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## Higher Dimensional Algebroids and Crossed Complexes

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## HIGHER DIMENSIONAL ALGEBROIDS AND <br> CROSSED COMPLEXES

## Thesis submitted to the University of Wales in support of the application for the Degree of Philosophiae Doctor

By
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## Supervised by Professor R.Brown

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## DECLARATION

The work of this thesis has been carried out by the candidate and contains the results of his own investigations. The work has not already been accepted in substance for any degree, and is not being concurrently submitted in candidature for any degree . All sources of information have been acknowledged in the text.

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G. H. MOSA

## SUMMARY

The equivalence between the category of crossed modules (over groups) and the category of special double groupoids with connections and with one vertix was proved by R.Brown and C.B.Spencer , ALSO, C.B.Spencer and Y.L.Wong have shown that there exists an equivalence between the category of 2-categories and the category of double categories with connections.
R. Brown and P.J.Higgins have generalised the first result : they proved that there exists an equivalence between the category of $\omega$-groupoids and the category of crossed complexes (over groupoids).

In this thesis we develop a parallel theory in a more algebraic context , with expectation of applications in non-abelian homological and homotopical algebra. We prove an equivalence between the category of crossed modules (over algebroids) and the category of special double algebroids with connections . Moreover we prove a similar result for the 3-dimensional case, that is, we prove that there exists an equivalence between the category ( $\underline{\text { Crs }})^{3}$ of 3 -truncated crossed complexes and the category $(\underline{\omega-A l g})^{3}$ of 3-tuple algebroids .

Also we end this work by giving a conjecture for the higher dimensional case. In particular, we have

Theorem: The functors $\gamma, \lambda$ form an adjoint equivalence

$$
y: \underline{D A}!\leftrightarrow \underline{C}: \lambda
$$

where $\underline{D A}^{\text {! }}$ is the category of special double algebroids with connections and $\underline{C}$ is the category of crossed modules over algebroids.

Theorem: The functors $\gamma, \lambda$ form an adjoint equivalence

$$
y:(\underline{\omega-A l g})^{n} \leftrightarrow(\underline{\text { Chs }})^{n}: \lambda
$$

for $n=3,4$.
Finally we give a conjecture whose validity would be sufficient for the general equivalence of categories of $\omega$-algebroids and crossed complexes.

In chapter $V I$ we explain some results which have been obtained in the case of groupoids and higher dimensional groupoids , and suggest the possibility of obtaining similar results in the case of algebroids and higher dimensional algebroids .

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## INTRODUCTION

## A. AIMS AND BACKGROUND:

1. Overall aim:

There are many useful analogies between the theory of groups and the theory of algebras which are exploited for example in homological algebra . Some interesting generalisations of groups are groupoids, crossed modules, crossed complexes, double groupoids and $\omega$-groupoids, dating respectively from 1926 [Brandt - 1] , 1946 [J.H.C.Whitehead 1,2], 1949 [Blakers - 1] , 1965 [Ehresmann - 1] and 1977 [Brown-Higgins - 8 ] Corresponding to groupoids as generalisations of groups , we have algebroids as generalisations of algebras, a theory due to B.Mitchell (1972) . There are also notions of crossed modules of algebras. But a theory of double and $n$-tuple algebroids does not seem to be available, and it is our aim to investigate this idea.

In order to see the motivation for this investigation and the kind of result to be expected, we first recall some facts on the group case.
2. Crossed modules, crossed complexes in groups
and $\omega$-groupoids:
First, a group homomorphism $a: M \rightarrow P$ is said to be a crossed $P$-module (in groups) if there is given an action of $P$ on $M$, ( $p, m$ ) $\rightarrow P_{m}$ which satisfies the following axioms :
(i) $\partial\left(P_{m}\right)=p(\partial m) p^{-1}$ (ii) $\partial m_{m}=m m^{\prime} m^{-1}$ for $m, m$, $\in M$ and $p \in P$. Standard examples of crossed modules are :

1) the inclusion $N \rightarrow P$ of a normal subgroup $N$ of the group $P$, with the action of $P$ on $N$ given by conjugation ;
2) the zero morphism $0: M \rightarrow P$ in which $M$ is a P-module in the usual sense;
3) the boundary map $a \quad \pi_{2}\left(X, Y, x_{0}\right) \rightarrow-\pi_{1}\left(Y, x_{0}\right)$ from the second relative homotopy group to the fundamental group with the standard action of $\pi_{1}\left(Y, X_{0}\right)$ on $\pi_{2}\left(X, Y, x_{0}\right)$.

As this last example suggests, crossed modules can be used to model certain homotopy types. In fact from the standpoint of homotopy theory, crossed modules should be viewed as "2-dimensional groups" . It is reasonable to ask then, what are the n-dimensional groups (or crossed modules) ?
J.H.C.Whitehead gave a partial answer to this by introducing what he called a "homotopy system", but which are now called crossed complexes . These gadgets consist of a sequence of groups

$$
\ldots{ }_{n}^{\partial_{n}} c_{n} \xrightarrow{\partial_{n-1}} c_{n-1} \xrightarrow{\partial_{n}-2} \ldots{ }^{\partial_{3}} c_{3} \xrightarrow{\partial_{2}} c_{2} \xrightarrow{\partial_{1}} c_{1} \xrightarrow{\rightarrow} c_{0}
$$

where $C_{0}$ is a single point and satisfy the axioms ;
i) $a_{1}$ is a crossed module ;
ii) $C_{n}$ is abelian for $n \geqslant 3$;
iii) $\partial^{2}=0$;
iv) $C_{1}$ acts on $C_{n}, n \geqslant 2$ and $a_{1} C_{2}$ acts trivially on $C_{n}$ for n $\geqslant 3$.

The standard example of a crossed complex is obtained from a pointed filtered space (c.f. [Br-3]).

Work in homotopy theory has developed the well known notion of "groupoids", which are categories in which every arrow is invertible.

Since a crossed module has been considered as a "higher dimensional group", the question aries : what is a higher dimensional groupoid? Ehresmann [Eh-l] has defined the notion of double groupoid. R-Brown and C.B.Spencer have proved that there exists an equivalence between the category of crossed modules (over groups) and the category of double groupoids with special connections and one vertex . But the general case has been defined in [B-Hi-8] ; namely they have defined $\omega$-groupoids and crossed complexes (over groupoids) by using the cubical set notion. Moreover they have proved in [B-Hi-2] there exists an equivalence between the category of $\omega$-groupoids and the category of crossed complexes (over groupoids).

The above discussion of the development of group theory in this direction is summarised in the diagram

Groups $\rightarrow$ Groupoids $\rightarrow$ Double groupoids $\rightarrow$ $\rightarrow$ groupoids .
There are in fact a remarkable collection of equationally defined categories of (many-sorted) algebras which are nontrivially equivalent to $\omega$-groupoids . These are summarised in the following diagram :

in which the arrows denote explicit functors which are equivalence of categories. The symbols in square brackets give references to the proofs.

## 3. Crossed modules and Crossed complexes over algebras:

The work of [Ge-l] essentially involves the notion of crossed modules in associative and commutative algebras under a different name Also the work of [K-L-l] in algebraic K-theory has introduced crossed modules of Lie algebras : The definition of crossed modules in associative algebras is given on page (9,10).

The notion of crossed modules of algebras has been generalised to crossed complexes over algebras [Po-3], namely ;

Let $R$ be a commutative ring and let $K$ be an $R$-algebra. A crossed complex of R-algebras is a sequence of R-algebras

$$
\ldots \xrightarrow{C_{n}} \xrightarrow{\partial_{n}} C_{n-1} \xrightarrow{-\cdots} \stackrel{\partial_{2}}{-} C_{2} \xrightarrow{\partial_{1}} K
$$

in which
i) $a_{1}$ is a crossed $K$-module,
ii) $C_{i}$ for idl is a $\begin{aligned} & \text { K-module on } \\ & \alpha \text { ond } \\ & \partial_{i} \text { preserves the actions }\end{aligned}$ iii) for $i \geqslant 1, a_{i+1} \partial_{i}=0$.

Now one can ask, what are the higher dimensional
algebras ? In this thesis we shall give a partial answer to this question and we will give some extra conjectures .

## 4. Algebraic geometry:

The idea of this work arose from the consideration of bringing crossed module ideas into commutative algebra and algebraic geometry ; namely an ideal in a polynomial ring corresponds to an affine algebraic variety . Crossed modules in commutative algebras are generalisations of ideals.

One would like to know the geometric analogue of a crossed module, but nothing seems to be known on this question.

The original motivation for this thesis was to see if it would be easier to find analogues of "double commutative algebroids" in algebraic geometry, assuming such were equivalent to crossed modules. This lead to the problem of finding analogues for algebroids of the work of Brown Higgins on $\omega$-groupoids, and this problem has since occupied our full attention.

There are still many problems in relating this work to algebraic geometry, but we believe this will eventually be possible.

## B. STRUCTURE AND MAIN RESULTS:

In chapter $I$ we give an example to show how algebras are appropriately generalised to algebroids and we show that the category of R-algebroids is a monoidal closed category . We give the definition of a crossed module over an associative algebra and introduce the definition of a crossed module over an algebroid . Also we deduce some properties of crossed modules similar to the well known properties of crossed modules over groups.

In chapter II, we define an algebroid in one higher dimension . In fact we introduce the notion of a double algebroid by using double categories ; namely a double R-algebroid $D$ is four related R-algebroids

$$
\begin{aligned}
& \left(D, D_{1}, \partial_{1}^{i}, \varepsilon_{1},+_{1}, *_{1}, .\right),\left(D_{2}, D_{2}, \partial_{2}^{i}, c_{2},+_{2}, *_{2}, \cdot_{2}\right) \\
& \left(D_{1}, D_{0}, \delta_{1}^{i}, c_{1}+, *, .\right),\left(D_{2}, D_{0}, \delta_{2}^{i}, c_{,}, *, \ldots\right)
\end{aligned}
$$

where $i=0,1$ and these algebroids satisfy the following
axioms :
i) $\varepsilon_{2}^{i} a_{2}^{j}=\varepsilon_{1}^{j} a_{1}^{i} \quad$ in $\in\{0,1\}$
ii) $\partial_{2}^{i}\left(\alpha{ }_{2} \beta\right)=\partial{ }_{2}^{i} \alpha+\partial_{2}^{i} \beta \quad, \partial_{2}^{i}\left(\alpha+{ }_{2} \beta\right)=\partial_{1}^{i} \alpha+\partial{ }_{1}^{i} \beta$

$$
\partial_{2}^{i}\left(\alpha *_{1} \beta\right)=\partial_{2}^{i} \alpha * a_{2}^{i} \beta, \partial_{1}^{i}\left(\alpha *_{2} \beta\right)=\partial_{1}^{i} \alpha * \partial_{1}^{i} \beta
$$

for $i=0,1, \alpha, \beta \in D$ and both sides are defined.
iii) $r \cdot \rho_{1}\left(\alpha+_{2} \beta\right)=\left(r \rho_{1} \alpha\right)+_{2}\left(r \rho_{1} \beta\right)$,

$$
\begin{aligned}
& r \cdot{ }_{2}\left(\alpha+{ }_{1} \beta\right)=\left(r \cdot{ }_{2} \alpha\right)+_{1}\left(r \cdot{ }_{2} \beta\right) \\
& r \cdot{ }_{1}\left(\alpha *_{2} \beta\right)=\left(r \cdot{ }_{2} \alpha\right) *_{2}\left(r \cdot_{1} \beta\right) \\
& r \cdot{ }_{2}\left(\alpha *_{1} \beta\right)=\left(r \cdot_{2} \alpha\right) *_{1}\left(r \cdot_{2} \beta\right) \\
& r \cdot{ }_{1}\left(s \cdot{ }_{2} \alpha\right)=s \cdot{ }_{2}\left(r \cdot{ }_{1} \alpha\right)
\end{aligned}
$$

for $\alpha, \beta \in D, r, s \in R$ and both sides are defined.
iv) $\left(\alpha+{ }_{1} \beta\right)+{ }_{2}\left(y+{ }_{1} \delta\right)=(\alpha+2 y)+1(\beta+2 \delta)$
$\left(\alpha *_{1} \beta\right) *_{2}\left(\gamma *_{1} \delta\right)=\left(\alpha *_{2} y\right) *_{1}\left(\beta *_{2} \delta\right)$
$\left(\alpha+{ }_{1} \beta\right) *_{2}\left(\gamma+{ }_{1} \delta\right)=\left(\alpha *_{2} \gamma\right)+_{1}\left(\beta *_{2} \varepsilon\right)$
$\left(\alpha+_{2} \beta\right) *_{1}\left(\gamma+_{2} \delta\right)=\left(\alpha *_{2} \gamma\right)+_{2}\left(\beta *_{1} \delta\right)$
for $\alpha, \beta, \gamma, \varepsilon \in D$ and both sides are defined.
v) $c_{1}\left(a+a_{1}\right)=c_{1} a+c_{2} c_{1}, c_{2}\left(b+b_{1}\right)=c_{2} b+c_{2} b_{1}$
for $a, a_{1} \in D_{1}, b, b_{1} \in D_{2}$ and the additions are defined . (t)
Thus we get a category of double R-algebroids DA .
We can ask now what is the relation between the category of crossed modules (over algebroids) and the category of double R-algebroids. At this stage we prove the following ;
Proposition: If $D$ is a double R-algebroid, then we have two crossed modules associated with D. That is, there exist two functor from the category of double R-algebroids to the category of crossed modules (over algebroids).
(t) $\varepsilon_{1}(a * b)=\varepsilon_{1} a *_{2} \varepsilon_{1} b, \varepsilon_{2}(a * b)=\varepsilon_{2} a * \varepsilon_{2} b, \varepsilon_{1}(r \cdot a)=r_{2} \varepsilon_{1} a$, $\varepsilon_{2}(r \cdot a)=r_{i} \varepsilon_{2} a, \mid L_{x} \varepsilon_{v i} \Gamma^{\prime} b_{x}=\varepsilon_{1} \varepsilon x$.

In the end of this chapter we give some examples on this notion.

In chapter III, we define the notion of a special double R-algebroid (this is a double R-algebroid with $D_{1}=D_{2}$ ) and we define a "thin" structure on $D$ which is a morphism $\theta: G D_{1} \rightarrow D$ (where $O D_{1}$ is a double $R$-algebroid with commuting squares). An element $e\left(a_{b}^{c} d\right)$ is called thin , where $a, b, c, d \in D_{1}$. Also we define a connection on $D$ to be $a$ pair of functions $\Gamma, \Gamma^{\prime}: D_{1} \rightarrow D$ which satisfy
i) $\Gamma^{\prime} a *_{2} \Gamma a=\varepsilon_{1} a, \Gamma^{\prime} a *_{1} \Gamma a=\varepsilon_{2} a$
ii) $\Gamma^{\prime}(a b)=\left(\Gamma^{\prime} a *_{1} \varepsilon_{1} a\right) *_{2}\left(\varepsilon_{2} a *_{2} \Gamma^{\prime} b\right)$
$\Gamma(a b)=\left(\Gamma a *_{1} \varepsilon_{2} b\right) *_{2}\left(c_{1} b *_{1} \Gamma b\right)$
iii) $\Gamma^{\prime}\left(a+a_{1}\right) *_{2}\left(\alpha+_{1} \beta\right) *_{2} \Gamma\left(d+d_{1}\right)=\left(\Gamma{ }^{\prime} a *_{2} \alpha *_{2} \Gamma d\right)+_{2}$ $\left(\Gamma^{\prime} a_{1} *_{2} \beta *_{2} \Gamma d_{1}\right)$
where $\alpha, \beta \in D$ with boundaries ( $a_{b}^{c} d$ ), ( $a_{1}{ }_{b}^{c} d_{1}$ ) respectively

$$
\begin{aligned}
& \text { iv) } \Gamma^{\prime} r a *_{2}(r \cdot 1 \alpha) *_{2} \Gamma r d=r \cdot 2\left(\Gamma^{\prime} a *_{2} \alpha *_{2} \Gamma d\right)=\Gamma^{\prime} a *_{2} \\
& (r \cdot 2 \alpha) *_{2} \Gamma d .
\end{aligned}
$$

Theorem 1 : Let $D$ be a special double R-algebroid with connection $5^{\prime}, \Gamma$. Then there is morphism of special double R-algebroids $e: ~ Q D_{1} \longrightarrow D$ which is the identity on $D_{1}$ and $\Gamma_{a}=\theta\left(a_{1}^{a} l\right), \Gamma^{\prime} b=\theta\left(l_{b}^{l} b\right)$, where $a, b \in D_{1}$.

In fact, the reason for defining these two structures on D is that ;

It is easier to deal with connection than with thin structure.

Also we define a morphism between two special double R-algebroids with connection . Thus we have a category DA! of special double R-algebroids with connection. Then we get a functor from the category of special double R-algebroids with connection to the category of crossed modules.

Now, to get a functor from the category of crossed modules to the category of special double $R$-algebroids with connection we introduce the notion of a folding operation": $\Phi$ which has the effect of "folding" all edges $\alpha \in D$ onto the edge $a_{1}^{0} \alpha$. We prove;

Proposition: There exists a functor from the category of crossed modules to the category of special double R-algebroids with connections .

In fact, we prove
Theorem 2: These Categories are equivalent.
Finally we introduce the notion of a reflection on a special double $R$-algebroid which gives an equivalence between the two algebroid structures .

In chapter IV, we define an $\omega$-algebroid (without connections) by using the cubical complex idea namely ;

An w-algebroid (without connections) $A=\left\{A_{n} ; \partial_{i}^{\alpha}, c_{i}\right\}$
is a cubical complex and for $n \geqslant 1, A_{n}$ has $n$ algebroid structures over $A_{n-1}$ of the form $\left(A_{n},+_{i}, *_{i}, y_{i}, \partial_{i}^{0}, \partial_{i}^{1}, c_{i}\right)$ related appropriately to each other and to $a_{i}^{0}, \partial_{i}^{1}, c_{i}$. Thus we can define finite dimensional versions of the above definition. Therefore we get ;

Algebras $\rightarrow$ Algebroids $\rightarrow$ Double algebroids $\rightarrow$ w-Algebroids, (the arrows give the generalisation of these notions). Also we define a crossed complex $M$ (over algebroid) to consist of a sequence of morphisms of R-algebroids over $M_{0}$

M:

$$
\ldots \xrightarrow{M_{n}} \xrightarrow{\delta} M_{n-1} \xrightarrow{\delta} \ldots \ldots-\stackrel{\delta}{\rightarrow} M_{2} \xrightarrow{\delta} M_{1}
$$

satisfying the relations;
i) each $\delta: M_{n} \rightarrow M_{n-1}, n \geqslant 2$ is the identity on $M_{0}$.
ii) $M_{1}$ operates on the right and on the left on each $M_{n}$
( $n>2$ ) , by actions $(a, m) \nrightarrow a_{m},(m, b) \neq m b$,
whenever $m \in M_{n}(x, y)$, a $\in M_{1}(w, x)$, $b \in M_{1}(y, z)$ and $\delta$ preserves these actions.
iii) If $m \in M_{n}(x, y), m^{\prime} \in M_{2}(y, z), m^{\prime \prime} \in M_{2}(w, x)$, then
$m^{\delta m}= \begin{cases}0_{x z} & \text { if } n \geqslant 3 \\ m m & \text { if } n=2\end{cases}$
$\delta_{m} \|_{m}= \begin{cases}0_{w y} & \text { if } n \geqslant 3 \\ m "_{m} & \text { if } n=2\end{cases}$
Finally we prove that ;
Theorem 3: There exists a functor $y$ from the category of w-algebroids (without connections) to the category of crossed complexes (over algebroids) .

In chapter $V$, $\$ 1$ we define an $\omega$-algebroid with connections and the morphisms between them and also we give the definition of a finite dimensional versions of an $\omega$-algebroid.

In $\$ 2$ we introduce the notion of "folding operation" $\$$, which has a similar effect to the folding operation in the two dimensional case . Also we give the relations between this operation and the axioms of 3 and 4 - tuple algebroids, that is,

Proposition: Let a $\in A_{n}$. Then $\Phi$ a belongs to the associated crossed complex $\gamma$ A .

Proposition: i) If $a, b \in A_{n}$ with $\partial_{j}^{\alpha}=\partial_{j}^{\alpha}$, for $\alpha=0,1$,
then $\Phi(\mathrm{a}+\mathrm{j} \mathrm{b})=\Phi \mathrm{a}+{ }_{\mathrm{n}} \Phi \mathrm{b}$.
ii) For $n=3,4$, if $a, b \in A_{n}$ with $\partial_{j}^{1} a=a_{j}^{0} b$, then
$\Phi\left(a *_{j} b\right)=u_{j}{ }_{(\Phi b)}+_{n}(\Phi a)^{v_{j}}{ }^{b}$
where $u_{j} a=\partial_{1}^{0} \ldots \partial_{j-1}^{0} \partial_{j+1}^{0} \ldots \partial_{n}^{0} a$ and $v_{j} b=\partial_{1}^{1} \ldots \partial_{j-1}^{1} \partial_{j+1}^{1} \ldots \partial_{n}^{1} b$.
iii) If a $\in A_{n}$ and $r \in R$, then

$$
\Phi\left(r \cdot j^{a}\right)=r \cdot{ }_{n} \Phi a
$$

Proposition: 1) For $n=3$, let a $\in A_{2}$. Then
$\Phi c_{i} a=\Phi \Gamma_{j} a=\Phi \Gamma_{j}^{\prime} a=0$ in dimension 3 for $1 \leqslant i \leqslant 3$ and
$1 \leqslant j \leqslant 2$.
2) For $n=4$, let $a \in A_{3}$. Then
$\Phi c_{i} a=\Phi \Gamma_{j} a=\Phi \Gamma_{j}^{\prime} a=0$ in dimension 4 for $1 \leqslant i n<4$ and l < $\mathbf{j}$ 6 3 .

