2: ((.375,.832),(.375,.75),(.25,.75),(.25,.875),(.343,.875).

PolyLine3: (.375,0),(.375,.25),(.5,.25),(.5,0). PolyLine4: (.25,0),(.25,.375).

PolyLine5: (.625,0),(.625,.25),(.75,.25),(.752,0). PolyLine6: (0,0),(.125,0),(.125,.502).

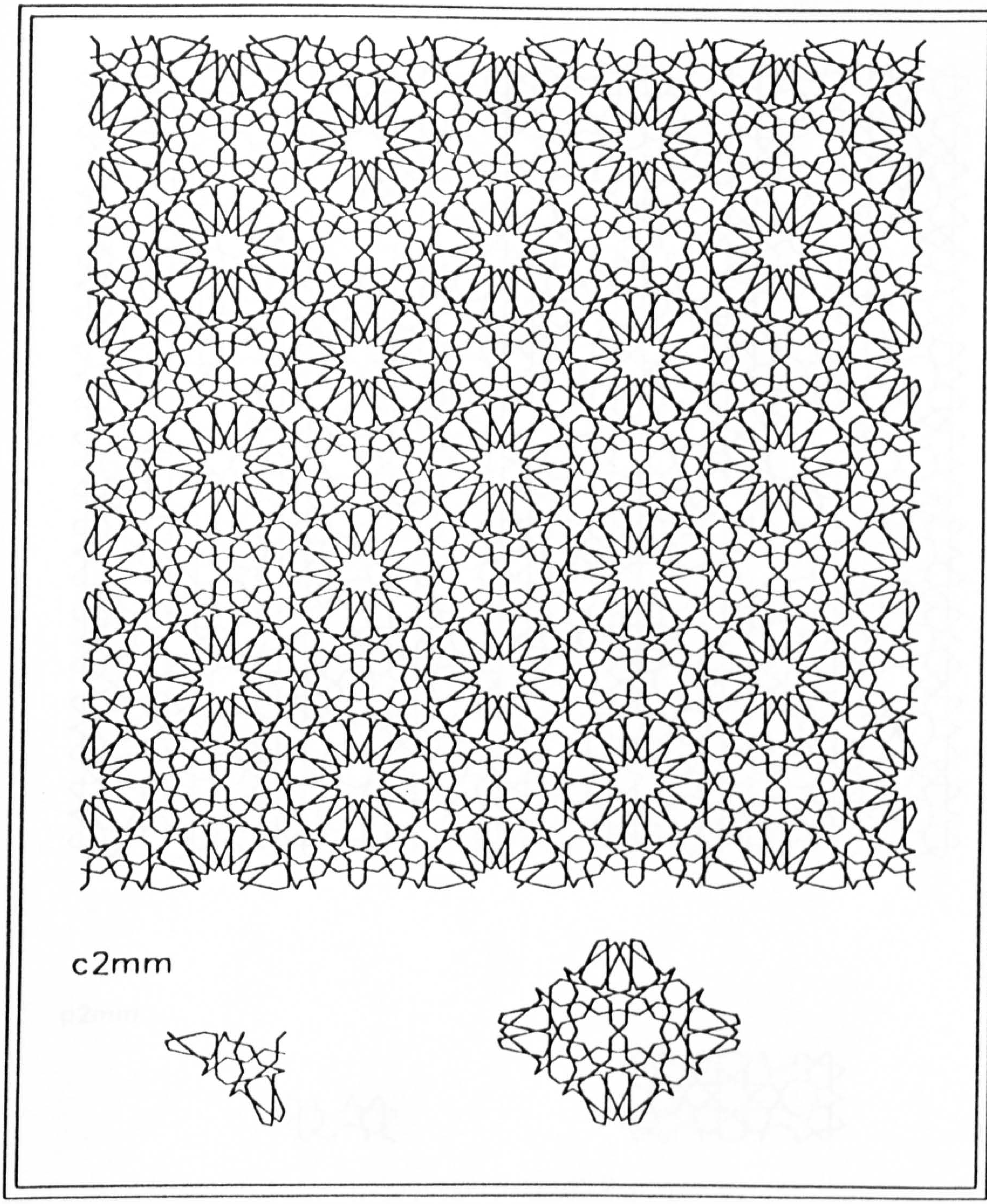
PolyLine7: (.533,.625),(.25,.625),(.25,.5),(.627,.5). PolyLine8: (.125,.375),(.72,.375).

PolyLine9: (.88,.16),(.88,.125),(.907,.125).



Template Motif Data

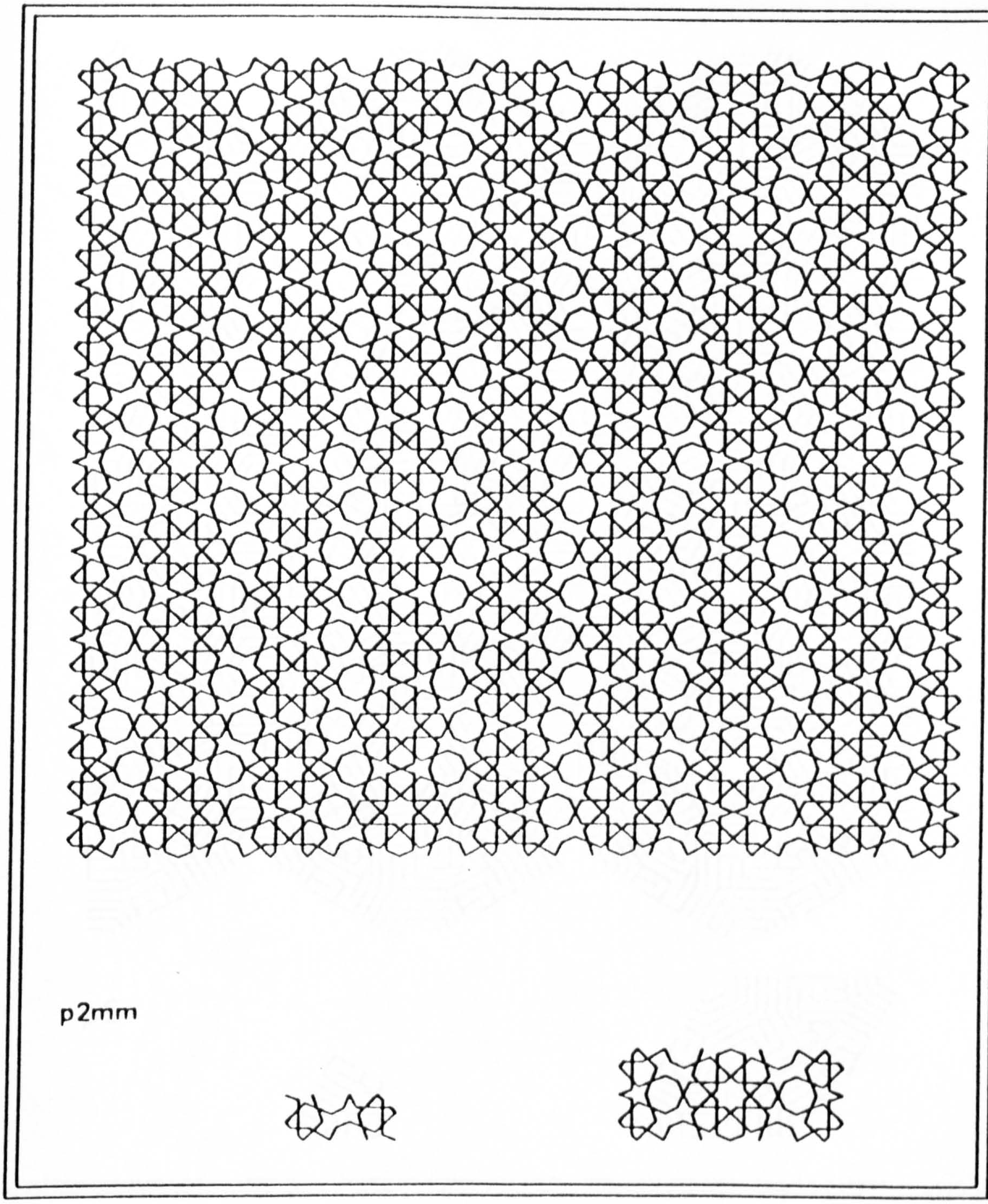
PolyLine1: (.785,.287),(.5,.5),(.334,.334),(.334,.25),(.287,.215).



Template Motif Data

PolyLine1: (.075,.393),(.222,.279),(.2,.235),(.288,.303),(.362,.285),(.426,.333),(.5,.316).

PolyLine2: (.5,.395),(.471,.367),(.471,.292),(.355,.198),(.421,.062),(.458,.055),(.46,.24),(.5,.252),(.409,.276) (.373,.346),(.295,.364),(.28,.395),(.233,.297),(.079,.333),(.082,.329),(.067,.363),(.207,.395),(.252,.385) (.252,.232),(.288,.203),(.437,.237),(.5,.103),(.485,.066),(.338,.186),(.3,.16),(.349,.253),(.317,.323),(.35,.395).

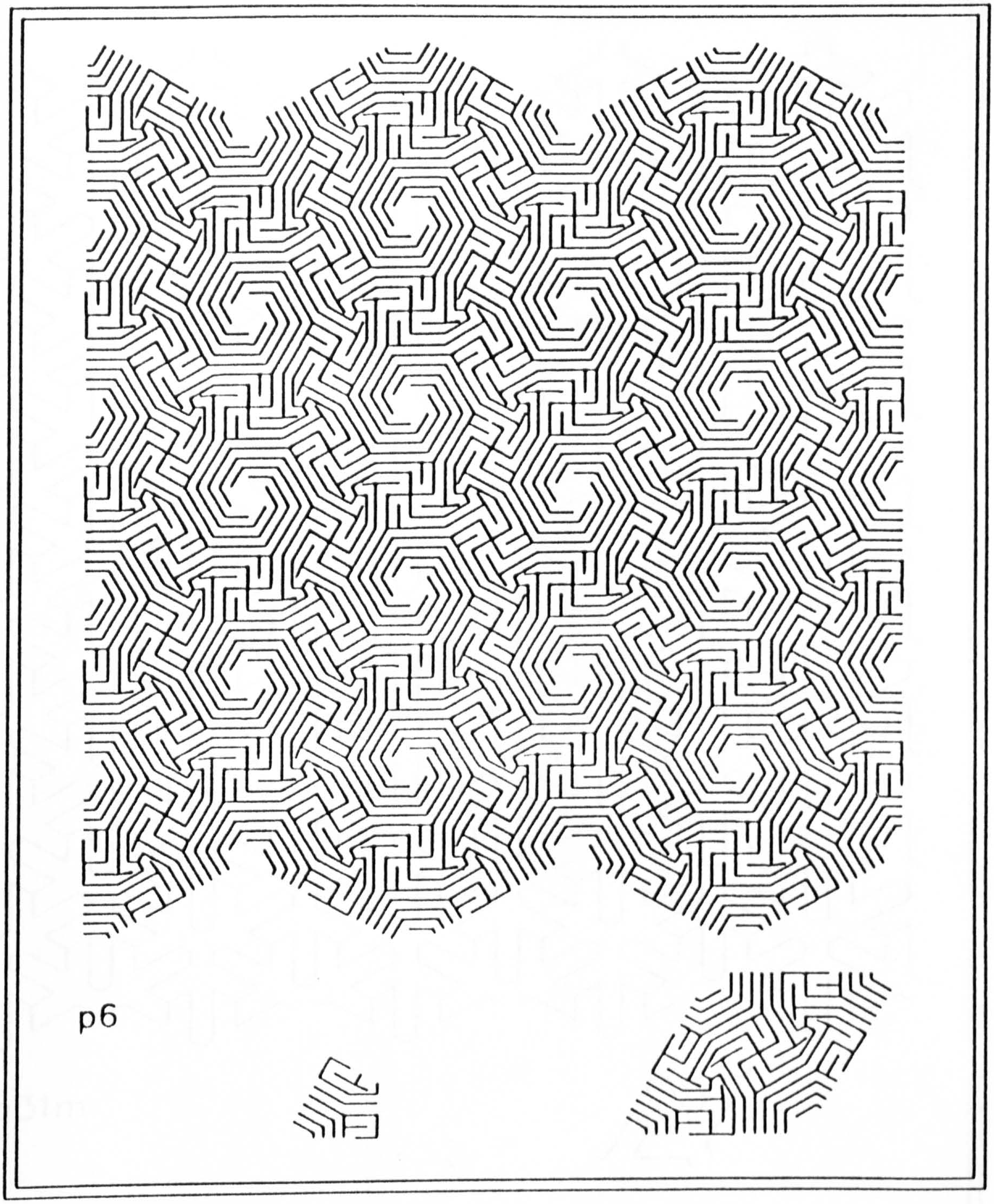


Template Motif Data

PolyLine1: (.5,0),(.439,.026),(.439,.18),(.414,.205),(.329,.118),(.373,0).

PolyLine2: (0,.205),(.061,.181),(.061,.024),(.083,0),(.169,.086),(.12,.205).

PolyLine3: (.024,.057),(0,.082),(.088,.169),(.208,.121),(.264,.146),(.289,.205),(.317,.149),(.471,.149),(.5,.124) (.408,.037),(.29,.086),(.231,.06),(.204,0),(.108,.057),(.024,.057).



Template Motif Data

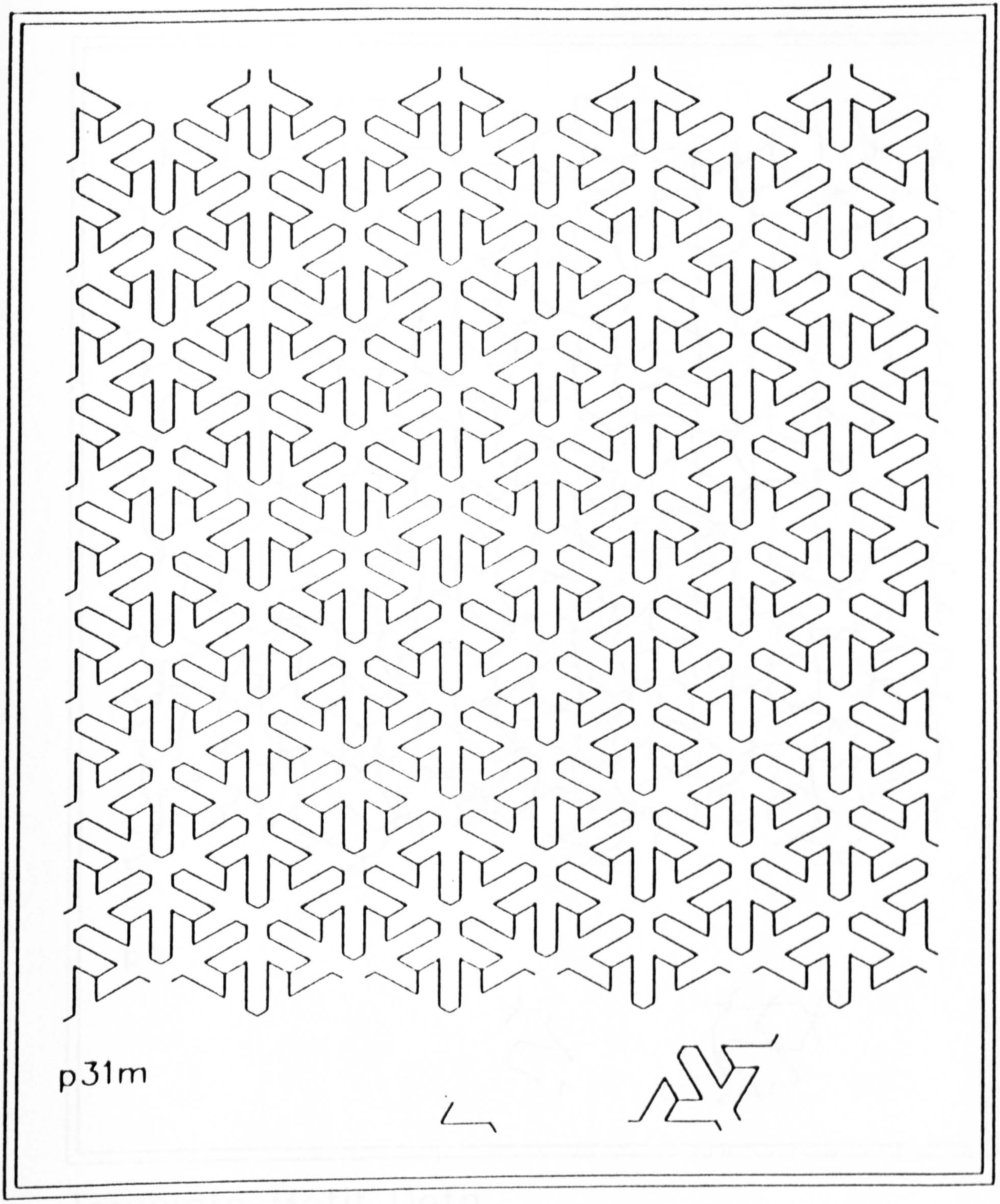
PolyLine1: (.161,0),(.161,.031),(.054,.093). PolyLine2: (.321,.124),(.5,.124). PolyLine3: (.214,0),(.214,.062),(.08,.139).

PolyLine4: (.321,.062),(.446,.062). PolyLine5: (.382,0),(.5,0),(.5,.124). PolyLine6: (.321,0),(.321,.124),(.134,.232).

PolyLine7: (.268,0),(.268,.093),(.107,.186). PolyLine8: (.5,.186),(.321,.186),(.161,.278).

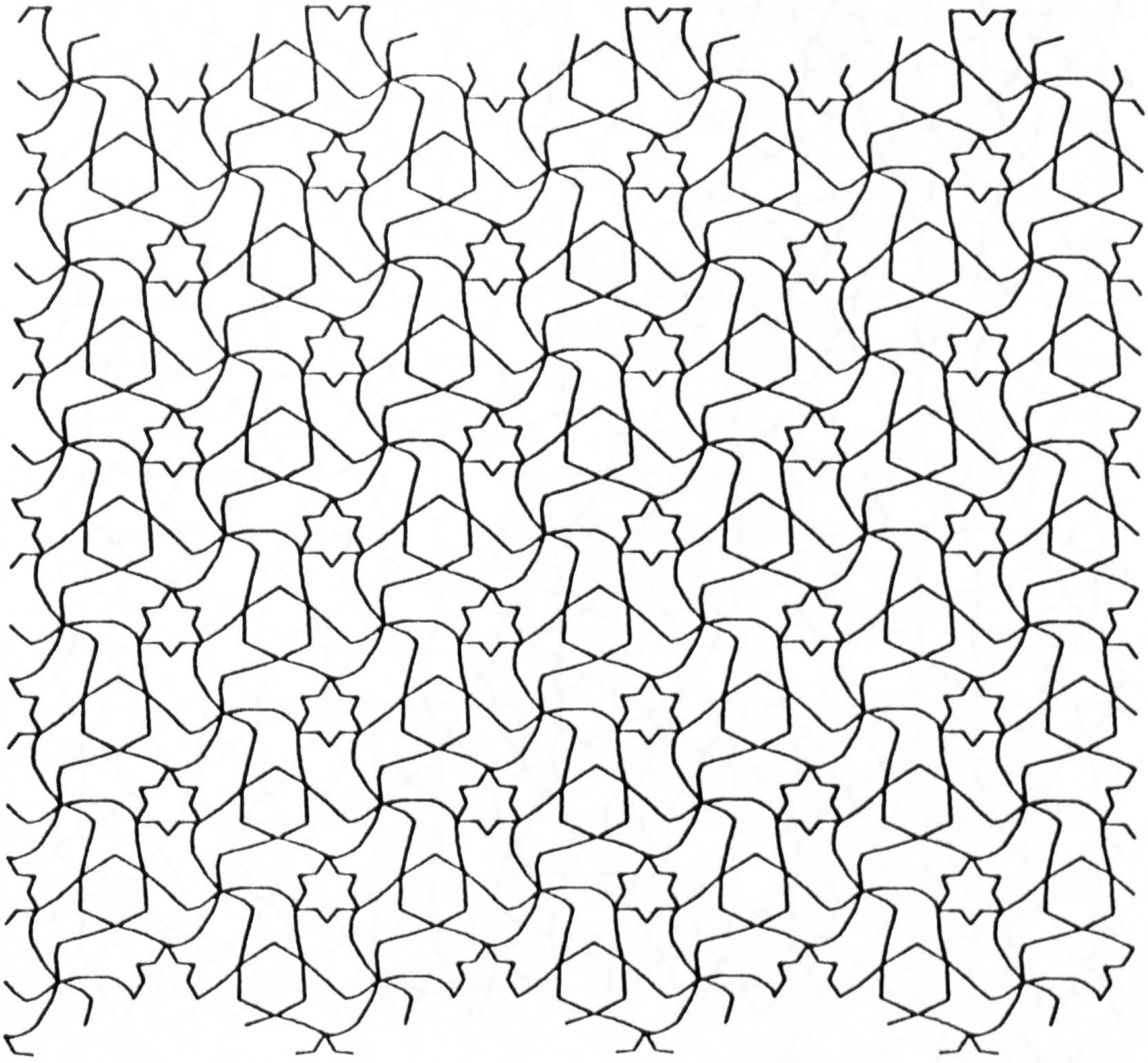
PolyLine9: (.446,.255),(.375,.288),(.268,.348). PolyLine10: (.38,.288),(.411,.34).

