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**Journal of Symbolic Computation**

*DOI:*
10.1016/S0747-7171(02)00103-7

Published: 01/01/2003

Peer reviewed version

Cyswllt i'r cyhoeddiad / Link to publication

*Dyfyniad o'r fersiwn a gyhoeddwyd / Citation for published version (APA):*
Wensley, C. D., & Brown, R. (2003). Computation and homotopical applications of induced crossed modules. *Journal of Symbolic Computation*, *35*, 59-72. https://doi.org/10.1016/S0747-7171(02)00103-7

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28. Apr. 2024

*Article Submitted to Journal of Symbolic Computation*
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# Computation and Homotopical Applications of Induced Crossed Modules

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**Abstract**

We explain how the computation of induced crossed modules allows the computation of certain homotopy 2-types and, in particular, second homotopy groups. We discuss various issues involved in computing induced crossed modules and give some examples and applications.

## Introduction

The interactions between topology and combinatorial and computational group theory are largely based on the fundamental group functor

$$\pi_1 \ : \ \text{(based spaces)} \ \to \ \text{(groups)} \ .$$

At the beginning of the 20th century there was an aim to to generalise the non commutative fundamental group to higher dimensions, hopes which seemed to be dashed in 1932 by the proof that the definition of higher homotopy groups $\pi_n$ then proposed by Čech led to commutative groups for $n \geqslant 2$.

Nonetheless, in the late 1930s and 1940s J.H.C. Whitehead developed properties of the second relative homotopy group functor

$$\begin{aligned} \Pi_2 \ : \ \text{(based pairs of spaces)} \ &\to \ \text{(crossed modules)} \ , \\ (X, A, a) \ &\mapsto \ (\partial : \pi_2(X, A, a) \to \pi_1(A, a)) \ , \end{aligned}$$

where $a \in A \subseteq X$ (see Section 4). Mac Lane and Whitehead showed in 1950 (22) that crossed modules modelled homotopy 2-types (3-types in their notation) and evidence has grown that crossed modules can be regarded as '2-dimensional groups'. Part of this evidence is the 2-dimensional version of the Van Kampen

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Theorem proved by Brown and Higgins in 1978 (8), which allows new computations of homotopy 2-types and so second homotopy groups. This result should be seen as a higher dimensional, non commutative, local-to-global theorem, illustrating themes in Atiyah's article (4). It is interesting to note that the computation of these second homotopy groups is obtained through the computation of a larger non commutative structure. This work also throws emphasis on the problem of explicit computation with crossed modules, the discussion of which is the theme of this paper.

Our main emphasis in this paper is on induced crossed modules, which were defined in (8) and studied further in papers by the authors (13; 14). Given the crossed module $\mathcal{M} = (\mu : M \to P)$ and a morphism of groups $\iota : P \to Q$, the *induced crossed module* $\iota_*\mathcal{M}$ has the form $(\partial : \iota_*M \to Q)$, a crossed module over $Q$, and comes with a morphism of crossed modules $(\iota_*, \iota) : \mathcal{M} \to \iota_*\mathcal{M}$ :

$$\begin{array}{ccc} M & \xrightarrow{\iota_*} & \iota_* M \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\partial} \\ P & \xrightarrow[\iota]{} & Q \end{array} .$$

Their study requires a solution to many of the general computational problems of crossed modules.

In the case $\mu = 0$, when $M$ is simply a $P$-module, $\iota_*M$ is the usual induced $Q$-module $M \otimes_{\mathbb{Z}P} \mathbb{Z}Q$.

Even in the case $M = P,\ \mu = \mathrm{id}_P$, we know of no relation between the induced crossed module $(\partial : \iota_*P \to Q)$ and other standard algebraic constructions, although, interestingly, im $\partial = N^Q(\iota P)$ the normal closure of $\iota P$ in $Q$. Thus the induced crossed module construction replaces this normal closure by a bigger group on which $Q$ acts, and which has a universal property not usually enjoyed by $N^Q(\iota P)$.

A long-term project at Bangor is the development of a share library for the computational group theory program GAP (16), providing functions to compute with these higher-dimensional structures. The first stage of this project saw the production of the library XMod1, containing functions for crossed modules and their derivations and for $\mathrm{cat}^1$-groups and their sections. The manual for XMod1 was included in (26) as Chapter 73. In particular, Alp (1) enumerated all isomorphism classes of $\mathrm{cat}^1$-structures on groups of order at most 47. This library has recently been rewritten for GAP4, with XMod2 included with the 4.3 release. Related libraries include Heyworth's IdRel (17) for computing identities among the relators of a finitely presented group, and Moore's GpdGraph and XRes (23) for computing with finite groupoids; group and groupoid graphs; and crossed resolutions. These libraries are available at the HDDA website (18).