Also we define a thin structure on As follows ;
let a $\in A_{n}$, then a is called thin if and only if $\Phi_{a}=0$.
In $\$ 3$ we construct the coskeleton in terms of "shells" for an n-tuple algebroid and we define $\partial_{i}^{\alpha}, \varepsilon_{i}, \Gamma_{i}, \Gamma_{i}$ and the operations on $\quad \square A_{n}$ to prove the following ;

Proposition: If ( $A_{n}, \ldots, A_{0}$ ) is an $n$-tuple algebroid, then ( $\left(A_{n}, A_{n}, \ldots, A_{0}\right.$ ) is an ( $n+1$ )-tuple algebroid.

Proposition: Let $A$ be an $\omega$-algebroid and let $\underline{M}=\gamma \underline{A}$ be its associated crossed complex. Let a $\in \square_{n-1}$ and $\varepsilon \in M_{n}(u, v)$ where $u=\beta_{0} a, v=\beta_{1} a$. Then a neccessary and sufficient
condition for the existence of $b \in A_{n}$ such that $\underset{b}{ }=a$ and $\Phi b=\varepsilon$ is that $\delta \varepsilon=\delta \Phi$ da . Further if bexist, it is unique. In $\$ 4$ we construct a functor $\lambda$ from the category of 3-truncated crossed complexes to the category of 3-tuple algebroids by using the folding operation. Also we prove that ;

Theorem 4: The functors $\gamma, \lambda$ form an adjoint equivalence

$$
y:(\underline{\omega-A l g})^{3} \longleftrightarrow(\underline{\text { Crs }})^{3}: \lambda
$$

## CHAPTER I

## R-ALGEBROIDS

## 0. INTRODUCTION :

We begin this chapter by defining $R$-algebroids and their morphisms. These have been studied in several papers, $[\mathrm{Po}-1],[\mathrm{Mi}-1],[\mathrm{Mi}-2],[\mathrm{Mi}-3],[\mathrm{A}-1]$.

For instance B.Mitchell [Mi-1,2,3] has given a categorical definition of R-algebroids, and obtained some interesting results on these gadgets. His definition is the following .

Let $R$ be a commutative ring . An R-category $A$ is a category equipped with an $R$-module structure on each hom set such that composition is R-bilinear. An R-functor is a functor $T: A \rightarrow B$ between $R$-categories such that the maps

$$
T: A\left(a_{1}, a_{2}\right) \cdots B\left(T a_{1}, T a_{2}\right)
$$

are R-linear.
In the language of enriched categories, one can define an R-category to be a category which is enriched over the closed category of R-modules . An R-category with one object is an associative R-algebra with identity.

An R-algebroid $A$ is a small R-category. If $A$ and $A^{\prime}$ are R-algebroids, define $A \otimes_{R^{\prime}} A^{\prime}$ by $O b\left(A \Theta_{R^{\prime}} A^{\prime}\right)=O b A \times O b A^{\prime}$,

$$
A \otimes_{R^{\prime}} A^{\prime}\left((a, b),\left(a^{\prime}, b^{\prime}\right)\right)=A\left(a, a^{\prime}\right) \otimes A^{\prime}\left(b, b^{\prime}\right) .
$$

Composition is the unique $R$-bilinear map satisfying

$$
\left(a a^{\prime}\right)\left(b \otimes b^{\prime}\right)=a b \otimes a^{\prime} b^{\prime} .
$$

The enveloping $R$-algebroid of an $R$-algebroid $A$ is

$$
A^{e}=A \theta_{R} A^{O p}
$$

An R-algebroid $A$ is separable if $A$ considered as its own hom functor is projective as an $A^{e-m o d u l e . ~ I t ~ i s ~ c e n t r a l ~ i f ~}$ the map $R \rightarrow \operatorname{Hom}_{\Lambda} e^{(A, A)}$ is an isomorphism.

Two R-algebroids are Morita equivalent if their module categories are R-equivalent.

Before we state the first result of [Mi-3], let us give the definition of the Brauer group of the commutative ring .

Let $R$ be a commutative ring and let $V(R)$ denote the isomorphism classes of all algebras having $R$ as center and which are separable over $R$. Let $V_{0}(R)$ be the subset of $V(R)$ consisting of the algebras $H^{\prime \prime} m_{R}(E, E)$ where $E$ is any finitely generated projective faithful $R$-module. One can prove that $V(R), V_{0}(R)$ are closed under the operation of tensor product over $R$ (see [A-G-1]).

Define an equivalence relation in $V(R)$ as follows : if $\delta_{1}, \delta_{2}$ are in $V(R)$, then $\delta_{1}$ is equvalent to $\delta_{2}$ if there are algebras $\Delta_{1}$ and $\Delta_{2}$ in $V_{0}(R)$ such that $\delta_{1} \otimes_{R} \Delta_{1} \cong \delta_{2} \otimes_{R} \Delta_{2}$. Let $B(R)$ denote the set of equivalence classes of $V(R)$. Then $B(R)$ is an abelian group [A-G-l].

Now we are ready to state the result given in [Mi-3]; namely that the Morita class of an R-algebroid A is an element of $B(R)$ if and only if $A$ is central, separable and equivalent to an algebra.

One of the reasons to generalise algebras to algebroids is that an R-algebroid $A$ which is only separable need not be equivalent to an algebra. Thus algebroids give a new direction in the theory of separability.

All the above material has been given in [Mi-1,2,3]. In [Po-l], T.Porter has defined an R-algebroid in a slightly different setting . He has defined an R-algebroid $A$ on a fixed set of "objects" $A_{0}$ to be a disjoint family of R-modules, so that $A$ need not have identities . Also he defined an action of an R-algebroid on a "C-structure". Finally he defined a crossed module and linked crossed modules with internal groupoids . More precisely, he proved that in the category of R-algebroids over a fixed set, any internal category is an internnal groupoid.

Now we move from this setting to say that it is well known that groups are appropriately generalised to groupoids , (see for example [Br-l],[Hi-l]). As explained above algebras are appropriately generalised to algebroids ; we give an example in section 1 to illustrate this. Moreover we give the definition of a tensor product between two $R$-algebroids and reprove the known fact that the category of algebroids is a monoidal closed category [Mi-l].

In sections 2 and 3 we give the definition of a crossed module over an associative algebra (see for example [Ge-l], [K-L-1], [E1-1]) and introduce the notion of crossed module over an algebroid. Also we give some properties similar to those well known for crossed modules over groups.

## 1. R-ALGEBROIDS:

The material of this section may be found in [Mi-1], [Mi-2] , [Po-1]. We shall give the definition of an

R-algebroid $A$ on a set of "objects" $A_{0}$ in the following way : Recall that $A$ is called a directed graph over a set $A_{0}$ if there are given functions $\partial^{0}, \partial^{1}: A \rightarrow A_{0}, C: A_{0} \rightarrow A$, called respectively the source, target and unit maps, such that $\quad \partial^{O_{C}}=\partial^{1} c=1_{A_{0}}$. Then we write
$A(x, y)=\left\{a \in A: \partial^{0} a=x, \partial^{2} a=y\right\}$, and write $l_{x}$ for $e_{x}$. If a $\in A(x, y)$, we also write $a: x-\rightarrow y$.

An R-algebroid ( $A, A_{0}, \partial^{0}, \partial^{2}, c,+,$. ) (which is abbreviated to A) is a directed graph $A$ over $A_{0}$ together with for all $x, y, z \in A_{0} ;$
i) an $R$-module structure on each $A(x, y)$,
ii) an R-bilinear function, called composition,

$$
\left.\begin{array}{r}
*: \Lambda(x, y) \times \Lambda(y, z) \\
(a, b) \quad \rightarrow A(x, z) \cdot \\
\end{array}\right)
$$

The only axioms are that composition is associative, and that the elements $l_{x}, x \in \Lambda_{0}$, act as identities for composition : if $a: x \rightarrow y$, then $l_{x} * a=a * l_{y}=a$. Thus the composition makes $A$ into a small category.

A morphism $f: A \rightarrow B$ of R-algebroids $A, B$ is a functor of the underlying categories which is also R-linear on each $A(x, y) \rightarrow B(f x, f y)$. The set of all morphisms. $A \rightarrow B$ is written $\operatorname{Hom}_{R}(A, B)$. Note that a morphism $f: A \rightarrow B$ preserves the identities.

The zero of $A(x, y)$ is written 0 , or $0_{x y}$ if additional clarity is required. As usual , bilinearity implies $a * 0=0,0 * a=0$, whenever these are defined.

## Examples:

1) If $A_{0}$ has exactly one object, then an R-algebroid over $A_{0}$ is an R-algebra.
2) If $A$ is an R-algebroid over $A_{0}$ and $x \in A_{0}$, then $A(x, x)$ is an R-algebra.

We now come to one of the most important features of the category of R-algebroids namely that it has an internal hom functor.

Let $A, B$ be $R$-algebroids . Suppose given $f, g \in \operatorname{Hom}_{R}(A, B)$; we define Hom(f,g) to be the set of all "natural
transformations " $f \rightarrow g$, that is, the set of all functions $b: A_{0} \rightarrow B$ such that $b x \in B(f x, g x), x \in A_{0}$, and for all $x, y \in A_{0}$ and $a \in A(x, y)$ the following square

commutes. Then Hom(f,g) is given the structure of R-module $b y\left(r b+r^{\prime} b^{\prime}\right) x=r b x+r^{\prime} b^{\prime} x$, whenever $x \in A_{0}$ and $r, r^{\prime} \in R$. There is a bilinear composition

$$
\begin{array}{r}
\text { Hom }(f, g) \times \underline{\operatorname{Hom}(g, h)} \rightarrow \underline{\operatorname{Hom}(f, h)} \\
\left(b, b^{\prime}\right) \cdots b * b^{\prime}
\end{array}
$$

Where $\left(b * b^{\prime}\right) x=b x * b^{\prime} x$. Then we get ;
Proposition 1.1.1: With the above structure, the family

$$
\underline{\operatorname{Hom}}(A, B)=\{\underline{\operatorname{Hom}}(f, g)\}_{f, g} \in \operatorname{Hom}_{R}(A, B)
$$

is an R-algebroid.
A special case is when $A, B$ are R-algebras ; we still get an R-algebroid Hom ( $A, B$ ) and this is one of the motivating examples for considering the extension from R-algebras to R-algebroids.

Definition 1.1.2: If $A, B$ are two R-algebroids over $A_{0}, B_{0}$ respectively, we define the tensor product $A \otimes_{R} B$ over $A_{0} \times B_{0}$
to be the family of $R$-modules

$$
\left\{A(x, y) \otimes_{R} B(u, v): x, y \in A_{0}, u, v \in B_{0}\right\}
$$

with composition ( $a^{\prime} \otimes b^{\prime}$ ) * (a © $b$ ) $=\left(a^{\prime} * a\right)\left(b^{\prime} * b\right)$.
Lemma 1.1.3: Let $A, B$ be R-algebroids over $A_{0}, B_{o}$ respectively. Then $A \otimes_{R} B$ is an $R$-algebroid over $A_{0} \times B_{0}$.

Proposition 1.1.4: Let $A, B, C$ be R-algebroids. Then there is a natural isomorphism between $\operatorname{Hom}_{R}\left(A \theta_{R} B, C\right)$ and
$\operatorname{Hom}_{R}(A, \underline{H o m}(B, C))$.
Proof:
Define a map $\eta: \operatorname{Hom}_{R}\left(A \otimes_{R} B, C\right) \rightarrow \operatorname{Hom}_{R}(A, \underline{H o m}(B, C))$ as follows: if $\Phi: A \otimes_{R} B \rightarrow C$, then $\Pi(\Phi): A \rightarrow H o m(B, C)$ and if $x \in O b(A)$, then $n(\Phi)(x)$ is to be a morphism $B \rightarrow C$, given on objects by $y \rightarrow \Phi(x, y)$ and on arrow $b: y \rightarrow y^{\prime}$ by
$(n(\Phi)(x))(b)=\Phi\left(1_{x} \otimes b\right)$. If a is an arrow in $A$, then
$\eta(\Phi)(a) \in \operatorname{Hom}(B, C)$ which is given on objects by
$y \rightarrow \phi\left(a \otimes l_{y}\right)$ and on arrows $b: y \rightarrow y^{\prime}$ by
$(\eta(\Phi)(a))(b)=\Phi(a \otimes b)$.
Define a map $n^{\prime}: \operatorname{Hom}_{R}(A, \underline{H o m}(B, C)) \rightarrow \operatorname{Hom}_{R}\left(A \theta_{R} B, C\right)$ as follows :
if $\Psi: A \rightarrow \underline{H o m}(B, C)$, then $\eta^{\prime}(\psi): A \otimes_{R^{B}} \rightarrow C$. If $(x, y)$ is an object in $O b(A) \times O b(B)$, then we define $\eta^{\prime}(\Psi)(x, y)=\Psi(x)(y)$ and if a $\otimes b$ is an arrow in $A \otimes_{R} B$ such that a : $x \rightarrow x^{\prime}, b: y \rightarrow y^{\prime}$, then $\Psi(x), \Psi\left(x^{\prime}\right): B \rightarrow C$ and so $\Psi(a): \Psi(x) \rightarrow \Psi\left(x^{\prime}\right)$ and $\Psi(x)(b): \eta^{\prime} \Psi(x, y)^{\prime} \rightarrow \eta^{\prime} \Psi\left(x, y^{\prime}\right)$, $\Psi\left(x^{\prime}\right)(b): \eta^{\prime} \Psi\left(x^{\prime}, y\right) \rightarrow \eta^{\prime} \Psi\left(x^{\prime}, y^{\prime}\right) \ldots$

Thus we get the diagram in $C$


Define $\eta^{\prime} \Psi(a \otimes b)=\Psi(a)\left(y^{\prime}\right) \psi(x)(b)=\Psi\left(x^{\prime}\right)(b) \Psi(a)(y)$.
Now we want to show that $n n^{\prime}=1, n^{\prime} n=1$. For $n n^{\prime}=1$, let $\Psi: A \rightarrow H O M(B, C)$ and let $(x, y) \in O b(A) \times O b(B)$, then $\Pi \Pi^{\prime}(\Psi)(x, y)=\eta(\Psi(x)(y))=\Psi(x, y)$. If $a \otimes b \in A \otimes_{R} B$, then $\eta \eta^{\prime}(\Psi)(a \otimes b)=\Pi\left(\Psi(a)\left(y^{\prime}\right) \Psi(x)(b)\right)=\Psi(a \otimes 1 y) \Psi\left(l_{x} \otimes b\right)=\psi(a \otimes b)$. Thus $n n^{\prime}=1$.

For $n^{\prime} n=1$, let $\Phi: A \theta_{R} B \rightarrow C$ and let $x \in O b(A)$, then $\left(\eta^{\prime}(\eta(\phi))(x)\right)(y)=\eta^{\prime}(\phi(x, y))=\Phi(x)(y)$ for $y \in O b(B)$ and $\eta^{\prime}((n(\Phi))(x))(b)=\eta^{\prime}\left(\Phi\left(1_{x} \otimes b\right)\right)=\Phi\left(l_{x}\right)\left(y^{\prime}\right) \Phi(x)(b)=\Phi(x)(b)$ for $b: y \rightarrow y^{\prime} \in B$.
Hence $\eta^{\prime}((\eta(\Phi))(y))=\eta^{\prime}\left(\Phi\left(a \otimes 1_{y}\right)\right)=$ $\Phi(a)(y) \Phi(x)\left(l_{y}\right)=\Phi(a)(b)$, whenever $a: x \rightarrow x$, and $b: y \rightarrow y^{\prime}$. That is, the category of R-algebroids can be given the structure of a monoidal closed category.

For other properties of the category of R-algebroids which are not valid in the category of R-algebras see. [ $M-1,2,3]$.

If the unit map is omitted from the algebroid structure then we obtain an R-algebroid (without identities) -
Remark 1.1.5: Let A, B be algebroids (without identities) and let $M\left(A_{0}, B_{0}\right)$ denote the set of functions $A_{0} \rightarrow B_{0}$. Let be the function

$$
\begin{gathered}
\theta: \operatorname{Hom}_{R}(A, B) \rightarrow M\left(A_{0}, B_{0}\right) \\
f(
\end{gathered}
$$

Then each fibre $e^{-1}(h)=\operatorname{Hom}_{R}(A, B ; h)$ can be given the structure of $R$-module by $(f+g) a=f a+g a,(r f) a=f(r a)$, for alla $\boldsymbol{C} \boldsymbol{A}, r \in R$.

## 2. CROSSED MODULES (OVER ASSOCIATIVE ALGBBRAS):

The general concept of crossed module originates in the work (1949) of J.H.C.Whitehead [Wh-l],[Wh-2] in algebraic topology. There the crossed modules were free crossed modules of groups. Also the notion of crossed module has been studied by Peiffer [Pe-l] and Reidemeister [R-1], and they have defined identities among relations . For further detail see the survey of Brown-Huebschmann [B-Hu-1]. In the group case, a crossed module generalises the concepts of a normal subgroup and that of an ordinary module.

The work of [K-L-1] in algebraic K-theory has introduced crossed modules of Lie algebras. In fact they have studied a fibration in lie-algebras and they found that the induced map of the fibration gives a crossed module. The early work of [Ge-l], [L-1] and [L-S-l] essentially involves the notion of crossed modules in associative algebras and commutative algebras under different names, which thęy use to define cohomology groups of algebras . Also [L-R-l] has analysed crossed modules in associative algebras, and the general case of crossed modules in a category of interest $C$ has been discussed in [Po-2]: he has proved that "the category of
internal categories in a category of interest $C$ is equivalent to the category of crossed modulesin C. For the precise result, see [Po-2].

In this section, we give the definition of crossed module in the category of associative algebras in order to set the stage for the definition of crossed module over algebroid in the next section.

Fix a commutative ring $R$ (with unit), and let al be the category of associative algebras over $R$.
We define now an associative action in the category AL as follows:

Let $A, M$ be associative algebras over $R$. An associative action of $A$ on $M$ is a pair of maps

$$
\begin{array}{ll}
A \times M \rightarrow M & , \\
(a, m) \rightarrow A \rightarrow M \\
a_{m} & , \\
(m, a) \rightarrow m^{a}
\end{array}
$$

such that $M$ is a left and right $A$-module (bi-A-module), that is,
i) $\left(m+m^{\prime}\right)^{a}=m^{a}+m^{\prime} a \quad, \quad a\left(m+m^{\prime}\right)=a_{m}+a_{m}$,
ii) $m^{a+a^{\prime}}=m^{a}+m^{a^{\prime}} \quad ; \quad a^{\prime} a^{\prime} m=a_{m}+a^{\prime} m$,
and satisfy the conditions :
iii) $\left(m \cdot m^{\prime}\right)^{a}=m \cdot m^{\prime a} \quad ; \mathbf{a}^{a}\left(m \cdot m^{\prime}\right)=a_{m} \cdot m^{\prime}$,
iv) $m^{a a^{\prime}}=\left(m^{a}\right)^{a^{\prime}} \quad, \quad \mathbf{a} a^{\prime} m^{a}\left(a^{\prime} m\right),\left(a_{m}\right)^{a^{\prime}}={ }^{a}\left(m^{a^{\prime}}\right), 1_{m}=m^{1}=m$
v) $r\left(m^{a}\right)=m^{r a}=(r m)^{a}, r\left(a_{m}\right)=r a_{m}={ }^{a}(r m)$, for all $r \in R, m, m^{\prime} \in M$, and $a, a \in A$.

A crossed module in AL is an associative algebra morphism $\mu: M \rightarrow A$ with an associative action of $A$ on $M$ such that :
i) $\mu\left(a_{m}\right)=a \cdot(\mu m), \mu\left(m^{a}\right)=(\mu m) \cdot a$
ii) $\mu_{m_{m}}=m m^{\prime} \quad, \quad m_{m} \mu^{\prime}=m m^{\prime}$
for all $m, m^{\prime} \in M$ and $a \in A$.
Examples: 1) Let $A$ be an associative R-algebra and let $I$ be a two-sided ideal in $A$. Let $i: I \rightarrow A$ be the inclusion map , then i with action of $A$ on $I$ given by multiplication is a crossed module.
2) Let $A, M$ be associative algebras and let $M$ be a bi A-module. Then the zero map from $M$ to $A$ is a crossed module with the action given by bimodule structure.

Now we move on and in the next section to give the definition of crossed module (over an algebroid) by using the above definition.

## 3. CROSSED MODULES (OVER ALGEBROIDS):

In the previous section, we defined a crossed module in the context of associative algebras. In this section we define a crossed module over an algebroid.