PolyLine11: (.321,.186),(.161,.278),(.5,.235),(.348,.235),(.187,.328),(.25,.433).



Template Motif Data

PolyLine1: (.5,.289),(.389,.096),(.833,.096),(.889,0).



p3

Template Motif Data

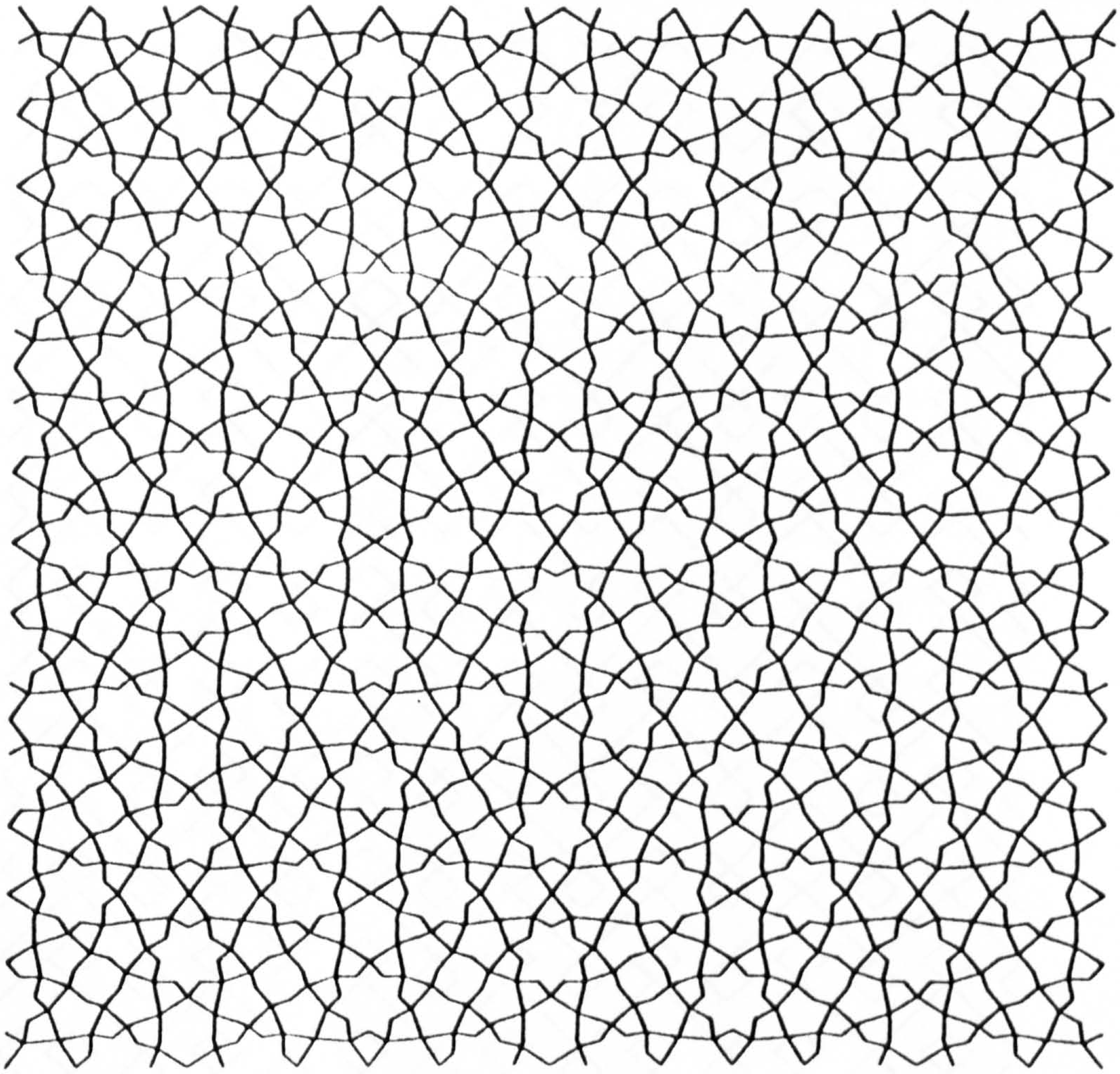
PolyLine1: (.5,.75),(.601,.693),(.601,.808). PolyLine2: (1,.117),(.91,.157),(.899,.058).

PolyLine3: (.5,.289),(.530,.366),(.554,.422),(.601,.469),(.671,.455),(.751,.433),(.915,.401),(1,.315).

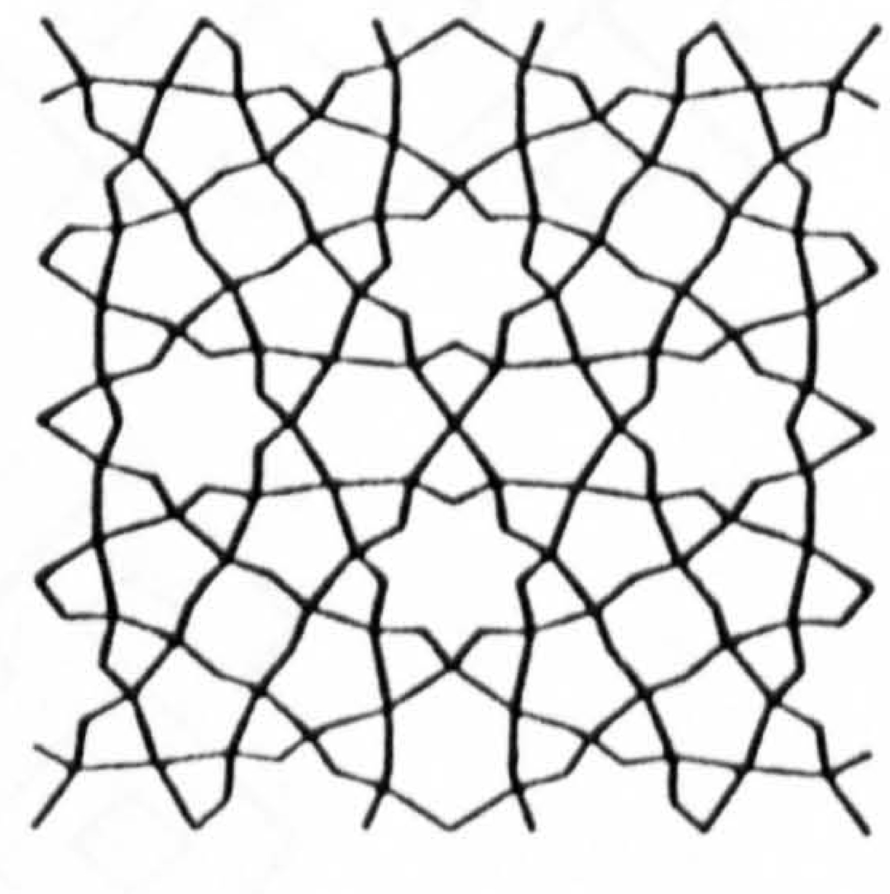
PolyLine4: (.767,.712),(.718,.701),(.601,.693),(.554,.676),(.526,.643),(.5,.601).

PolyLine5: (1,.401),(.915,.401),(.791,.583),(.845,.0667).

PolyLine6: (.5,.289),(.61,.25),(.740,.175),(.771,.135).



p4gm



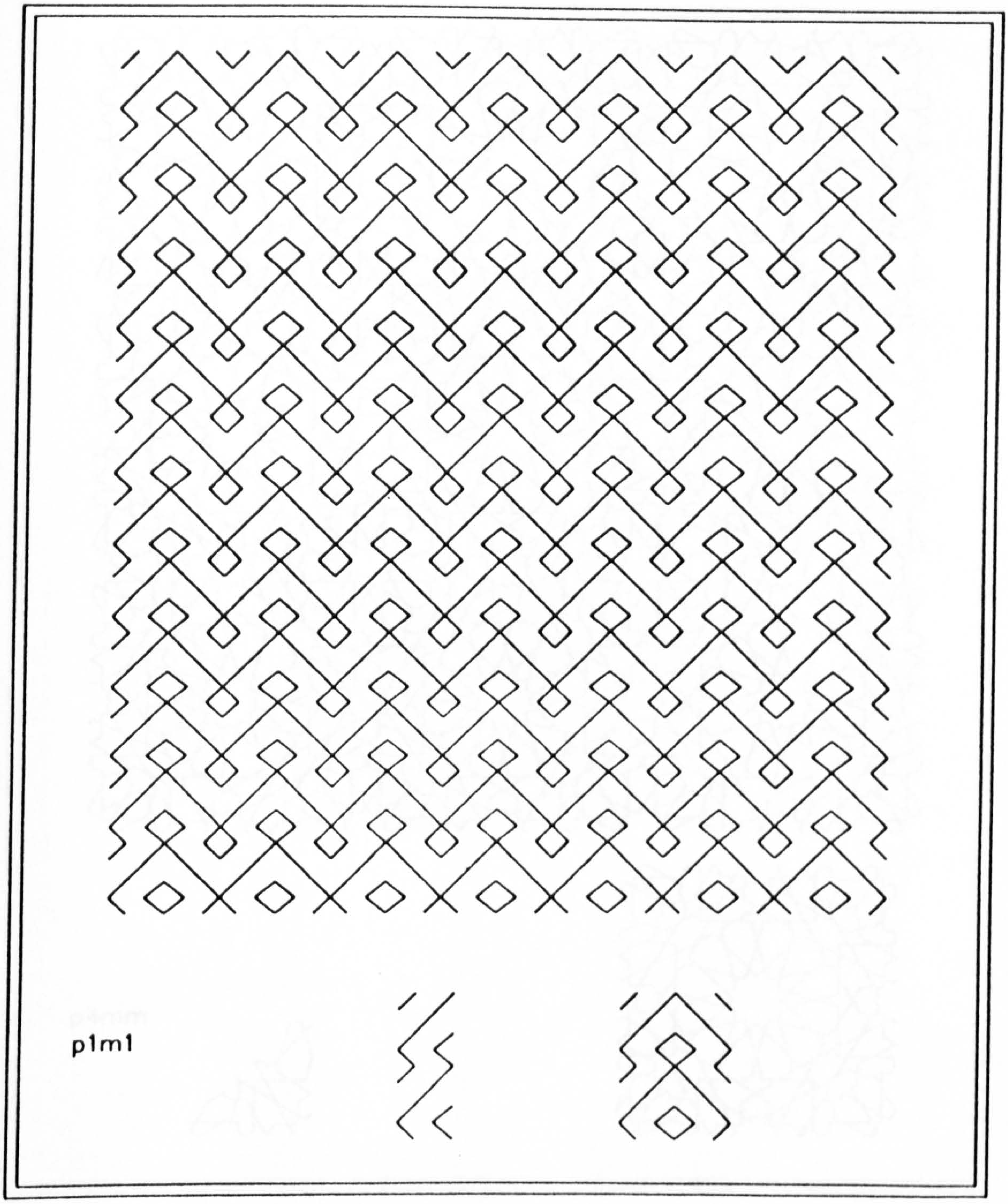
Template Motif Data

PolyLine1: (0,.1),(.052,.072),(.147,.079),(.236,.093),(.295,.083),(.332,.118),(.37,.131).

PolyLine2: (.203,.297),(.187,.275),(.172,.231),(.117,.163),(.081,.131),(.052,.072),(0,0).

PolyLine3: (.073,.428),(.075,.352),(.095,.258),(.080,.198),(.117,.183),(.147,.079),(.196,0),(.234,.036),(.236,.093)
(.27,.171),(.298,.204). PolyLine4: (.4,0),(.421,.053),(.427,.075).

PolyLine5: (.5,0),(.421,.053),(.361,.070),(.332,.118),(.27,.172),(.219,.189),(.172,.231),(.095,.258)
(.036,.257),(0,.302),(.075,.352),(.13,.371).



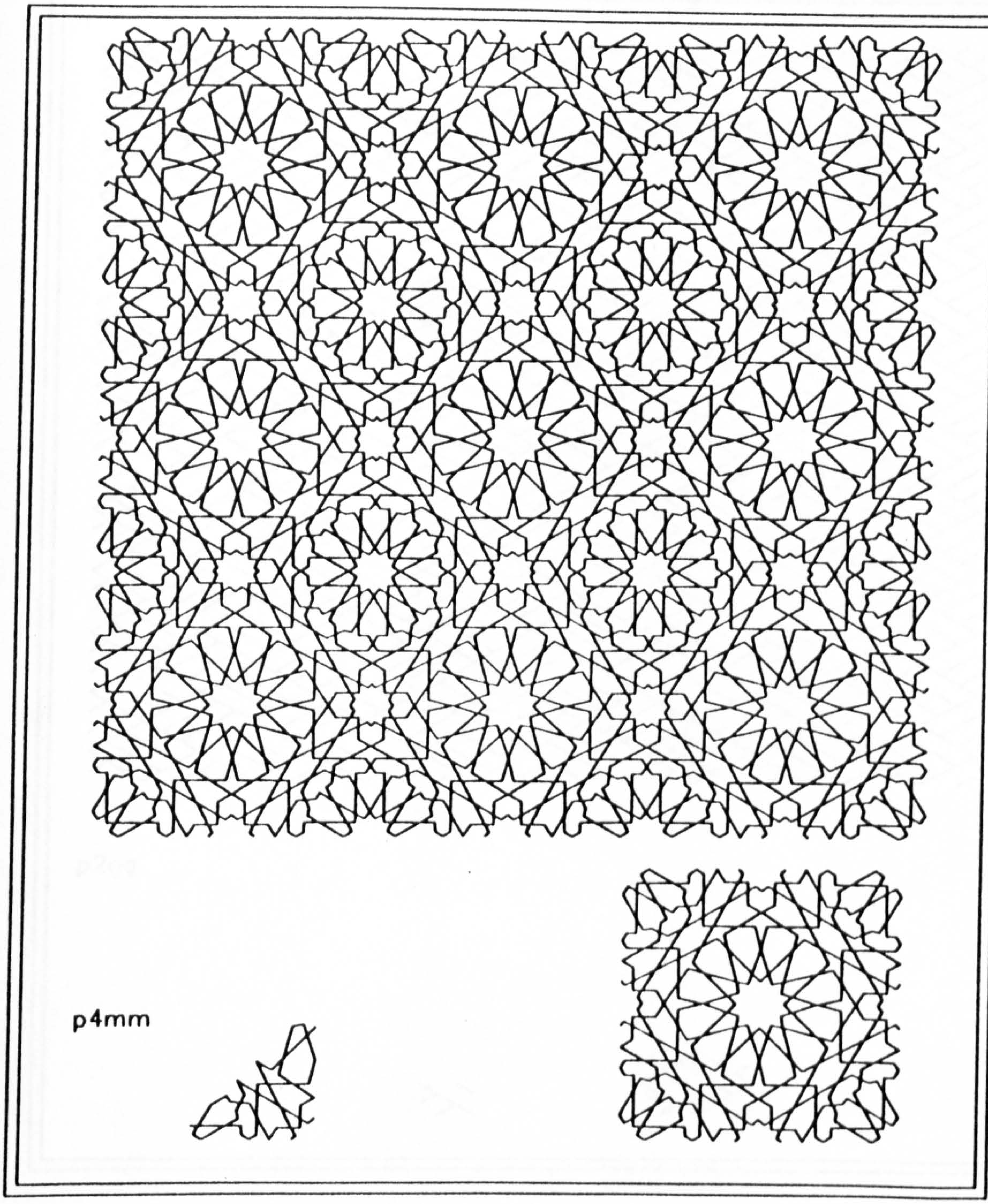
Template Motif Data

PolyLine1: (.15,0),(0,.15),(.5,.666),(.319,.816),(.5,.966).

PolyLine2: (0,.516),(.15,.666),(0,.816),(.5,1.332).

PolyLine3: (.5,0),(.319,.15),(.5,.3).

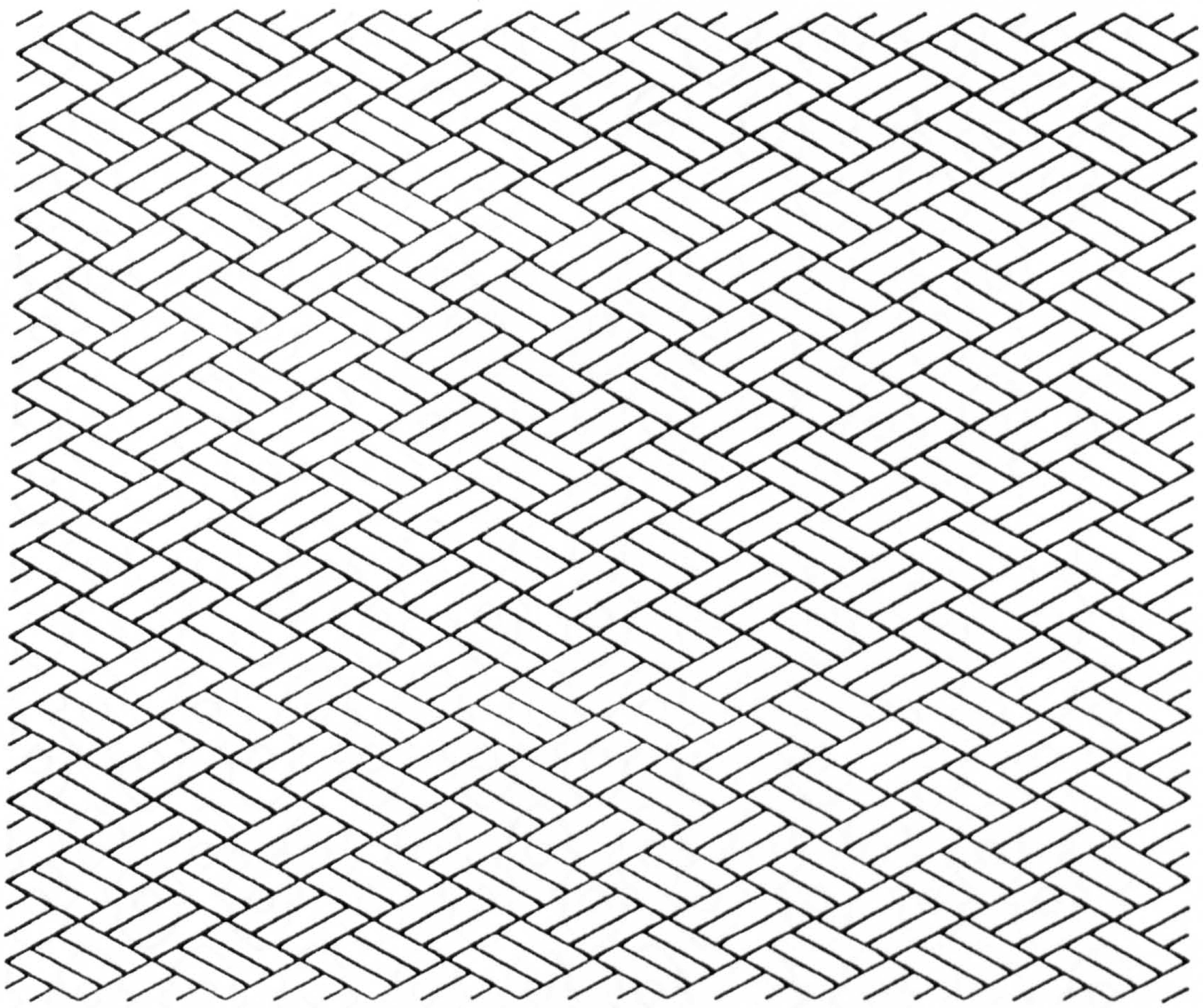
PolyLine4: (0,1.182),(.15,1.332).



Template Motif Data

PolyLine1: (.5,.414),(.344,.256),(.287,.287),(.434,.013),(.417,0).