## 1. Crossed modules

A *crossed module* $\mathcal{M}$ (over $P$) consists of a morphism of groups $\mu : M \to P$, called the *boundary* of $\mathcal{M}$, together with an action of $P$ on $M$, written $(m,p) \mapsto m^p$, satisfying for all $m, n \in M,\ p \in P$ the axioms:

$$CM1) \quad \mu(m^p) = p^{-1}(\mu m)p\ , \qquad CM2) \quad n^{\mu m} = m^{-1}nm\ .$$

When CM1) is satisfied, but not CM2), the structure is a *pre-crossed module* (10; 19), having a *Peiffer subgroup* $C$ generated by *Peiffer commutators* $\langle m,n\rangle = m^{-1}n^{-1}m\,n^{\mu m}$, and an associated crossed module $(\mu' : M/C \to P)$ with $\mu'$ induced by $\mu$.

Some standard algebraic examples of crossed modules are:

(i) normal subgroup crossed modules $(i : N \to P)$ where $i$ is an inclusion of a normal subgroup, and the action is given by conjugation;

(ii) automorphism crossed modules $(\chi : M \to \mathrm{Aut}(M))$ in which $(\chi m)(n) = m^{-1}nm$;

(iii) abelian crossed modules $(0 : M \to P)$ where $M$ is a $P$-module;

(iv) central extension crossed modules $(\mu : M \to P)$ where $\mu$ is an epimorphism with kernel contained in the centre of $M$.

For our purposes, an important standard construction is the *free crossed $Q$-module*

$$\mathcal{F}_\omega \;=\; (\partial : F(\omega) \to Q)$$

on a function $\omega : \Omega \to Q$, where $\Omega$ is a set and $Q$ is a group. The group $F(\omega)$ has a presentation with generating set $\Omega \times Q$ and relators

$$(m,q)^{-1}\,(n,p)^{-1}\,(m,q)\,(n,pq^{-1}(\omega m)q) \quad \forall\, m,n \in \Omega,\ p,q \in Q\ .$$

The action is given by $(m,q)^p = (m,qp)$ and the boundary morphism is defined on generators by $\partial(m,q) = q^{-1}(\omega m)q$. This construction will be seen later as a special case of an *induced* crossed module. The reader should be warned that the group $F(\omega)$ can be very far from a free group: in fact, if $\omega$ maps all of $\Omega$ to $\{1_Q\}$, then $F(\omega)$ is just the free $Q$-module on the set $\Omega$, and in particular is a commutative group.

The major geometric example of a crossed module can be expressed in two ways. Let $(X,A,a)$ be a based pair of spaces, with $a \in A \subseteq X$. The *second relative homotopy group* $\pi_2(X,A,a)$ consists of homotopy classes rel $J^1$ of continuous maps

$$\alpha \;:\; (I^2, \dot{I}^2, J^1) \;\to\; (X,A,a)$$

where $I = [0,1]$ and $J^1 = (I \times \{0,1\}) \cup (\{1\} \times I) \subset I^2$. Each such $\alpha$ is a map from the unit square $I^2$ to the space $X$ mapping three sides of the square to the point $a$ and the fourth side to a loop at $a$. Whitehead showed in (28) that there is a crossed module $\Pi_2(X, A, a)$ with boundary map

$$\partial \; : \; \pi_2(X, A, a) \to \pi_1(A, a), \quad \alpha \mapsto \beta = \alpha(I \times \{0\}) \; .$$

The image of $\alpha_1 \in \pi_2(X, A, a)$ under the action of $\beta_2 \in \pi_1(A, a)$ is illustrated in the right-hand square of Figure 1.

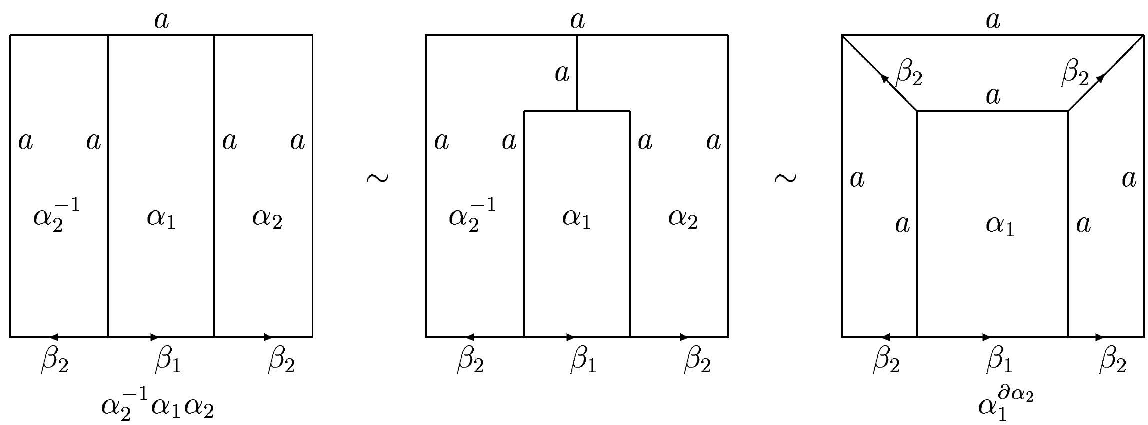



**Figure 1:** Verification of CM2) for $\Pi_2(X, A, a)$ .

Whitehead's main result in (27; 28; 29) was:

Theorem1.1(Whitehead) If $X$ is obtained from $A$ by attaching 2-cells, then $\pi_2(X, A, x)$ is isomorphic to the free crossed $\pi_1(A, x)$-module on the attaching maps of the 2-cells.