Let $A_{0}$ be a set and let $A, M$ be two R-algebroids over $A_{0}$, where $M$ need not have identities . Suppose $A$ operates on $M$ on the right and on the left as follows :

Let $m: x \rightarrow y \in M$ and $a \in A(w, x)$, $b \in A(y, z)$, then we denote the right action $b y m^{b} \in M(x, z)$, and the left action by $a_{m} \in M(w, y)$ as shown in the diagram below

$$
\begin{array}{cc}
w-\frac{q}{\rightarrow}-x & y-\xrightarrow{-m}-y \\
a_{\text {m }} \in M(w, y) & m b \in M(x, z) \\
\text { left action } & \text { right action }
\end{array}
$$

such that these actions satisfy the following axioms ; (1.3.1)

$$
\begin{equation*}
\left(a_{m}\right)^{b}=a^{\left(m^{b}\right)} \tag{1.3.1}
\end{equation*}
$$

$\left(m^{a}\right)^{b}=m^{a b}$
$a\left(m m^{\prime}\right)=a_{m} m^{\prime}, \quad b\left(a_{m}\right)=b a_{m},\left(m m^{\prime}\right)^{a}=m m^{\prime}$,
$m^{a+b}=m^{a}+m^{b}, \quad a+b_{m}=a_{m}+b_{m}$
$\left(m+m_{1}\right)^{b}=m^{b}+m_{1} b, a\left(m+m_{1}\right)=a_{m}+a_{m_{1}}$
$(r m)^{b}=r m^{b}=m^{r b}, \quad \mathbf{a}(r m)=r^{a_{m}}=r_{m}$
We assume that $A$ has an identity
$l_{x_{m}}=m=m^{1} y$
for all $a, b \in A, m_{1}, m_{1} \in M$ and $x, y \in A_{0}$. Thus we get :
$\frac{\text { Definition 1.3.2: }}{\text { has an identity }}$ Let $A, M$ be two R-algebroids over Ao such that $A$ morphism $\mu: M \rightarrow A$ is called a crossed module if there are actions of $A$ on $M$ satisfying the above axioms and also the following axioms :
$\mu\left(m^{b}\right)=(\mu m) b, \mu\left(a_{m}\right)=a(\mu m)$
$m_{m}{ }^{\prime}=\mu_{m} \mu^{\prime}=\mu_{m}$,
for all $a, b \in A, m, m^{\prime} \in M$ and both sides are defined.
Definition 1.3.3: A morphism of crossed modules $(\alpha, \beta):(A, M, \mu) \rightarrow\left(A^{\prime}, M^{\prime}, \mu^{\prime}\right)$ is two algebroid morphisms $\alpha: A \rightarrow A^{\prime}, \beta: M \rightarrow M^{\prime}$ such that $\alpha \mu=\mu^{\prime} \beta$ and $\beta\left(a_{m}\right)=\alpha_{\beta m}, \beta\left(m^{b}\right)=\beta_{m} \alpha b$, for all $a, b \in \Lambda, m \in M$ and $\alpha: A \rightarrow A^{\prime}$ is to preserve identities. Thus we have a category $C$ of crossed modules (over algebroids).

To give examples of such crossed modules, we define a subalgebroid and two-sided ideal.
Definition 1.3.4: Let $A$ be an R-algebroid over $A_{0}$. A subalgebroid $A^{\prime}$ is a disjoint family of R-submodules

$$
\left\{A^{\prime}(x, y) \subseteq A(x, y)\right\} x, y \in A_{0}
$$

with units and each R-bilinear function

$$
A^{\prime}(x, y) \times A^{\prime}(y, z) \rightarrow A^{\prime}(x, z)
$$

is the restriction of the R-bilinear function

$$
A(x, y) \times A(y, z) \cdots A(x, z)
$$

Definition 1.3.5: Given an R-algebroid $A$ over $A_{0}$, a two-sided ideal $I$ in $A$ is a family of R-submodules

$$
\{I(x, y) \subseteq A(x, y)\}_{x, y \in A_{0}}
$$

such that $I$ satisfies the axiom:
if $a \in I(x, y), b \in A(z, x), c \in A(y, w)$, then $b a \in I(z, y)$ and ac $\in I(x, w)$.

Example: Let $A$ be an R-algebroid over $A_{0}$ and suppose $I$ is a two-sided ideal in $A$. Let $i: I \rightarrow A$ be the inclusion morphism and let $A$ operate on $I$ by
(i) $a^{c}=a c$ (ii) $b_{a}=b a$, for $a l l a \in I, b, c \in A$.

Then i:I $\rightarrow A$ is a crossed module. Clearly I is an R-algebroid (without identities) .
Remark 1.3.6: Let $f: A \rightarrow B$ be an algebroid morphism, where $A, B$ are defined over the same set $A_{0}$ and $O b(f)=l_{A_{0}}$. Then ker $f=\left\{a \in A(x, y): f a=0_{x y}\right.$ for all $\left.x, y \in A_{0}\right\}$ is a two-sided ideal in $A$.
Proposition 1.3.7: Let $\mu: M \rightarrow A$ be a crossed module of algebroids. Then $\operatorname{Im} \mu=\{\mu \mathrm{m}: \operatorname{m}\}$ is a two-sided ideal in A.

Proof: Let $a \in I m \mu$, so there is meM such that $\mu m=a$, for some a $\in A$. Let $b \in A$ such that $a b$ is defined, then $a b=\mu m b=\mu\left(m^{b}\right)$. Thus $a b \in \operatorname{Im} \mu$ and similarly ca $\in \operatorname{Im} \mu$, for ceA and ca is defined.

Let $I$ be a two-sided ideal in $A$. Then we can define quotient $R$-modules $A(x, y) / I(x, y)$ for all $x, y \in A_{0}$. Then there is an R-bilinear morphism
$*: A(x, y) / I(x, y) \times A(y, z) / I(y, z) \rightarrow A(x, z) / I(x, z)$
and associativity holds .
Then we get an R-algebroid $A / I$ which is the family of quotient $R$-modules

$$
\left\{A(x, y) / I(x, y): x, y \in A_{0}\right\}
$$

We call it the quotient $R$-algebroid and then there is a canonical mapping $p: A \rightarrow A / I$ of R-algebroids. Also we have an exact sequence

$$
0 \rightarrow I \xrightarrow{\mathbf{i}} A-\mathbf{D}_{\rightarrow} A / I \rightarrow 0
$$

Thus for any crossed module ( $A, M, \mu$ ), there is an exact sequence $\quad 0 \rightarrow$ ker $\mu \longrightarrow M \longrightarrow$ Im $\mu \rightarrow 0$

We can state some properties of algebroids.
i) Since $\operatorname{Im} \mu$ is a two-sided ideal, then coker $\mu=A / \mu M$ exist and hence there is an exact sequence

Im $\mu \rightarrow A \rightarrow$ coker $\mu$.
ii) Since $\mathrm{mm}^{\prime}=\mu \mathrm{m}_{\mathrm{m}}{ }^{\prime}$, and if $\mu \mathrm{m}=0$, then $\mathrm{m}, \mathrm{M}=0$ and M. $m=0$. Thus $m \in \operatorname{Ann}(M)$ (Ann means annihilator) and clearly Ann(M) is a subalgebroid of $M$. In particular.ker $\mu$. ker $\mu=0$. iii) Coker $\mu=A / \operatorname{Im} \mu$ acts on ker $\mu$.
iv) Let $0 \rightarrow K \rightarrow M-P \rightarrow B \rightarrow$ be a central extension, that is, it is a short exact sequence such that if $k \in$ and $m \in M$, then $k m=m k=0$. Then $p: M \rightarrow A$ can be give the structure of a crossed module.

Proof: For any a $\in A$, let sadenote an element of $M$ such that $p s a=a(t h u s s$ is a section of $p$ ).

Define actions of $A$ on $M$ as follows:
$\mathbf{a}_{\mathrm{m}}=(\mathrm{sa}) \mathrm{m} \quad, \quad \mathrm{m}^{b}=\mathrm{m}(\mathrm{sb})$.
First, to show that these actions are well-defined
Let $s^{\prime}$ be a section of $p$ and let $m_{1} \in M$, then we want to prove that
$\mathbf{a}_{\mathrm{m}}=(\mathrm{sa}) \mathrm{m}=\left(\mathrm{s}^{\prime} \mathrm{a}\right) \mathrm{m}$ for a $\in \mathrm{A}$.
Let $m_{1} \in M$, then $p m_{1}=a$ and hence $m_{1}-s a, m_{1}-s^{\prime} a \in K$. So $\left(m_{1}-s a\right) m=\left(m_{1}-s^{\prime} a\right) m=0$, then $(s a) m=\left(s^{\prime} a\right) m$. So the left action is well-defined. We can prove similarly that the right action is well-defined. It is clear that these actions satisfy the axioms for a crossed module. $\quad$ a

## DOUBLE R-ALGEBROIDS

## O. INTRODUCTION:

We begin this chapter by showing how to mimic the idea given in chapter $I$ in one higher dimension. That is, we look for "algebroids in two dimensions" . So we need two different additions and compositions .

In fact, we make an analogy to the idea given by R.Brown "Higher dimensional group theory" [Br-2] to define double R-algebroids .

In section 2 we prove that there exist two functors from the category of double R-algebroids to the category of crossed modules. Also we give examples of double R-algebroids in the third section.

## 1. DEFINITIONS:

The notion of double category has occured often in the literature (see for example, [Be-1],[Gr-1],[Ma-1],[Wy-1], $[K-S-1],[B-S-1],[S-W-1]$ and is due originally to Ehresmann [Eh-l]) . In this section we study an object with more structure than a double category, which we call a double R-algebroid

To define double R-algebroids, we start to give in some detail the definition of double category ;

Definition 2.1.1: [Eh-1],[B-S-1] By a double category $D$ is meant four related categories

$$
\begin{aligned}
& \left(D_{1} D_{1}, \partial_{1}^{0}, \partial_{1}^{1}, *_{1}, c_{1}\right),\left(D, D_{2}, \partial_{2}^{0}, \partial_{2}^{1}, *_{2}, c_{2}\right) \\
& \left(D_{1}, D_{0}, \delta_{1}^{0}, \delta_{1}^{1}, *, C\right),\left(D_{2}, D_{0}, \varepsilon_{2}^{0}, \delta_{2}^{1}, *, C\right)
\end{aligned}
$$

as partially shown in the diagram

and satisfying the rules (i-v) given below. The elements of $D$ will be called squares, of $D_{1}, D_{2}$ horizontal and vertical edges respectively, and of $D_{o}$ points or objects . We will assume the relation :
i) $\delta_{2}^{i} a_{2}^{j}=\delta_{1}^{j} a_{1}^{i}$
$i, j=0,1$
and this allows us to represent a square $\alpha \in D$ as having boundary edges pictured as

while the edges are pictured as


From now on we will write the boundary of a square as $\underline{\partial}$ (the square) for example the boundary of $\alpha$ is written as
$\underline{\partial} \alpha=\left(\partial_{2}^{O}, \alpha{ }_{\partial_{1}^{1} \alpha}^{\partial_{1}^{O} \alpha} \partial_{2}^{1} \alpha\right)$.
ii) $\partial_{2}^{i}\left(c_{1} a\right)=c \delta_{1}^{i} a$

$$
i=0,1
$$

$$
\partial_{1}^{j}\left(\dot{c}_{2} b\right)=c \delta_{2}^{j_{b}}
$$

$$
j=0,1 .
$$

So the identities $\varepsilon_{1}$ a,$\varepsilon_{2}$ form squares which have boundaries $\underline{a}\left(c_{1} a\right)=\left(c_{x}^{a} c_{y}\right), \underline{a}\left(c_{2} b\right)=\left(b c_{c}^{c z} b\right)$.
iii) $\varepsilon_{1} c x=\varepsilon_{2} c x$
iv) $a_{2}^{i}(\alpha *$
$\beta)=\partial_{2}^{i} \alpha * \partial_{2}^{i} \beta$
$i=0,1$
$\partial_{1}^{j}\left(\alpha *_{2} \beta\right)=\partial_{1}^{j} \alpha * \partial_{1}^{j} \beta \quad j=0,1$
for all $\alpha, \beta \in D$ such that both sides are defined.
$v)$ (The interchange law)

$$
\left(\alpha *_{1} \beta\right) *_{2}\left(y *_{1} \delta\right)=\left(\alpha *_{2} y\right) *_{1}\left(\beta *_{2} \delta\right),
$$

whenever $\alpha, \beta, \gamma, \varepsilon \in D$ and both sides are defined.
Definition 2.1.2: $A$ double R-algebroid $D$ is four related R-algebroids

$$
\begin{aligned}
& \left(D, D_{1}, \partial_{1}^{0}, \partial_{1}^{1}, c_{1},+_{1}, *_{1}, H_{1}\right),\left(D, D_{2}, \partial_{2}^{0}, \partial_{2}^{1}, c_{2},++_{2}, *_{2}, D_{2}\right) \\
& \left(D_{1}, D_{0}, \delta_{1}^{0}, \delta_{1}^{1}, c,+, *, .\right) \quad\left(D, D, \delta_{2}^{0}, \delta_{2}^{1}, c_{1}+, *_{,}\right)
\end{aligned}
$$

as shown in the diagram

and satisfying the rules given below.

The elements of $D$ will be called squares, of $D_{1}, D_{2}$ horizontal and vertical edges respectively and of $D_{0}$ the set of "objects".
(2.1.3)

$$
\delta_{2}^{i} \partial_{2}^{j}=\delta_{2}^{j} \partial_{1}^{i} \quad i, j \in\{0,1\}
$$

Then we can represent a square $\alpha$ as having boundary edges given by

where the edges pictured as
$\varepsilon_{1}^{0} a \ldots-a_{1} \ldots \varepsilon_{1}^{1} a$
a $\in D_{1}$

$$
\int_{\delta_{2}^{1} b}^{\delta_{2}^{0} b}
$$

$$
\mathrm{b} \in \mathrm{D}_{2}
$$

First, we assume on $D$ four operations $+_{1}, *_{1},+_{2}, *_{2}$ defined in the following way :

Let $\alpha, \beta, \gamma, \delta, \zeta \in D$ have boundaries given by
$\underline{\partial} \alpha=\left(a_{b}^{c} d\right), \underline{\partial} \beta=\left(a_{1}^{c} d_{1}\right), \underline{\partial} y=\left(a^{\prime} e^{b} d^{\prime}\right), \underline{\partial} \delta=\left(a{ }_{b_{1}}^{c} d\right)$
and $\partial \zeta=\left(\mathrm{d}_{\mathrm{b}}, \mathrm{e}\right)$.
Then $\alpha+_{1} \beta, \alpha *_{1} \gamma, \alpha+_{2} \delta, \alpha *_{2} 5$ have boundary edges in the form
$\underline{a}\left(\alpha+_{1} \beta\right)=\left(a+a_{1} \underset{b}{c} d+d_{1}\right), \underline{a}\left(\alpha *_{1} y\right)=\left(a a^{\prime} e^{c} d d^{\prime}\right)$,
$\underline{a}(\alpha+2 \varepsilon)=\left(a \underset{b+b_{1}}{c+c_{1}} d\right), \underline{a}\left(\alpha *_{2} \quad \varsigma\right)=\left(a_{b b^{\prime}}^{c} e\right)$.
So we are ready to give more rules for double R-algebroid (2.1.4)
$\partial_{2}^{i}\left(\alpha+{ }_{2} \beta\right)=\partial_{2}^{i} \alpha+\partial_{2}^{i} \beta \quad i=0,1$
$\partial_{1}^{i}\left(\alpha+{ }_{2} \beta\right)=a_{1}^{i} \alpha+a_{1}^{i} \beta \quad i=0,1$
$\partial_{2}^{i}\left(\alpha *{ }_{2} \beta\right)=\partial_{2}^{i} \alpha * \partial_{2}^{i} \beta$
$\partial_{1}^{i}\left(\alpha *_{2} \beta\right)=a_{1}^{i} \alpha * \partial_{1}^{i} \beta \quad i=0,1 \quad$ (2.1.4)(iv)
for all $\alpha, \beta \in D$ and both sides are defined.
(2.1.5)

We have two scalar multiplications ; for $\propto \in D$ as above and $r \in R$, so we define $r \cdot 1 \propto, r \cdot 2 \propto$ to have boundary edges in the form
$\underline{\partial}\left(r \cdot 1_{1} \alpha\right)=\left(r_{b}^{c} r d\right), \underline{\partial}(r \cdot 2 \alpha)=\left(a^{r c} d\right)$.
These multiplications are to satisfy the following axioms :


$r \cdot 1(s \cdot 2 \alpha)=s \cdot 2(r \cdot 1 \alpha)$
for all $\alpha, \beta \in D, r, s \in R$ and both sides are defined.
These rules make sense in terms of boundaries, for example, let $\alpha, \beta \in D$ have boundaries given by
$\underline{\partial} \alpha=\left(a_{b}^{c} d\right), \underline{\partial} \beta=\left(a_{b_{1}}^{c_{1}} d\right)$, then $\underline{\partial}(r .1 \alpha)=\left(r a{ }_{b}^{c} r\right)$,

and $\underline{\partial}\left[\left(r ._{1} \alpha\right)+_{2}(r \cdot 1 \beta)\right]=\left(r a \underset{b+b_{1}}{c+c_{1}} r\right)$, that is,
$r \cdot 1\left(\alpha+_{2} \beta\right)=(r \cdot 1 \alpha)+_{2}(r \cdot 1 \beta)$ in terms of boundaries. (2.1.6) (The interchange laws) :
$\left(\alpha+{ }_{1} \beta\right)+{ }_{2}\left(\gamma+{ }_{1} \delta\right)=\left(\alpha+{ }_{2} \gamma\right)+{ }_{1}\left(\beta+{ }_{2} \delta\right)$
which is diagrammatically as shown below :

$\left(\alpha *_{1} \beta\right) *_{2}\left(\gamma *_{1} \delta\right)=\left(\alpha *_{2} \gamma\right) *_{1}\left(\beta *_{2} \delta\right)$

$(\alpha+1 \beta) *_{2}\left(\gamma+_{1} \delta\right)=\left(\alpha *_{2} \gamma\right)+_{1}\left(\beta *_{2} \delta\right)$
which is diagrammatically given by

$\left(\alpha+_{2} \beta\right) *_{1}\left(\gamma+_{2} \delta\right)=\left(\alpha *_{1} \gamma\right)+_{2}\left(\beta *_{1} \delta\right) \quad(2.1 .6)(i v)$.
The explanation is similar to that for the interchange law (iii), whenever $\alpha, \beta, \gamma, \delta \in D$ and both sides are defined. (2.1.7)

We assume that each of the algebroid structures has identities and then $\varepsilon_{1}$, $\varepsilon_{2}$ give these identities in the following way ;
given $a \in D_{1}(x, y)$, $b \in D_{2}(x, y)$, then $c_{2} a, c_{2} b$ having boundaries given by ;
$\underline{\partial}\left(c_{1} a\right)=\left(1_{x}^{a} l_{y}\right), \underline{\partial}\left(c_{2} b\right)=\left(b^{1_{x}} b\right)$, such that $c_{1}, c_{2}$ are algebroid morphisms $\quad \varepsilon_{1}: D, \longrightarrow D ; \Sigma_{2}: D_{2} \longrightarrow D$.

We shall need later some simple facts on zero elements namely;

Remark 2.1.8: If $x, y \in D_{0}$, then we write 0 or $0_{x y}$ for the zero elements in both $D_{1}(x, y)$ and $D_{2}(x, y)$. However if $c \in D_{1}(x, y)$, b $\in D_{1}(z, w)$, then we have a set $D^{2}(c, b)=\left(a_{1}^{0}\right)^{-1}(c) \cap\left(a_{1}^{1}\right)^{-1}(b)$ and this set has zero which we write $0_{c b}^{1}$. The boundaries of this element are given by

and it is clear that $0_{c b}^{1}$ is the zero for $+_{2}$ in $D^{1}(c, b)$, where $D^{2}(c, b)$ is the set of arrows in direction 1 , from $c$ to b .

Also we can get a square $0_{a d}^{2}$ with boundaries given in the form

which is the zero for $+_{2}$ in $D^{2}(a, d)$. Notice that, if $\propto \in D$ is given by

then $0_{c b}^{2} *_{1} \alpha=0_{\text {ce }}^{2}, 0_{a d}^{2} *_{2} \alpha=0_{a f}^{2}$ by distributivity.
Definition 2.1.9: A morphism between two double R-algebroids D , B (over the same set of objects) is a triple of functions
$\Psi_{1}: D \rightarrow E, \Psi_{1}: D_{1} \rightarrow E_{1}, \Psi_{2}: D_{2} \rightarrow \mathrm{E}_{2}$ which together preserves all structures. Thus we get a category of double R-algebroids. Also we can define a morphism between two double R-algebroids on different sets of objects, by using the definition given in chapter lection 1 . Let us denote the category of double R-algebroids (over different sets of objects) by DA.

## 2. FUNCTORS (DOUBLE ALGEBROIDS) $\rightarrow$ (CROSSED MODULES):

In chapter 1 section 2 and in the previous section, we have defined two categories namely the category of crossed modules $C$ and the category of double R-algebroids DA.

In this section, we make an analogy with the result given in ([B-S-1], proposition 1 ) that is, we want to show how to obtain from a double R-algebroid two crossed modules (over algebroids) . We start with the main result of this section namely ;

Proposition 2.2.1: If $D$ is a double R-algebroid, then we have two crossed modules associated with D.

Proof: First, let $A_{0}=D_{0}$ (the set of objects of $D$ ), and $A_{2}=D_{1}$, the algebroid of arrows of $D_{2}$. We take $M_{2}$ to
consist of squares $\beta$ with boundary of the form $\binom{1}{0}$,
that is,
$M_{2}(x, y)=\left\{\beta \in D: \partial_{1}^{0} \beta=m, \partial_{1}^{1} \beta=0_{x y}, \partial_{2}^{0} \beta=1_{x}, \quad \partial_{2}^{1} \beta=1_{y}\right\}$.
We define,$+ *$, on $M_{2}$ by $\beta+\beta_{1}=\beta+2 \beta_{1}, \beta * \beta^{\prime}=\beta *_{2} \beta^{\prime}$ and $r$. $\beta=r .2 \beta$, whenever $\beta_{1} \beta_{1}, \beta^{\prime} \in M_{2}$ and $r \in R$. Thus $M_{2}$ is an R-algebroid over $A_{0}$. Let $\beta \in M_{2}$ as above and let $a^{\prime} \in A_{2}(y, z)$, $a \in A_{2}(w, x)$. So we get two squares in the form


Then we define the right and the left actions of $A_{2}$ on $M_{2}$ by the formulae

$$
\beta^{a}=\beta *_{2} c_{1} a^{\prime}, \quad a_{\beta}=c_{1} a *_{2} \beta \text { as shown below: }
$$



We now prove that these actions satisfy the axioms for crossed modules (1.3.1)(i-iv).