PolyLine2: (.052,.052),(.238,.052),(.238,.018),(.266,0),(.293,.018),(.293,.207),(.479,.207),(.5,.295),(.457,.424)
(.416,.416),(.368,.24),(.5,.156),(.451,.128),(.451,.045),(.367,.045),(.34,0),(.266,.133),(.197,.139),(.182,.161)
(.129,.129),(.064,.02),(.093,0),(.226,.077),(.219,.105),(.247,.166).

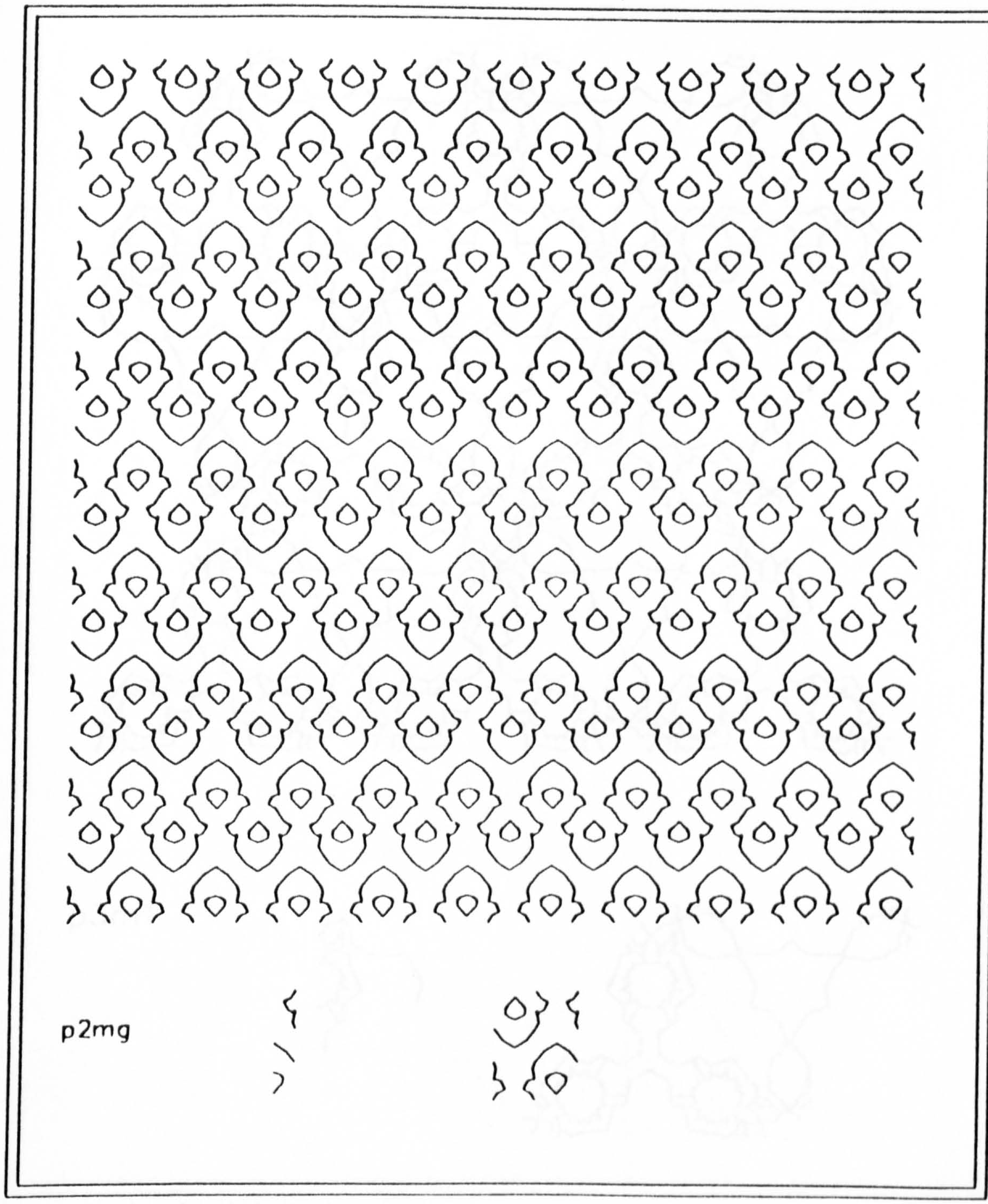


p2gg

Template Motif Data

PolyLine1: (.5,0),(0,.288). PolyLine2: (.5,.191),(.328,.288).

Polyline3: (0,0.99),(.166,.192). Polyline4(.168,0),(.333,.096).

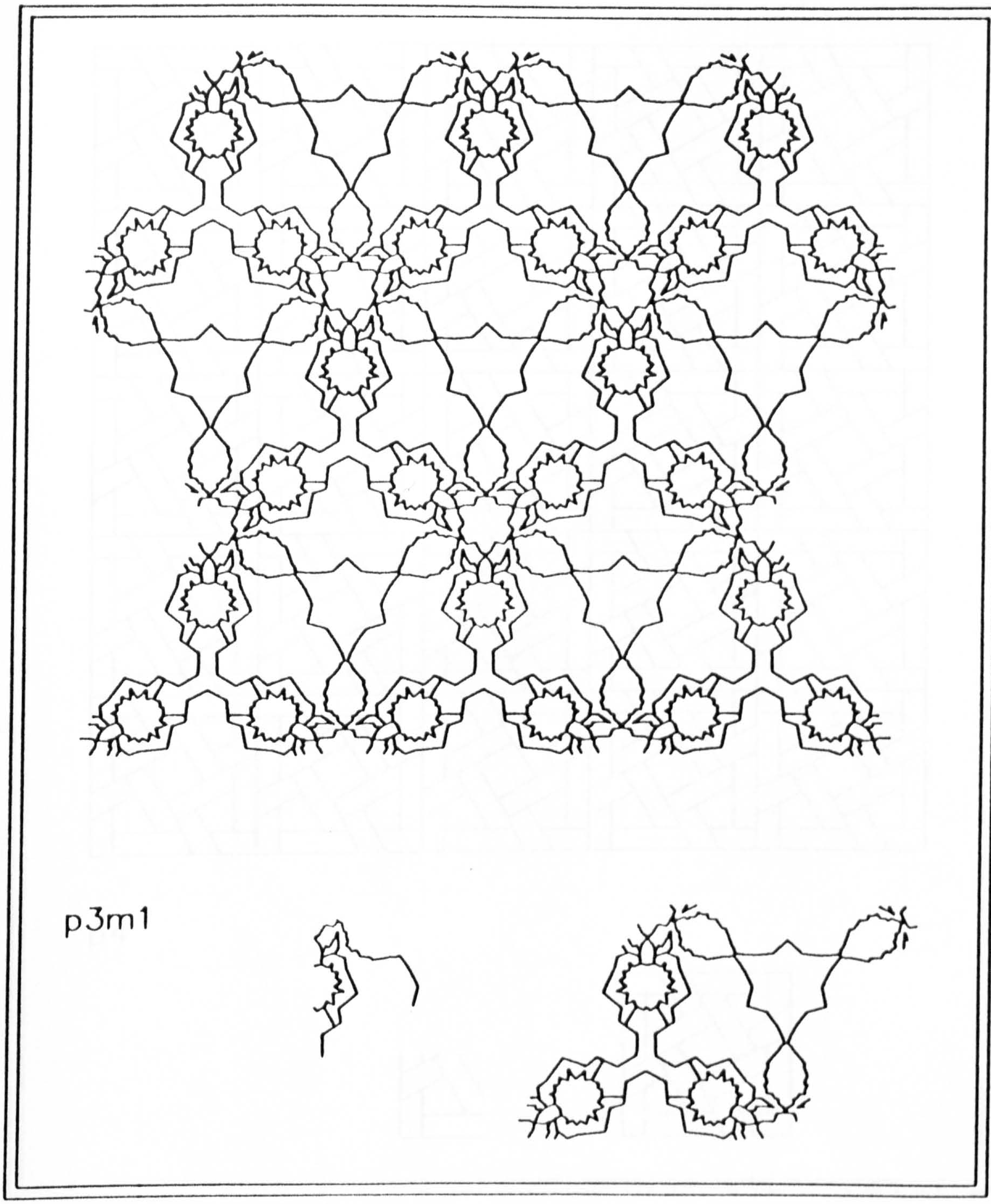


Template Motif Data

PolyLine1: (0,0),(.041,.064),(.108,.1),(.07,.172),(.025,.195),(0,.184),(.025,.289),(0,.348).

PolyLine2: (.19,.478),(.08,.552).

PolyLine3: (.19,.944),(.104,.847),(.099,.79),(.141,.763),(.19,.746).

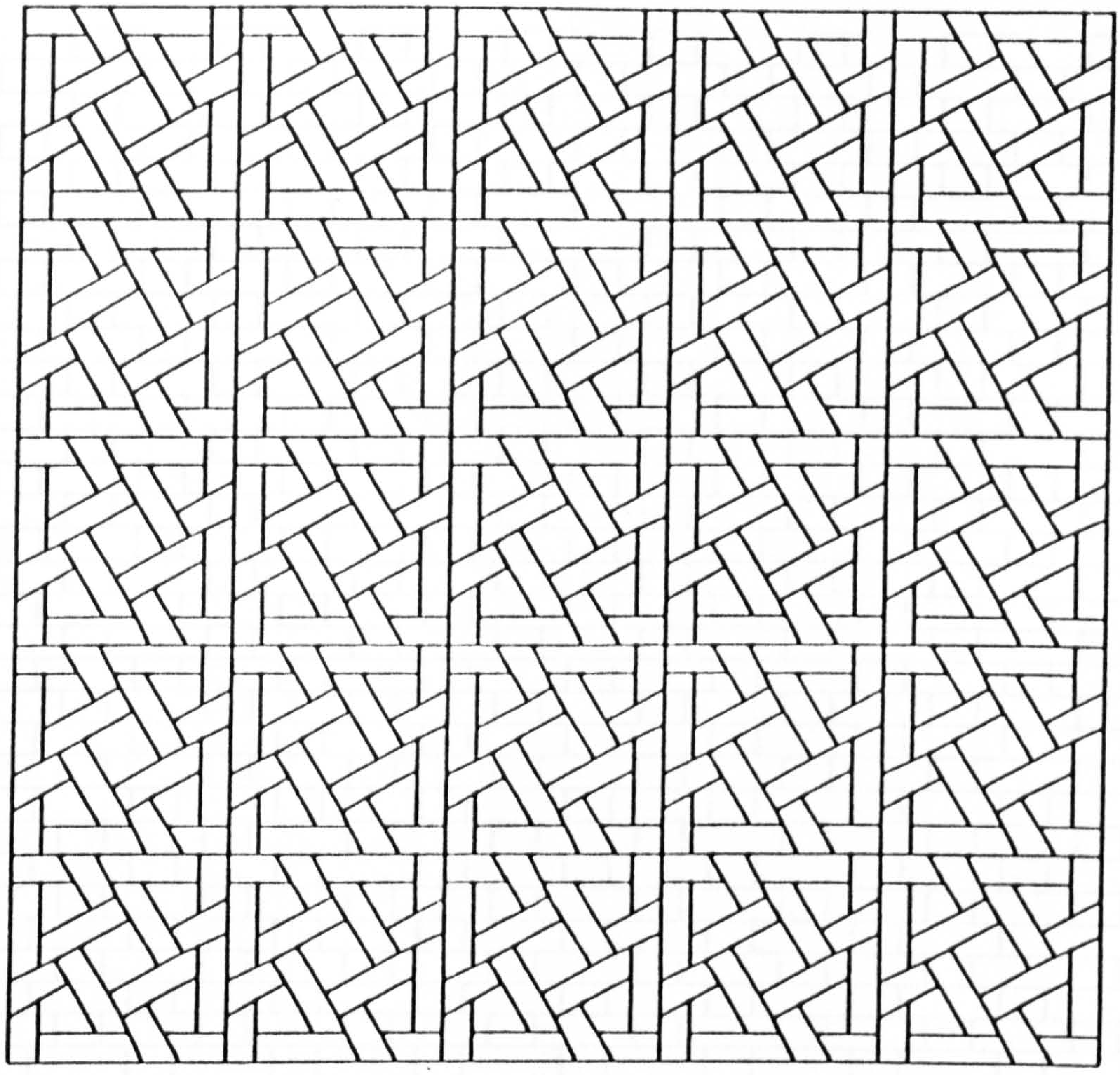


Template Motif Data

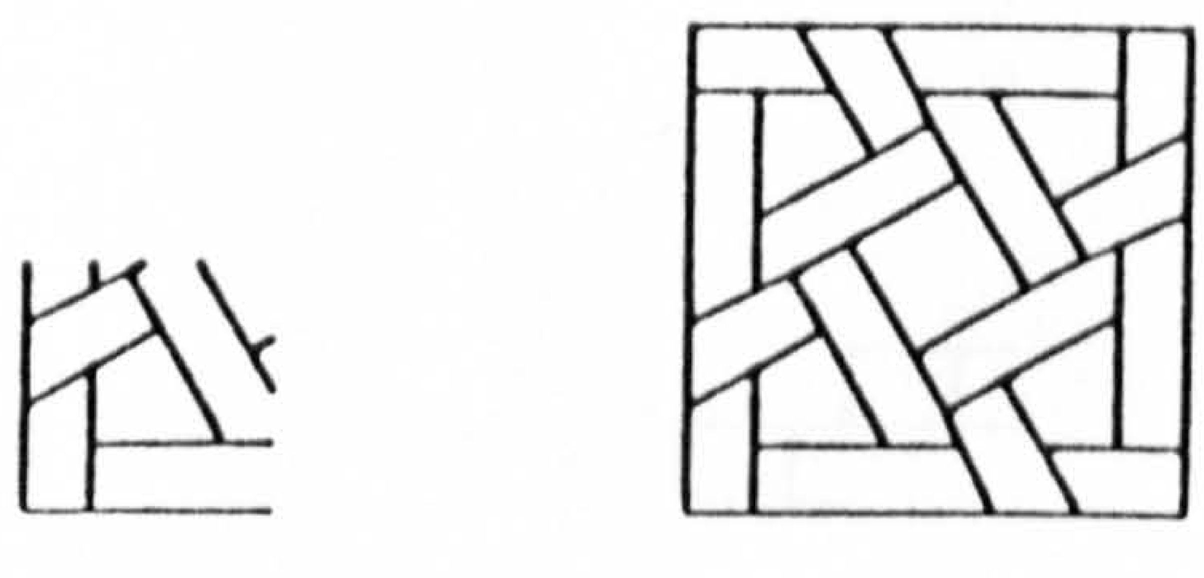
PolyLine1: (.858,.495),(.876,.566),(.848,.625),(.828,.657),(.808,.683),(.792,.677),(.77,.679),(.747,.673),(.71,.666) (.702,.677),(.665,.68),(.626,.698),(.647,.704),(.63,.702),(.62,.738),(.613,.755),(.616,.768),(.602,.773),(.598,.805) (.579,.815),(.567,.79),(.539,.797),(.518,.762),(.5,.749),(.52,.723),(.527,.698),(.526,.67),(.524,.65),(.449,.639) (.564,.653),(.561,.625),(.585,.621),(.577,.601),(.6,.588),(.574,.567),(.607,.55),(.567,.542),(.576,.5),(.547,.514) (.54,.494),(.577,.421),(.532,.389),(.334,.308). PolyLine2: (.54,.498),(.513,.478),(.5,.49).

PolyLine3: (.525,.677),(.589,.718),(.582,.762),(.57,.742),(.552,.715),(.541,.42),(.526,.694).

PolyLine4: (.569,.69),(.608,.675),(.615,.635),(.656,.543),(.589,.486),(.6,.436),(.532,.389).



p4



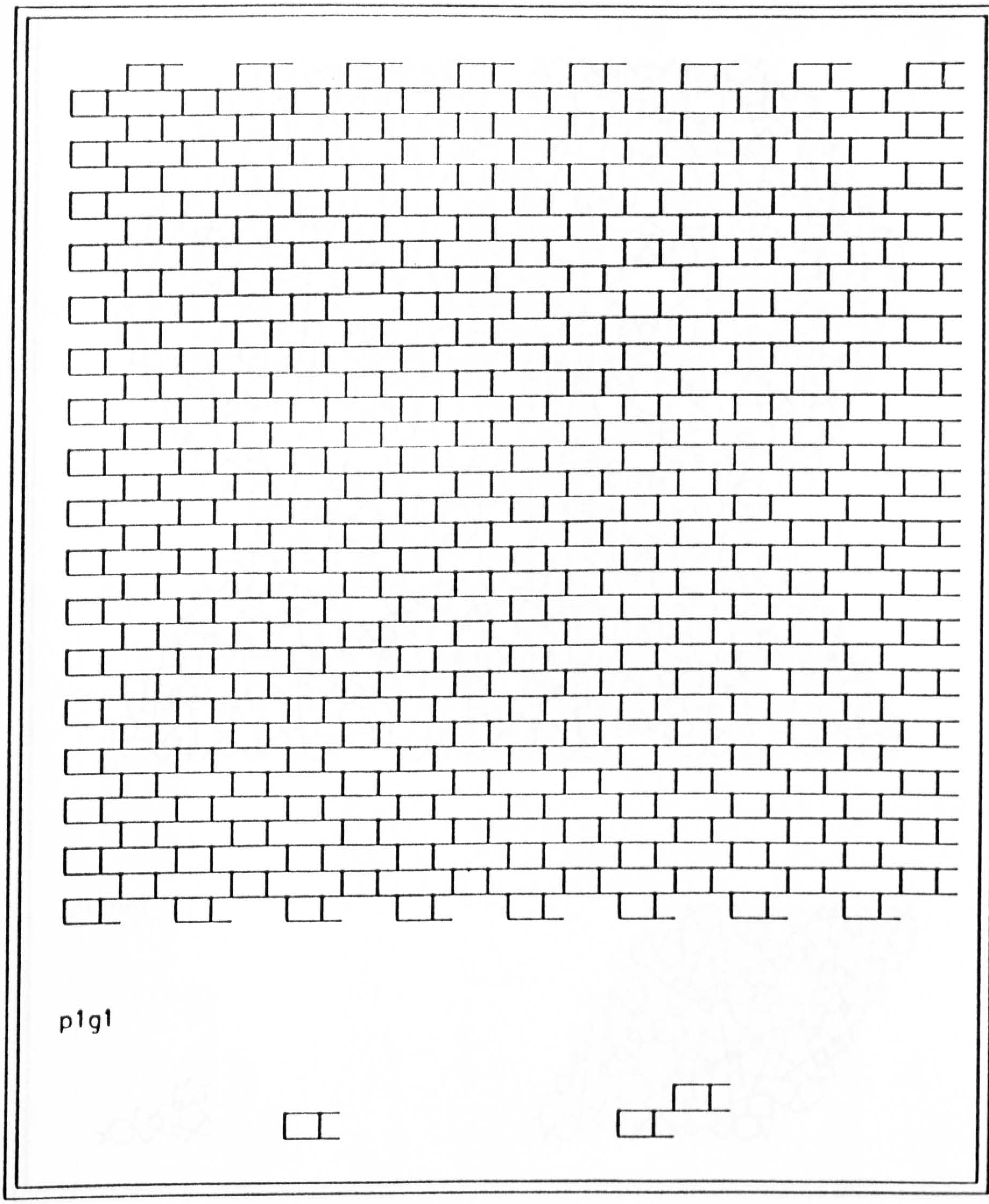
Template Motif Data

PolyLine1: (.452,.319),(.5,.347). PolyLine2: (.347,.5),(.5,.238). PolyLine2: (0,.5),(0,.383).

PolyLine4: (.136,.5),(.136,.451). PolyLine5: (.238,.5),(0,.383),(0,0),(.5,0)

PolyLine6: (.203,.482),(.402,.136). PolyLine7: (.5,.136),(.136,.136),(.136,.288). PolyLine8: (0,.211),(.271,.368).