Later Quillen observed that if $F \to E \to B$ is a based fibration, then the induced morphism of fundamental groups $\pi_1 F \to \pi_1 E$ may be given the structure of a crossed module. This fact is of importance in algebraic $K$-theory.

We also note the following fact, shown in various texts on homological algebra or the cohomology of groups, e.g. (6), and which we relate to topology in section 4:

**1.1** *A crossed module $\mathcal{M} = (\mu : M \to P)$ determines algebraically a cohomology class*

$$k_{\mathcal{M}} \in H^3(\text{coker } \mu, \text{ker } \mu),$$

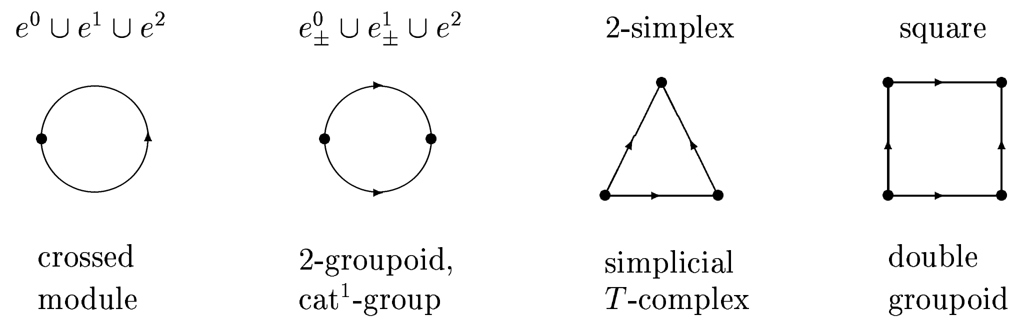
*called the $k$-invariant of $\mathcal{M}$, and all elements of this cohomology group have such a representation by a crossed module.*

## 2. Other structures equivalent to crossed modules

One aspect of the problem of higher dimensional group theory is that, whereas there is essentially only one category of groups, there are at least five categories of equationally defined algebraic structures which are equivalent to crossed modules, namely:

- cat$^1$-groups (21);
- group-groupoids (11);
- simplicial groups with Moore complex of length 1, (21);
- reduced simplicial $T$-complexes of rank 2, (15; 3; 24);
- reduced double groupoids with connection (12).

These categories have various geometric models. The 2-cells of some of these are illustrated in the following pictures:




There is also a polyhedral model, which allows rather general kinds of geometric objects (20).

Thus, for computation in "2-dimensional group theory", decisions must be made as to which category to use to represent a given object, and to compute constructions. One reason for computing with the crossed module format is that this is closer to the familiar realm of groups, for which many computational procedures and systems have been found and constructed. Part of the interest in computations with crossed modules is that such computations will also yield computations of these other structures, and this makes them more familiar and understandable.

### 2.1. Cat$^1$-groups

In a cat$^1$-group $\mathcal{C} = (e; t, h : G \to R)$ the *embedding* $e : R \to G$ is a monomorphism while the *tail* and *head* homomorphisms $t, h : G \to R$ are surjective and satisfy:

$$CAT1) \quad te = he = \mathrm{id}_R\,, \qquad CAT2) \quad [\ker t, \ker h] = \{1_G\}\,.$$

When CAT1) is satisfied, but nor CAT2), the structure is a *pre-cat$^1$-group* with Peiffer subgroup $[\ker t, \ker h]$. A cat$^1$-group $\mathcal{C}$ determines a crossed module $(\partial : S \to R)$ where $S = \ker t$ and $\partial = h|_S$. Conversely, a crossed module $(\mu : M \to P)$ determines a cat$^1$-group $(e; t, h : P \ltimes M \to P)$ where $t(p, m) = p$ and $h(p, m) =$

$p(\mu m)$. The axiom $he = \mathrm{id}_R$ is equivalent to CM1) for a crossed module, while CAT2) is equivalent to CM2). When $\mu$ is the inclusion of the trivial subgroup in $P$, the associated cat$^1$-group $\mathcal{C}_P$ has $e = t = h = \mathrm{id}_P$.

Note also that the semidirect product $P \ltimes M$ admits a groupoid structure with $t, h$ as source and target, and composition $\circ$ where $(p, m) \circ (p(\mu m), n) = (p, mn)$, making $P \ltimes M$ a group-groupoid, i.e. a group internal to the category of groupoids. This notion has a long history: the result that crossed modules are equivalent to group-groupoids goes back to Verdier, seems first to have been published in (11), and is used in (5). The holomorph $\mathrm{Aut}(M) \ltimes M$ of a group $M$ is the source of the cat$^1$-group associated to the automorphism crossed module $(\chi : M \to \mathrm{Aut}(M))$.