Axioms (1.3.1)(i-ii) , follow directly from the associativity of $\boldsymbol{*}_{2}$.
(1.3.1)(iii)
$\beta^{\mathrm{a}+\mathrm{b}}=\beta^{\mathrm{a}}+\beta^{\mathrm{b}}, \quad \mathrm{a}+\mathrm{b}_{\beta}=\mathrm{a}_{\beta}+\mathrm{b}_{\beta}$
$\left(\beta+\beta_{1}\right)^{b}=\beta^{b}+\beta_{1}{ }^{b},{ }^{a}\left(\beta+\beta_{1}\right)=a_{\beta}+{ }^{a} \beta_{1}$

## Proof:

$$
\begin{array}{rlr}
\beta^{a+b} & =\beta *_{2} c_{1}(a+b) \quad \text { by definition } \\
& =\beta *_{2}\left[c_{1} a+c_{2} \varepsilon_{1} b\right] \quad \text { by (2.1.7) } \\
& =\left(\beta *_{2} c_{1} a\right)+_{2}\left(\beta *_{2} \varepsilon_{1} b\right) \text { by distributivity } \\
& =\beta^{a}+\beta^{b} .
\end{array}
$$

We prove similarly that $a^{a+b} b_{\beta}=a_{\beta}+b_{\beta},\left(\beta+\beta_{1}\right)=\beta^{b}+\beta_{1} b$, ${ }^{a}\left(\beta+\beta_{1}\right)=a_{\beta}+{ }^{a} \beta_{1}$.
For (1.3.1)(iv), namely (r. $\boldsymbol{\beta}^{a}=\mathbf{r} . \beta^{a}=\beta^{\text {ra }}$ for all $r \in R$,

$$
\begin{aligned}
(r, \beta) a & =(r \cdot 2 \beta) *_{2} c_{1} a & \text { by definition } \\
& =r \cdot 2\left(\beta *_{2} c_{1} a\right) & \text { from bilinearity } \\
& =r \cdot 2 \beta^{a}=r \cdot \beta^{a} . &
\end{aligned}
$$

Also by the definition and bilinearity, we get $(r, \beta)^{a}=\beta r a$. Clearly (l.3.1)(iv) is satisfied.

Define now a map $\mu_{2}: M_{2} \rightarrow A_{2}$ by $\mu_{2} \beta=a_{1}^{0} \beta$
It is clear that $\mu_{2}$ is an algebroid morphism.
Finally, to prove that $\left(\Lambda_{2}, M_{2}, \mu_{2}\right)$ is a crossed module, it suffices to verify the axioms (1.3.2)(i-ii) ; namely $\mu_{2}\left(\beta^{a}\right)=\left(\mu_{2} \beta\right) a \quad, \mu_{2}\left({ }^{a} \beta\right)=a\left(\mu_{2} \beta\right), \beta \beta^{\prime}=\beta^{\mu_{2} \beta^{\prime}}=\mu_{2} \beta_{\beta^{\prime}}$. The first part is clear. Thus we just want to show that $\beta \beta^{\prime}=\beta^{\mu_{2} \beta^{\prime}}=\mu_{2} \beta_{\beta^{\prime}}$.

Suppose $\beta$, $\beta^{\prime}$ have boundaries in the form
$\left(\begin{array}{ll}1 & \mathrm{~m} \\ 0 & 1\end{array}\right),\binom{\mathrm{m}^{\prime}}{0}$. Then

$$
\begin{aligned}
\beta * \beta^{\prime} & =\beta *_{2} \beta^{\prime} \quad \text { by definition } \\
& =\left(\varepsilon_{1} \mathrm{~m} *_{1} \beta\right) *_{2}\left(\beta^{\prime} *_{1} \varepsilon_{1} 0\right) \text { by the identity rule } \\
& =\left(\varepsilon_{1} \mathrm{~m} *_{2} \beta^{\prime}\right) *_{1}\left(\beta *_{2} \varepsilon_{1} 0\right) \text { by }(2.1 .6)(i i) .
\end{aligned}
$$

Since $\beta *_{2} \varepsilon_{1} 0=c_{1} 0$ by remark (2.1.8), we have
$\beta * \beta^{\prime}=\left(c_{1} \mathrm{~m} *_{2} \beta^{\prime}\right) *_{2} c_{1} 0=c_{1 m} *_{2} \beta^{\prime}={ }^{m} \beta^{\prime}=\mu_{2} \beta_{\beta^{\prime}}$
by the definition and remark (2.1.8).
We can use similar argument to get
$\beta * \beta^{\prime}=\beta^{\mu_{2} \beta^{\prime}}$, as shown in the diagram below

$\beta * \beta^{\prime}=\beta *_{2} \beta^{\prime}=1$| m | $\beta^{\prime}$ |
| :---: | :---: |
| 0 | 0 |$=$



$=\beta^{m^{\prime}}=\beta^{\mu_{1} \beta^{\prime}}$.
Then we get a crossed module $\left(A_{2}, M_{2}, \mu_{2}\right)$.
For the second crossed module, we assume $A_{1}=D_{2}$ and take $M_{1}$ to consist of squares $\beta$ with boundary of the form (m $\left.\begin{array}{ll}1 & 0\end{array}\right)$, that is,
$M_{1}(x, y)=\left\{\beta \in D: \partial_{2}^{0} \beta=m, \quad \partial_{2}^{2} \beta=0_{x y}, \partial_{1}^{0} \beta=1_{x}, \quad \partial_{1}^{1} \beta=1_{y}\right\}$
and clearly $M_{1}$ is an $R$-algebroid over $A_{0}$ by $\beta+\beta_{1}=\beta+\beta_{1}$, $\beta * \beta^{\prime}=\beta *_{1} \beta^{\prime}$ and $r \cdot \beta=r \cdot 1 \beta$. Then we can use similar argument as above to get a crossed module ( $A_{1}, M_{1}, \mu_{1}$ ). This is the complete proof of the proposition.

The next section gives examples of double R-algebroids.

## 3. EXAMPLES:

We give in this section three examples of double algebroids.

1) Let $B$ be an R-algebroid over $B_{0}$. Then we can construct $a$ double R-algebroid $D=\square B$ of commuting squares in $B$ such that $D$ and $B$ have the same set of objects (i.er $D_{0}=B_{0}$ ).

Let $D_{1}=D_{2}=B$ be the horizontal and vertical algebroid structures, and let $D$ consist of quadruples
$x=(a, c d)$ for $a, b, c, d \in B$ and $c d=a b$.

Thus $\propto$ is determined by its boundary edges .
We define now $+_{1},+_{2}, *_{1}, *_{2}, \cdot 1, \cdot 2$ on $D$ in the following way :

Let $\alpha=\left(\begin{array}{ll}a & c \\ b\end{array}\right), \beta=\left(\begin{array}{lll}a_{1} & c & d_{1}\end{array}\right), \gamma=\left(\begin{array}{ll}a & c_{1} \\ b_{1}\end{array}\right)$,
$\varepsilon=\left(a^{,}, b^{\prime} d^{\prime}\right), \quad \varepsilon=\left(d_{b} c^{\prime}, e\right)$, then we define
$\alpha+{ }_{2} \beta=\left(\begin{array}{ll}a+a_{1} & c \\ b\end{array} d+d_{1}\right) \quad, \quad \alpha+2 y=\left(\begin{array}{l}\left.a \begin{array}{c}c+c_{1} \\ b+b_{1}\end{array} d\right), ~\end{array}\right.$
$\alpha *_{2} \varepsilon=\left(a a^{\prime}, e^{c} d d^{\prime}\right), \alpha *_{2} 5=\left(a c c^{\prime}, e\right)$. If $r \in R$,
we define $r \cdot 2 \alpha=(r a \underset{b}{c} r d)$ and $r \cdot 2 \alpha=\left(\begin{array}{l}r c \\ r b \\ d\end{array}\right)$.
It is clear that these operations are well-defined, for example $+_{1}$, since $\alpha, \beta \in D$, then $a b=c d$ and $a_{1} b=c d_{1}$ hence $\left(a+a_{1}\right) b=c\left(d+d_{1}\right)$, so $\alpha+{ }_{1} \beta \in D$.

Now we want to show that this structure satisfies the axioms for double algebroids.

It is obvious that this structure satisfies the axioms (2.1.4)(i-iv), (2.1.3), (2.1.7) and (2.1.5) (i-iii).

Thus it is enough to satisfy the axioms (2.1.6) (i-iv) or (2.1.6)(i), let $\alpha, \beta, \gamma, \delta \in D$ have boundaries given by $\alpha=\left(a_{b}^{c} d\right), \beta=\left(a_{1}^{c} d_{1}\right), y=\left(a_{b}^{c_{1}} d\right), \delta=\left(a_{1} c_{b_{1}} d_{1}\right)$, so
$\alpha+{ }_{1} \beta=\left(a+a_{1} \underset{b}{c} d+d_{1}\right), \gamma+1_{1} \delta=\left(a+a_{1}{\underset{b}{b_{1}}}_{c_{1}}^{d+d_{1}}\right)$,
$\alpha+{ }_{2} \gamma=\left(a \underset{b+b_{1}}{c+c_{1}} d\right)$ and $\beta+{ }_{2} \varepsilon=\left(a_{1}^{c+c_{1}} d_{1}\right)$ and then
$(\alpha+1 \beta)+2\left(\gamma++_{2} \varepsilon\right)=\left(a+a_{2} \underset{b+b_{1}}{c+c_{1}} d+d_{1}\right)$, and
$\left(\alpha+{ }_{2} \gamma\right)+{ }_{1}\left(\beta+{ }_{2} \delta\right)=\left(a+a_{1} \quad c+c_{1} d+d_{1}\right)$.
So $\left(a+a_{1}\right)\left(b+b_{1}\right)=\left(c+c_{1}\right)\left(d+d_{1}\right)$ (since $a b=c d, a_{1} b=c d_{1}$, $a b_{1}=c_{1} d$, and $a_{1} b_{1}=c_{1} d_{1}$. The explanation for (2.1.4)(ii-iv) is similar to that of (2.1.4)(i).

Thus the structure $\square B$ with these operations does satisfy the rules for a double algebroid.
2) Let $B$ be an R-algebra and let $B_{1}, B_{2}$ be two subalgebras of $B$ - Define $D=\theta\left(B_{1}, B_{2}\right)$ to be the set of commuting squares $\alpha=(a c d)$, for $a, d \in B_{1}, c, b \in B_{2}$ and $a b=c d$. Let $D_{0}$ b
$=\{*\}$. If we define the operations $+_{1},+_{2}, *_{1}, *_{2}$,
$\cdot 2$, $\cdot 2$ on $D$ in a similar way to that in example (l), we get a double R-algebroid.
3) A generalisation of example (2) is: if $B$ is an R-algebra and $B_{1}, B_{2}$ are subalgebras of $B$ and given homomorphisms $\Phi: B_{1} \rightarrow B, \Psi: B_{2} \rightarrow B$.

Define now, $D_{0}=\{*\}$ and $D_{1}=B_{1}, D_{2}=B_{2}$ and $D$ to
consist of quadruples (ad), for $a, d \in B_{1}, c, b \in B_{2}$ such
that $(\Phi a)(\psi b)=(\Psi \mathrm{c})(\Phi \mathrm{d})$.
We define $+_{1},+_{2}, *_{1}, *_{2}, r_{1}, r_{2}$ on $D$ in the following Way :
for $+_{1}$, let $\alpha=\left(a_{b}^{c} d\right), \beta=\left(a_{1}^{c} d_{1}\right)$, then
$\alpha+{ }_{2} \beta=\left(a+a_{1}{ }_{b}^{c} d+d_{1}\right)$. So we want to show that
$\left(\Phi\left(a+a_{1}\right)\right)(\psi b)=(\psi c)\left(\Phi\left(d+d_{1}\right)\right)$, and this equation follows from these two equations $(\Phi a)(\psi b)=(\Psi c)(\Phi d)$ and $\left(\Phi \mathrm{a}_{1}\right)(\Psi \mathrm{b})=(\Psi \mathrm{c})\left(\Phi \mathrm{d}_{1}\right)$ and $\Phi$ is a morphism.

For $+_{2}, *_{1}, *_{2}, e_{1}, \mu_{2}$, we can define these operations similar as in example (2) by using the fact that $\Phi, \Psi$ are algebra morphisms.

Clearly the above structure does satisfy the axioms of a double R-algebroid, Moreover, the two associated crossed modules of the above double algebroid are : essentially
i) the first crossed module is given/by the morphism

$$
\begin{aligned}
& \mathrm{B}_{2} \xrightarrow{\mathrm{I}} \mathrm{~B}_{2} \\
& \text { (1 } \left.\begin{array}{l}
\mathrm{c} \\
0
\end{array}\right) \rightarrow c \mathrm{c} \text { essentially }
\end{aligned}
$$

ii) The second crossed module is given/by the morphism

$$
\begin{aligned}
& \mathrm{B}_{1} \xrightarrow{\mathrm{I}} \mathrm{~B}_{1} \\
& 1 \\
& \text { (a } \\
& 1
\end{aligned}
$$

# THE EQUIVALENCE BETWEEN THE CATEGORY C OF CROSSED <br> MODULES AND THE CATEGORY DA! OF SPECIAL DOUBLE <br> ALGEBROIDS WITH CONNECTIONS 

## 0. INTRODUCTION:

R. Brown and C.B. Spencer $[B-S-1]$ have defined a functor
(crossed modules over groups) $\rightarrow$ (double groupoids), and they showed that this gives an equivalence between the category of crossed modules over groups and the category of special double groupoids with special connections and one vertex . The structure of connection on a double groupoid was shown in [ $\mathrm{B}-\mathrm{H}-1$ ] to be equivalent to a structure of thin squares, and a convenient notation for thin squares was later developed and exploited by R.Brown [Br-2] . Also [S-1] proved an equivalence between 2-categories and double categories with connections. Thin structures on double categories were exploited in [S-W-1] . Finally, it was proved in [B-H-2] that crossed modules over groupoids are equivalent to double groupoids with connections ; indeed this is a special case of an equivalence between crossed complexes (over groupoids) and $\omega$-groupoids .

Our programme is to prove results parallel to the above in the context of algebroids rather than groupoids ; that is we would like to prove that there exist an equivalence between w-algebroids and crossed complexes (over algebroids) .

Rather than move to the general case immediately, we give in this, chapter the case $n=2$, that is, for double algebroid . This will familiarise the reader with the techniques
involved. Also some of our lemmas for $n=2$ will be applied in the general case, and the complications of their proof makes it easier to give the case $n=2$ when the notation is simpler than in general.

As explained in the Introduction, in this thesis we do not acheive the general result, but we do obtain a lot of information on the general situation and complete results for $n=2,3,4$.

## 1. THIN STRUCTURES AND CONNECTIONS:

We will use the example which was given in chapter $2 \$ 3$ in order to define the extra structure needed later (we should mention that the example of $0 B$ given before is analogous to the example of double category due to Ehresmann [Eh-l]). But before that we start to define a special double algebroid. Definition 3.1.1: Let $D$ be a double R-algebroid. We say that $D$ is a special double $R$-algebroid if $D_{1}=D_{2}$.

Refering to the definition (2.1.9), a morphism ( $\Psi, \Psi_{1}, \Psi_{2}$ ) of double algebroids such that $\Psi_{2}=\Psi_{2}$ is called a morphism of special double algebroids.

Suppose given a special double algebroid $D$. Then there will be squares of $D$ with commuting boundary, that is, with edges given by

and for which $a b=c d$. Examples of such squares are degenerate squares;


Among the others there seems no way to distinguish any one from another . We therefore impose on $D$ an additional structure of "thin" squares.

Definition 3.1.2: Let $D$ be a special double algebroid. A thin structure on $D$ is a morphism $\theta: D_{1} \rightarrow-D$ of special double algebroids such that $\theta$ is the identity on $D_{2}$. Hence
$\underline{\partial} \theta\left(a_{b}^{c} d\right)=\left(a_{b}^{c} d\right)$. An element $\theta\left(a_{b}^{c} d\right)$ is called thin , and
is often written simply (a d), when $\theta$ is clear from the b
context.
Remark 3.1.3: Because $\theta$ is a morphism any composite of thin squares is thin ; any sum of thin squares is thin ; any scalar multiple of a thin square is thin. Thin squares should be thought of as generalisations of identity elements $\varepsilon_{1} a, \varepsilon_{2} a$ in a special double algebroid.

Instead of thin structures, one can use an alternative further structure on $D$, namely a connection ( $\Gamma$, $\Gamma^{\prime}$ ) . This will be important later for generalisation to higher dimensions.
(3.1.4)(i) for any a $\epsilon \mathrm{D}_{2}(\mathrm{x}, \mathrm{y})$, then ra , r'a have edges given by

(Clearly these two squares are commutative).
We assume the following axioms: for all $a, b \in D_{2}$ such that $a b$ is defined
$\Gamma^{\prime} a *_{2} \Gamma a=c_{1} a$
$\Gamma^{\prime} a *_{1} \Gamma a=c_{2} a$
$\Gamma^{\prime}(a b)=\left(\Gamma^{\prime} a *_{2} c_{1} a\right) *_{2}\left(c_{2} a *_{1} \Gamma^{\prime} a\right)=\Gamma^{\prime} a *_{2}\left(c_{2} a *_{2} \Gamma^{\prime} b\right)$
$\Gamma(\mathrm{ab})=\left(\begin{array}{llll}\Gamma \mathrm{a} & *_{1} & c_{2} \mathrm{~b}\end{array}\right) *_{2}\left(\begin{array}{lll}c_{1} \mathrm{~b} & *_{2} & \Gamma \mathrm{~b}\end{array}\right)=\left(\begin{array}{lll}\Gamma \mathrm{a} & *_{1} & c_{2} \mathrm{~b}\end{array}\right) *_{2} \Gamma \mathrm{~b}$
for all $x \in D_{0}$, we have $\left[{ }^{\prime} l_{x}=\left[I_{x}=c_{1} c_{x}\right.\right.$.
Definition 3.1.4(b): Let $D$ be a special double algebroid with a weak connection ( $\Gamma, \Gamma^{\prime}$ ) . We said that ( $\Gamma, \Gamma^{\prime}$ ) is a connection on D if it satisfy these extra axioms :
(3.1.4)(iv) Let $\alpha, \beta, \gamma \in D$ have boundaries given by
$\underline{\partial} \alpha=\left(a_{b}^{c} d\right), \underline{\partial} \beta=\left(a_{1}^{c} d_{1}\right), \underline{\partial} y=\left(a_{b_{1}}^{c_{1}} d\right) ;$
then we have
$\Gamma^{\prime}\left(a+a_{1}\right) *_{2}\left(\alpha+{ }_{1} \beta\right) *_{2} \Gamma\left(d+d_{1}\right)=\left(\Gamma^{\prime} a *_{2} \alpha *_{2} \Gamma d\right)+_{2}$ ( $\left.\Gamma{ }^{\prime} a_{2} *_{2} \beta *_{2} \Gamma d_{1}\right)$.
$\Gamma^{\prime}\left(c+c_{1}\right) *_{1}\left(\alpha+_{2} y\right) *_{1} \Gamma\left(b+b b_{1}\right)=\left(\Gamma^{\prime} c *_{1} \alpha *_{1} \Gamma b\right)+_{2}\left(\Gamma^{\prime} c_{1}\right.$ $\left.*_{1} \beta *_{1} \Gamma b_{1}\right)$
(3.1.4)(v) Let $r \in R$ and $\alpha \in D$ with boundary given by
$\partial \alpha=\left(a_{b}^{c} d\right) ;$ then we have
「'ra $*_{2}\left(r ._{1} \alpha\right) *_{2}$ 「rd $=r \cdot 2\left(\Gamma\right.$ 'a $\left.*_{2} \propto *_{2} \Gamma \mathrm{~d}\right)$,
$\Gamma^{\prime} \mathrm{rc} *_{1}(\mathrm{r} \cdot \mathrm{2} \alpha) *_{1} \Gamma \mathrm{rd}=\mathrm{r} \cdot 2\left(\Gamma^{\prime} \mathrm{c} *_{1} \propto *_{1} \Gamma \mathrm{~b}\right)$.
These axioms (3.1.4)(ii-iv) make sense in terms of
boundaries, as shown in the diagrams below :
let $a: x \rightarrow y, b: y \rightarrow z$ for $x, y, z \in D_{0} ;$ then the axiom (3.1.4)(ii) can be pictured as


The axiom (3.1.4)(iii) is pictured as ;


The axiom (3.1.4)(iv), is pictured as


The axiom (3.1.4)(v), is pictured as ;
the left hand side is ;


The other side is ;
r. 2

. Thus the boundaries are equal.

Remark 3.1.5: The axioms (3.1.4)(i,ii,iii) are essentially the axioms for connection on a double category given in [S-1]. These axioms involve only the composition and not the additions or scalar multiplications of the algebroid structure . But the axioms (3.1.4)(iv,v) give relations between ( $\left[, \Gamma^{\prime}\right.$ ) and the additions and scalar multiplications. These axioms are equivalent to conditions on the folding operation given later in $\$ 3.2$ and are not used until that section .

We go back to define a morphism between two special double algebroids with connections.

Definition 3.1.6: A morphism $\Psi: D \rightarrow E$ of special double algebroids with connections ( $\Gamma, \Gamma^{\prime}$ ) , ( $\Delta, \Delta^{\prime}$ ) is said to preserve the connections if and only if
$\Delta \Psi_{2}=\psi, \Gamma, \Delta^{\prime} \Psi_{2}=\psi \Gamma^{\prime}$.

Such morphisms form the morphisms of the category of special double algebroids with connections, denoted by DA! •

We gave in proposition (2.2.1) a functor from double algebroids to crossed modules (over algebroids), associating to $D$ the crossed module $(A, M, M)$ with $A=D_{2}$ and $M$ consisting of squares with boundary of the form ( $\begin{aligned} & \text { m } \\ & 0\end{aligned}$ ). We have a
forgetful functor $\underline{D A}$ : (special double algebroids with connection) $\rightarrow$ (double algebroids) . The composite functor DA! $-\rightarrow$ (crossed modules) will be written as $y$.

Notice that in a special double algebroid, a thin structure implies a connection satisfying (3.1.4)(i,ii,iii) where $\Gamma(a)=\theta\left(a_{1}^{a} 1\right), \Gamma^{\prime}(a)=\theta\left(\begin{array}{l}l \\ a\end{array} a\right)$. To complete the
equivalence between these two structures, we prove first that in a special double algebroid a thin structure may be weak
recovered from a/connection satisfying only (3.1.4)
(i,ii,iii) . This result leads us to use connections instead of thin structures . The idea particularly in higher dimensions has been given in [B-H-l] in the double groupoid case, and partially in [S-1], [S-W-l], for double categories.