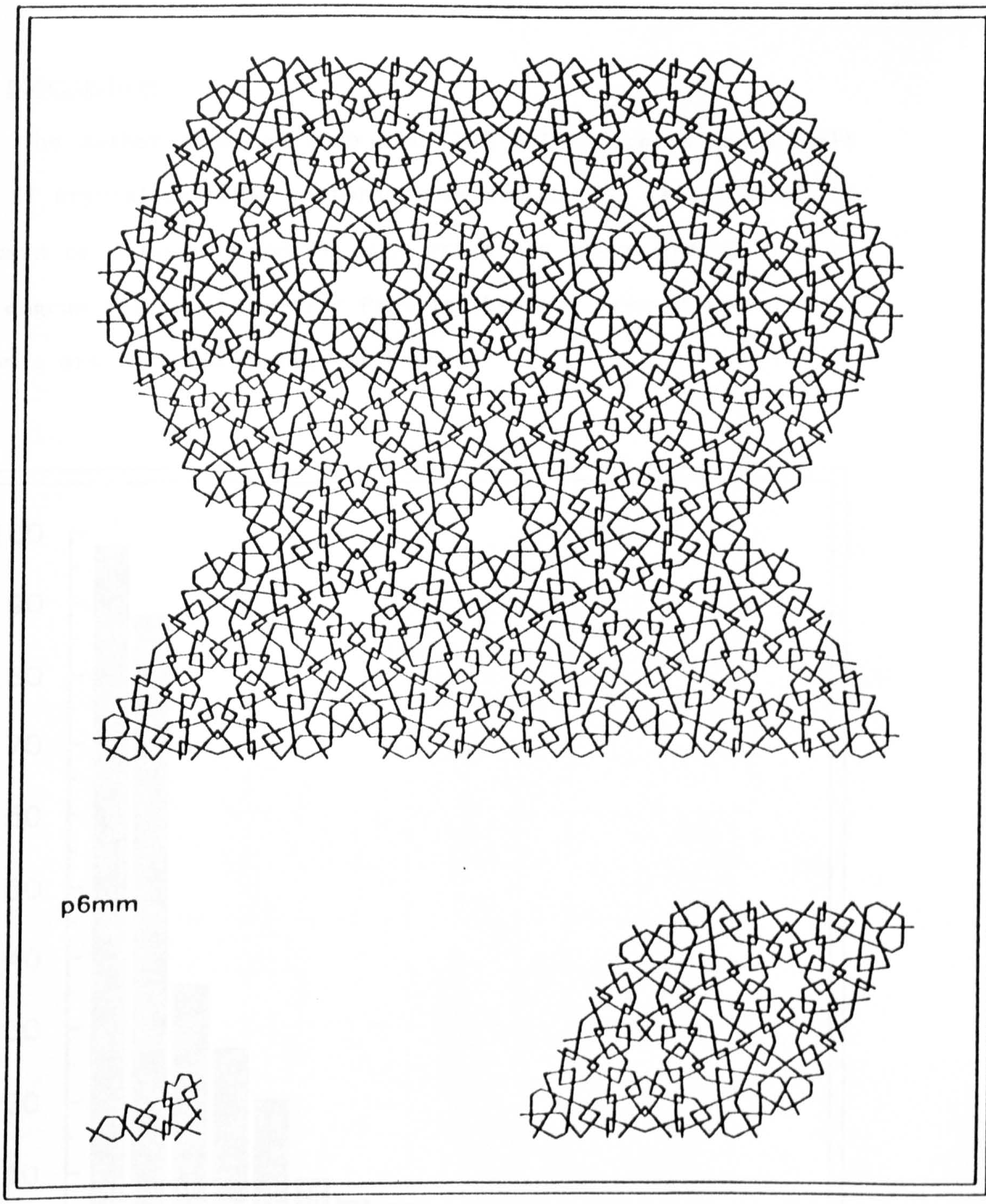
PolyLine9: (.136,0),(.136,.136).



Template Motif Data

PolyLine1: (.5,.234),(0,.234),(0,0),(.5,0).

PolyLine2: (.319,.234),(.319,0).



Template Motif data

PolyLine1: (.408,0),(.5,.114). PolyLine2: (.5,.022),(.388,.072),(.388,0).

PolyLine3: (.5,.076),(.429,.178),(.5,.221).

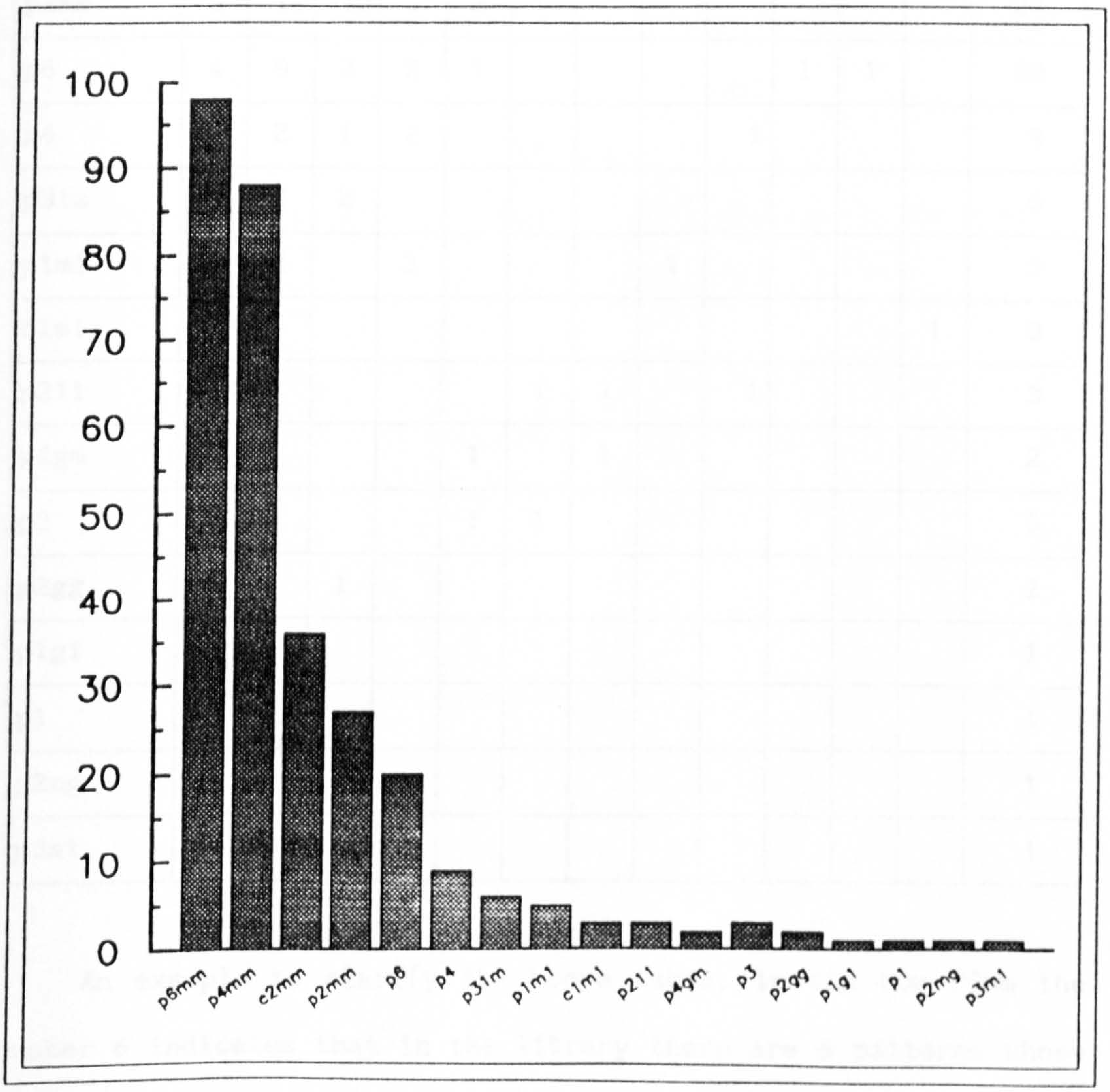
PolyLine4: (.421,.243),(.457,.238),(.485,.159),(.365,.123),(.372,.214),(.404,.198),(.421,.243).

PolyLine5: (.393,.158),(.388,.02),(.282,.089),(.322,.142),(.393,.158).

PolyLine6: (.1,0),(.158,.092),(.199,.092),(.269,0),(.339,.088),(.232,.138),(.218,.019),(.181,0),(.087,.052).

4.3 <u>CONCLUSIONS</u>

The author examined more than 300 Islamic patterns and all the 17 crystallographic patterns were found. The distribution of numbers of pattern found in each group are shown graphically in the diagram below. We see that P6mm is the most favored symmetry in Islamic art followed closely by P4mm.



The table below gives data for the number of polylines that occur in template motifs and also the number of patterns found in each group.

Crystall graphic group	Number of templates contains polylines												Total in group
	1	2	3	4	5	6	7	8	9	10	11	12	
p6mm	48	30	11	6	1	1	1						98
p4mm	20	37	17	10		2	2						88
c2mm	12	7	8		3	3	1	1	1				36
p2mm	3	11	10	1	2								27
p6	4	9	2	2	1					1	1		20
p4	3	2	1	2					1				9
p31m	2	2	2										6
p1m1		2		2				1					5
c1m1	2											1	3
p211						1	1		1				3
p4gm					1		1						2
p3		1			1	1							3
p2gg	1		1										2
p1g1		1											1
p1	1												1
p2mg			1										1
p3m1				1									1

An example to clarify the above table, in the row P6mm the number 6 indicates that in the library there are 6 patterns whose data is made up of 4 polylines.

4.4 LIBRARY OF ISLAMIC PATTERNS

This library named the Islamic Data Libraray (IDL) contains the template data for more than 300 patterns which were extracted as described above. The data is kept in the directory c:\IDL in files whose names have the following structure

SYMG.NUM

When **SYMG** is made up of up to 4 characters which specify the symmetry group of the pattern and **NUM** comprises up to 2 digits which stand for the pattern number. For example

P4MM.26

stands for the pattern number 26 with symmetry group **P4MM**.

5- THE PROGRAM

ISLAMIC GEOMETRIC PATTERNS

5.1 OVERVIEW

The analysis of Islamic patterns described in the last chapter led to a library of template motif data for more than 300 patterns. From this data the patterns can be created efficiently with this minimum information and using the algorithms developed in chapter three.

In this chapter we describe an interactive program ISLAMIC GEOMETRICAL PATTERNS (IGP), which was written to utilize our data. Although the program was written primarily to generate the Islamic geometrical design studied by us, it is in fact a general purpose program capable of generating the full set of plane crystallographic patterns from template motif data given in a file or created interactively. The program also allows for interactive modification of designs produced from library data.

The program was written in Turbo Pascal language and makes use of the Pluto II Graphic system. This system manufactured by Electronic Graphic LTD uses an Intel 8088 processor dedicated to graphics alone. The display frame buffer in its highest resolution mode has 768 X 576 pixels over 256 colors at any one time from a palette of over 16 million colors. The Pluto system in which the program was implemented was driven by a Viglen II IBM AT compatible machine with a VGA graphics card.

Of course, it would have been much preferable to have written the program so that it was machine independent. However, the graphics standard GKS was not available to the author and indeed is still not available on micro with Pascal binding at the time of writing this thesis. The hardware and software utilized by us was the best that was available to us and was the reason for our choice.

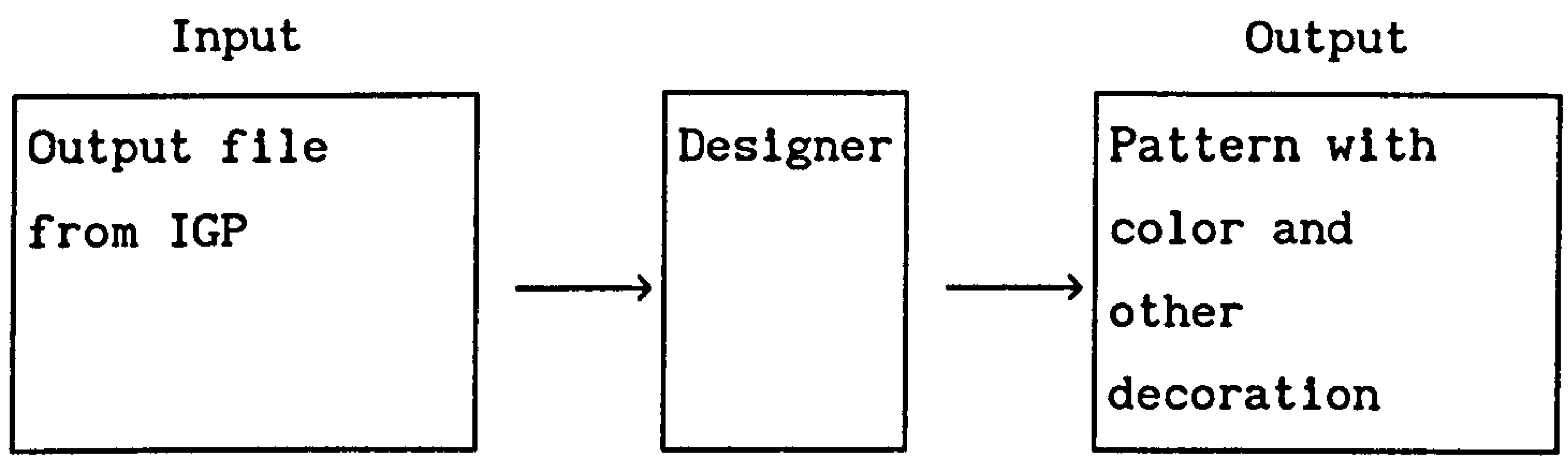
In this chapter we shall first give the overall structure of the program. Next we shall describe the method of its execution and will give examples of output produced by the program. The program listing and the numerical data are attached in a floppy disk but we shall give a brief description of the UNITS, TYPES, PROCEDURES and FUNCTIONS.

5.2 PROGRAM STRUCTURE

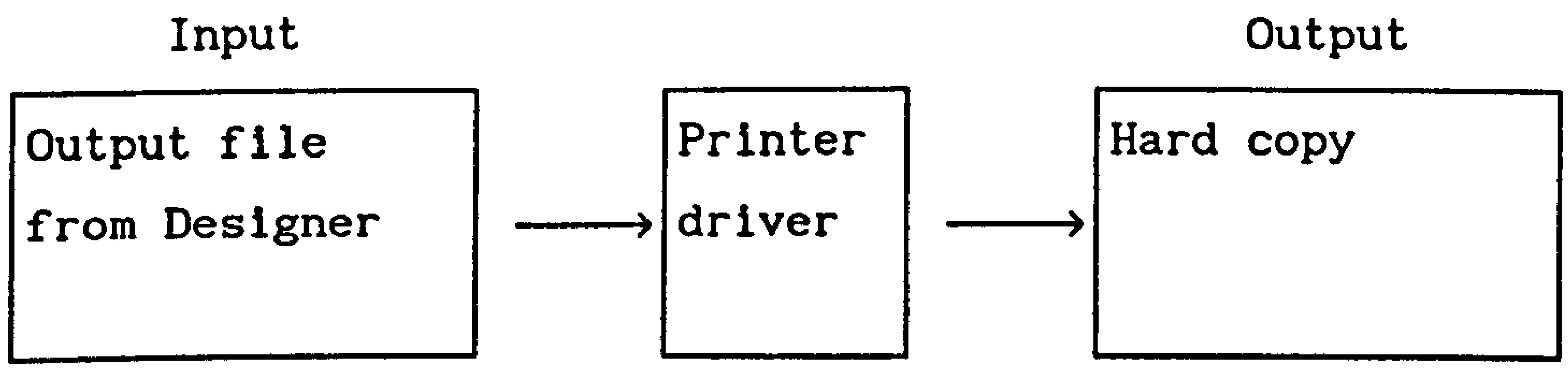
The general structure of the program is shown below.

Input				Output
Symmetry group; Net data; N. of repeats; Template motif data from library or produced interctively;	⟶	IGP	⟶	Islamic library pattern or user designs crystallographic pattern;

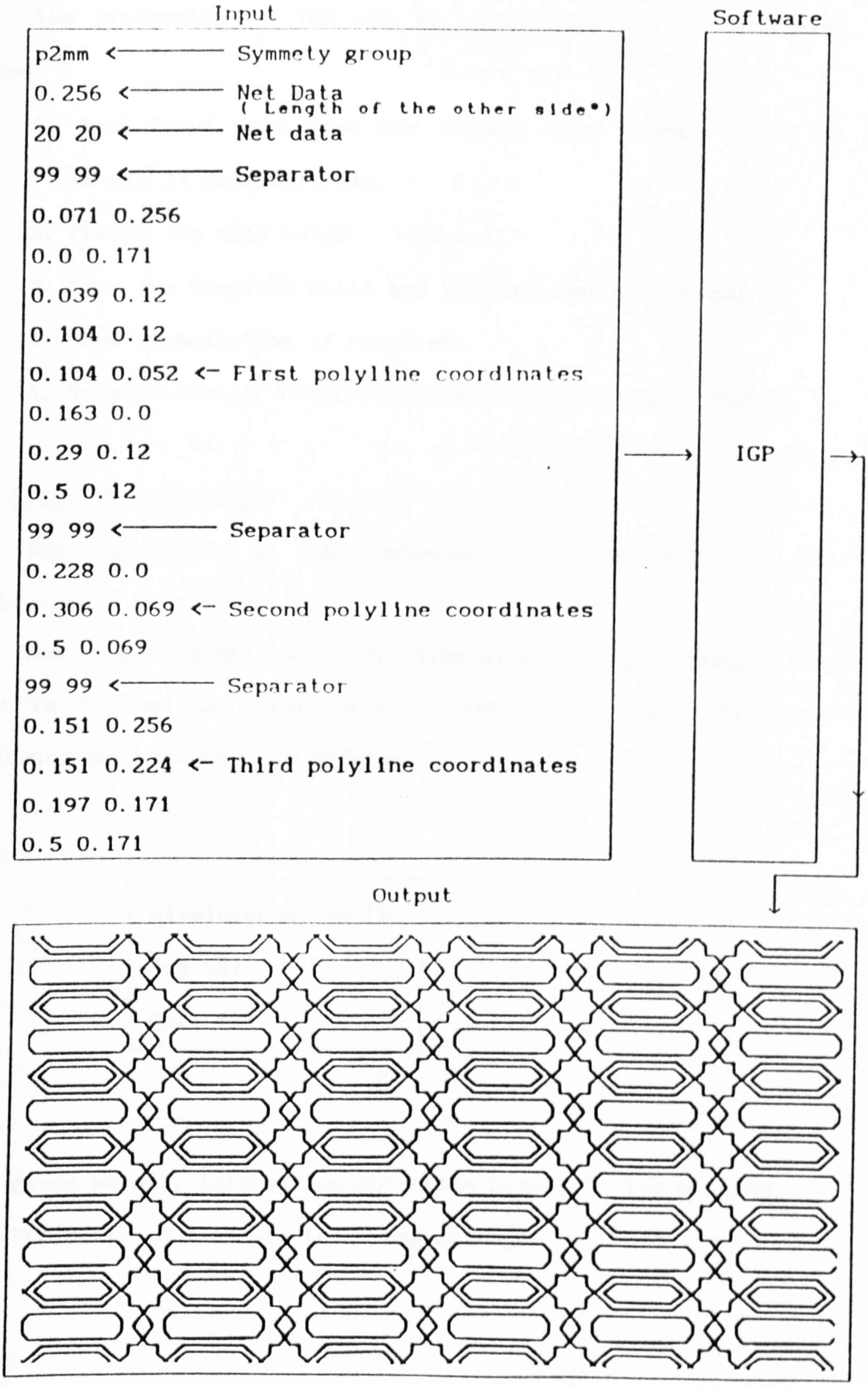
The output for IGP can be fed to the Designer Package produced by the same company Electronics Graphics Ltd, which produce Pluto. This package allows for extensive interactive facilities for coloring and other modifications.



The output from the Designer can than be utilized to print hard copy on a color printer using a suitable driver. The color output included in this thesis was produced on a Digital Laser Jet 250.



Fig(1) shows a typical example of input data required by IGP and the corresponding output.



fig(1)

* The side of the unitmotif in X direction is always taken as unity

The processing in IGP can be summarized in the following steps.

1. **Read input data from the islamic data library (IDL) or create it interactively.**
2. **Create the unit motif**
3. **Show the template motif and the unit motif if required.**
4. **Show tessellation if required.**
5. **Save pattern if required for decoration or hard copy.**

5.3 STEPS IN EXECUTION

The execution of the program is summarized in the following.

The program starts with the display of a logo message, the user is invited to press carriage return to start. Next the following menu choices are offered.

GiveInformation (I)
Run (R)
Quit (Q)

I, gives helpful information about the nature of the program.

The choice R leads to following message.

1- View an Islamic library pattern?

2- View your own pattern?

3- Create or modify pattern?

Choose 1, 2 or 3 :

The choice 1 allows the user to view the patterns kept in IDL. If desired the user can decorate the pattern and produce a hard copy (explained in help information). The choice 2 allows the same as choice 1 on a pattern which is not part of IDL. The choice 3 allows for the creation of a new pattern or the modification of a pattern which may be from IDL.

If the choice 1 is made then the execution proceeds in the following way.

A: Pick group.

B: Pick pattern number.

C: Show template motif and unit motif (as in fig(2)).