Now a colimit of cat$^1$-groups $\mathrm{colim}_i(e_i; t_i, h_i : G_i \to R_i)$ is easy to describe. One takes the colimits $G',\ R'$ of the underlying groups $G_i,\ R_i$, and finds that the endomorphisms $e_i,\ t_i,\ h_i$ induce endomorphisms $e' : R' \to G'$ and $t', h' : G' \to R'$ satisfying axiom CAT1). The required colimit is the cat$^1$-group $\mathcal{C}'' = (e''; t'', h'' : G'' \to R')$ which has $G'' = G'/[\ker t', \ker h']$ and $e'', t'', h''$ induced by $e', t', h'$.

When $\mathcal{C} = (e; t, h : G \to R)$ and $\iota : R \to Q$ is an inclusion, the induced cat$^1$-group $\iota_*\mathcal{C}$ is obtained as the pushout of cat$^1$-morphisms $(e, \mathrm{id}_R) : \mathcal{C}_R \to \mathcal{C}$ and $(\iota, \iota) : \mathcal{C}_R \to \mathcal{C}_Q$ . See Alp (1), (2) for further details.

Further investigation is needed to see whether the use of cat$^1$-groups can be shown to be more efficient than the direct method for the computation of some colimits of crossed modules, particularly induced crossed modules,. The procedure has three stages: convert a crossed module $\mathcal{M}$ to a cat$^1$-group $\mathcal{C}$; calculate $\iota_*\mathcal{C}$; then convert $\iota_*\mathcal{C}$ to $\iota_*\mathcal{M}$.

## 3. Computing colimits of crossed modules

The homotopical reason for interest in computing colimits of crossed modules is the 2-dimensional Van Kampen Theorem (2-VKT) due to Brown and Higgins (8). The formulation and proof of this theorem was found through the notion of double groupoid with connection, since such structures yield an appropriate algebraic context in which to handle both "algebraic inverses to subdivision", and the "homotopy addition lemma" (which gives a formula for the boundary of a 3-cube).

One form of the 2-VKT states that Whitehead's *fundamental crossed module* functor

$$\Pi_2 \ : \ \text{(based pairs of spaces)} \to \text{(crossed modules)}$$

preserves certain colimits. So for the calculation of certain homotopy invariants, we need to know how to calculate colimits of crossed modules. To this end, we start by using some elementary category theory.

The forgetful functor (crossed modules) $\to$ (groups), $(\mu : M \to P) \mapsto P$, has a right adjoint $P \mapsto (i : P \to P)$, and so preserves colimits. This shows how to compute the 1-dimensional part of the colimit crossed module in terms of colimits of groups.

The aim now is to transfer the problem to computing colimits of crossed modules over a fixed group $P$. To do this, suppose given a morphism of groups $\iota : P \to Q$. Then there is a pullback functor

$$\iota^* \ : \ \text{(crossed modules over } Q) \to \text{(crossed modules over } P) \ .$$

This functor has a left adjoint

$$\iota_* \ : \ \text{(crossed modules over } P) \to \text{(crossed modules over } Q) \ ,$$

which gives our induced crossed module. This construction can be described as a "change of base" (7). To compute a colimit $\text{colim}_i(\mu_i : M_i \to P_i)$, one forms the group $P = \text{colim}_i P_i$, and uses the canonical morphisms $\phi_i : P_i \to P$ to form the family of induced crossed $P$-modules $((\mu_i)_* : (\phi_i)_* M_i \to P)$. The colimit of these in the category of crossed $P$-modules is isomorphic to the original colimit. Now if $M'$ is the colimit in the category of groups of the $(\phi_i)_* M_i$, then there is a canonical morphism $M' \to P$ and an action of $P$ on $M'$. The resulting $(M' \to P)$ is a pre-crossed module, and quotienting by its Peiffer subgroup gives the required crossed module.

Presentations for induced crossed modules were given in (8), and more recently families of explicit examples have been computed, partly by hand and partly using GAP (13). Computation of induced crossed modules is here reduced to problems of computation in combinatorial group theory. A key fact which makes one expect successful computations is that if $(\mu : M \to P)$ is a crossed module with $M$ finite, and if $\iota : P \to Q$ is a morphism of finite index, then the induced crossed $Q$-module $\iota_* M$ is also finite (13, Theorem 2.1).

**Example 3.1** When $\mu : M \to P$ and $\iota : P \to Q$ are subgroup inclusions, there are complete descriptions of $\iota_* M$ in the following cases:

(i) If $\iota$ is surjective then $\iota_* M \cong M/[M, \ker \iota]$, ((8, Proposition 9)).

(ii) If $M$ is abelian and $\iota\mu(M)$ is normal in $Q$ then $\iota_* M$ is abelian and is the usual induced $Q$-module $M \otimes_{\mathbb{Z}P} \mathbb{Z}Q$, ((13, Corollary 1.6)).

(iii) If $M$ and $P$ are normal subgroups of $Q$ then $\iota_* M \cong M \times (M^{\text{ab}} \otimes I(Q/P))$, where $I$ denotes the augmentation ideal. If in addition $M = P$ then $\iota_* P \cong P \times (P^{\text{ab}})^{[Q:P]-1}$, ((14, Theorem 1.1)).