Theorem 3.1.7: Let $D$ be a special double algebroid with connection $\Gamma$, $\Gamma^{\prime}$. Then there is a morphism of special double algebroids $\theta: \theta D_{1} \rightarrow D$, which is the identity on $D_{1}$
and such that $\Gamma a=\theta\left(a_{l}^{a} 1\right), \Gamma^{\prime} a=\theta\left(l_{b}^{l} b\right)$.

Proof: For any $a, b, c, d \in D_{1}$ satisfying $c d=a b$, define functions $\theta_{1}, \theta_{2}: \square D_{1} \longrightarrow D$ by

$$
\begin{aligned}
& \theta_{1}\left(a_{b}^{c} d\right)=\left(c_{1} c *_{2} \Gamma^{\prime} d\right) *_{1}\left(\Gamma a *_{2} c_{1} b\right), \\
& \theta_{2}\left(a{ }_{b}^{c} d\right)=\left(c_{2} a *_{1} \Gamma^{\prime} b\right) *_{2}\left(\Gamma c *_{1} c_{2} d\right) .
\end{aligned}
$$

The two definitions make sense in terms of boundaries; Appendix $I$ give diagrams for these definitions and for the proof of the next lemma.

Lemma 3.1.8: The two definitions $\theta_{1}, \theta_{2}$ are equivalent, that is, $\theta_{1}=\theta_{2}$.
Proof: Let $a, b, c, d \in D_{1}$ be such that $c d=a b$, then

$$
\theta_{1}\left(a_{b}^{c} d\right)=\left(\varepsilon_{1} c *_{2} \Gamma^{\prime} d\right) *_{1}\left(\Gamma a *_{2} c_{1} b\right)
$$

$$
=\left(\varepsilon_{1} c *_{2} \Gamma \Gamma^{\prime} d\right) *_{2} \varepsilon_{1} a b *_{2}\left(\Gamma a *_{2} \varepsilon_{1} b\right) \text { by the identity rule }
$$

$$
=\left(\varepsilon_{1} c *_{2} \Gamma^{\prime} d\right) *_{1}\left(\Gamma^{\prime} a b *_{2} \Gamma c d\right) *_{2}\left(\Gamma a *_{2} c_{1} b\right)
$$

$$
\text { by }(3.1 .4)(\mathrm{ii}) \text { and } \mathrm{cd}=\mathrm{ab}
$$

$$
=\left(c_{1} c *_{2} \Gamma^{\prime} d\right) *_{1}\left\{[ \Gamma ^ { \prime } a * _ { 2 } ( c _ { 2 } a * _ { 1 } \Gamma ^ { \prime } b ) ] * _ { 2 } \left[\left(\Gamma c *_{1} c_{2} d\right) *_{2}\right.\right.
$$

$$
[d]\} *_{1}\left(\left[a *_{2} \varepsilon_{1} b\right) \quad b y(3.1 .4)(i i i)\right.
$$

$$
=\left(c_{1} c *_{2} \Gamma^{\prime} d\right) *_{1}\left\{\left[\Gamma^{\prime} a *_{2}\left(c_{2} a *_{1} \Gamma^{\prime} b\right) *_{2}\left(\Gamma c *_{1} c_{2} d\right)\right] *_{2}\right.
$$

$$
[\mathrm{d}\} *_{1}\left(\left[a *_{2} c_{1} b\right) \quad\right. \text { by associativity }
$$

$$
=\left\{\left\{c_{1} c *_{1}\left[\Gamma^{\prime} a *_{2}\left(c_{2} a *_{1}\left[{ }^{\prime} b\right) *_{2}\left(\Gamma c *_{1} c_{2} d\right)\right]\right\} *_{2}\right.\right.
$$

$$
\left(\Gamma^{\prime} d *_{1}[d)\right] *_{1}\left(\left[a *_{2} \varepsilon_{1} b\right) \text { by }(2.1 .6)(i i)\right.
$$

$$
=\left[\varepsilon_{2} c *_{1}\left[\Gamma^{\prime} a *_{2}\left(\varepsilon_{2} a *_{1} \Gamma^{\prime} b\right) *_{2}\left(\left[c *_{1} \varepsilon_{2} d\right)\right]\right] *_{1}\right.
$$

$$
\left(\Gamma a *_{2} \varepsilon_{1} b\right) \quad \text { by }(3.1 .4)(i i)
$$

$$
=\varepsilon_{1} c *_{1}\left\{\left\{\Gamma^{\prime} a *_{2}\left[\left(\varepsilon_{2} a *_{1} \Gamma^{\prime} b\right) *_{2}\left(\Gamma c *_{1} \varepsilon_{2} d\right)\right]\right\} * 1\right.
$$

$$
\left(\left[a *_{2} \varepsilon_{1} b\right)\right\} \quad \text { by associativity }
$$

$$
=\varepsilon_{1} c *_{2}\left\{( \Gamma ^ { \prime } a * _ { 2 } \Gamma a ) * _ { 2 } \left[\left[\left(\varepsilon_{2}^{a} *_{1} \Gamma^{\prime} b\right) *_{2}\left(\Gamma c *_{2} c_{2} d\right)\right] *_{1}\right.\right.
$$

$$
\left.\left.\left.\varepsilon_{1} b\right]\right\}\right) \quad \text { by (2.1.6)(ii) }
$$

$$
\begin{aligned}
& =\varepsilon_{1} c *_{1}\left[\left(c_{2} a *_{1} \Gamma^{\prime} b\right) *_{2}\left(\left[c *_{1} c_{2} d\right)\right] *_{1} c_{2} b\right. \text { by (3.1.4)(ii) } \\
& =\left(c_{2} a *_{1} \Gamma^{\prime} b\right) *_{2}\left(\Gamma c *_{1} c_{2} d\right) \text { by the identity rule } \\
& =\theta_{2}(a \quad d) \text {. This is the complete proof of the lemma. }
\end{aligned}
$$

Now we continue to prove theorem (3.1.7), that is, we prove e satisfies the following; where $\theta=\theta_{1}=\theta_{2}$ :
i) $\theta(a$ $\left.{ }_{1}^{1} a\right)=(a$ 1 a) $\quad \theta\left(1_{a}^{a}\right.$
$1)=\left(1_{a}^{a}\right.$
 iii) $\theta\left(a_{b}^{c} d\right)+{ }_{1} \theta\left(a_{1} \underset{b}{c} d_{1}\right)=\theta\left(a+a_{1} \underset{b}{c} d+d_{1}\right)$; iv) $\theta\left(a a_{b}^{c} d\right)+{ }_{2} \theta\left(a{ }_{b_{1}}^{c_{1}} d\right)=\theta\left(a c_{b+b_{2}}^{c+c_{1}} d\right)$; v) $r \cdot 1 \theta\left(a_{b}^{c} d\right)=\theta\left(r_{b}^{c} r d\right)$;
vi) $r \cdot 2 \theta\left(a_{b}^{c} d\right)=\theta\left(a r_{r b}^{r c} ;\right.$
vii) $\theta\left(a \begin{array}{l}c \\ b\end{array}\right) *_{2} \theta\left(d_{b}^{\prime}, e\right)=\theta\left(a c c^{\prime}, d\right) ;$
viii) $\theta\left(a a_{b}^{c} d\right) *_{1} \theta\left(a^{\prime}, d^{\prime}\right)=\theta\left(a a^{\prime} e^{c} d d^{\prime}\right)$.

The proof of (i), (ii) are easy . To prove (iii), we use the interchange law (2.1.6)(iii), distributive law, (2.1.7) and $\theta=\theta_{2}$;
$\theta_{2}\left(a a_{b}^{c} d\right)+_{1} \theta_{2}\left(a_{1}^{c} d_{1}\right)=\left[\left(\varepsilon_{2}^{a} *_{1} \Gamma^{\prime} b\right) *_{2}\left(\left[c *_{1} c_{2} d\right)\right]+{ }_{1}\right.$
$\left[\left(\begin{array}{lllll}c_{2} a_{1} & *_{1} & \Gamma^{\prime} b\end{array}\right) *_{2}\left(\Gamma c *_{1} c_{2} d_{1}\right)\right]$

$$
\begin{aligned}
& =\left[\left(c_{2}{ }^{a} *_{1} \Gamma^{\prime} b\right)+_{1}\left(\varepsilon_{2}{ }_{2} *_{2} \Gamma^{\prime} b\right)\right] *_{2}\left[\left(\left[c *_{1} \varepsilon_{2} d\right)+{ }_{1}\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
& \text { by (2.1.6)(iii) }
\end{aligned}
$$

$$
\begin{aligned}
& \text { by distributivity } \\
& =\left(c_{2}\left(a+a_{1}\right) *_{1} \Gamma^{\prime} b\right) *_{2}\left(\Gamma c *_{1} \varepsilon_{2}\left(d+d_{1}\right)\right) \quad b y(2.1 .7) \\
& =\theta_{2}\left(a+a_{1} \underset{b}{c} d+d_{1}\right) .
\end{aligned}
$$

To prove (iv), we use (2.1.7),(2.l.6)(iv), distributivity, and $\theta=\theta_{1}$;
$\theta_{1}(a \underset{b}{c} d)+_{2} \theta_{1}\left(a_{b_{1}}^{c_{1}} d\right)=\left[\left(c_{1} c *_{2}[\prime d) *_{1}\left(\Gamma a *_{2} c_{1} b\right)\right]+{ }_{2}\right.$
$\left[\left(c_{1} c_{1} *_{2}\left[\Gamma^{\prime} d\right) *_{2}\left(\Gamma a *_{2} c_{1} b_{1}\right)\right]\right.$
$=\left[\left(c_{1} c *_{2} \Gamma^{\prime} d\right)+_{2}\left(c_{1} c_{1} *_{2} \Gamma^{\prime} d\right)\right] *_{1}\left[\left(\Gamma a *_{2} c_{1} b\right)+_{2}\right.$ ( $5 \mathrm{a} *_{2} \mathrm{c}_{1} \mathrm{~b}_{1}$ )] by (2.1.6)(iv)
$=\left[\left(\varepsilon_{1} c+c_{2} c_{1}\right) *_{2} \Gamma^{\prime} d\right] *_{1}\left[\Gamma a *_{2}\left(\varepsilon_{1} b+_{2} c_{1} b\right)\right]$ by distributivity
$=\left(c_{1}\left(c+c_{1}\right) *_{2} \Gamma^{\prime} d\right) *_{1}\left(\Gamma a *_{2} c_{1}\left(b+b_{1}\right)\right) \quad b y(2.1 .7)$
$=\theta_{1}\left(a^{c+c_{1}} d\right)$. $b+b_{1}$

To prove (v), we use the rule (2.1.5)(ii) and $\theta=\theta_{2}$;

$$
\begin{aligned}
& \theta_{2}(r a c r d)=\left(c_{2} r a *_{1} \Gamma^{\prime} b\right) *_{2}\left(\Gamma c *_{2} c_{2} r d\right) \\
& =\left(r \cdot 1 c_{2} a *_{1} \Gamma^{\prime} b\right) *_{2}\left(\left[c *_{1} r \cdot 1 \quad c_{2} d\right)\right. \\
& =\left(r \cdot 1\left(c_{2} a *_{1} \Gamma^{\prime} b\right)\right) *_{2}\left(r \cdot 1\left(\Gamma c *_{1} c_{2} \dot{d}\right)\right) \\
& =r \cdot 1\left[\left(c_{2} a *_{1} \Gamma^{\prime} b\right) *_{2}\left(\left[c *_{1} c_{2} d\right)\right] \quad b y(2.1 .5)(i i)\right. \\
& =r \cdot 1 \theta\left(a_{b}^{c} d\right) .
\end{aligned}
$$


by using (2.1.5) (ii) and $\theta=\theta_{1}$.
For (vii), we use the interchange law (2.1.6)(ii), the identity rule, the associativity, (3.1.4)(ii), the equality $c^{\prime} d^{\prime}=a b d^{\prime}=a a^{\prime} e$ and $\theta=\theta_{1}$;
$\theta\left(a c^{c \prime}, e\right)=\left(c_{1} c c^{\prime} *_{2} \Gamma^{\prime} e\right) *_{1}\left(\Gamma a *_{2} c_{1} b b^{\prime}\right)$
$=\left(\varepsilon_{1} c *_{2} \varepsilon_{1} c^{\prime} *_{2} \Gamma^{\prime} e\right) *_{1} \varepsilon_{1} c d b^{\prime} *_{1}\left(\Gamma a *_{2} \varepsilon_{1} b *_{2} \varepsilon_{1} b^{\prime}\right)$ by the identity rule
$=\left(\varepsilon_{1} c *_{2}\left(\varepsilon_{1} c^{\prime} *_{2} \Gamma^{\prime} e\right)\right) *_{1}\left(\varepsilon_{2} c *_{2} \varepsilon_{1} d *_{2} c_{2} b^{\prime}\right) *_{1}$ ( $\left.\left(\Gamma a *_{2} c_{1} b\right) *_{2} c_{1} b^{\prime}\right)$ by the associativity $=\left[c_{1} c *_{2}\left(c_{1} c \prime *_{2} \Gamma^{\prime} e\right)\right] *_{1}\left[\left(c_{1} c *_{2} \Gamma^{\prime} d\right) *_{2}\left(\left[d *_{2} c_{1} b^{\prime}\right)\right] *_{1}\right.$ $\left[\left(\Gamma a *_{2} c_{1} b\right) *_{2} \varepsilon_{1} b^{\prime}\right]$ by (3.1.4)(ii)
$=\left[\left(c_{2} c *_{1}\left(c_{1} c *_{2} \Gamma^{\prime} d\right)\right) *_{2}\left(\left(\varepsilon_{1} c^{\prime} *_{2} \Gamma^{\prime} e\right) *_{1}\left(\left[\mathrm{~d} *_{2} c_{1} b^{\prime}\right)\right)\right]\right.$
$*_{1}\left[\left(\left[a *_{2} c_{1} b\right) *_{2} c_{1} b^{\prime}\right]\right.$ by (2.1.6)(ii)
$=\left[\left(\varepsilon_{1} c *_{2} \Gamma^{\prime} d\right) *_{1}\left(\Gamma a *_{2} \varepsilon_{1} b\right)\right] *_{2}\left[\left(\varepsilon_{1} c^{\prime} *_{2} \Gamma^{\prime} e\right) *_{1}\right.$ ([d $\left.\left.*_{2} \varepsilon_{1} b^{\prime}\right) *_{1} \varepsilon_{1} b^{\prime}\right]$ by the identity rule and (2.1.6)(ii)
$=\left[\left(c_{1} c *_{2} \Gamma^{\prime} d\right) *_{1}\left(\Gamma a *_{2} c_{1} b\right)\right] *_{2}\left[\left(c_{1} c^{\prime} *_{2} \Gamma^{\prime} e\right) *_{1}\right.$ ([d $\left.\left.*_{2} c_{1} b^{\prime}\right)\right]$ by the identity rule
$=\theta_{1}\left(a_{b}^{c} d\right) *_{2} \theta_{1}\left(d_{b}^{c}, e\right)$.
We can prove similarly that $\theta\left(a_{b}^{c} d\right) *_{1} \theta\left(a^{,} e^{b}\right)=$
$\theta\left(a a^{c} d d^{\prime}\right)$, by using (3.1.4)(ii), the, identity rule,
the interchange law (2.1.6)(ii), cdd' $=a b d^{\prime}=a a^{\prime} e$ and $\theta=\theta_{2}$.

Then $\theta$ is morphism. This is the complete proof of the theorem.

We move now in the next section to construct a functor $\lambda$ : C $\rightarrow$ DA! by using a "folding" operation , whose definition involves the connections.

## 2. THE FOLDING OPERATION:

In this section, we introduce on squares of a special double algebroid with connections $D$ an operation which has the effect of "folding" all edges of $\alpha \in D$ onto the edge $\partial_{2}^{0} \alpha$. This operation $\Phi$ transforms $\alpha$ into an element of the associated crossed module $\gamma \mathrm{D}$.

We define $\Phi: D \rightarrow D$ in the following way;
given $\alpha \in D$ with boundary edges in the form


We define

$$
\Phi \alpha=\left(\Gamma^{\prime} a *_{2} \alpha *_{2} \Gamma d\right)-2 \varepsilon_{1} a b
$$

It is easy to check that this composition and subtraction are defined. Simply, if $\alpha$ as above, then $\Phi \alpha$ has boundary in the form


Thus $a_{1}^{0} \Phi \alpha=c d-a b, \partial_{1}^{1} \Phi \alpha=0, \partial_{2}^{0} \Phi \alpha=1, \partial_{2}^{1} \Phi \alpha=1$
And hence $\Phi \propto \in Y D$.

Proposition 3.2.1: $\Phi \alpha=\alpha$ if and only if $\alpha$ is in $\gamma \mathrm{D}$. In Particular $\Phi^{2} \alpha=\Phi \alpha$ for all $\alpha \in D$.
Proof: If $\Phi \alpha=\alpha$, then $\alpha$ has boundary edges given by (1 $\left.\begin{array}{c}m \\ 0\end{array}\right)$, for $m \in D_{1}=D_{2}$ and then $\alpha \in \gamma D$ (by the construction given in chapter II) . The converse is clear . We now develop relations between $\Phi$ and the operations of the special double algebroid $D$.
First, let $0^{2}=c_{1} 0 \in D$, as in the diagram


Proposition 3.2.2: Let a $\in D_{1}(x, y)$, then
i) $\Phi \Gamma^{\prime} a=0^{2}, ~ \Phi \Gamma a=0^{2}$,
ii) $\Phi \varepsilon_{1} a=0^{2}, \Phi \varepsilon_{2} a=0^{2}$.

Proof: i) Since a $\in D_{1}(x, y)$, then ria has boundary in the form

$$
1_{x}^{x} \overbrace{r^{\prime} a}^{a}{ }^{1_{x}}
$$

$$
\begin{aligned}
\text { And then } \Phi \Gamma^{\prime} a & =\left(\Gamma^{\prime} l_{x} *_{2} \Gamma^{\prime} a *_{2} \Gamma a\right)-2 c_{1} a \\
& =c_{1} a-2 c_{1} a \\
& =0^{2}
\end{aligned}
$$

We can prove similarly that $\Phi \Gamma a=0^{2}$.
ii) Since a $\in D_{1}(x, y)$, then $\varepsilon_{1} a$ has boundary edges given by

and then $\Phi c_{1} a=\left(\Gamma^{\prime} l_{x} *_{2} c_{1} a *_{2} \Gamma l_{y}\right)-_{2} c_{1} a=0^{2}$. Similarly We can prove that $\Phi c_{2}{ }^{a}=0^{2} b y(3.1 .4)(i i)$.