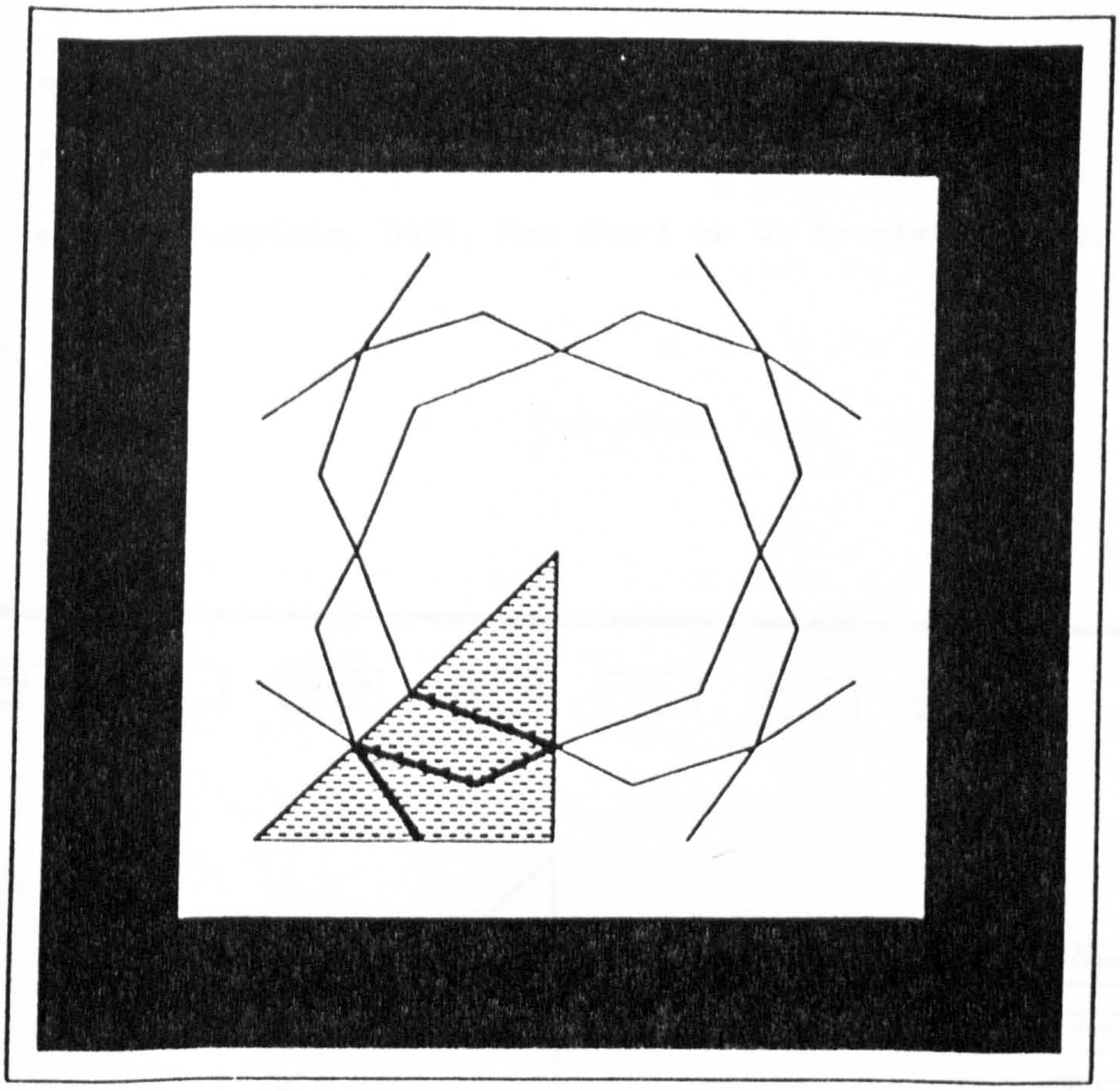
D: Show pattern.

E: Quit, New start or Go to step C.

F: Run DESIGNER to decorate (See help information for steps F, G and H).

G: Save.

H: Produce hard copy.



fig(2)

If the choice 2 is made then the execution proceeds as in the choice 1 but it will ask the user to 'Give file name' instead of A and B.

If the choice 3 is made then the execution proceeds in the following way

a: Pick group.

b: Enter extra data if required (A menu appears as shown in fig(3)).

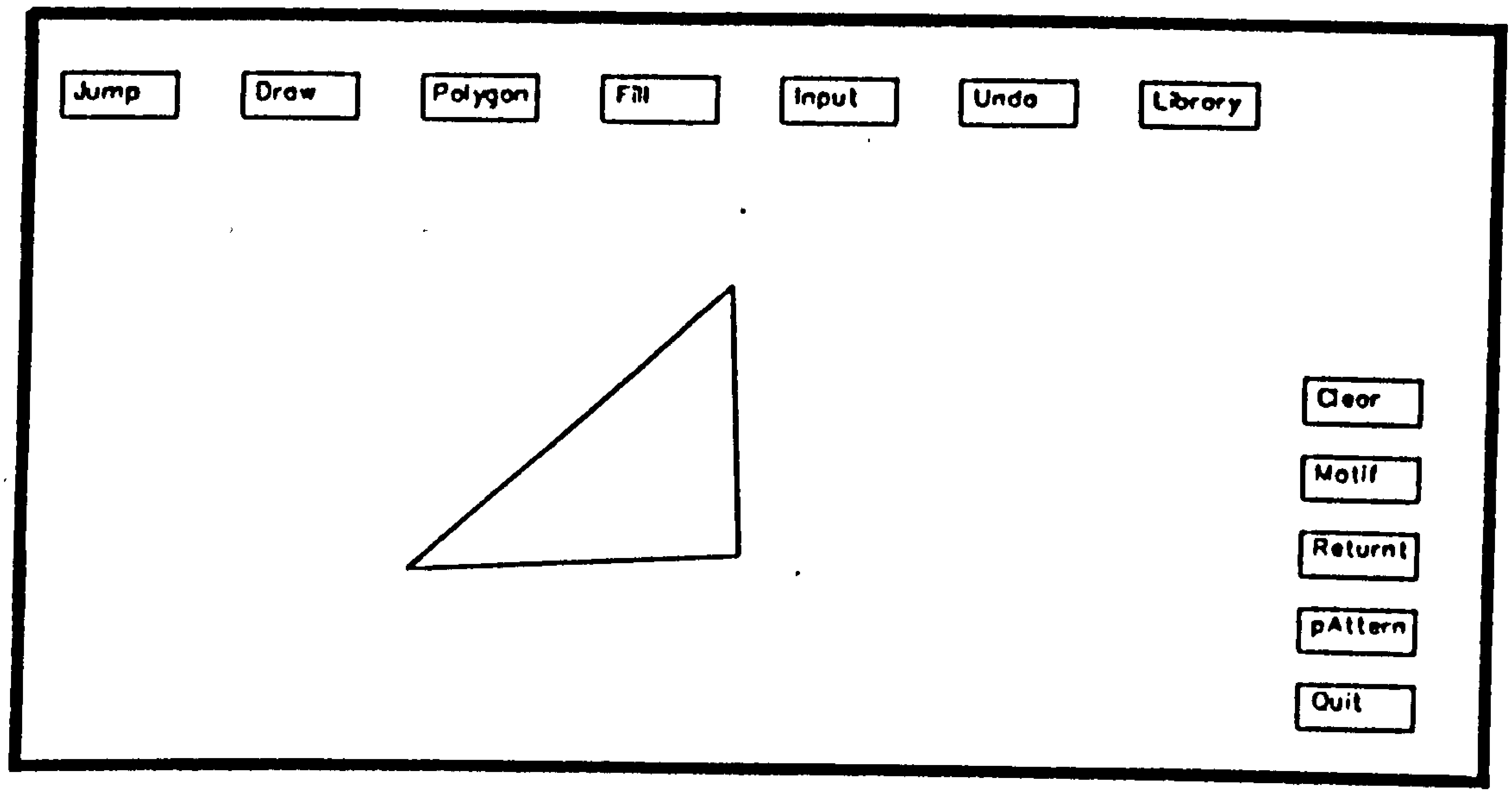
c: Construct template motif.

d: construct unit motif and show the template & unit motif.

e: Enter No. repeats in X and Y.

f: Display pattern.

g: Save template, Quit, New start or Go to step e or d.



fig(3)

5.4. INTERACTION

Fig(3) shows the complete structure of the main menu used by IGP. It occupies the top and right side of the graphics monitor while the drawing stage is active. This menu provides for interactive construction and modification of the template motif. The user can access the menu items in two ways:

1) Keyboard:

To select an item, simply press the key corresponding to the letter shown in capitals in the item name, or, move the cursor onto the item using the arrow keys. The item is highlighted and the user can than press ENTER.

2) Mouse:

A cursor is moved using the mouse in the usual way. Simply press the right button to move the highlighted bar one step to the right on the menu items, or, move the cursor onto the item to activate it. Again, the item is highlighted and the user can then press ENTER.

5.5. MENU ITEMS

The menu remains active on the graphic screen until the user selects the pAttern or the Quit option. Below is an explanation of each of the items.

5.5.1 JUMP

Selecting Jump allows the user to reposition the current coordinates of the cursor on the screen. Reference markers, (crosses) are drawn at these points to show the points picked in a construction. These markers are automatically deleted as soon as they are no longer needed.

5.5.2 DRAW

Selecting Draw from the menu allows the user to add lines to the template motif from the previous cursor position to it's current position.

5.5.3 POLYGON

The user selects this item when he wants to draw a line from the current position of the cursor to close a set of lines to form a polygon.

5.5.4 FILL

To fill a polygon with color the user should select this item then press ENTER. He is then asked to select the vertices forming the polygon and press ENTER when finished. Next, a color menu appears on Pluto screen and the message ' Choose color - Press [CR] to accept, ESC to abort'. As the cursor in moved across the color menu the polygon is filled with the corresponding color. Finally, the user selects the color required from this menu by position the cursor at a color and pressing ENTER. To fill another polygon the user should follow the same procedure.

5.5.5 INPUT

This option is utilized when the user wants to type in the coordinates, rather than generate them using the mouse. The question appears

(J)ump or (D)raw?

The user selects J if he requires to move to a new position without drawing and selects D if he requires to draw a line between the previous position and the new position. He is then asked to enter the coordinates.

This mode of input is repeated by continuing to press ENTER and is completed when the user selects a new item from the menu.

5.5.6 UNDO

Selecting Undo allows the user to remove lines from the template motif. The lines are removed in reverse order to which they were added so that the latest addition is the one which is removed at every use of Undo.

5.5.7 LIBRARY

The library item allows the user to call up a template motif. This template motif can be one of the Islamic library (IDL) or a template motif which has been created by the user at some earlier time and stored in a save file. On selecting this option, the following question will appear

Enter file name?

Simply type the file name of the save file. The saved template will then appear in the template motif region on the screen (See section 4.4 for information on (IDL)).

5.5.8 CLEAR

This option clears the screen and initializes the indices of the array which draws the coordinates of the template motif. The user can then start to draw a new template motif.

5.5.9 MOTIF

Motif is used when the drawing of the template is finished and it is required to see the unit motif.

5.5.10 RETURNT

ReturnT option allows the user to go back to the template motif and to modify it.

5.5.11 PATTERN

This option allows the user to view the complete periodic pattern.

5.5.12 QUIT

Used to quit IGP.

5.6 SUMMARY OF UNITS, TYPES, PROCEDURES AND FUNCTIONS

Having described the structure and the method execution of IGP, we shall now introduce the reader to the units, procedures and functions used in the program. Also, we list the type declarations utilized. The listing is arranged alphabetically.

5.6.1 DICTIONARY OF UNITS

It is appropriate to mention here that the units Doc, Crt and Graph of Turbo Pascal version IV have been utilized.

AidCmotf :

Purpose : provides

(1) Isometry transformation procedures.

(2) Procedures needed to load, save and draw

patterns.

Used In : CrUnMotf, AidIGPPr, IGP.

AidIgpPr :

Purpose : (1) Set up initial value and menu data.

(2) Display menu to create unit motif on the Pluto screen.

(3) Call up pattern from IDL.

Used In : IGP.

AidPlInt :

Purpose : (1) Plot menu item names.

(2) Shows the cursor coordinates while it is moved on the screen.

Used In : DisCurso, DisMenus.

CrUnMotf :

Purpose : Uses the data of the template motif and the isometry transformations of a specific group to generate the data of the unit motif.

Used In : AidIGPPr, IGP.

DataStru :

Purpose : Set up the main linked list data structure of the program.

Used In : DisPolyg, CrUnMotf, AidCmotf, AidIGPPr, IGP.

DevCurso :

Purpose : Handles input from the mouse and the keyboard.

Used In : DisCurso.

DisCurso :

Purpose : Used to

(1) Define the area where the items are on the screen and plot the current coordinates of the cursor on the Pluto screen.

(2) Create and control the movement of the cursor using the mouse and the keyboard.

Used In : DisPolyg, AidIGPPr.

DisImage :

Purpose : Saves a part of the image temporarily in the memory to provide part of the screen for the display of the color menu. Replaces the image back on the screen when the user removes the color menu, freeing the memory.

Used In : LcolorTa.

DisMenus :

Purpose : Defines the menu items and puts them on the Pluto screen, Highlights an item to confirm selection.

Used In : AidIGPPr.

DisPolyg :

Purpose : Provides the following facilities

(1) Draws the required polygon.

(2) Fills Polygon.

Used In : AidCmotf, AidIGPpr.

FiMotif :

Purpose : Reads the boundary data of the template motif and set up transformations used to create the unit motif data from the template motif data.

Used In : CrUnMotf, AidIGPPr, IGP.

GrafIntf :

Purpose : Used for plotting the Logo message, Help information and menus on the PC screen.

Used In : IGP.

LcolorTa :

Purpose : (1) Load color menu file from the hard disk.

(2) Display color menu on the Pluto screen.

Used In : AidIGPPr, IGP.

PlutIntf :

Purpose : Contains the Graphics Interface for Pluto II.

Used In : DisPolyg, AidPlInt, RealGraph, DisCurso, LcolorTa, DataStru, DisMenus, AidIGPPr, AidCmotf, FiMotif, DisImage, IGP.

RealGraph :

Purpose : Provides the procedures to

(1) Set up upto 50 windows and mappings to these windows.

(2) Make a particular window active.

(3) Draw lines, clear the window and draw a border in the active window .

Used In : DisPolyg, DisCurso, AidCmotf, AidIGPPr, IGP.

5.6.2 DICTIONARY OF TYPES

In this section, we list the type declarations which are utilized in the program IGP. The dictionary is provided to make the code of the program more easily comprehensible to a reader. On the left is given the type identifier and on the right is given the name of the unit where the identifier is declared. This is followed by the syntax, some helpful remakes where necessary and the names of any other units where the type is used.

Action DataStru

Syntax Action =(Jump, Draw, Polygon, Fill);

Purpose These are some of the item of the menu.

Used In AidIGPPr.

Boundray FiMotif

Syntax Boundray =Array [1..4,Xcoord..Ycoord] of Real;

Purpose This is used to define the vertices of the template motif boundary.

Used In IGP, AidIGPPr.

Cell FiMotif

```
Syntax      Cell =Record
             Group              : String[10];
             NumSubCells        : 0..MaxSubCells;
             GeneratorRegion    : Region;


             SubCells             : Array [1..MaxSubCells] of
             Subcell  CellFile = FILE of Cell;
             End;
```

Purpose Used to store the unit motif information.

Used In AidIGPPr, CrUnMotf.

CellPart FiMotif

```
Syntax      CellPart =Record
             Source                : 0..MaxSubCells;
             CASE  Move           : Transformation Of
                     Identity    : ();
                     Rotation    : ( AroundX, AroundY,
                                          Angle : Real);
                     Reflection : (X0, Y0, X1, Y1 : Real);
                     Shift       : (Dx, Dy : Real);
                     Scale       : (Aboutx,  Abouty,
                                      Sx, Sy  : Real);
```

Greflection: (Gx0, Gy0, Gx1, Gx2, Gdx,

Gdy :Real);

End;

Purpose Generates the unit motif from the information on a specific transformation and a specific template.

Coords FiMotif

Syntax coords =(Xcoord, Ycoord, Zcoord);

Purpose Coordinate of the picture points. Zcoord is not used but it is included for future work.

ColourPlane PlutIntf

Syntax ColourPlane =(Blue, Green, Red);

Used In LcolorTa.

Device DisCurso

Syntax Device =(KeyBoard, Mouse);

Purpose Devices used as an input.

FigureColourMap DataStru

Syntax	FigureColourMap =Array [0..255] of Integer;
Purpose	Maximum number of colors is 256.
Used In	AidCmotf.

FileName PlutoIntf

Syntax	String[125];
Purpose	Text.
Used In	AidCmtf, AidIGPPr, Fimotif, IGP.

InputData DisCurso

Syntax

```
InputData = Record
 PlutoX,
 PlutoY,
 Event      : Integer;
 Case From  : Device Of
 keyboard   : (Code              : Integer);
 Mouse      : (Left  ,  Right    : Boolean);
 End;
```

Purpose	Store the position of the cursor in Pluto screen.
Used In	DisPolyg, AidIGPPr.

LookUpTable PlutIntf

Syntax LookUpTable =File Of Lut;

Purpose Used to access the look up table.

Used In LcolorTa.

Lut PlutIntf

Syntax Lut =Array [0..255] of LutEntry;

Purpose Used to access the look up table.

Used In LcolorTa.

LutEntry PlutIntf

Syntax LutEntry =Array [ColourPlane] of Integer;

Purpose Used to access the look up table.

Used In LcolorTa.

Menu DisMenus

Syntax Menu =Record

```
Heading        : string[20];
EventNo,Items  : Integer;
Fore,Back      : Integer;
```

```
Item          : Array[1..10] of MenuItem;
Keys          : Set Of Char;
Current       : Integer;
End;
```

Purpose Defines the main menu.

Used In AidIGPPr.

MenuItem DisMenus

Syntax

```
MenuItem =Record
PosX,PosY : integer;
Name      : String[10];
EventNo   : Integer;
Keys      : Set OF Char;
End;
```

Purpose Used to define each item of the menu for which we need the position of the item on the screen, the name of the item, the number and the key associated with each item.

OverLap RealGraph

Syntax Set of Side;

Remark Determines the clipping region.

where Side = (Left,Right,Bottom,Top);

Pointer DataStru

```
Syntax      Pointer =^Point;
                Point =Record
                   X, Y         : real;
                   Prior, Next  : Pointer;
                   Move         : Action;
                   polyline     : Integer;
                   colour       : Integer;
                 End;
              FigureFile = File of point;
```

Purpose   The main data structure of the program.

Used In   AidIGPPr, AidCmotf, DisPolyg, CrUnMotf, IGP.

Raster DisImages

```
Syntax      Raster  =^Block;
                 Block =Record
                    B     : ^Integer;
                    size  : Word;
                    Nexit : Raster;
                 End;
```

Purpose   Used to get, save, put and free the image.

Used In   LcolorTa.

Side	RealGraph

Syntax	Side =(Left, Right, Bottom, Top);
Purpose	Used in defining the clipping region.

Transformation	FiMotif

Syntax	Transformation =(Identity, Rotation, Reflection, Shift, Scale, Greflection);
Purpose	Isometry transformation for a specific template to generate the unit motif.

5.6.3 DICTIONARY OF PROCEDURES AND FUNCTIONS

Below is an alphabetical list of all the procedures and functions used in IGP.

PROCEDURE ActiveWindow	RealGraph

Syntax	ActiveWindow(Active : INTEGER);
Purpose	Used to make a specific window.

PROCEDURE Addto AidIgpPr

Syntax Addto(Move : action; XPo,yPo : REAL);

Purpose Add new option and coordinate to the list of the data figure.

PROCEDURE Box GrafIntf

Syntax Box(Tx,Ty,Bx,By :INTEGER);

Purpose To draw box.

PROCEDURE ChiExitFile IGP

Syntax ChiExitFile(VAR ChoiceFile :FileName);

Purpose Searches the file enterd by the user, if there is no such file the user is asked to enter the file name again.

PROCEDURE ClearWindow; RealGraph

Syntax Clearwindow;

Purpose Saves the current color and finds the background color. Fills the window with the background color and sets the color back to the current color.

PROCEDURE ChangeCursor DisCurso

Syntax ChangeCursor(Symno : INTEGER);

Purpose Erases present cursor and replaces it with the new cursor SymNo. If new cursor can't be drawn on the screen then the call is ignored.