(iv) If $M = P = C_2$, the cyclic group of order 2, $\mu = \text{id}_P$, and $\iota : C_2 \to D_{2n}$ is the inclusion to a reflection in the dihedral group $D_{2n}$, then $\iota_* P \cong D_{2n}$ ((13, Example 1.4)). The action is not the usual conjugation: when $n$ is odd the boundary is an isomorphism, but when $n$ is even the kernel and cokernel are isomorphic to $C_2$. □

## 4. Homotopical applications

As explained in the Introduction, the fundamental crossed module functor $\Pi_2$ assigns a crossed module $(\partial : \pi_2(X, A, a) \to \pi_1(A, a))$ to any based pair of spaces

$(X, A, a)$. Theorem C of (8) is a 2-dimensional Van Kampen type theorem for this functor. We will use the following consequence:

Theorem4.1*((8), Theorem D)* Let $(B, V, b)$ be a cofibred pair of spaces, let $f : V \to A$ be a based map, and let $X$ be the pushout $A \cup_f B$ in the left-hand diagram below. Suppose also that $A, B, V$ are path-connected, and $(B, V, b)$ is 1-connected. Then the based pair $(X, A, a)$ is 1-connected and the right-hand diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & A \\ {\scriptstyle\subseteq}\downarrow & & \downarrow \\ B & \longrightarrow & X \end{array} \qquad\qquad \begin{array}{ccc} \pi_2(B, V, b) & \xrightarrow{\lambda_*} & \pi_2(X, A, a) \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ \pi_1(V, b) & \xrightarrow[\lambda]{} & \pi_1(A, a) \end{array}$$

presents $\pi_2(X, A, a)$ as the crossed $\pi_1(A, a)$-module $\lambda_*(\pi_2(B, V, b))$ induced from the crossed $\pi_1(V, b)$-module $\pi_2(B, V, b)$ by the group morphism $\lambda : \pi_1(V, b) \to \pi_1(A, a)$ induced by $f$.

As pointed out earlier, when $P$ is a free group on a set $\Omega$ and $\mu$ is the identity, the induced crossed module $\iota_* P$ is the free crossed $Q$-module on the function $\iota|_\Omega : \Omega \to Q$. Thus Theorem 4.1 implies Whitehead's Theorem as stated in Theorem 1.1. A considerable amount of work has been done on this case, because of the connections with identities among relations, and methods such as transversality theory and "pictures" have proved successful ((10; 25)), particularly in the homotopy theory of 2-dimensional complexes (19). However, the only route so far available to the wider geometric applications of induced crossed modules is Theorem 4.1. We also note that this Theorem includes the relative Hurewicz Theorem in this dimension, on putting $A = \Gamma V$, and $f : V \to \Gamma V$ the inclusion.

We will apply this Theorem 4.1 to the *classifying space of a crossed module*, as defined by Loday in (21) or Brown and Higgins in (9). This classifying space is a functor $B$ assigning to a crossed module $\mathcal{M} = (\mu : M \to P)$ a based $CW$-space $B\mathcal{M}$ with the following properties:

**4.1** *The homotopy groups of the classifying space of the crossed module $\mathcal{M} = (\mu : M \to P)$ are given by*

$$\pi_i(B\mathcal{M}) \;\cong\; \begin{cases} \operatorname{coker}\mu & \text{for } i = 1\ , \\ \ker\mu & \text{for } i = 2\ , \\ 0 & \text{for } i > 2\ . \end{cases}$$

*The first Postnikov invariant of $B\mathcal{M}$ is precisely the $k$-invariant of $\mathcal{M}$ as in 1.1.*

**4.2** *The classifying space $BP = B(i : 1 \to P)$ is the usual classifying space of the group $P$, and $BP$ is a subcomplex of $B\mathcal{M}$. Further, there is a natural isomorphism of crossed modules*

$$\Pi_2(B\mathcal{M}, BP, x) \;\cong\; \mathcal{M}\ .$$

**4.3** *If $X$ is a reduced CW-complex with 1-skeleton $X^1$, then there is a map*

$$X \;\to\; B(\Pi_2(X, X^1, x))$$

*inducing an isomorphism of $\pi_1$ and $\pi_2$.*

It is in these senses that it is reasonable to say, as in the Introduction, that crossed modules model all based homotopy 2-types.

We now give two direct applications of Theorem 4.1.

Corollary4.1 Let $\mathcal{M} = (\mu : M \to P)$ be a crossed module, and let $\iota : P \to Q$ be a morphism of groups. Let $\beta : BP \to B\mathcal{M}$ be the inclusion. Consider the pushout

$$\begin{array}{ccc} BP & \xrightarrow{\beta} & B\mathcal{M} \\ {\scriptstyle B\iota}\downarrow & & \downarrow \\ BQ & \xrightarrow[\beta']{} & X \end{array} .$$

Then the fundamental crossed module of the based pair $(X, BQ, x)$ is isomorphic to the induced crossed module $(\partial : \iota_* M \to Q)$, and this crossed module determines the 2-type of $X$. In particular, the second homotopy group $\pi_2(X, x)$ is isomorphic to $\ker \partial$.