The following proposition is the main technical work required for the proof of the equivalence of categories given in the next sections.
Proposition 3.2.3: Let $\alpha, \beta \in D$ and $r \in R$, then the following hold whenever each left-hand side is defined :
i) $\Phi\left(\alpha+{ }_{1} \beta\right)=\Phi \alpha+{ }_{2} \Phi \beta$,
ii) $\Phi\left(\alpha+{ }_{2} \beta\right)=\Phi \alpha+{ }_{2} \Phi \beta$,
iii) $\Phi\left(\alpha *_{1} \beta\right)=\left(\Phi \alpha *_{2} c_{1} \partial_{2}^{1} \beta\right)+{ }_{2}\left(c_{1} \partial_{2}^{0} \alpha *_{2} \Phi \beta\right)$,
iv) $\Phi\left(\alpha *_{2} \beta\right)=\left(c_{1} \partial_{1}^{0} \alpha *_{2} \Phi \beta\right)+{ }_{2}\left(\Phi \alpha *_{2} c_{1} \partial_{1}^{1} \beta\right)$,
v) $\Phi(r \cdot 1 \alpha)=r \cdot 2 \Phi \alpha, \Phi(r \cdot 2 \alpha)=r \cdot 2 \Phi \alpha$.
(Appendix II gives diagrams for the proof of the above Proposition) .
Proof: i) If $\alpha, \beta$ have boundaries given by
$\underline{\partial} \alpha=\left(a_{b}^{c} d\right), \underline{\partial} \beta=\left(a_{1}^{c} d_{i}\right)$,
then $\Phi\left(\alpha+{ }_{1} \beta\right)=\left[\Gamma^{\prime}\left(a+a_{1}\right) *_{2}\left(\alpha+{ }_{1} \beta\right) *_{2}\left[\left(d+d_{1}\right)\right]-2\right.$ $c_{1}\left(a+a_{1}\right) b$
$=\left[\left(\Gamma^{\prime} a *_{2} \alpha *_{2} \Gamma d\right)+_{2}\left(\Gamma^{\prime} a_{1} *_{2} \beta *_{2}\left[d_{1}\right)\right]-2\left[c_{2} a b+c_{2} a_{1} b\right]\right.$ by (3.1.4)(iv)
$=\left[\left(\Gamma^{\prime} a *_{2} \alpha *_{2} \Gamma d\right)-2 c_{1} a b\right]+_{2}\left[\left(\Gamma^{\prime} a_{1} *_{2} \beta *_{2} \Gamma d_{1}\right)-2 \varepsilon_{1} a_{1} b\right]$
$=\Phi \alpha+{ }_{2} \Phi \beta$.
ii) This follows from the algebroid rules for $+_{2}, *_{2}$. iii) If $\alpha, \beta$ having boundaries given by
$\underline{\partial} \alpha=(a \underset{b}{c} d), \underline{\partial} \beta=\left(a^{\prime}, d^{\prime}\right)$, then $\alpha *_{1} \beta$ has boundary
edges in the form $\underline{\partial}\left(\alpha *_{1} \beta\right)=$ (aa, $\left.e^{c} d d^{\prime}\right)$.
Then $\Phi\left(\alpha *_{1} \beta\right)=\left(\Gamma^{\prime} a a^{\prime} *_{2}\left(\alpha *_{2} \beta\right) *_{2}\right.$ Id') $-_{2} c_{1} a a^{\prime} e$ $=\left\{\left[\Gamma^{\prime} a *_{1}\left(c_{1} a *_{2} \Gamma^{\prime} a^{\prime}\right)\right] *_{2}\left(\alpha *_{1} \beta\right) *_{2}\left[\left(\Gamma d *_{2} c_{1} d^{\prime}\right) *_{1}\right.\right.$ [d]\} $-2 \varepsilon_{1} a a^{\prime} e$ by (3.1.4)(ii)
$=\left\{\left[\Gamma^{\prime} a *_{2} \alpha *_{2}\left(\Gamma d *_{2} c_{1} d^{\prime}\right)\right] *_{1}\left[\left(c_{1} a *_{2} \Gamma^{\prime} a^{\prime}\right) *_{2} \beta *_{2}\left[d^{\prime}\right]\right\}\right.$ $-_{2} e_{1 a a}$ e by (2.1.6)(ii) and the associativity
$=\left\{\left[\left(\Gamma^{\prime} a *_{2} \alpha *_{2} \Gamma d\right) *_{2} \varepsilon_{1} d^{\prime}\right] *_{1}\left[\varepsilon_{1} a *_{2}\left(\Gamma^{\prime} a{ }^{\prime} *_{2} \beta *_{2} \Gamma d^{\prime}\right)\right\}-2\right.$ ( $\left.c_{1 a \in} e^{\prime} *_{1} \varepsilon_{1} a a^{\prime} e\right)$ by the associativity and the identity rule $=\left\{\left[\left(\Gamma^{\prime} a *_{2} \alpha *_{2} \Gamma d\right) *_{2} c_{1} d^{\prime}\right]-_{2} c_{1} a a^{\prime} e\right\} *_{1}\left\{\left[c_{1} a *_{2}\left(\Gamma^{\prime} a^{\prime} *_{2}\right.\right.\right.$ $\left.\left.B *_{2}\left[d^{\prime}\right)\right]-2 \varepsilon_{1} a a^{\prime} e\right\} \quad b y(2.1 .6)(i v)$
$=\left\{\left[\left(\Gamma^{\prime} a *_{2} \alpha *_{2} \Gamma d\right) *_{2} c_{1} d^{\prime}\right]-2 c_{1} a b d^{\prime}+_{2} c_{1}\left(a b d^{\prime}-a a^{\prime} e\right)\right\} *_{1}$ $\left\{c_{1} a *_{2}\left[\left(\Gamma^{\prime} a^{\prime} *_{2} \beta *_{2}\left[d^{\prime}\right)-_{2} \varepsilon_{1} a^{\prime} e\right]\right\}\right.$ by distributivity $=\left\{\left\{\left[\left(\Gamma^{\prime} a *_{2} \alpha *_{2}[d)-2 \varepsilon_{1} a b\right] *_{2} \varepsilon_{1} d^{\prime}\right\}+c_{2} \varepsilon_{1}\left(a b d^{\prime}-a a^{\prime} e\right)\right\} *_{1}\right.$ $\left\{c_{1 a} *_{2} \Phi \beta\right\} \quad$ by distributivity
$=\left[\left(\Phi \alpha *_{2} c_{1} d^{\prime}\right)+{ }_{2} c_{1}\left(a b d^{\prime}-a a^{\prime} e\right)\right] *_{2}\left[\varepsilon_{1} 0+{ }_{2}\left(\varepsilon_{1} a *_{2} \Phi \alpha\right)\right]$
by the identity rule
$=\left[\left(\Phi \alpha *_{2} \varepsilon_{1} d^{\prime}\right) *_{2} c_{1} 0\right]+_{2}\left[c_{1}\left(a b d^{\prime}-a a^{\prime} e\right) *_{2}\left(c_{1} a *_{2} \Phi \beta\right)\right]$

$$
\text { by }(2.1 .6)(i v)
$$

$=\left(\Phi \alpha *_{2} \varepsilon_{1} d^{\prime}\right)+_{2}\left(c_{1} a *_{2} \Phi \beta\right)$ by the identity rule
$=\left(\phi \alpha *_{2} c_{1} \partial_{2}^{1} \beta\right)+_{2}\left(c_{1} \partial_{2}^{0} \alpha *_{2} \Phi \beta\right)$.
iv) If $\alpha, \beta$ have boundaries given by
$\underline{\partial} \alpha=\left(\begin{array}{ll}\mathrm{a} & \mathrm{d} \\ \mathrm{b}\end{array}\right), \underline{\partial} \beta=\left(\mathrm{d}_{\mathrm{b}}, \mathrm{e}\right)$,
then $\alpha *_{2} \beta$ has boundary edges in the form (a le $\begin{gathered}\left.c b^{\prime}, e\right) .\end{gathered}$ Now we compute $\Phi\left(\alpha *_{2} \beta\right)=\left[\Gamma^{\prime} a *_{2}\left(\alpha *_{2} \beta\right) *_{2}[e]-c_{2} c_{1} a b b\right.$, $=\left[\left(\Gamma^{\prime} a *_{2} \alpha\right) *_{2} c_{2} d *_{2}\left(\beta *_{2} \Gamma e\right)\right]-2 c_{1} a b b$, by associativity and the identity rule

$$
\begin{aligned}
& =\left[\left(\Gamma^{\prime} a *_{2} \alpha\right) *_{2}\left(\Gamma^{\prime} d *_{2} \Gamma d\right) *_{2}\left(\beta *_{2} \Gamma e\right)\right]-c_{2} c_{1} a b b^{\prime} \\
& \text { by (3.1.4)(i) } \\
& =\left\{\left[c_{1} c *_{1}\left(\Gamma^{\prime} a *_{2} \alpha\right)\right] *_{2}\left(\Gamma^{\prime} d *_{1} \Gamma d\right) *_{2}\left[\left(\beta *_{2} \Gamma e\right) *_{1} c_{1} b^{\prime}\right]\right\} \\
& -_{2} c_{1} a b b \text { b by the identity rule } \\
& =\left\{\{ ( c _ { 1 } c * _ { 2 } \Gamma ^ { \prime } d ) * _ { 1 } [ ( \Gamma ^ { \prime } a * _ { 2 } \alpha ) * _ { 2 } \Gamma d ] \} * _ { 2 } \left[\left(\beta *_{2} \Gamma e\right) *_{1}\right.\right. \\
& \left.\left.c_{1} b^{\prime}\right]\right\}-2 c_{1} a b b \text { by (2.1.6)(ii) } \\
& =\left\{\left[c _ { 1 } c * _ { 2 } \Gamma ^ { \prime } d * _ { 2 } \beta * _ { 2 } [ e ] * _ { 2 } \left[\left(\Gamma^{\prime} a *_{2} \propto *_{2}\left[d *_{2} c_{2} b^{\prime}\right]\right\}-2\right.\right.\right. \\
& \text { ( } c_{1} a b b^{\prime} *_{1} c_{1} a b b^{\prime} \text { ) by (2.1.6)(ii) and associativity } \\
& =\left\{\left[\left(\varepsilon_{1} c *_{2} \Gamma^{\prime} d *_{2} \beta *_{2} \Gamma e\right)-c_{2} c_{1} c d d^{\prime}\right]+_{2} c_{1}\left(c d b^{\prime}-a b b^{\prime}\right)\right\} \\
& *_{1}\left\{\left[\left(\Gamma^{\prime} a *_{2} \alpha *_{2} \Gamma d *_{2} \varepsilon_{1} b^{\prime}\right)-_{2} c_{1} a b b^{\prime}\right]\right\} \text { by (2.1.6)(iv) } \\
& =\left\{\left\{c_{1} c *_{2}\left[\left(\Gamma^{\prime} d *_{2} \beta *_{2} \Gamma e\right)-c_{2} c_{1} d b^{\prime}\right]\right\}+_{2} \varepsilon_{1}\left(c d b^{\prime}-a b b^{\prime}\right)\right\} *_{1} \\
& \left.\left\{\left(\Gamma \prime a *_{2} \alpha *_{2} \Gamma d\right)-2 c_{1} a b\right) *_{2} c_{1} b{ }^{\prime}\right\} \text { by the identity rule } \\
& =\left[\left(c_{1} c *_{2} \Phi \beta\right)+_{2} c_{1}\left(c d b^{\prime}-a b b^{\prime}\right)\right] *_{1}\left[\Phi \alpha *_{2} c_{1} b^{\prime}\right] \\
& =\left[\left(\varepsilon_{1} c *_{2} \Phi \beta\right)+_{2} \varepsilon_{1}\left(c d b^{\prime}-a b b^{\prime}\right)\right] *_{1}\left[c_{1} 0+_{2}\left(\Phi \alpha *_{2} c_{1} b^{\prime}\right)\right] \\
& =\left[\left(c_{1} c *_{2} \Phi \beta\right) *_{1} c_{1} 0\right]+_{2}\left[c_{1}\left(c d b^{\prime}-a b b^{\prime}\right) *_{1}\left(\Phi \alpha *_{2} c_{1} b^{\prime}\right)\right] \\
& \text { by (2.1.6)(iv) }
\end{aligned}
$$

$=\left(c_{1} c *_{2} \Phi \beta\right)+_{2}\left(\Phi \alpha *_{2} c_{1} b^{\prime}\right)$ by the identity rule
$=\left(c_{1} \partial_{1}^{0} \alpha *_{2} \Phi \beta\right)+{ }_{2}\left(\Phi \alpha *_{2} c_{1} \partial_{1}^{1} \beta\right)$.
It is clear that (v) is satisfied by using (3.1.4)(v) for the first rule, and the algebroid laws for the second. This completes the proof of the proposition .

We are ready now to construct a functor say $\lambda$ from the category $C$ of crossed modules (over. algebroids) to the category DA! of special double algebroids with connections.

## 3. THE FUNCTOR $\lambda: C \rightarrow$ DA!:

In this section, we construct a special double algebroid With connections from a crossed module (over an algebroid) by Using the folding operation.

Let ( $A, M, \mu$ ) be a crossed module (over an algebroid), and let $D_{0}=A_{0}$ (the set of objects), $D_{1}=D_{2}=A$ (the algebroid of arrows of $A$ ). Since $\square D_{1}=\square A$ is a special double algebroid With thin structure, then the folding operation applies to it and so for $a \in \square A$ with boundary edges $\left(\begin{array}{lll}a_{1} & a_{3} & a_{4}\end{array}\right)$, we have $\partial_{1}^{0} \Phi_{a}=a_{3} a_{4}-a_{1} a_{2}$. We let $D$ be given by $D=\left\{(\underline{a}, \zeta): \underline{a} \in D_{1}, \zeta \in M\right.$ such that $\left.\mu \zeta=\partial_{1}^{0} \Phi_{\underline{a}}\right\}$. Thus we Can define the maps $c_{j}, a_{1}^{i}, a_{2}^{i}, \Gamma, \Gamma$ (for $j=1,2$ and $i=0,1$ ) in the following way :
if $a_{1} \in D_{1}$, define $c_{j} a_{1}=\left(c_{i} a_{1}, 0\right)$, where $c_{i}$ is defined by (2.1.7) Clearly $\varepsilon_{j^{a}} \in \mathcal{D}$ (since $\Phi \varepsilon_{j^{a}}=0^{2}$ ). Also define $\partial_{1}^{i}, \partial_{2}^{i}: D \rightarrow D_{1}=D_{2}$ by: if( $a, 5$ ) $\in D$, then the boundary edges of $(\underline{a}, 5)$ are to be those of $\underline{a}, i . e . \underline{\partial}(\underline{a}, 5)=\underline{a}$ a . Define $\Gamma a_{1}=\left(\Gamma a_{i} ; 0\right), \Gamma^{\prime} a_{1}=\left(\Gamma^{\prime} a_{1}, \Delta\right)$; where $\Gamma, \Gamma^{\prime}$ are defined by (3.1.4) .

We define now some algebraic structure on elements of $D$. First we define two additions ; namely $+_{2},+_{2}$. For $+_{1}$, let $(\underline{a}, 5),(\underline{b}, n) \in D$ with $a_{1}^{i} \underline{a}=a_{1}^{i} \underline{b}$; then we define $(\underline{a}, \boldsymbol{s})+_{1}(\underline{b}, n)=(\underline{a}+1 \underline{b}, s+n)$. For $+_{2}$, let $(\underline{a}, 5),(\underline{b}, n) \in D$ with $\partial_{2}^{j} \underline{a}=a_{2}^{j} \underline{b}$; we define
$(\underline{a}, 5)+_{2}(\underline{b}, n)=(\underline{a}+2 \underline{b}, 5+n)$.
We define two scale $r$ multiplications: let (a, s) $\in D$ and $r \in R$;

Note that these definitions make sense. Thus we have
$\partial_{1}^{0} \Phi\left(\underline{a}+{ }_{i} \underline{b}\right)=\partial_{1}^{0} \Phi \underline{a}+{ }_{2} \partial_{1}^{0} \Phi \underline{b}=\mu 5+\mu \eta=\mu(5+\eta)$,
$\partial_{1}^{0} \Phi\left(r r_{1} \underline{a}\right)=\partial_{1}^{0}\left(r r_{2} \Phi \underline{\text { a }}\right)=r \cdot \partial_{1}^{0} \Phi \underline{a}=r \cdot \mu 5=\mu(r \cdot 5)$.
Next, we define two compositions :
let $(\underline{a}, 5),(\underline{b}, \pi) \in D$ with $\partial_{2}^{1} \underline{a}=\partial_{2}^{\circ} \underline{b}$; then we define
$(\underline{a}, 5) *_{1}(\underline{b}, n)=\left(\underline{a} *_{1} \underline{b}, \varsigma^{a_{2}^{1} \underline{b}}+{ }_{2}^{\alpha_{2} \underline{\underline{a}}} n\right)$. If $(\underline{a}, 5),(\underline{b}, n) \in D$ With $\partial_{2}^{1} \underline{a}=a_{2}^{0} \underline{b}$, then we define
$(\underline{a}, \underline{s}) *_{2}(\underline{b}, n)=\left(\underline{a} *_{2} \underline{b}, a_{1}^{0} \underline{a} n+5^{\partial_{1}^{1} \underline{b}}\right)$.
We must verify the appropriate boundary condition, we have $\partial_{1}^{0} \Phi\left(\underline{a} *_{1} \underline{b}\right)=\partial_{1}^{0}\left[\left(\Phi \underline{a} *_{2} c_{1} a_{2}^{1} \underline{b}\right)+_{2}\left(c_{1} a_{2}^{0} \underline{a} *_{2} \Phi \underline{b}\right)\right.$
$=\left(\partial_{1}^{0} \Phi \underline{a} *_{2} \partial_{1}^{0} c_{1} \partial_{2}^{1} \underline{b}\right)+_{2}\left(\partial_{1}^{0} c_{1} \partial_{2}^{0} \underline{\underline{a}} *_{2} \partial_{1}^{0} \Phi \underline{b}\right)$
$=\left(\mu \varphi *_{2} \partial_{2}^{1} \underline{b}\right)+_{2}\left(\partial_{2}^{0} \underline{a} *_{2} \mu n\right)=\mu\left(c^{\partial_{2}^{1} \underline{b}}+\partial_{2}^{0} \underline{\underline{a}} n\right)$
by (1.3.1)(iii) and (i.3.2)(i).

Thus we are ready to give the main result of this section.
Proposition 3.3.1: The above structure'is a special double algebroid with connections.
Proof: First, we want to verify that $\left(+_{1}, *_{1}, \mu_{1}\right)$ and $\left({ }_{2}, *_{2}, 2_{2}\right)$ each give an algebroid structure, that is, $*_{1}$, $*_{2}$ are R-bilinear morphisms and satisfy the associative Condition. It is clear that $*_{1}$ is an R-bilinear morphism.

Thus $\left(+_{1}, *_{1}, H_{1}\right)$ an R-algebroid if $*_{1}$ satisfies associativity . Let $(\underline{a}, \underline{s}),(\underline{b}, n),(\underline{c}, \xi) \in D$. Then
$\left[(\underline{a}, \varsigma) *_{1}(\underline{b}, n)\right] *_{1}(\underline{c}, \xi)=\left[\underline{a} *_{1} \underline{b}, s^{a_{2}^{1} \underline{b}}+{ }_{2}^{\partial_{2}^{0}}{ }_{n]} *_{1}(\underline{c}, \xi)\right.$
$=\left[\left(\underline{a} *_{1} \underline{b}\right) *_{1} \underline{c},\left(5^{\partial_{2}^{1} \underline{b}}+\partial_{2}^{0} \underline{a}\right)^{\partial_{2}^{1} \underline{c}}+\partial_{2}^{0}\left(\underline{a} *_{1} \underline{b}\right) \varepsilon\right]$. $O_{n}$ the other hand ;
$(\underline{a}, \underline{s}) *_{1}\left[(\underline{b}, n) *_{1}(\underline{c}, \varepsilon)\right]=(\underline{a}, \underline{s}) *_{1}\left[\left(\underline{b} *_{1} \underline{c}\right), n^{a_{2}^{1} \underline{c}}+{ }_{2}^{\partial_{2}^{0} \underline{b}} \varepsilon\right]$
$=\left[\underline{a} *_{1}\left(\underline{b} *_{1} \underline{c}\right), 5^{\partial_{2}^{1}\left(\underline{b} *_{1} \underline{c}\right)}+{ }_{2}^{\partial_{2}^{0} \underline{a}}\left(\eta^{\partial_{2}^{1} \underline{c}}+{ }^{\partial_{2}^{0} \underline{b}} \varepsilon\right)\right]$. Clearly ( $\underline{a} *_{1} \underline{b}$ ) $*_{1} \underline{c}=\underline{a} *_{1}\left(\underline{b} *_{1} \underline{c}\right)$. To prove that
$\left(5^{\partial_{2}^{1} \underline{b}}+\partial_{2}^{0} \underline{a}\right)^{a_{2}^{1} \underline{c}}+\partial_{2}^{0}\left(\underline{a} *_{1} \underline{b}\right) \varepsilon=5^{\partial_{2}^{1}\left(\underline{b} *_{1} \underline{c}\right)}+$



$=\left(\varepsilon^{\partial_{2}^{1} \underline{b}}+\partial_{2}^{0} \underline{a}\right)^{\partial_{2}^{1} \underline{c}}+\partial_{2}^{0}\left(\underline{a} *_{1} \underline{b}\right)_{\varepsilon}$
by (1.3.1)(i ,iii)
$=$ left hand side. Then
$\left[(\underline{a}, \varsigma) *_{1}(\underline{b}, n)\right] *_{1}(\underline{c}, \varepsilon)=(\underline{a}, \varsigma) *_{1}\left[(\underline{b}, n) *_{1}(\underline{c}, \underline{\xi})\right]$. The verification of the associativity with respect to $*_{2}$ is similar to that of $*_{1}$. Thus $\left(t_{2}, *_{2}, \therefore_{2}\right)$ is an
R-algebroid. So we get algebroid structures for each of these $t_{\text {wo }}$ kind of operations .

Next, we want to verify the relations between these operations, and the rules for connections .

For the rules (2.1.3) , (2.1.4) the proofs are obvious,
since $D_{1}=D_{2}$. Now we verify the rule (2.1.5)(i-iii).
Let $(\underline{a}, 5),(\underline{b}, n) \in D$, then $(\underline{a}, \underline{c})+_{2}(\underline{b}, \eta)=(\underline{a}+2 \underline{b}, s+\pi)$ and hence $r \cdot{ }_{1}\left(\underline{a}+{ }_{2} \underline{b}, s+n\right)=\left(r \cdot_{1}(a+2 \underline{b}), r \cdot{ }_{1}(5+n)\right)$
$=\left[\left(r \cdot \underline{\underline{a}}+{ }_{2} r \cdot \underline{r}\right),((r \cdot s)+(r \cdot n))\right]$
$=((r \cdot 1 \underline{a}),(r \cdot 5))+_{2}((r \cdot 1 \underline{b}),(r \cdot n))$
$=r r_{1}(\underline{a}, ~ ¢)+_{2} r ._{1}(\underline{b}, n)$.
We prove similarly that
$r \cdot{ }_{2}\left[(\underline{a}, 5)+_{1}(\underline{b}, n)\right]=r \cdot 2(\underline{a}, 5)+_{1} r \cdot 2(\underline{b}, n)$,
if ( (a, 5 ) $+_{2}(\underline{b}, n$ ) is defined. Thus the rule (2.1.5)(i) is satisfied.

For (2.1.5)(ii), suppose given ( $\mathbf{a}, 5$ ), ( $\mathbf{b}, \boldsymbol{n}$ ) such that $(\mathrm{a}, 5) *_{2}(\underline{b}, n)$ is defined. Then
$r \cdot \cdot_{1}\left[(\underline{a}, s) *_{2}(\underline{b}, \eta)\right]=r \cdot 1\left[\left(\underline{a} *_{2} \underline{b}\right),{ }^{a_{1}^{o} \underline{a}} n+5^{\partial_{2}^{1} \underline{b}}\right]$
$=\left(r ._{1}\left(\underline{a} *_{2} \underline{b}\right), r \cdot\left({ }_{1}^{\partial_{1}^{0} \underline{a}}{ }_{n}+5^{\partial_{1}^{1} \underline{b}}\right)\right)$
$=\left(\left(r ._{1} \underline{a} *_{2} r \cdot 1 \underline{b}\right),\left(r \cdot{ }_{1}^{0} \underline{a} n\right)+\left(r \cdot \varepsilon^{a_{1}^{1} \underline{b}}\right)\right)$ by (1.3.1)(iv)
$=\left[\left((r \cdot 1\right.\right.$ a $\left.) *_{2}\left(r \cdot{ }_{1} \underline{b}\right)\right),\left(\partial_{1}^{o}(r ; \underline{a})(r \cdot n)+r_{2}(r \cdot 5)^{\partial_{1}^{1}(r} ; \underline{b}\right)$

> by bilinearity

$=\left(r ._{1}(\underline{a}, ~ s)\right) *_{2}\left(r ._{1}(\underline{b}, n)\right)$. Similarly for the second part of (2.1.5) (ii).