PROCEDURE ClipPoint RealGraph

Syntax

```
ClipPoint(VAR  xs,ys,xf,yf  :  REAL;
               VAR  Edges        : Overlap);
```

Purpose Pushes the point (Xf,Yf) into the window to produce a new (Xf,Yf) if necessary.

PROCEDURE ClipTest   RealGraph

Syntax   ClipTest(VAR Wx,Wy : REAL; VAR Outside : Overlap);

Purpose   Decides if a point in world coordinates is inside or outside the window. If it is outside then determines the side on which it lies.

FUNCTION CopyFigure   AidCmotf

Syntax   CopyFigure(VAR Figure : POINTER) : POINTER;

Purpose   Copies the data for a figure to apply an isometry transformation.

PROCEDURE CreateFigure   CrUnMotf

Syntax   CreateFigure(VAR Figure : POINTER; VAR C : Cell);

Purpose   Used to generate the unit motif.

PROCEDURE Crosscursor   DisCurso

Syntax   Crosscursor(Col : INTEGER);

Purpose   Sets the cursor to '+' i.e., Ascii character 43 of the Pluto default symbol partition 255.

PROCEDURE CrossCoordinate AidIgpPr

Syntax CrossCoordinate(VAR CrossX,CrossY : REAL);

Purpose Shows reference markers on the Pluto screen.

PROCEDURE CursorColour DisCurso

Syntax CursorColour(Col : INTEGER);

Purpose Sets new color.

PROCEDURE CursorInquire DisCurso

Syntax CursorInquire(VAR curInfo : curarray);

Purpose Returns current cursors information.

PROCEDURE CursorStep DisCurso

Syntax CursorStep(Inc : INTEGER);

Purpose Used to control the steps of the cursor movement on the screen.

PROCEDURE DefineEventArea DisCurso

Syntax	DefineEventArea(N,dx0,dx1,dy0,dy1 : INTEGER; State : BOOLEAN);
Purpose	Defines an area of the screen to be the event number N and draws a rectangle around it.

PROCEDURE DelCrossCoordinate AidIgpPr

Syntax	DelCrossCoordinate(VAR CrossX,CrossY : REAL);
Purpose	Deletes reference markers from the Pluto screen.

PROCEDURE DeleteFigure AidCmotf

Syntax	DeleteFigure(VAR Figure : POINTER);
Purpose	Frees the memory occupied by a figure.

PROCEDURE DeviceInput DisCurso

Syntax	DeviceInput(VAR Data : InputData);
Purpose	Moves the cursor around the screen until a key or mouse bottom is pressed or an event occurs.

PROCEDURE DrawCursor; DisCurso

Syntax DrawCursor;

Purpose Draws the cursor at current position . Used to reactivate cursor after a call to ERASE-CURSOR.

PROCEDURE DrawFigure AidCmotf

Syntax DrawFigure(VAR Figure : POINTER;Sx,Sy : REAL; VAR Map : FigureColourMap);

Purpose Put the figure on Pluto screen after shifting by (Sx,Sy).

PROCEDURE DrawFramPattern IGP

Syntax DrawFramPattern(Fram1,Fram2 :INTEGER);

Purpose Sets and draws a frame around the pattern.

PROCEDURE DrawTo RealGraph

Syntax DrawTo(Wx,Wy : REAL);

Purpose Draws a line from the current pen position to the point (Wx,Wy) in world coordinates, clipping if necessary.

PROCEDURE Draw2DPattern IGP

Syntax Draw2DPattern(NXRepeatation, NYrepeatation : INTEGER; Mesh : char);

Purpose Fills the Pluto screen with pattern.

PROCEDURE EndCursors DisCurso

Syntax EndDisCursors;

Purpose Releases the partition used by the cursor so that it can be used again.

PROCEDURE EraseCursor DisCurso

Syntax EraseCursor;

Purpose The code is the same as DrawCursor, It removes the cursor from the screen.

PROCEDURE EraseF Fimltif

Syntax EraseF;

Purpose Used to remove files from hard disk.

PROCEDURE Event DisCurso

Syntax Event(VAR EventNumber, PosX, PosY : INTEGER;

VAR Occurence : BOOLEAN);

Remark Checks whether cursor is in any of the event boxes.

PROCEDURE FiMaltif FiMltif

Syntax FiMoltif(VAR SetFile: FileName; SideX, SideY :REAL);

Purpose To get the unit motif information from a file.

PROCEDURE FreeImage DisImages

Syntax FreeImage(VAR Start : Raster);

Purpose Deletes the stored image from the memory after showing it on the Pluto screen.

PROCEDURE FristSelectOption IGP

Syntax FristSelectOption(VAR Cha :CHAR);

Purpose Used in getting the option 'I', 'R' or 'Q' at the start of the program.

PROCEDURE FristTypeMenu GrafIntf

Syntax FristTypeMenu;

Purpose Puts the first menu on the PC screen which involves Give Information, Run and Quit.

PROCEDURE GeNePatt AidIgpPr

Syntax GeNePatt(VAR SetDat :FileName; VAR POINTER;
Data :POINTER);

Purpose Used to create a new pattern.

PROCEDURE GeOlPatt AidIgpPr

Syntax GeOlPatt(VAR SetDat :FileName)};

Purpose Use to show pattern from library.

PROCEDURE GetColour LcolorTa

Syntax GetColour(ColourNumber: INTEGER;
VAR R,B,G : INTEGER);

Purpose Return the number associated with a color.

PROCEDURE GetFillLibrary AidIgpPr

Syntax GetFillLibrary(VAR FillLibrary :FileName; VAR CrystGroup : STRING; VAR XDim,YDim :INTEGER);

Purpose Used in calling a library pattern.

PROCEDURE GetImage DisImages

Syntax GetImage(x,y,width,height : INTEGER;
VAR Start : Raster);

Purpose Saves the image to memory when calling the color menu.

PROCEDURE GetKey DevCurso

Syntax GetKey(VAR Pressed :BOOLEAN; VAR Code :INTEGER);

Purpose Used to link the keyboard with the cursor.

PROCEDURE GetMeshPattern IGP

Syntax GetMeshPattern(VAR Xsteps,Ysteps,
InXstep,InYstep :INTEGER);

Purpose Gets the repetition in X and Y and calls PrintMeshPattern to echo on the PC screen.

PROCEDURE GetMouse DevCurso

Syntax GetMouse(VAR X,Y :INTEGER;

VAR Left, Center, Right :BOOLEAN);

Remark Return mouse position and button status.

PROCEDURE GetPolygon DisPolyg

Syntax Getpolygon(VAR figure,first,last : POINTER);

Purpose Draws a polyline. Data is any REAL valued structure which holds successive (x,y) values, these are the vertices of the polyline (see also FILL). To draw a polygon set the last point equal to the first. The current position is unaltered.

PROCEDURE GetTempletFil Fimltif

Syntax GetTempletFil(VAR TheFile :FileName;

GetSx,GetSy :REAL);

Purpose Used to load unit motif information from file.

PROCEDURE GiveINformation GrafIntf

Syntax GiveINformation;

Purpose Gives help information about the program.

PROCEDURE GraphWindow RealGraph

Syntax GraphWindow(M : INTEGER;x0,x1,y0,y1 : REAL);

Purpose Defines an area of the graphics screen to be a window. [X0,X1] and [Y0,Y1] lie in the range [0,1], the origin being taken to be the top left hand corner. The Y axis points downwards. Thus X0=0, X1=0.5, Y0=0 Y1=0.5 and M=5 will define window number 5 to be the top left hand quarter of the device screen.

FUNCTION GReflect AidCmotf

Syntax GReflect(VAR Figure :POINTER;
Gx0,Gy0,Gx1,Gy1,Gdx,Gdy :REAL);

Purpose Produces a glide reflection for a given figure about a given line by agiven distance. (Gx0,Gy0) and (Gx1,Gy1) are two points on the line. (Gdx,Gdy) represents the glide distance.

PROCEDURE HighlightItem DisMenus

Syntax HighlightItem(VAR M : Menu; ItemNumber : INTEGER);

Purpose Highlights an item when it is chosen by the mouse or the keyboard.

FUNCTION InEventArea DisCurso

Syntax InEventArea(VAR EventNumber, PosX, PosY : INTEGER):BOOLEAN;

Remark True If the current Cursor Position is inside an event box.

PROCEDURE Initcursor DisCurso

Syntax Initcursor(Code , Col : INTEGER);

Purpose Used in cursor initialization.

PROCEDURE InitialGraphPc GrafIntf

Syntax InitialGraphPc;

Purpose Inititalises the graphics system and puts the hardware into graphics mode.

PROCEDURE InitialiseGraphics RealGraph

Syntax	InitialiseGraphics(Mode : INTEGER);
Purpose	Initialises Pluto, sets the current work partition, clears Pluto screen, initialises window number 0 to be the whole screen and makes it the active window. If the value of the Mode equals zero the Pluto screen is set to resolution (767W X 575H) else to (767W X 287).

PROCEDURE InitMouse DevCurso

Syntax	InitMouse(VAR Present :BOOLEAN)
Remark	Initializes the mouse software and hardware.

PROCEDURE InitPlutoMouse DevCurso

Syntax	InitPlutoMouse(VAR Mode :INTEGER; VAR DriverPresent :BOOLEAN);
Remark	Puts mouse in Pluto and limits the movement of the cursor to the Pluto screen.

PROCEDURE InitTempalateMotifDraw AidIGPPr

Syntax InitTempalteMotifDraw;

Purpose Sets up initial values and menu data.

PROCEDURE Jumpto RealGraph

Syntax Jumpto(X,Y : REAL);

Purpose Moves the cursor to the point (X,y) in world coordinates.

PROCEDURE Library AidIgpPr

Syntax
```
Library(State          :STRING;
        VAR LibFile    :FileName;
        VAR GroupNa    :STRING;
        VAR Dnet,YDnet :INTEGER);
```

Purpose Allows the user to call up a pattern file from library.

FUNCTION LoadFigure AidCmotf

Syntax LoadFigure(Fname : Filename) : POINTER;

Purpose Loads a file used to save a figure.

PROCEDURE Logo DevCurso

Syntax Logo;

Purpose Plots Logo massage on PC screen.

PROCEDURE MapWindow RealGraph

Syntax MapWindow(M : INTEGER; x0,x1,y0,y1 : REAL);

Purpose Sets up a mapping on window number M which is defined by a call to GraphWindow. Unlike GraphWindow the origin is taken to be at the bottom left hand corner and the Y axis points upwards.

FUNCTION MatchKeyToItem DisMenus

Syntax MatchKeyToItem(VAR Code : CHAR; VAR M : Menu) : INTEGER;

Purpose Match the key board with the menu items.

PROCEDURE MaxEvents DisCurso

Syntax MaxEvents(N : INTEGER);

Purpose Sets the maximum number of the menu items.

PROCEDURE MoveCursor DisCurso

Syntax MoveCursor(NewX,NewY : INTEGER);

Purpose If both NewX and NewY are within the border, draw cursor at the new position after erasing the old position. If it is outside the screen then ignore the call.

FUNCTION NearestPoint DisPolyg

Syntax NearestPoint(VAR Figure : POINTER; VAR X, Y :REAL)
: POINTER;

Remark Find the nearest coordinate point on the vertex of the figure.

PROCEDURE Perform AidIgpPr

Syntax Perform(Choice : INTEGER; VAR Px,Py : INteger;
VAR MutifData :POINTER);

Purpose Set up menu item and carry out a choice.

PROCEDURE PlutoRealNumber AidPlInt

Syntax PlutoRealNumber(Number : REAL; Width,Decimals : INTEGER);

Purpose To plot the coordinates of the current cursor position on the Pluto screen.

PROCEDURE PlutoWritechar AidPlInt

Syntax PlutoWritechar(Character : STRING);

Purpose Write a sequence of characters on the Pluto screen using the Pluto procedure Pchar.

PROCEDURE PolyFiller DisPolyg

Syntax PolyFiller(VAR Col :INTEGER; VAR Start :POINTER);

Purpose To fill in the current window the chosen polygon with the chosen color (See GetPolygon).

PROCEDURE PutImage DisImages

Syntax PutImage(X,Y,Width,Height : INTEGER; VAR Start : Raster);

Purpose Put the image back from memory when the color menu is removed.

PROCEDURE PrintMeshPattern IGP

Syntax PrintMeshPattern(VAR Step : INTEGER;
OurChoice : INTEGER);

Remarkes Print the repetition in X and Y on the PC screen.

FUNCTION Reflect AidCmotf

Syntax Reflect(VAR Figure : POINTER; x0,y0,x1,y1 : REAL)
: POINTER;

Purpose Reflect a given figure about a given line. (X0,Y0) and (X1,Y1) are two distinct points on the line.

PROCEDURE RemovePalette LcolorTa

Syntax RemovePalette;

Purpose Store the background image when the color menu is ON, and take it back when the color menu is OFF.

PROCEDURE RestoreMode GrafIntf

Syntax RestoreMode;

Purpose	Set the current background color, restore the original mode before graphics was initialized and free the graphics memory on the heap.

FUNCTION Rotate AidCmotf

Syntax	Rotate(VAR Figure : POINTER; X,Y,Phi : REAL) : POINTER;
Purpose	Rotate the figure around (x,y) with Phi degrees.The positive direction is auticlockwise.

PROCEDURE SaveFigure AidCmotf

Syntax	SaveFigure(Fname : FileName; VAR Figure : POINTER);
Purpose	Save the data of a given figure in the Fname file.

PROCEDURE SaveFile IGP

Syntax	SaveFile(VAR NameFile :STRING; VAR SaveFile :POINTER);
Purpose	Save the pattern into disk ASCII format.

PROCEDURE SaveLut LcolorTa

Syntax SaveLut(F : FileName);

Purpose Save the color menu to disk when it is turned off.

FUNCTION Scale AidCmotf

Syntax Scale(VAR Figure : POINTER;
AboutX,AboutY,Sx,Sy : REAL) : POINTER;

Purpose Scale a figure relative to a the point (AboutX, AboutY) by the scale vector (Sx,Sy).

PROCEDURE SecondSelectOption IGP

Syntax SecondSelectOption(VAR Cha :CHAR;
WayMenu :BOOLEAN);

Purpose To offer the user various choices, e.g. Start again, Quit etc, and removes files from hard disk.

PROCEDURE SecondTypeMenu GrafIntf

Syntax SecondTypeMenu(VAR SelectOption :BOOLEAN);

Purpose Puts the seconed menu on the PC screen which involves New start, Quit ect.

PROCEDURE Setcolour LcolorTa

Syntax Setcolour(colournumber: INTEGER; r,b,g : INTEGER);

Purpose Sets the number associated with a color.

PROCEDURE SetLut LcolorTa

Syntax SetLut(f : filename; wp : INTEGER);

Purpose Loads the file from disk related to color menu.

PROCEDURE SetMeshPattern IGP

Syntax SetMeshPattern(VAR TypeMesh : STRING; VAR DXmesh,DYmesh :INTEGER);

Purpose Set the length of the unit motif in both direction for each crystallographic group.

PROCEDURE SetMouseLimit DevCurso

Syntax SetMouseLimit(X0,Y0,X1,Y1 :INTEGER)

Purpose Set the movement of cursor limited by min/max values for horizontal/vertical.

PROCEDURE SetMousePosition DevCurso

Syntax SetMousePosition(X,Y :INTEGER)

Purpose Set mouse position on the screen.

PROCEDURE SetPixelToMickey DevCurso

Syntax SetPixelToMickey(X,Y :INTEGER)

Purpose Set the mickey to pixel ratio.

FUNCTION SetSelectGroup IGP

Syntax SetSelectGroup(VAR NumberGroup :INTEGER) : FileName;

Purpose Map numbers to group and set the group selected.

FUNCTION Shift AidCmotf

Syntax Shift(VAR Figure : POINTER; Dx,Dy : REAL)
: POINTER;

Purpose Shift a given figure relatively by (dx,dy).

PROCEDURE ShowColour LcolorTa

Syntax ShowColour(ColourNumber : INTEGER;
TopLhX,TopLhY,Size : INTEGER);

Purpose Define the area occupied by each color on the color menu.

PROCEDURE ShowCoords DisCurso

Syntax ShowCoords(PosX,PosY : INTEGER);

Purpose Return current position of cursor.

PROCEDURE ShowMenu DisMenus

Syntax ShowMenu(VAR M : Menu);

Purpose Put the menu on the Pluto screen.

PROCEDURE ShowPAlette LcolorTa

Syntax ShowPAlette(toplhx, toplhy, size, First, Last, across : INTEGER);

Purpose Show color menu in specific size and order.

PROCEDURE ShowPattern IGP

Syntax ShowPattern;

Purpose Show pattern if required.

PROCEDURE ShowTemplateMotif AidIGPPr

Syntax ShowTemplateMotif(TypeLine :STRING);

Purpose Display the template motif area bounded with a dotted line.

PROCEDURE ShowTemplateUnit IGP

Syntax ShowTemplateUnit(VAR ShowAgain :BOOLEAN);

Purpose Show the Template unit motif and unit motif on the Pluto screen if required.

PROCEDURE SizeFram IGP

Syntax SizeFram(Distance :INTEGER);

Purpose Set the thickness of the frame.

FUNCTION Splice AidCmotf

Syntax Splice(VAR Figure1 , Figure2 : POINTER) : POINTER;

Purpose Compine two figures into a single figure.

PROCEDURE TemplateMotifEnq AidIgpPr

Syntax TemplateMotifEnq(VAR BounaryGenerator : boundary);

Purpose Displays the template motif.

PROCEDURE TypeSelectGroup IGp

Syntax TypeSelectGroup;

Purpose Print select group menu on the the PC screen which associates a number with crystallographic group.

PROCEDURE TypeSelectPattern IGP

Syntax TypeSelectPattern(VAR CrystallGroup : INTEGER);

Purpose Print the number of the patterns available in IDL for each group.

PROCEDURE Undo AidIGPPr

Syntax Undo;

Purpose Allows to removed of the last graphic primitive.

PROCEDURE WhatToDo AidIgpPr

Syntax WhatToDo(VAR D : InputData; VAR MuData :POINTER);

Purpose Allows the user to chose the device ' Mouse or keyboard' for the input of data.

PROCEDURE WindowFrame RealGraph

Syntax WindowFrame(M : INTEGER);

Purpose Draw frame around the current window.

PROCEDURE WindowToWorld RealGraph

Syntax WindowToWorld(VAR PX,Py : INTEGER;VAR Wx,Wy :REAL);

Purpose Convert screen coordinates to world coordinates.

PROCEDURE WinEnq RealGraph

Syntax WinEnq(M : INTEGER; VAR WinInfo : WinData);

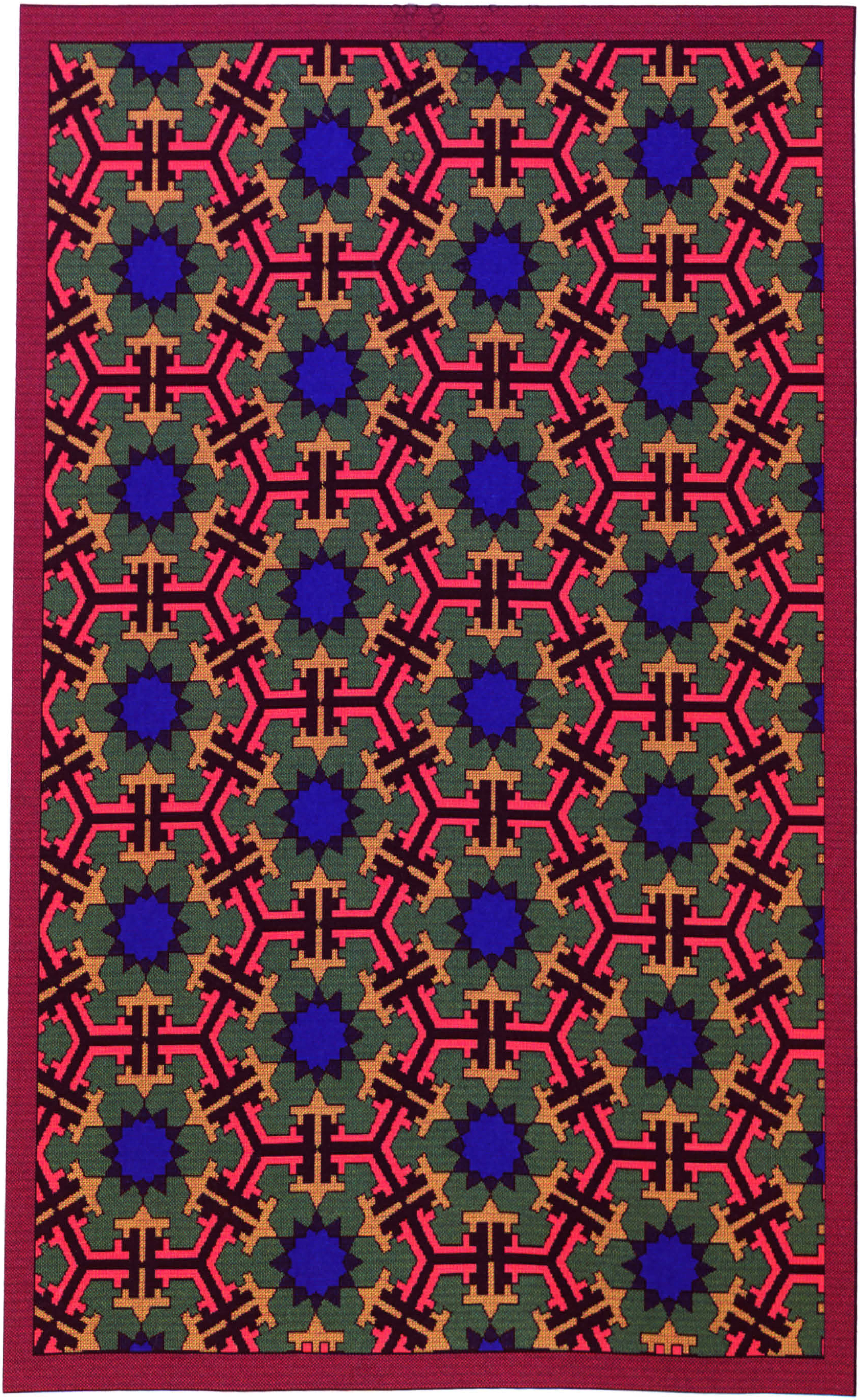
Purpose Return information on the normalized device coordinates of window M and the mapping associated with it.