*Proof:* The first statement is immediate from Theorem 4.1. The second statement follows from results of (9), since the morphism $Q \to \pi_1(X)$ is surjective. The final statement follows from the homotopy exact sequence of $(X, BQ, x)$. □

*Remark:* An interesting special case of the last Corollary is when $\mathcal{M}$ is an inclusion of a normal subgroup, since then $B\mathcal{M}$ is of the homotopy type of $B(P/M)$. So we have determined the 2-type of a homotopy pushout

$$\begin{array}{ccc} BP & \xrightarrow{Bp} & BR \\ {\scriptstyle B\iota}\downarrow & & \downarrow \\ BQ & \xrightarrow[p']{} & X \end{array}$$

in which $p : P \to R$ is surjective. □

Corollary4.2 Let $\iota : P \to Q$ be a morphism of groups, and let $\Gamma BP$ denote the cone on $BP$. Then the fundamental crossed module $\Pi_2(BQ \cup_{B\iota} \Gamma BP, BQ, x)$ is isomorphic to the induced crossed module $(\partial : \iota_* P \to Q)$. In particular the second homotopy group $\pi_2(BQ \cup_{B\iota} \Gamma BP, x)$ is isomorphic to $\ker \partial$.

We also note that in determining the crossed module representing a 2-type we are also determining the first Postnikov invariant of that 2-type. However it may be more difficult to describe this invariant as a cohomology class, though this is done in some cases in (13; 14).

# 5. Computational issues

We now consider some features of the function `InducedXMod` as implemented in XMod2. The method selection mechanism of GAP4 allows for special methods when $\iota$ is surjective or injective, and for the cases listed in example 3.1.

Recall from Proposition 9 of (8) that when $\iota : P \to Q$ is a surjection then $\iota_* M \cong M/[M,K]$, where $K = \ker \iota$ and $[M,K]$ denotes the subgroup of $M$ generated by the $m^{-1}m^k$ for all $m \in M, k \in K$. When $\iota$ is neither surjective nor injective, we obtain a factorisation $\iota = \iota_2 \circ \iota_1$ with $\iota_1$ surjective and $\iota_2$ injective, and construct the induced crossed module in two stages:

$$\begin{array}{ccccc} M & \xrightarrow{(\iota_1)_*} & (\iota_1)_* M & \xrightarrow{(\iota_2)_*} & \iota_* M \\ \mu \downarrow & & \partial_1 \downarrow & & \downarrow \partial \\ P & \xrightarrow[\iota_1]{} & \operatorname{im} \iota & \xrightarrow[\iota_2]{} & Q\ . \end{array}$$

The first stage is easily constructed as a quotient group, so in the following subsections we restrict to the case when both $\iota$ and $\mu$ are subgroup inclusions.

Note that computation of free crossed modules, as described in section 1, is in general difficult since the groups are usually infinite, and is not attempted in the current version of the package.

### 5.1. Copower of groups

The construction of induced crossed modules, described in (8; 13), involves the copower $M \vec{*} T$, namely the free product of groups $M_t,\ t \in T$, each isomorphic to $M$. Here $T$ is a transversal for the right cosets of $P$ in $Q$, in which the representative of the subgroup $P$ is taken to be the identity element. The group $M_t = \{(m,t) \mid m \in M\}$ has product $(m,t)(n,t) = (mn,t)$ and $Q$ acts by $(m,t)^q = (m^p, u)$ where $tq = (\iota p)u$ in $Q$. The map $\delta' : M \vec{*} T \to Q$ is defined by $(m,t) \mapsto t^{-1}(\iota\mu m)t$.

The GAP function `IsomorphismFpGroup` enables the construction of finitely presented groups $FM, FP, FQ$ isomorphic to the groups $M, P, Q$; monomorphisms $F\mu : FM \to FP,\ F\iota : FP \to FQ$ mimicing the inclusions $M \to P \to Q$; and an action of $FP$ on $FM$. If $FM$ has $\gamma$ generators then a finitely presented group $FC$, isomorphic to $M \vec{*} T$ and with $\gamma\,|T|$ generators, may be constructed using functions in the GAP Tietze package ((16), Chapter 46). The relators of $FC$ comprise $|T|$ copies of the relators of $FM$, suitably renumbered.

### 5.2. Tracing Tietze transformations

Let $\Omega$ be a generating set for $FM$ and let $\Omega^{FP}$ be the closure of $\Omega$ under the action of $FP$. Then $\iota_*(M) \cong FC/FN$ where $FN$ is the normal closure in $FC$ of the Peiffer elements

$$\langle (n,s),(m,t) \rangle = (n,s)^{-1}(m,t)^{-1}(n,s)(m,t)^{\delta'(n,s)} \quad (m,n \in \Omega^{FP},\ s,t \in T). \quad (1)$$

The homomorphism $\iota_*$ is induced by the projection $\mathrm{pr}_1 m = (m, 1_{FQ})$ onto the first factor, and the boundary $\delta$ of $\iota_*\mathcal{M}$ is induced from $\delta'$ as shown in the following diagram:

$$\begin{array}{ccc} FM & \xrightarrow{\iota_*} & FC/FN \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\delta} \\ FP & \xrightarrow[\iota]{} & FQ \end{array} \quad .$$

Thus a finitely presented group $FI \cong \iota_* M$ is obtained by adding to the relators of $FC$ further relators corresponding to the list of elements in equation (1), and the presentation may be simplified by applying Tietze transformations.