Finally, for (2.1.5)(iii), given ( $\mathrm{a}, \mathrm{s}$ ) $\in \mathrm{D}$, then




Next, we want to verify the interchange laws (2.1.6)(i-iv).
For (2.1.6)(i), let $(\underline{a}, 5),(\underline{b}, n),(\underline{c}, \varepsilon),(\underline{d}, \Psi) \in D$ such that
$(\underline{a}, 5)+_{1}(\underline{b}, \eta),(\underline{a}, 5)+_{2}(\underline{c}, \varepsilon),(\underline{b}, \eta)+_{2}(\underline{d}, \Psi)$,
$(\underline{\xi}, \xi)+_{1}(\underline{d}, \psi)$ are defined, then
$\left[(\underline{a}, \underline{5})+_{1}(\underline{b}, n)\right]+{ }_{2}\left[(\underline{c}, \underline{\varepsilon})+_{1}(\underline{d}, \psi)\right]=(\underline{a}+\underline{b}, 5+n)+2$ $(\underline{\varepsilon}+1 \underline{d}, \varepsilon+\psi)$
$=\left[\left(\underline{a}+{ }_{1} \underline{b}\right)+_{2}\left(\underline{c}+{ }_{1} \underline{d}\right),(s+\eta)+(\varepsilon+\psi)\right]$
$=[(\underline{a}+2 \underline{c})+1(\underline{b}+2 \underline{d}),(s+\varepsilon)+(n+\psi)]$
$=(\underline{a}+2 \underline{c}, s+\varepsilon)+1(\underline{b}+2 \underline{d}, n+\psi)$
$=\left((\underline{a}, \underline{s})+{ }_{2}(\underline{c}, \underline{\xi})\right)+1_{1}\left((\underline{b}, n)+{ }_{2}(\underline{d}, \Psi)\right)$.
For (2.1.6)(ii), let $(\underline{a}, \underline{s}),(\underline{b}, n),(\underline{c}, \varepsilon),(\underline{d}, \psi) \in D$ such that
$\left(\underline{a}\right.$, s) $*_{1}(\underline{b}, n),(\underline{a}, 5) *_{2}(\underline{c}, \varepsilon),(\underline{b}, n) *_{2}(\underline{d}, \psi),(\underline{c}, \varepsilon) *_{1}(\underline{d}, \psi)$
ore defined, then
$\left((\underline{a}, \varsigma) *_{1}(\underline{b}, n)\right) *_{2}\left((\underline{c}, \underline{\varepsilon}) *_{1}(\underline{d}, \psi)\right)=$
$\left(\underline{Q} *_{1} \underline{\mathrm{~b}}, 5^{\partial_{2}^{1} \underline{\mathrm{~b}}}+\partial_{2}^{0} \underline{\mathrm{a}} n\right) *_{2}\left(\underline{c} *_{1} \underline{\mathrm{~d}}, \varepsilon^{\partial_{2}^{1} \underline{\mathrm{~d}}}+{ }_{2}^{\partial_{2}^{0} \underline{c}} \Psi\right)=$
$l\left(\underline{a} *_{1} \underline{b}\right) *_{2}\left(\underline{c} *_{1} \underline{d}\right), a_{1}^{0}\left(\underline{a} *_{1} \underline{b}\right)\left(\varepsilon^{\partial_{2}^{2} \underline{d}}+\partial_{2}^{0} \underline{c} \psi\right)+$
$\left.\left(c^{\partial^{2} \underline{b}}+\partial_{2}^{o} \underline{a} n\right)^{a_{1}^{1}\left(\underline{c} *_{1} \underline{d}\right)}\right]$
$=\left[\left(\underline{a} *_{2} \underline{\mathrm{c}}\right) *_{1}(\underline{\mathrm{~b}} * 2 \underline{\mathrm{~d}}), \partial_{1}^{0} \underline{\mathrm{a}}\left(\varepsilon^{\partial_{2}^{1} \underline{\mathrm{~d}}}+\partial_{2}^{0} \underline{\mathrm{c}} \psi\right)+\right.$

 $\left.5^{\partial_{2}^{2} \underline{b} a_{1}^{1} \underline{d}}+\partial_{2}^{0}{ }^{0}\left(n^{a_{1}^{1} \underline{d}}\right)\right] \quad$ by $(1.3 .1)(i, i i i)$.
On the other hand;
$\left[(\underline{\mathrm{A}}, \mathrm{s}) *_{2}(\underline{\mathrm{c}}, \underline{\varepsilon})\right] *_{1}\left[(\underline{\mathrm{~b}}, \mathrm{n}) *_{2}(\underline{\mathrm{~d}}, \Psi)\right]=$

$1\left(\underline{\mathrm{a}} *_{2} \underline{\mathrm{c}}\right) *_{1}\left(\underline{\mathrm{~b}} *_{2} \underline{\mathrm{~d}}\right),\left({ }^{\partial_{1}^{\mathrm{O}} \underline{\mathrm{a}}}+5^{\mathrm{a}_{2}^{1} \underline{\mathrm{c}}}\right)^{\mathrm{a}_{2}^{1}\left(\underline{\mathrm{~b}} *_{2} \underline{\mathrm{~d}}\right)}+$ $\left.\partial_{2}^{0}\left(\underline{\underline{G}} *_{2} \underline{\mathrm{c}}\right)\left(\partial_{1}^{\mathrm{O}} \underline{\mathrm{b}}+\eta^{\mathrm{a}_{1}^{1} \underline{\mathrm{~d}}}\right)\right]=$


In order for these to be equal, we need ;



i.e. $\varsigma^{\Phi{ }^{\Phi}}=\Phi_{\underline{\underline{d}}}$.

The last equation follows from the crossed module rule (1.3.2)(ii), since both sides are $5 * \psi$.
$F_{o r}(2.1 .6)(i \operatorname{ii}), \operatorname{let}(\underline{a}, 5),(\underline{b}, n),(\underline{c}, \varepsilon),(\underline{d}, \Psi) \in D$ such that
$(\underline{a}, \varsigma) *_{2}(\underline{b}, n),(\underline{c}, \xi) *_{2}(\underline{d}, \psi),(\underline{a}, \xi)+_{1}(\underline{c}, \varepsilon)$,
$(b, \eta)+_{1}(\underline{d}, \psi)$ are defined, then
$\left[(\underline{\mathrm{a}}, \boldsymbol{\rho}) *_{2}(\underline{\mathrm{~b}}, \mathrm{n})\right]+_{1}\left[(\underline{\mathrm{c}}, \boldsymbol{\varepsilon}) *_{2}(\underline{d}, \psi)\right]$

$=\left[\left(\underline{a} *_{2} \underline{b}\right)+\left(\underline{c} *_{2} \underline{d}\right),\left({ }_{1}^{a_{1}^{0} \underline{n}}+5^{\partial_{1}^{1} \underline{b}}\right)+\left(\partial_{1}^{0} \underline{c} \Psi+\varepsilon^{\partial_{1}^{1} \underline{d}}\right)\right]$

$$
\begin{aligned}
& =\left[(\underline{a}+1 \underline{c}) *_{2}\left(\underline{b}+_{1} \underline{d}\right),\left(^{\partial_{1}^{0} \underline{a}} n+\partial_{1}^{0} \underline{c} \psi\right)+\left(5^{\partial_{1}^{1} \underline{b}}+\varepsilon^{\partial^{1}}\right)\right] \\
& =\left[(\underline{a}+1 \underline{c}) *_{2}\left(\underline{b}+_{1} \underline{d}\right), \partial_{1}^{0} \underline{a}(n+\psi)+\left(5+\varepsilon^{0}\right)^{1}\right]
\end{aligned}
$$

by (1.3.1)(ii) and the above hypothesis

$$
\left.=\left[\left(\underline{a}+_{1} \underline{c}\right) *_{2}\left(\underline{b}+_{1} \underline{d}\right), a_{1}^{0}(\underline{a}+1 \underline{c})(n+\Psi)+(\zeta+\varepsilon)^{a_{1}^{1}(\underline{b}+} \underline{d}\right)\right]
$$

$$
=[\underline{a}+1 \underline{c}, \zeta+\xi] *_{2}[\underline{b}+1 \underline{d}, \eta+\Psi]
$$

We can use a similar argument to verify (2.1.6) (iv)

It is clear that $c_{j}$ satisfy the rule (2.1.7), and $\theta$ satisfy the conditions (3.1.4)(i-v). This completes the proof.

Thus any crossed module (over an algebroid) gives a special double algebroid with connections . If (A, M, $\mu$ ), ( $A^{\prime}, M^{\prime}, \mu^{\prime}$ ) are two crossed modules (over algebroids) and $(\alpha, \beta):(A, M, \mu) \rightarrow\left(A^{\prime}, M^{\prime}, \mu^{\prime}\right)$ is a morphism of crossed modules (over algebroids), then ( $\alpha, \beta$ ) determines a morphism $\lambda(\alpha, \beta)=\Psi: \lambda(A, M, \mu) \rightarrow \lambda\left(A^{\prime}, M^{\prime}, \mu^{\prime}\right)$ where $\Psi:\left(D_{1} D_{1}, D_{2}, D_{0}\right) \rightarrow\left(D^{\prime}, D_{1}^{\prime}, D_{2}^{\prime}, D_{0}^{\prime}\right)$ and $\Psi_{1}=\Psi_{2}=\alpha$, $\Psi\left(m ; a_{b}^{c} d\right)=\left(\beta m ; \alpha a_{\alpha b}^{\alpha c} \alpha d\right)$. This defines a functor
$\lambda: \underline{C} \rightarrow \underline{D A}!$, from the category $\underline{C}$ of crossed modules (over algebroids) to the category DA: of special double algebroids with connections .

## 4. THE EQUIVALENCE OF CATEGORIES:

In this section, we want to prove the main result, which is the equivalence of the two categories $C$, $\underline{D}$ ! .

Theorem 3.4.1: The functors $\gamma, \lambda$ defined previously form an equivalence.

```
y: \underline{DA! #--M C : \lambda.}
```

Proof: First, we want to prove that $\gamma \lambda$ is naturally equivalent to the identity, that is, $\gamma \lambda \cong 1$.

Let $(A, M, \mu)$ be an object of $C$ and let ( $\left.A^{\prime}, M^{\prime}, \mu^{\prime}\right)=$ $\gamma^{\gamma}(A, M, \mu)$. Then $A_{0}=A_{0}$ and $A=A^{\prime}$. It is clear that $M^{\prime}$ is defined on the same set of objects $A_{0}$. Define a map $g: M \rightarrow M$, by
$g(m)=\left(\mu m ; 1 \begin{array}{l}m \\ 0\end{array}\right)$, and let $I: A \rightarrow A$, be the identity map.
We want now to prove that $(I, g):(A, M, \mu) \rightarrow\left(A^{\prime}, M^{\prime}, \mu^{\prime}\right)$ is ${ }^{\text {a }}$ crossed module morphism, that is $I \mu=\mu$ ' $\delta$ and $g$ preserves the actions. Clearly $I, g$ are algebroid morphisms and $\mu^{\prime} g=I \mu$ - So it is enough to show that ( $I, g$ ) preserves the actions.

Take $m: x \rightarrow y \in M(x, y)$ and let $b: y \rightarrow z \in A(y, z)$. Thus

$$
\mathrm{g}\left(\mathrm{~m}^{\mathrm{b}}\right)=\left(\mathrm{m}^{\mathrm{b}} ; 1 \mathrm{n}^{(\mu \mathrm{m}) \mathrm{b}} \mathrm{l}\right) \quad \text { by }(1.3 .2)(\mathrm{i})
$$

$$
=\left(m ; 1 \int_{0}^{\mu \mathrm{m}} 1\right) *_{2} \varepsilon_{1} b=\left(m ; 1{\underset{0}{\mu m} 1)^{b}=g(m)^{b} .}^{\mu m} .\right.
$$

We prove similarly that $g\left(b_{m}\right)=b_{g}(m)$.
We define now a map ( $I, f$ ): ( $\left.A^{\prime}, M^{\prime}, \mu^{\prime}\right) \rightarrow(A, M, \mu)$ such that $(I, g),(I, f)$ are inverse to each other . Let $I: A, \rightarrow A$ be the identity map and define $f: M, \cdots M$ by $f\left(m ; 1 m_{0}^{\mu} 1\right)=m$.
Clearly If are algebroid morphisms and $\mu \mathrm{f}=\mu^{\prime}$. Thus (If) is a crossed module morphism if it preserves the action, that is.
let $\left(\mathrm{m} ; 1 \mathrm{l}_{0}^{\mu \mathrm{m}} \mathrm{l}\right) \in M^{\prime}$ and $b \in A^{\prime}$. Then

$$
f\left[\left(m ; 1 l_{0}^{\mu m} 1\right)^{b}\right]=f\left[\left(m ; 1 \sum_{0}^{\mu m} 1\right) *_{2} c_{1} b\right]=f[(m^{b} ; 1 \underbrace{(\mu m) b}_{0} 1)]
$$

$=f\left(m^{b} ; 1 c^{\mu\left(m^{b}\right)} 1\right)=m^{b}=\left[f\left(m ; 10_{0}^{\mu m} 1\right)\right]^{b}$.
It is clear that $(I, g),(I, f)$ are inverse to each other. Therefore $\gamma \boldsymbol{\lambda}$ is naturally equivalent to the identity.

Second, we want to show that $\lambda y$ is naturally equivalent to the identity, that is, $1 \cong \lambda y$.

Let $D$ be an object of $D A$ ! and let $E=\lambda y(D)$. Then $D_{0}=E_{0}$,
$D_{1}=D_{2}=E_{1}=E_{2}$. We define $n: D \rightarrow E$ to be the identity on $D_{0}$ and $D_{1}=D_{2}$ and on $D$ as follows :
let $\alpha \in D$, define $n(\alpha)=(\underline{\partial} \alpha, \Phi \alpha)$. First we prove;
Lemma 3.4.1: The map $n$ is a morphism of double R-algebroid With connections ( $\Gamma, \Gamma^{\prime}$ ) .

Proof: It suffices to prove that $\cap$ preserves $+_{1},+_{2}, *_{1}$, $*_{2}, \cdot 1, \cdot 2$, and the connection $\left[, \Gamma^{\prime}\right.$.
For $+_{1}$, let $\alpha, \beta \in D$ such that $\alpha+{ }_{1} \beta$ is defined, then $\eta\left(\alpha+{ }_{1} \beta\right)=\left[\underline{\partial}\left(\alpha+{ }_{1} \beta\right), \Phi\left(\alpha+{ }_{1} \beta\right)\right]=\left(\underline{\partial} \alpha+{ }_{1} \underline{\partial} \beta, \Phi \alpha+\Phi \beta\right)$ (since $\Phi a, \Phi \beta \in Y(D)$ )
$=(\underline{\partial} \alpha, \Phi \alpha)+_{1}(\underline{\partial} \beta, \Phi \beta)=\eta \alpha+_{1} \Pi \beta$.
We can prove similarly that $n\left(\alpha+{ }_{2} \beta\right)=n \alpha+{ }_{2} n \beta$, if $\alpha+{ }_{2} \beta$ is defined.

For $*_{1}$, let $\alpha, \beta \in D$ such that $\alpha, \beta$ have boundaries in the
$\operatorname{form}_{b}\left(a_{b}^{c} d\right),\left(a^{b} d^{\prime}\right)$ respectively, then
$\eta\left(\alpha *_{2} \beta\right)=\left[\underline{\partial}\left(\alpha *_{1} \beta\right), \phi\left(\alpha *_{1} \beta\right)\right]$
$=\left(\underline{\partial} \alpha *_{1} \underline{\partial} \beta,(\Phi a) d^{\prime}+a(\Phi \beta)\right.$ by (3.2.3)(iii).
$O_{n}$ the other hand;
$\eta_{\alpha} *_{1} \eta \beta=(\underline{\partial} \alpha, \Phi \alpha) *_{1}(\underline{\partial} \beta, \Phi \beta)$
$=\left(\underline{\partial} \alpha *_{1} \underline{\partial} \beta,(\Phi \alpha) d^{\prime}+a(\Phi \beta)\right)=n\left(\alpha *_{1} \beta\right)$.
We prove similarly that $n\left(\alpha *_{2} \beta\right)=n \alpha *_{2} n \beta$, if $\alpha *_{2} \beta$ is defined.
For . 1 , let $\alpha \in D$ and $r \in R$, then
$\eta(r \cdot 1 \propto)=(\underline{a}(r \cdot 1 \alpha), \Phi(r \cdot 1 \propto))=(r \cdot 1 \underline{\partial} \alpha, r \cdot 2 \Phi \alpha)$ by (3.2.3)(v)
$=(r \cdot 12 \alpha, r \cdot \Phi \alpha)$
(since $\Phi \propto \in \gamma D$ )
$=r \cdot 1(\underline{\partial} \alpha, \Phi \alpha)=r \cdot{ }_{2} n \alpha$. Similarly for $\cdot 2$, we get
$n(r \cdot z \alpha)=r \cdot r^{n \alpha}$.
Finally, for the connection $\Gamma, \Gamma^{\prime}$, let a $\in D_{1}=D_{2}$, so $\Gamma_{a} \in D$ and then $n(\Gamma a)=(\underline{a} \Gamma a, ~ \Phi \Gamma a)=\left(\underline{a} \Gamma a, 0^{2}\right)$ by (3.2.2)(i) $=$ ra . Similarly for $\Gamma^{\prime}$. This is the complete proof of the leman .

We continue now to prove the theorem . First, we define $\eta: E \rightarrow D$ to be the identity on $E_{0}$ and $E_{1}=E_{2}$ and on $E$ by the formulae :
$\eta^{\prime}(\alpha, 5)=\theta\left(1 \begin{array}{c}c \\ c d\end{array}\right) *_{1}\left[5+c_{1} a b\right] *_{1} \theta\left(a \quad \begin{array}{c}a b \\ b\end{array}\right)$ as shown below:


Whenever $(\alpha, 5)$ has boundary edges of the form ( $a c d$ ) and $t_{1}, t_{2}$ are abbreviations for the thin elements with boundaries ( ${ }^{\mathrm{cd}} \mathrm{d}$ ), $\left(\mathrm{a}_{\mathrm{b}}^{\mathrm{ab}} 1\right)$.
Lemma 3.4.3: The maps $n, n^{\text {' }}$ are inverse to each other, that is , (i) $n n^{\prime}=1$ (ii) $n^{\prime} n=1$.

Proof: (i) Let $(\alpha, 5) \in E$, with $\partial_{1}^{0} \Phi \alpha=\mu 5$ and $\alpha$ has boundary -55-
edges given by ( $a^{c} d$ ), then
b

$$
\begin{aligned}
& n n^{\prime}(\alpha, 5)=n\left[(1 \underset{c d}{c} d) *_{1}\left(\Phi \alpha+c_{2} \varepsilon_{1} a b\right) *_{1}\left(a{ }_{b}^{a b} l\right)\right] \\
& =n(\alpha)=(\underline{\partial} \alpha, \Phi \alpha) .
\end{aligned}
$$

It is clear that $\alpha, \partial \alpha$ havethe same boundary (a ${ }_{b}^{c} d$ ), and $\mu \Phi_{\alpha}=\mu \zeta$. Thus $\Pi \eta^{\prime}(\alpha, \zeta)=(\alpha, \zeta)$.
(ii) Let $\propto \in D$, where $\propto$ has boundary (a ${ }_{b}^{c} d$ ), so
$\eta^{\prime} \eta(\alpha)=\eta^{\prime}(\underline{\partial} \alpha, . \Phi \alpha)=\alpha$ (since $\underline{\partial} \alpha$, $\alpha$ have the same boundaries) . This is the complete proof of lemma (3.4.3). $\square$ This completes the proof that $\eta: D \rightarrow-\in$ is an isomorphism. The naturality of $n$ is clear . So we have proved the natural equivalence $1 \approx \lambda \gamma$.

We move on to give a property of these objects by using the above theorem.
5. REFLECTION:

In this section we use the above theorem to show that every object in DA! has a nice property called "reflection" : in a special double algebroid with connection the two algebroid structures are isomorphic.

This property has been given in the double groupoid case in [B-2] under the name "rotation". Reflections in double categories with connection have also studied in [S-1],[S-W-].