PROCEDURE WorldToWindow RealGraph

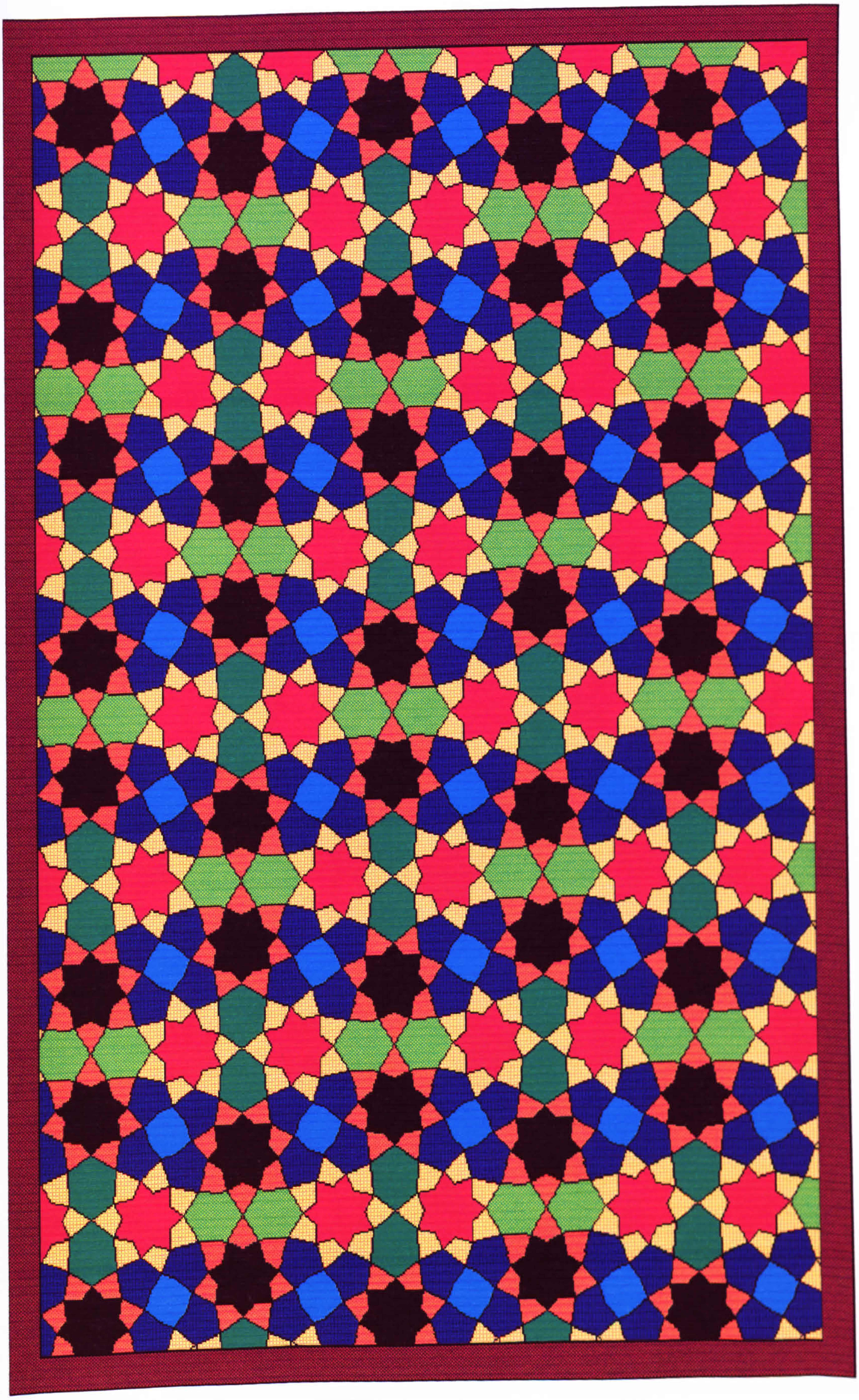
Syntax WorldToWindow(VAR Wx,Wy : REAL;
VAR Px,Py : INTEGER);

Purpose Convert world coordinates to screen coordinates.

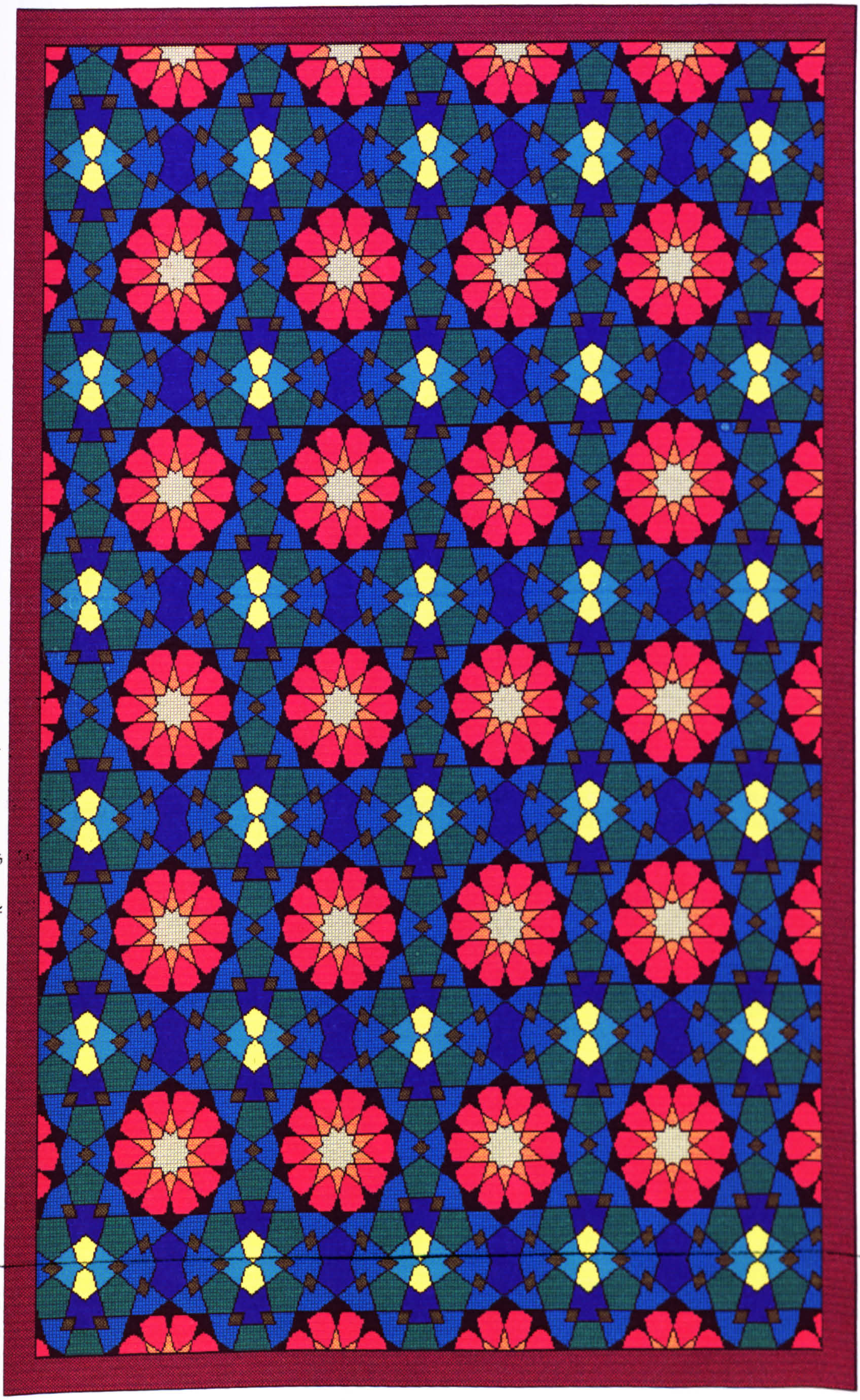
5.7 EXAMPLES OF ISLAMIC GEOMETRIC DESIGN