As well as returning an induced crossed module, the construction should return a morphism of crossed modules $(\iota_*, \iota) : \mathcal{M} \to \iota_*\mathcal{M}$. When Tietze transformations are applied to the initial presentation for $FI$, during the resulting simplification some of the first $\gamma$ generators may be eliminated, so the projection $\mathrm{pr}_{()}$ may be lost. In order to preserve this projection, and so obtain the morphism $\iota_*$, it is necessary to record for each eliminated generator $g$ a relator $gw^{-1}$ where $w$ is the word in the remaining generators by which $g$ was eliminated.

A significant advantage of GAP is the free availability of the library code, which enables the user to modify a function so as to return additional information. For the XMod1 version of the package, the Tietze transformation code was modified so that the resulting presentation contained an additional field `presI.remember`, namely a list of (at least) $\gamma \mid T \mid$ relators expressing the original generators in terms of the final ones. In more recent releases of GAP an equivalent facility has been made generally available using the `TzInitGeneratorImages` function.

### 5.3. Polycyclic groups

Recall that a polycyclic group is a group $G$ with power-conjugate presentation having generating set $\{g_1, \ldots, g_n\}$ and relations

$$\{g_i^{o_i} = w_{ii}(g_{i+1}, \ldots, g_n), \;\; g_i^{g_j} = w'_{ij}(g_{j+1}, \ldots, g_n) \;\; \forall\, 1 \leqslant j < i \leqslant n\}. \qquad (2)$$

These are implemented in GAP as `PcGroups` (see (16), Chapters 43,44). Since subgroups $M \leqslant P \leqslant G$ have induced power-conjugate presentations, if $T$ is a transversal for the right cosets of $P$ in $G$, then the relators of $M \vec{*} T$ are all of the form in (2). Furthermore, all the Peiffer relations in equation (1) are of the form $g_i^{g_j} = g_k^p$, so one might hope that a power conjugate presentation would result. Consideration of the cyclic-by-cyclic case in the following example shows that this does not happen in general.

**Example 5.1** Let $C_n$ be cyclic of order $n$ with generator $x$, and let $\alpha : x \mapsto x^a$ be an automorphism of $C_n$ of order $p$. Take $G = \langle g, h \mid g^p, h^n, g^{-1}h^{-1}g\,h^a \rangle \cong C_p \ltimes C_n$. When $M = P = C_n \lhd G$ cases (ii) and (iii) of example 3.1 apply, and $\iota_* C_n \cong C_n^p$.

It follows from the relators that $h^i g = g h^{ai},\; 0 < i < n,$ and that $h^{-1}(g h^{i(1-a)})h =$

$gh^{(i+1)(1-a)}$. So if we put $g_i = gh^{i(1-a)}$, $0 \leqslant i < n$, then $g_i^{g_j} = g_{[j+a(i-j)]}$. When $M = P = C_p = \langle g \mid g^p \rangle$ and $\iota : C_p \to G$, we may choose as transversal $T = \{1_G, h, h^2, \ldots, h^{n-1}\}$. Then $M \vec{*} T$ has generators $\{(g, h^i) \mid 0 \leqslant i < n\}$, all of order $p$, and relators $\{(g, h^i)^p \mid 0 \leqslant i < n\}$. The additional Peiffer relators in equation (1) have the form

$$(g, h^i)(g, h^j) = (g, h^j)(g^k, h^l) \quad \text{where} \quad h^i h^{-j} g h^j = g^k h^l$$

so $k = 1$ and $l = [j + a(i-j)]$. Hence $\theta : \iota_* M \to Q$, $(g, h^i) \mapsto g_i$ is an isomorphism, and $\iota_* \mathcal{M}$ is isomorphic to the identity crossed module on $Q$. Furthermore, if we take $M$ to be a cyclic subgroup $C_m$ of $C_p$ then $\iota_* \mathcal{M}$ is the normal subgroup crossed module $(i : C_m \ltimes C_n \to C_p \ltimes C_n)$. □

### 5.4. Identifying $\iota_* M$

From some of the special cases listed in example 3.1 and from other examples, we know that many of the induced groups $\iota_* M$ are direct products. However the generating sets in the presentations that arise following the Tietze transformation do not in general split into generating sets for direct summands, as the following simple case shows.

**Example 5.2** Let $Q = S_4$, the symmetric group of degree 4, and $M = P = A_4$, the alternating subgroup of $Q$ of index 2. Since the abelianisation of $A_4$ is cyclic of order 3, case (iii) in section 3 shows that $\iota_* M \cong A_4 \times C_3$. However a typical presentation for $A_4 \times C_3$ obtained from the program is

$$\langle x, y, z \mid x^3, y^3, z^3, (xy)^2, zy^{-1}z^{-1}x^{-1}, yzyx^{-1}z^{-1}, y^{-1}x^2y^2x^{-1} \rangle,$$

and one generator for the $C_3$ summand is $yzx^2$.

Using the function `IsomorphismPermGroup` we obtain a permutation group of degree 12 with generating set

$$\{(1,2,3)(4,10,8)(5,11,6)(7,12,9),\ (1,4,5)(2,6,7)(3,9,10)(8,12,11),\ (2,5,8)(3,4,7)(6,12,10)\}.$$

On applying `IsomorphismPcGroup` to the permutation group we obtain a 4-generator polycyclic group with composition series

$$A_4 \times C_3 \ \rhd\ A_4 \ \rhd\ C_2^2 \ \rhd\ C_2 \ \rhd\ I\ ,$$

where each subgroup drops the generator $g_i$, $i = 1 \ldots 4$ and $g_1 g_2 g_4$ is a generator for the normal $C_3$. In these representations the cyclic summand $\ker \partial = C_3$ remains hidden, and an explicit search among the normal subgroups must be undertaken to find it. □

## 6. Results

In this section we list the crossed modules induced from subgroups of groups of order at most 23 (excluding 16), except that the special cases mentioned earlier enable us to exclude abelian and dihedral groups; the case when $P$ is normal in $Q$; and the case when $Q$ is a semidirect product $C_m \ltimes C_n$.

In table 1 we assume given an inclusion $\iota : P \to Q$ of a subgroup $P$ of a group $Q$, and a normal subgroup $M$ of $P$. We list the isomorphism types of $\iota_* M$ and the kernel of $\partial : \iota_* M \to Q$. Recall that this kernel is realised as a second homotopy group in corollary 4.1. Labels $I, C_n, D_{2n}, A_n, S_n$ denote the identity, cyclic, dihedral, alternating and symmetric groups of order $1, n, 2n, n!/2$ and $n!$ respectively. The group $H_n$ is the holomorph of $C_n$ and $H_n^+$ is its positive subgroup in degree $n$, while $SL(2,3)$ and $GL(2,3)$ are the special and general linear groups of order 24 and 48. Labels of the form `[m,n]` refer to the $n$th group of order $m$ according to the GAP4 numbering.

**Table 1:**

| $\lvert Q\rvert$ | $M$ | $P$ | $Q$ | $\iota_* M$ | $\ker\partial$ |
|---|---|---|---|---|---|
| 12 | $C_2$ | $C_2$ | $A_4$ | $H_8^+$ | $C_4$ |
| | $C_3$ | $C_3$ | $A_4$ | $SL(2,3)$ | $C_2$ |
| 18 | $C_2$ | $C_2$ | $C_2 \ltimes C_3^2$ | $[54,8]$ | $C_3$ |
| | $S_3$ | $S_3$ | $C_2 \ltimes C_3^2$ | $[54,8]$ | $C_3$ |
| 20 | $C_2$ | $C_2$ | $H_5$ | $D_{10}$ | $C_2$ |
| | $C_2$ | $C_2^2$ | $D_{20}$ | $D_{10}$ | $I$ |
| | $C_2^2$ | $C_2^2$ | $D_{20}$ | $D_{20}$ | $I$ |
| 21 | $C_3$ | $C_3$ | $H_7^+$ | $H_7+$ | $I$ |

Table 2 contains the results of calculations with $Q = S_4$, where $C_2 = \langle(1,2)\rangle$, $C_2' = \langle(1,2)(3,4)\rangle$, and $C_2^2 = \langle(1,2),(3,4)\rangle$. The final column specifiess the automorphism group $\mathrm{Aut}(\iota_* M)$.

**Table 2:**

| $M$ | $P$ | $\iota_* M$ | $\ker\partial$ | $\mathrm{Aut}(\iota_* M)$ |
|---|---|---|---|---|
| $C_2$ | $C_2$ | $GL(2,3)$ | $C_2$ | $S_4C_2$ |
| $C_3$ | $C_3$ | $C_3\ SL(2,3)$ | $C_6$ | $[144,183]$ |
| $C_3$ | $S_3$ | $SL(2,3)$ | $C_2$ | $S_4$ |
| $S_3$ | $S_3$ | $GL(2,3)$ | $C_2$ | $S_4C_2$ |
| $C_2'$ | $C_2'$ | $[128,?]$ | $C_4C_2^3$ | |
| $C_2'$ | $C_2^2, C_4$ | $H_8^+$ | $C_4$ | $S_4C_2$ |
| $C_2'$ | $D_8$ | $C_2^3$ | $C_2$ | $SL(3,2)$ |
| $C_2^2$ | $C_2^2$ | $S_4C_2$ | $C_2$ | $S_4C_2$ |
| $C_2^2$ | $D_8$ | $S_4$ | $I$ | $S_4$ |
| $C_4$ | $C_4$ | $[96,219]$ | $C_4$ | $[96,227]$ |
| $C_4$ | $D_8$ | $S_4$ | $I$ | $S_4$ |
| $D_8$ | $D_8$ | $S_4C_2$ | $C_2$ | $S_4C_2$ |

An interesting problem is to obtain a clearer understanding of the geometric significance of these tables.

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