plate(1) p6mm

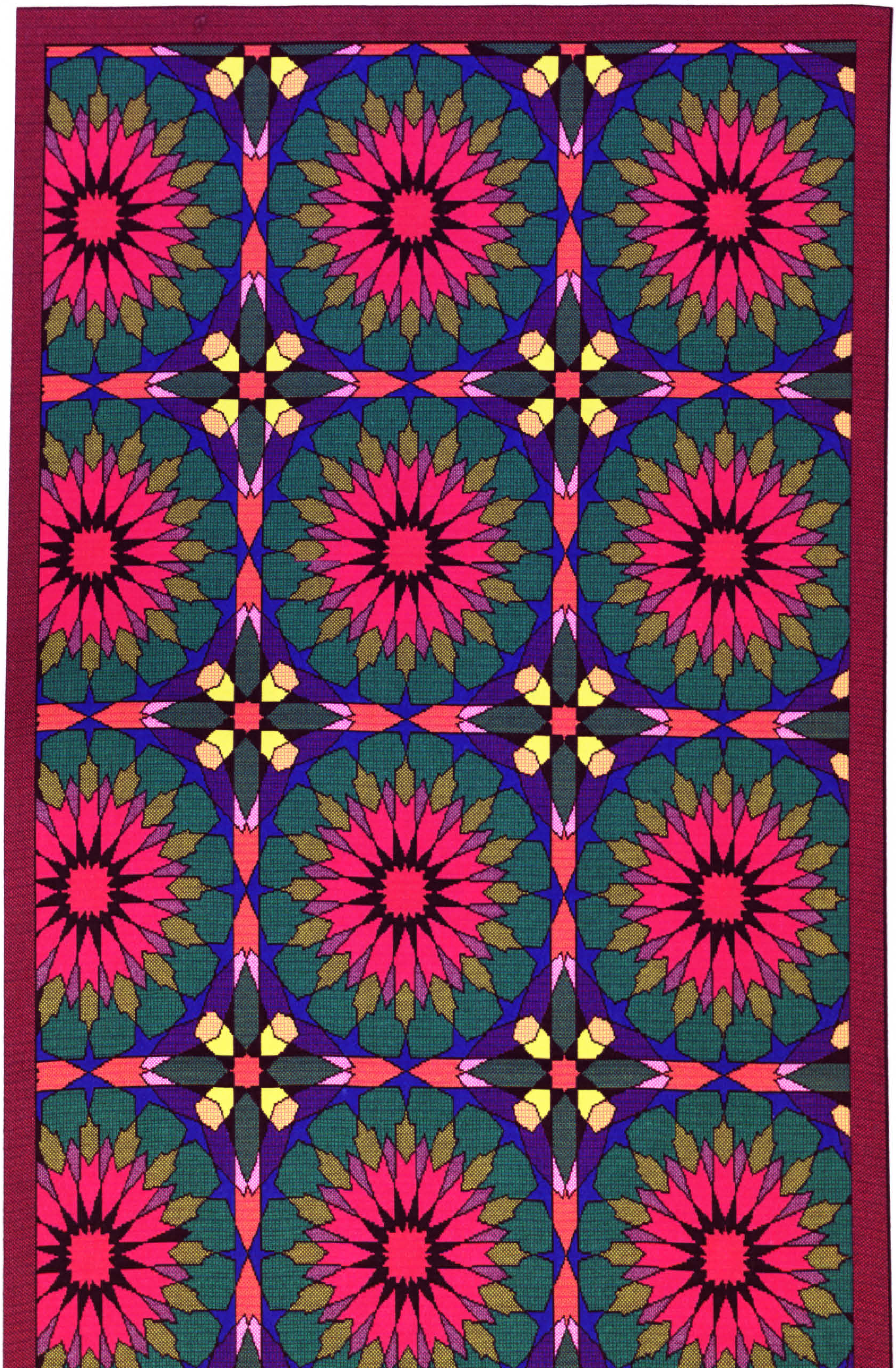


plate(2) p4gm

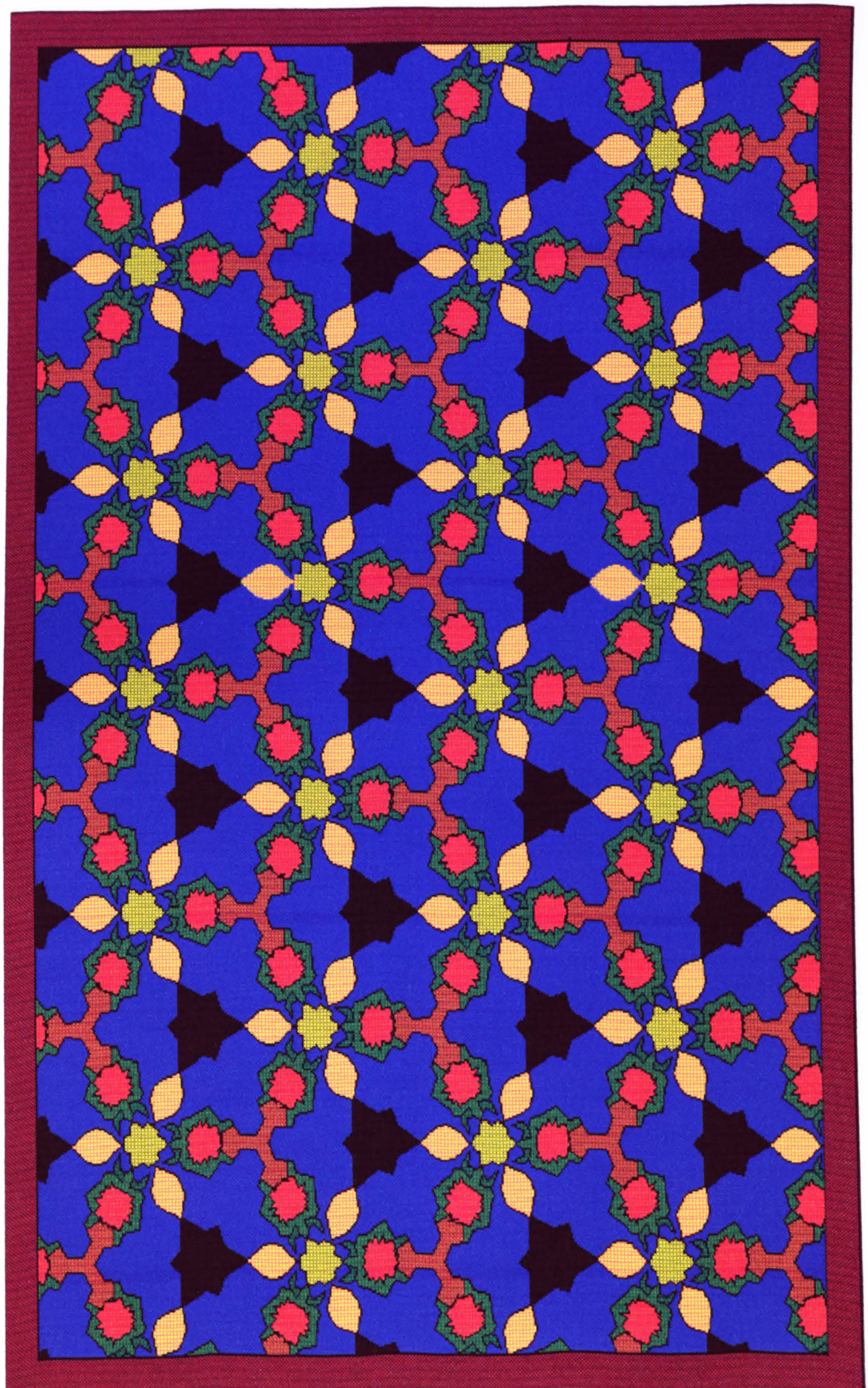


plate(3) p2mm

plate(4) p1m1

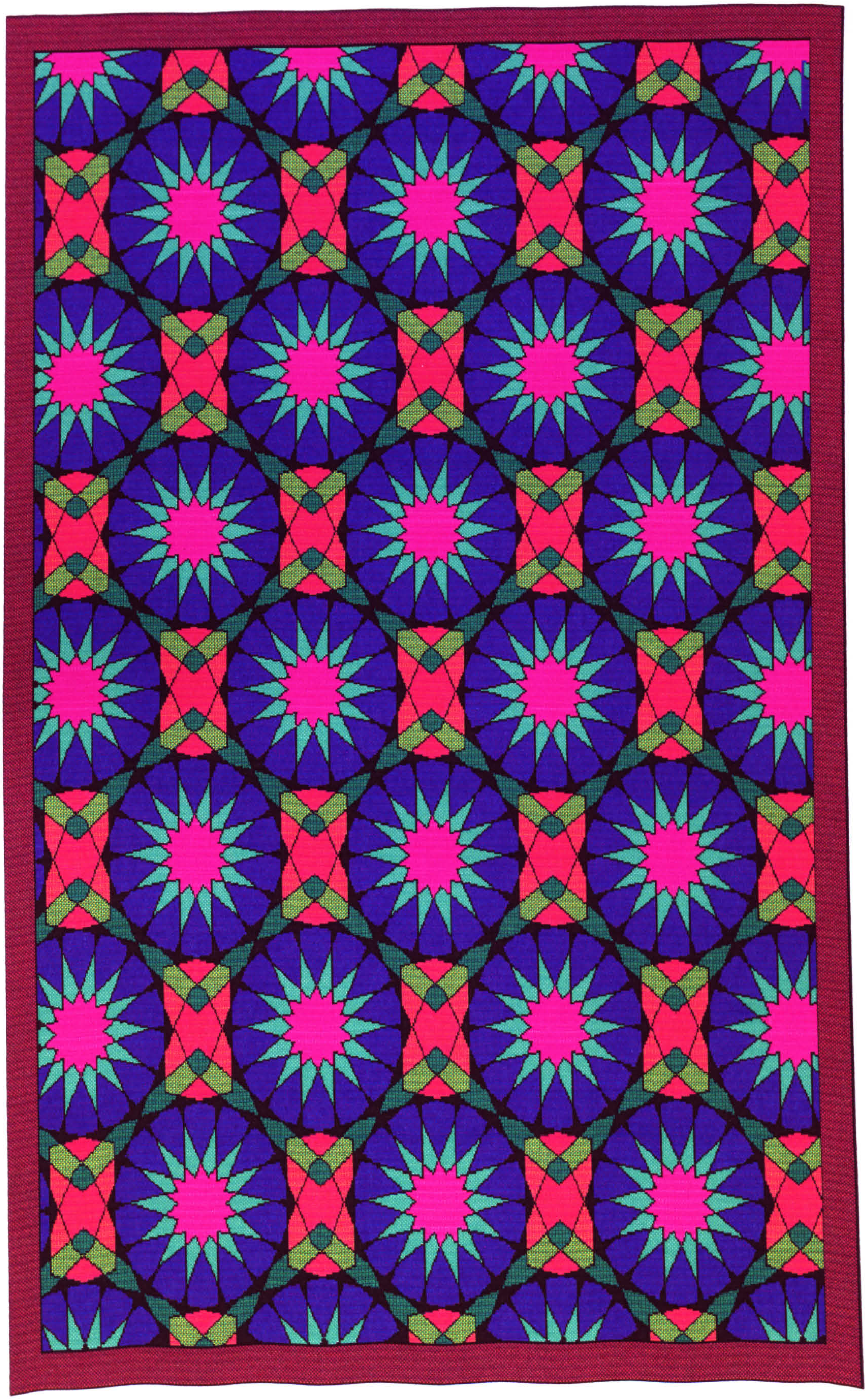


plate(5) p4mm

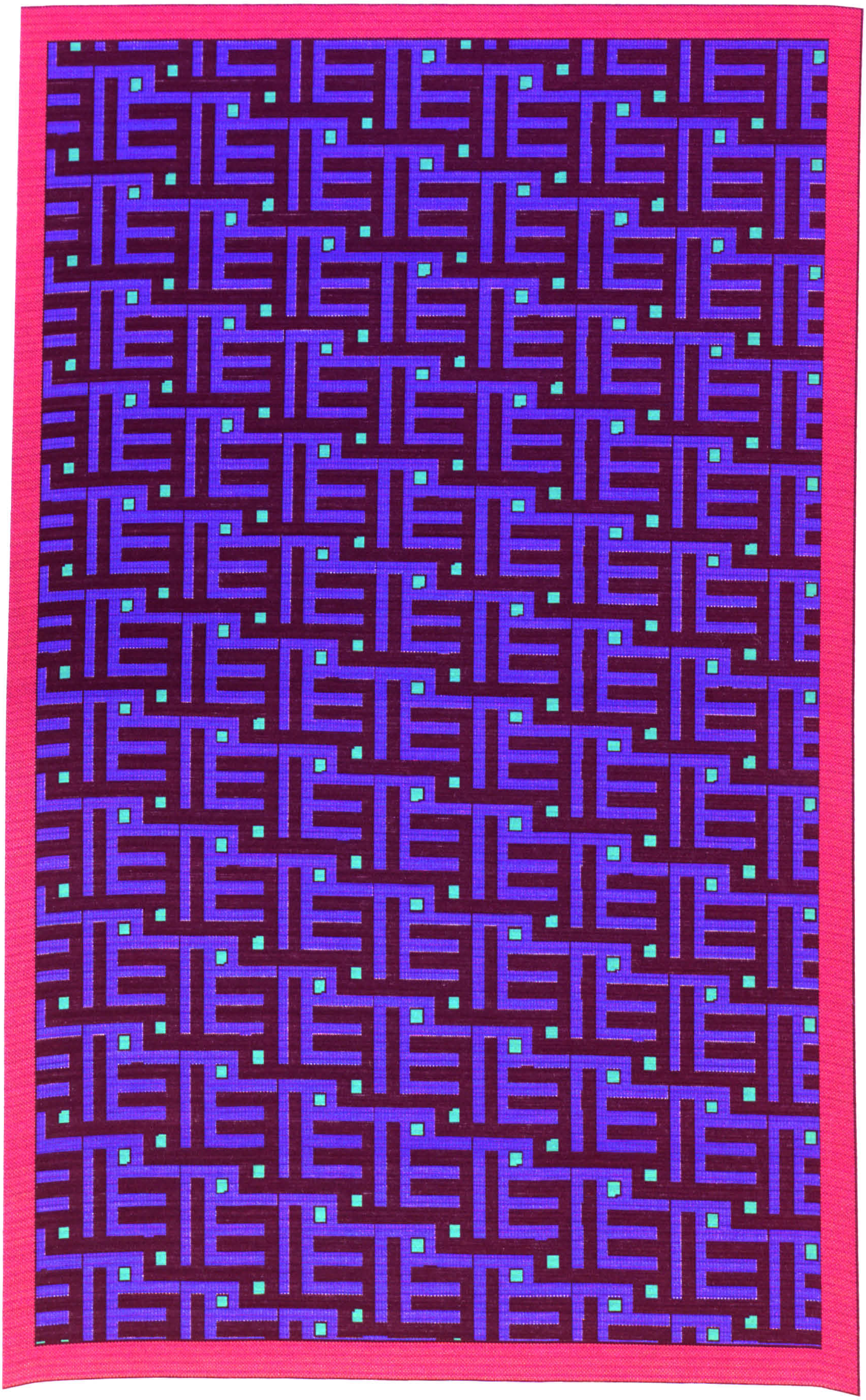


plate(6) p3m1

17



plate(7) c2mm



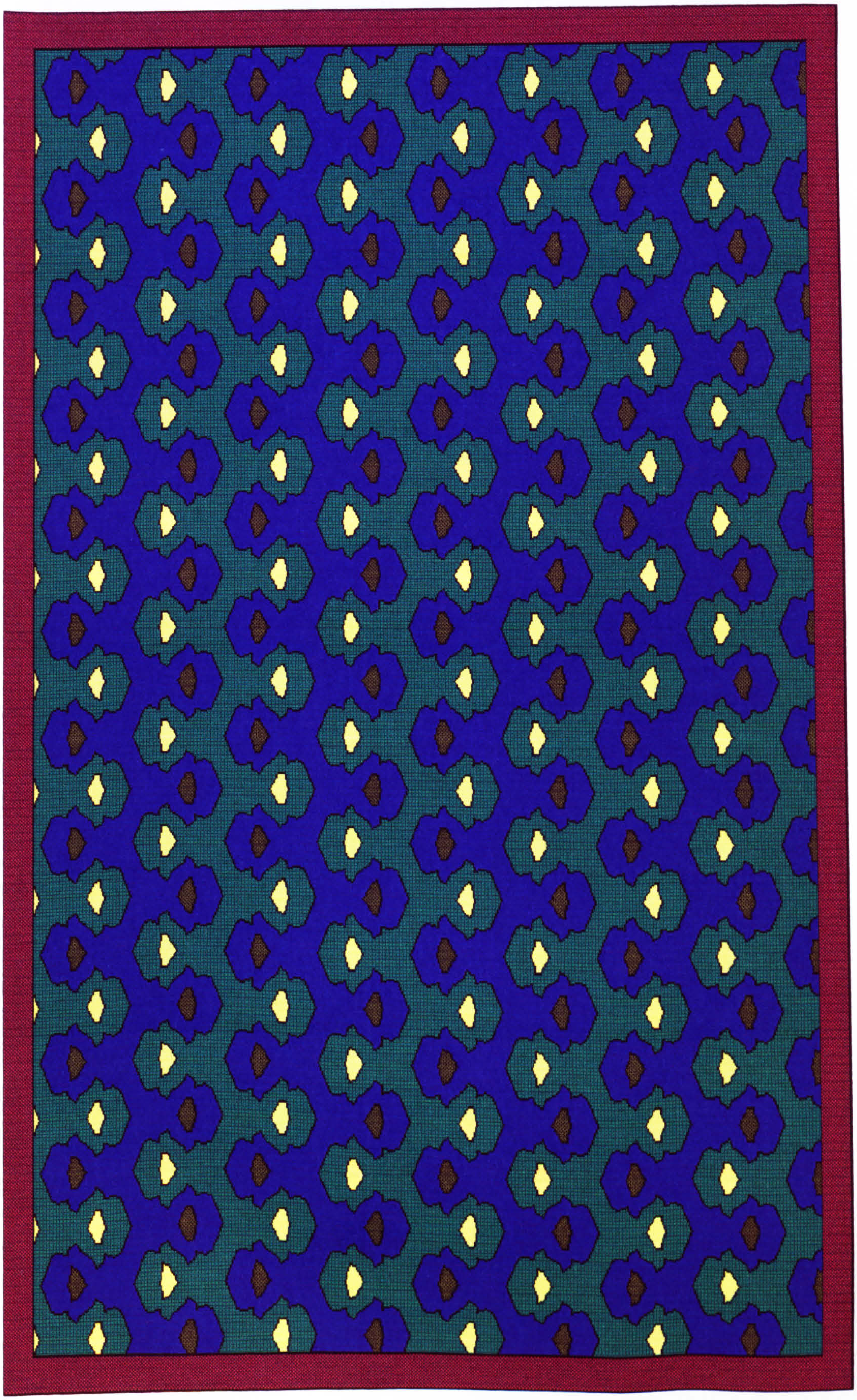
plate(8) p211

plate(9) p611

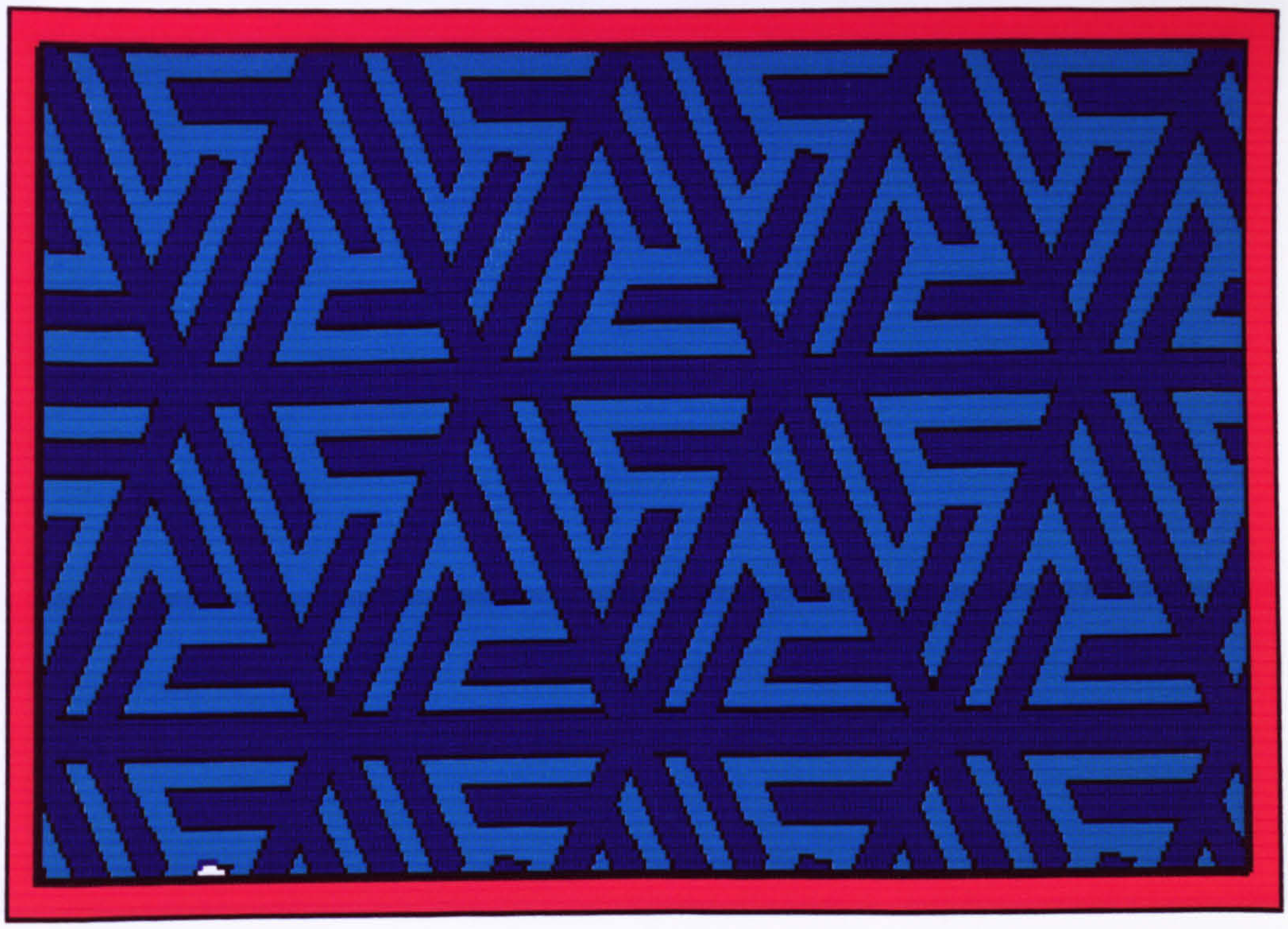
plate(10) p311

plate(11) c1m1

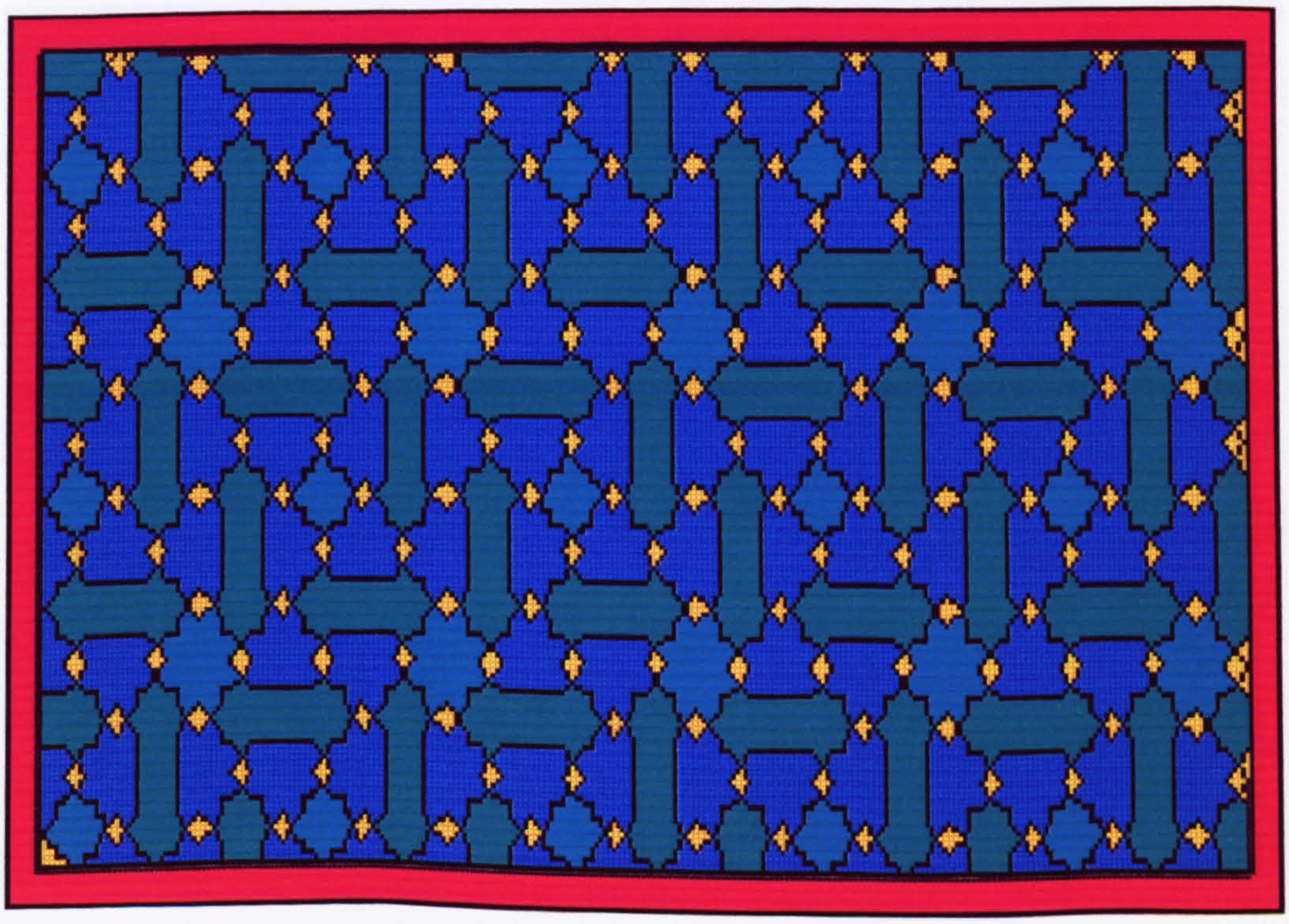
plate(12) p111



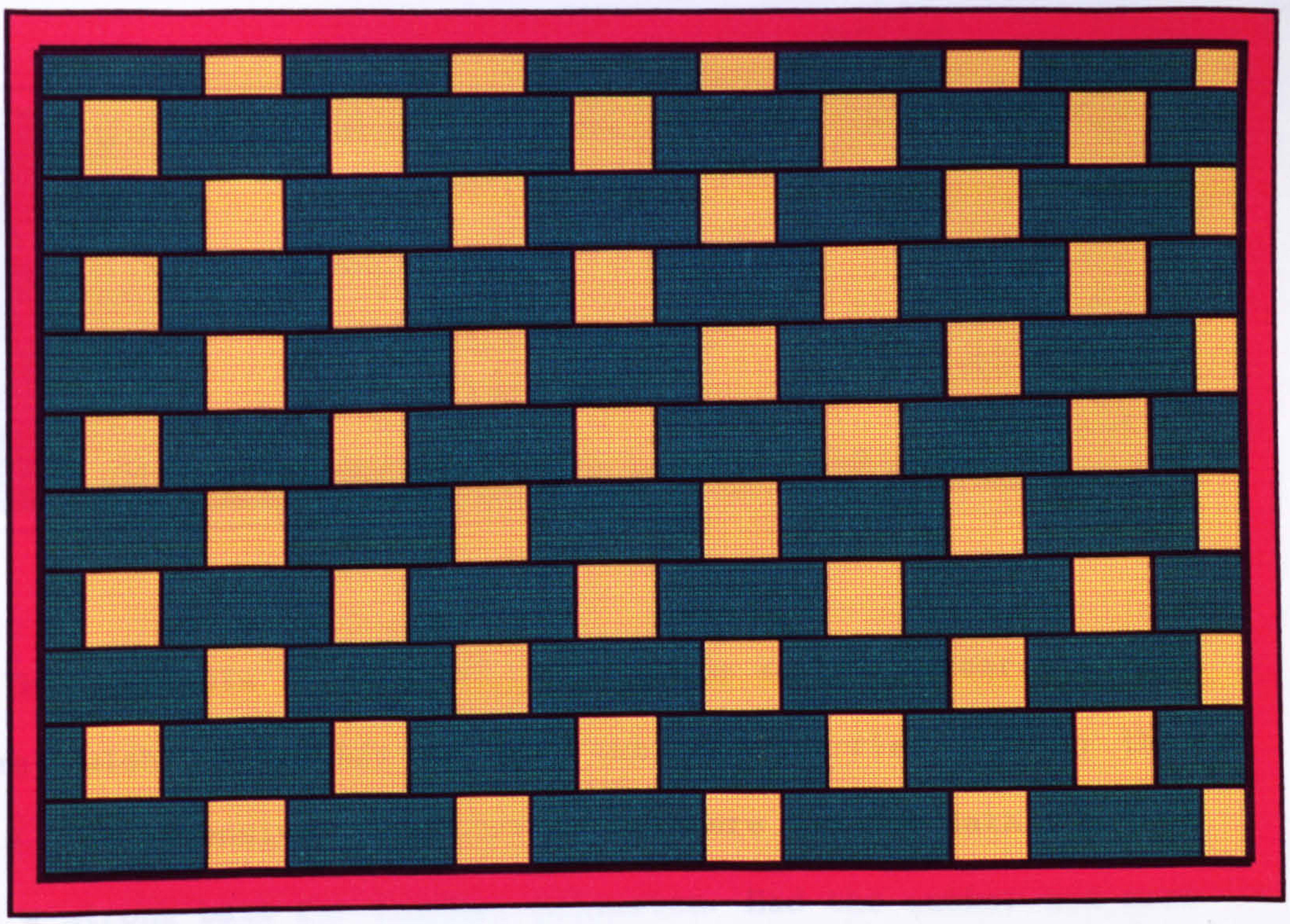
plate(13) p2mg



plate(14) p31m



plate(15) p411



plate(16) p1g1

plate(17) p2gg

APPENDIX

SUGGESTIONS FOR FURTHER DEVELOPMENTS OF THIS WORK

This appendix is provided to help any future research worker who may wish to further develop the work carried out in this thesis. The extensive study carried out here and the data provided offer a strong base for further extension. Had the author had more time he would like to have explored some of the following.

i) Calligraphic decoration of tilings:

A key feature of Islamic art is the use of calligraphy and clearly this is the must obvious extension that could be carried out to this work.

ii) Systematic exploration of color and color symmetry:

Color was explored extensively by the original Islamic artists. Computer graphics offers much greater possibilities. An interesting extension of this work would involve the development of group theory algorithms for color symmetry.

iii) Tilings in 3-D and on surfaces:

The 2-D work in here could be extended to tilings on 3-D surfaces.

iv) Using CAD CAM to design real objects:

The data produced here could be used in CAD CAM to manufacture real objects. This would have considerable commercial possibilities.

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