For each object $\left(D, \Gamma, \Gamma^{\prime}\right) \in \underline{D A}$ ! , there is a reflection $\rho: D \longrightarrow D$ such that on edges $\rho$ behaves as follows: let $\alpha$ be a square in $D$, pictured as

then $\rho \alpha$ is a square in the form

and $p \propto$ is defined by
$\rho \alpha=\left(c_{1} a *_{2} \Gamma^{\prime} b\right) *_{1}\left[\left(\varepsilon_{1} a b-2\left(\Gamma^{\prime} a *_{2} \alpha *_{2}[d)\right)+_{2} c_{1} c d\right] *_{1}\right.$ ( $\Gamma_{c} *_{2} \varepsilon_{1} d$ ), as shown diagrammatically;

${ }^{+} \quad 1\left[\begin{array}{c}c d \\ c_{1} c d \\ c d\end{array}\right] *_{1} c\left[\begin{array}{cc}c & c \\ {[c} & c_{1} d \\ l\end{array}\right.$,


Theorem 3.5.1: The reflection $P$ satisfies
i) $\rho\left(\Gamma_{a}\right)=\Gamma_{a}, \rho\left(\Gamma^{\prime} a\right)=\Gamma^{\prime} a, \rho\left(c_{1} a\right)=c_{1} a, \rho\left(c_{2} a\right)=c_{2} a$, for $a \in D_{2}$ or $D_{1}$.
ii) $\rho\left(\alpha+{ }_{1} \beta\right)=\rho \alpha+{ }_{2} \rho \beta, \rho\left(\gamma+{ }_{2} \delta\right)=\rho y+{ }_{1} \rho \delta$, whenever $\alpha+{ }_{1} \beta, \gamma+{ }_{2} \delta$ are defined.
iii) $\rho\left(\alpha *_{2} \beta\right)=\rho \alpha *_{2} \rho \beta, \rho\left(y *_{2} \delta\right)=\rho y *_{1} \rho \delta$, whenever $\alpha *_{1} \beta, y *_{2} \delta$ are defined.
iv) $\rho^{2}=i d$.
v) $\rho\left(r_{\cdot 1} \alpha\right)=r \cdot{ }_{2} \rho \alpha, \rho\left(r_{2} \alpha\right)=r \cdot_{1} \rho \alpha$, where $r \in R$. Proof: By theorem (3.4.1), we may assume that $D$ is the double algebroid arising from a crossed module $\mu: M \rightarrow A$. So if $\alpha^{\alpha} \in D$, we may write $\alpha=(m ; a \underset{b}{c} d)$, where $m \in M$, a,b,c,d $\in A$ and $\mu m=c d-a b$. We calculate now $\rho(\alpha)$ as follows :
$\left.\left(0 ; 1{ }_{\text {cd }}^{\text {cd }} 1\right)\right] *_{1}\left(0 ; c_{d}^{c d} 1\right)$

$*_{1}\left(0 ; c^{c d} 1\right)$
$=\left(0 ; 1_{a b}^{a}\right.$ b) $*_{1}\left(-m ; 1{ }_{c d}^{a b} 1\right) *_{1}\left(0 ; c_{d}^{c d} 1\right)=\left(-m ; c_{d}^{a} b\right)$. Now we verify the relations (inv).
i) $\rho(\Gamma a)=\rho\left(0 ; a_{1}^{a} 1\right)=\left(0 ; a \frac{a}{1} 1\right)=\Gamma a$ and by similar way for $r^{\prime} a, c_{1} a, c_{2} a$.
ii) Let $\alpha, \beta \in D$ with boundaries (a $\left.{ }_{b}^{c} d\right),\left(a_{1}^{c} d_{1}\right)$, then $\rho\left(\alpha+{ }_{1} \beta\right)=\left(-\left(m+m_{1}\right) ; c^{a+a_{1}} b\right)$. On the other hand ; $\rho(\alpha)+{ }_{2} P(\beta)=\left(-m ; c_{d}^{a} b\right)+_{2}\left(-m_{1} ; c_{d_{2}}^{a_{1}} b\right)=\left(-\left(m+m_{1}\right) ; c_{d+d_{1}}^{a+a_{1}} b\right)$ $=\rho\left(\alpha+{ }_{1} \beta\right)$. Thus $\rho\left(\alpha+{ }_{1} \beta\right)=\rho \alpha+{ }_{2} \rho \beta$. Also we prove similarly that $\rho(y+2 \delta)=\rho y+{ }_{1} \rho \delta$.
iii) Let $\alpha, \beta \in D$ with boundaries $\left(a_{b}^{c} d\right),\left(a^{b} d^{\prime}\right)$, then $P\left(\alpha *_{1} \beta\right)=\left(-\left(m m^{\prime}\right) ; c^{a a^{\prime}}, e\right)$. On the other hand; $\rho(\alpha) *_{2} \rho(\beta)=\left(-m ; c_{d}^{a} b\right) *_{2}\left(-m^{\prime} ; b^{a^{\prime}} \quad e\right)=\left(-\left(m m^{\prime}\right) ; c^{a a^{\prime}} \quad e\right)$. Thus $\rho\left(\alpha *_{1} \beta\right)=\rho \alpha *_{2} \rho \beta$. Similarly for $\rho\left(\gamma *_{2} \delta\right)=\rho \gamma *_{1} \rho \delta$. The calculation of (iv), (v) are easy to verify. Therefore $p$ satisfies the relations (inv).

## W-ALGEBROIDS (WITHOUT CONNECTIONS) AND

CROSSED COMPLEXES

## 0. INTRODUCTION:

In this chapter our aim is to prove that there exists a functor from the category of w-algebroids (without connections) to the category of crossed complexes (over algebroids) . Thus we should define an w-algebroid (without connections) and crossed complexes (over algebroids).

An analogous result has been given in [B-Hi-2] where they Proved that the existence of a similar functor in the groupoid case. In fact, they proved there exists an equivalence between the category of $\omega$-groupoids and the category of crossed complexes (over groupoids).

## 1. $\omega$-ALGEBROIDS (WITHOUT CONNECTIONS):

In order to define w-algebroids (without connections), we recall the definition of cubical complex (see, for example (B-Hi-2]).

A cubical complex $K$ is a graded set $\left(K_{n}\right)_{n \geqslant 0}$ with face maps ${ }_{i}^{\alpha}: K_{n} \rightarrow K_{n-1}(i=1,2, \ldots, n ; \alpha=0,1)$ and degeneracy maps $c_{i}: K_{n-1} \rightarrow K_{n} \quad(i=1,2, \ldots, n)$ satisfying the usual cubical relations namely
(4.1.1)

$$
\begin{equation*}
\partial_{i}^{\alpha} \partial_{j}^{\beta}=\partial_{j-1}^{\beta} \partial_{i}^{\alpha} \tag{i<j}
\end{equation*}
$$

$$
\begin{aligned}
& c_{i} c_{j}=c_{j+1} c_{i} \\
& \partial_{i}^{\alpha} \varepsilon_{j}=\left\{\begin{array}{lll}
c_{j-1} a_{i}^{\alpha} & (i\langle j) & (4.1 .1)(i i) \\
c_{j} \quad a_{i-1}^{\alpha} & (i>j) & (4.1 .1)(i i i) \\
i d & (i=j)
\end{array}\right.
\end{aligned}
$$

## Definition 4.1.3: An w-algebroid (without connections)

$A=\left\{A_{n} ; a_{i}^{\alpha}, c_{i}\right\}$ is a cubical complex and for $n \geqslant 1$, $A_{n}$ has $n$ algebroid structures over $A_{n-1}$ of the form $\left(A_{n},+_{i}, *_{i},{ }_{i}, \partial_{i}^{0}, \partial_{i}^{1}, \varepsilon_{i}\right)$ related appropriately to each other and to $a_{i}^{\alpha}, c_{i}$. More precisely we require the following axioms : (4.1.3) If $a, b \in A_{n}$, and $a+j b$ is defined (ie. for $\alpha=0,1$, $\partial_{j}^{\alpha}=\partial_{j}^{\alpha}$, then for $\alpha=0,1$
$\partial_{i}^{\alpha}\left(a+{ }_{j} b\right)= \begin{cases}\partial_{i}^{\alpha} a+{ }_{j-1} \partial_{i}^{\alpha} & (i<j) \\ \partial_{i}^{\alpha} a+{ }_{j} \partial_{i}^{\alpha} b & (i>j) \\ \partial_{i}^{\alpha} a & (i=j)\end{cases}$
$c_{i}\left(a+{ }_{j} b\right)= \begin{cases}c_{i} a+{ }_{j+1} c_{i} b & (i<j) \\ c_{i} a+{ }_{j} c_{i} b & (i \Delta j)\end{cases}$
(4.1.4) If $a, b \in A_{n}$, and $a * j b$ is defined (ie. $\partial_{j}^{o} b=a_{j}^{1} a$ ), then for $\alpha=0,1$
$\partial_{i}^{\alpha}\left(a *_{j} b\right)= \begin{cases}\partial_{i}^{\alpha} a *_{j-1} \partial_{i}^{\alpha} & (i<j) \\ \partial_{i}^{\alpha} a *_{j} a_{i}^{\alpha_{b}} & (i>j)\end{cases}$

$$
a_{j}^{0}\left(a *_{j} b\right)=a_{j}^{0} a, \partial_{j}^{1}\left(a *_{j} b\right)=a_{j}^{1} b
$$

$c_{i}\left(a *_{j} b\right)=\left\{\begin{array}{ll}c_{i} a & *_{j+1} c_{i} b\end{array}(i \leqslant j)\right.$
$c_{j} \partial_{j}^{0} a *_{j} a=a=a *_{j} c_{j} \partial_{j}^{1} a$
(4.1.5) If a $\in A_{n}$, and $r \in R$ then $r \cdot j$ a is always defined and

$$
\begin{align*}
& \partial_{i}^{\alpha}\left(r \cdot j^{a}\right)= \begin{cases}r \cdot j-1 \\
r a_{i}^{\alpha} a_{i}^{\alpha} & (i<j) \\
\partial_{i}^{\alpha} a_{i}^{\alpha} & (i>j) \\
(i=j)\end{cases}  \tag{4.1.5}\\
& \varepsilon_{i}\left(r \cdot j_{j}^{a}\right)= \begin{cases}r \cdot{ }_{j+1} c_{i}^{a} & (i \nless j) \\
r \cdot \varepsilon_{j}^{a} & (i>j)\end{cases}  \tag{4.1.5}\\
& r \cdot_{i}\left(a *_{j} b\right)=\left\{\begin{array}{ll}
\left(r \cdot_{i} a\right) *_{i} b=a *_{i}\left(r \cdot_{i}\right) & (i=j) \\
\left(r \cdot_{i}^{a}\right) *_{j}\left(r \cdot_{i} b\right) & (i \neq j)
\end{array}\right. \text { (4.1.5)(iii) } \\
& \left.r \cdot j_{i}\left(s \cdot{ }_{j} a\right)=s \cdot j^{(r} \cdot i^{a}\right) \tag{4.1.5}
\end{align*}
$$

Whenever $s \in R, b \in A_{n}$.
(4.1.6) (The interchange laws) : for $i \neq j$
$(a+i b)+j\left(c+{ }_{i}\right.$
$d)=(a+j$
c) $+_{i}\left(b+_{j}\right.$
d) (4.1.6)(i)
$\left(\mathrm{a} *_{i} \mathrm{~b}\right) *_{j}\left(\mathrm{c} *_{i}\right.$
$d)=(a *$
c) $*_{i}\left(b *_{j} d\right)$
(4.1.6)(ii)
$(a+i$
b) $*_{j}\left(c+_{i}\right.$
d) $=(a *$
c) $+_{i}(b *$
d) (4.1.6)(iii)

Whenever $a, b, c, d \in A_{n}$ and both sides are defined.
Note that for all $n \geqslant 2$ and $1 \leqslant i \leqslant n-1$, the pair $\left(A_{n}, A_{n-1}\right)$
With the two algebroid structures in directions $i$ and $i+1$ forms a double R-algebroid (without connections).

An $\underline{\omega}$-subalgebroid (without connections) of $\mathbb{A}$ is a cubical subcomplex closed under all operations $+_{j}, *_{j}, \cdot j$. Any set $S$ of elements of A generates an w-subalgebroid,
namely the intersection of all $\omega$-subalgebroids containing $S$. This $\omega$-subalgebroid can be built from $S$ by repeated applications of all the structure maps and operations.
Definition 4.1.7: A morphism between two w-algebroids (without Connections) ( $f: \underline{A} \rightarrow \underline{B}$ ) is a family of algebroid morphisms

$$
\left\{f_{n}: A_{n} \cdots B_{n}\right\}
$$

such that $f_{n}: A_{n} \rightarrow B_{n}$ is to commute with all the structures. We denote the resiting category of $\omega$-algebroids by ( $\omega$-Alg) . Clearly we can define finite dimensional versions of the above definitions.
Definition 4.1.8: An m-tuple algebroid (without connections) is an m-truncated cubical complex $\underline{A}=\left(A_{m}, A_{m-1}, \ldots, A O\right)$ Without connections, having $n$ algebroid structures in dimension $n(n \leqslant m)$, and satisfying all the laws for an W-algebroid (without connections) in so far as they make sense.

We describe now the zero and the identity elements in $A_{n}$. First, if $u, v \in A_{n-1}$, then $A_{n}^{j}(u, v)$ denotes the set of elements a $\in A_{n}$ such that $\partial_{j}^{0} a=u, \partial_{j}^{1} a=v$. This set has $a_{n}$ element $\boldsymbol{s}_{j}(u, v)$, where $\boldsymbol{s}_{j}: A_{n-1} \times A_{n-1} \rightarrow A_{n}$ namely, the zero for the $j-t h$ algebroid structure on $A_{n}$, so that if $a \in A_{n}^{j}(u, v)$, then $\zeta_{j}(u, v)+{ }_{j} a=a+{ }_{j} \zeta_{j}(u, v)=a$ and if $b \in A_{n}^{j}(v, w), c \in A_{n}^{j}(z, u)$, then $\zeta_{j}(u, v) *{ }_{j} b=\boldsymbol{\zeta}_{j}(u, w)$ and $c^{*}{ }_{j} \boldsymbol{\zeta}_{j}(u, v)=\zeta_{j}(z, v)$. If $n=1$, then $\boldsymbol{\zeta}_{j}(u, v)$ is written $0_{u v}$, as in chapter $I$. Also note that $u-{ }_{j} u=c_{j}\left(\partial_{j}^{o} u, \partial_{j}^{1} u\right)$.

The element $c_{j} u$ in $A_{n}^{j}(u, u)$, for $u \in A_{n-1}$, is the identity element at for the $j$-th algebroid structure on $A_{n}$. 2. CROSSED COMPLEXES:

We first consider some of the history of crossed complexes over groupoids.

As explained in [B-Hi-5], crossed complexes may be thought of as chain complexes with operators from a group (or
groupoid) but with non-abelian features in dimensions one and two . The crossed complex definition is motivated by the standard example of the homotopy crossed complex $\pi \underline{X}$ of a filtered space $\underline{X}$ :

A reduced crossed complex $\underline{M}$ is a crossed complex in which $M_{0}$ is a point. This structure was called "group system" by Blakers [Bl-1], and he used it to apply the homotopy addition lemma in his investigation of the relationship between the homology and homotopy groups of pairs.

Also J.H.C.Whitehead [Wh-1,2] studied reduced crossed complexes under the name of "homotopy systems" . He proved that the fundamental crossed complex $\pi \underline{X}$ of a CW-complex satisfies in each dimension a freeness condition. The paper [Wh-2] gives relations between homotopy systems and chain Complexes with operators - R.Brown and P.J.Higgins [B-Hi-4] Generalised these results to crossed complexes over groupoids. Also they proved in [B-Hi-2] an equivalence between the Category of crossed complexes over groupoids and the category of $\omega$-groupoids.

Huebschmann and others [Hu-1] have shown how crossed complexes may be used to give an interpretation of the Cohomology groups $H^{n}(G ; \pi)$ of a group $G$ with coefficients in a G-module $\pi$. Lue has explained in [L-2] how related ideas had been developed earlier for varieties of algebras rather than just groups.

In this section, we want to define a crossed complex over an algebroid by using ideas similar to those of [B-Hi-2]. Difinition 4.2.1: A crossed complex $\underline{M}$ (over an algebroid) Consists of a sequence of morphisms of R-algebroids over $M_{0}$

M :
satisfying the relations given below :
i) Each $\delta: M_{n} \rightarrow M_{n-1}, n \geqslant 2$, is the identity on $M_{0}$.
ii) The algebroid $M_{1}$ operates on the right and on the left on
each $M_{n}(n \geqslant 2)$ by actions written $(a, m) \cdots{ }^{a} m \in M_{n}(w, y)$; $(m, b) \cdots m^{b} \in M_{n}(x, z)$, if $m \in M_{n}(x, y)$, a $\in M_{1}(w, x)$, $b \in M_{1}(y, z)$ as shown below

$$
\begin{gathered}
x-\vec{T}-y \\
w_{a}^{-} x \quad y-\vec{b}^{-} z
\end{gathered}
$$

left action right action
$i_{i i}$ If $m \in M_{n}(x, y), m^{\prime} \in M_{2}(y, z)$, $m^{\prime \prime} \in M_{2}(w, x)$, then

$$
\begin{aligned}
& m^{\varepsilon_{m}}= \begin{cases}0_{x z} & \text { if } n \geqslant 3 \\
m m & \text { if } n=2\end{cases} \\
& \delta_{m}^{\prime \prime}= \begin{cases}0_{m y} & \text { if } n \geqslant 3 \\
m_{m}^{\prime \prime} & \text { if } n=2\end{cases}
\end{aligned}
$$

Thus $\delta: M_{2} \longrightarrow M_{1}$ is a crossed module.
iv) For $n>2, \varepsilon: M_{n} \rightarrow M_{n-1}$ preserves the actions of $M_{1}$, where $M_{1}$ acts on itself by composition.
v) $\delta \delta=0: M_{n} \rightarrow M_{n-2}$, for $n \geqslant 3$.

Definition 4.2.2: A morphism of crossed complexes $f: \underline{M} \rightarrow \underline{N}$ is a family of algebroid morphisms

$$
\left\{f_{n}: M_{n} \rightarrow N_{n} \mid n \geqslant l\right\}
$$

Which are compatible with the boundary maps $\delta: M_{n} \rightarrow M_{n-1}$, $N_{n} \rightarrow N_{n-1}$ and the action of $M_{1}, N_{1}$ on $M_{n}, N_{n}$ for $n \geqslant 2$. Thus we get a category of crossed complexes (over algebroids) denoted by (Mrs).

## 3. THE FUNCTOR ( $\omega-A 1 g$ ) $\rightarrow$ (CIs):

In chapter two section 2 we proved that there exists a functor from the category of double algebroids to the category of crossed modules (over algebroids).

In this section we prove our goal of this chapter, namely, there exists a functor from the category ( $\omega$-Alg) of Walgebroids (without connections) to the category (Ers) of crossed complexes, that is, there exists a functor say $y:(\underline{\omega-A l g}) \rightarrow(\underline{C r s})$.

For any $\omega$-algebroid $A$, we construct the crossed complex $M=\gamma \underline{A}$ associated with $A$ as follows :
let $M_{0}=A_{0}, M_{1}=A_{1}$ and $\delta^{\alpha}=\partial_{2}^{\alpha}: A=\Longrightarrow A_{0}(\alpha=0,1)$, the initial and final maps. For $n 22$ and $x, y \in M_{0}=A_{0}$, let $M_{n}(x, y)=\left\{a \in A_{n}: \partial_{i}^{\alpha} a=c_{i}^{n-2} 0_{x y}, i<n,(\alpha, i) \neq(0,1)\right.$ and $\left.\partial_{n^{a}}^{0}=c_{1}^{n-1} x \quad, \quad \partial_{n}^{1} a=c_{1}^{n-1} y\right\}$.
For example, if $n=2$, then $a \in M_{2}(x, y)$ if a has boundaries of the form

and for $n=3$ an element of $M_{3}(x, y)$ has edges and vertices of the following type


Theorem 4.3.1: The family $\left.\left\{M_{n}\right\}_{n}\right\rangle 0$ can be given the
structure of crossed complex with $\delta=a_{1}^{0}$; algebraic
operation on $M_{n}$ given by $+=+_{n}, *=*_{n}, \quad=\ln _{n}$ for $n \geqslant 1$ and action of $M_{1}$ on $M_{n}$ given by

for all $a, b \in M_{1}$ and $m \in M_{n}$ such that the compositions are defined.

Proof: Clearly the first axiom of the crossed complex is satisfied, since $M_{1}=A_{1}$. For the rest of the axioms, we Verify them in these two lemmas :
Lemma 4.3.2: For $n \geqslant 2$ and $x, y \in M_{0}$, then
i) if $m, m_{1} \in M_{n}(x, y)$, and $2 \leqslant j \leqslant n$, then $m+j m_{1}$ is defined, $m+_{j} m_{1}=m+m_{1}$, and $m t_{n} m_{1}$ belongs to $M_{n}(x, y)$. ii) if $m \in M_{n}(x, y), r \in R$, then $r \cdot j m$ defined and $r \cdot n \in \in M_{n}(x, y)$.
$\left.i_{i i}\right)$ if $m \in M_{n}(x, y)$, $m^{\prime} \in M_{n}(y, z)$, then $m *_{n} m^{\prime}$ is defined and $m *_{n} m^{\prime} \in M_{n}(x, z)$. If $n \geqslant 3$, then $m *_{n} m^{\prime}=\varepsilon_{1}^{n-1} 0_{x z}$. iv) if $n \geqslant 3$, then $+_{j}=+_{k}$ for $2 \leqslant j, k \leqslant n$.

Proof: i) Recall that $m+j m_{l}$ is defined if and only if
$\partial_{j}^{\alpha} m=\partial_{j}^{\alpha} m_{1}(\alpha=0,1)$. So for $2 \leqslant j<n$, we have
$\partial_{j}^{\alpha} m=\partial_{j}^{\alpha} m_{1}=c_{i}^{n-2} 0_{x y}$ (since $m, m_{1}$ belong to the
associated crossed complex), and if $j=n$, then
$\partial_{n}^{0} m=c_{1}^{n-1} x=\partial_{n}^{0} m_{1}$ and $\partial_{n}^{1} m=\varepsilon_{1}^{n-1} y=\partial_{n}^{1} m_{1}$. Thus
$\|_{j} \mathrm{~m}_{1}$ is defined for $2 \leqslant \mathrm{j} \leqslant \mathrm{n}$.
That $m+m_{1}=m+n m_{1}$ now follows in a standard way from the interchange law for ${ }^{+},{ }_{j}$,

We prove now $m+_{n} m_{1} \in M_{n}(x, y)$, so that we need to show
that $\partial_{i}^{\alpha}\left(m+_{n} m_{1}\right)=\varepsilon_{1}^{n-2} 0_{x y}$ for $i<n,(\alpha, i) \not(0,1)$ and
$\partial_{n}^{0}\left(m+_{n} m_{1}\right)=c_{1}^{n-1} x \quad, \partial_{n}^{1}\left(m+m_{1}\right)=c_{1}^{n-1} y \quad$.
For the first part;
$\partial_{i}^{\alpha}\left(m+m_{1}\right)=a_{i}^{\alpha} m{ }_{n-1} \partial_{i}^{\alpha} m_{1} \quad$ by (4.1.3)(i)
$=c_{1}^{n-2} 0_{x y}+{ }_{n-1} c_{1}^{n-2} 0_{x y}=c_{1}^{n-2} 0_{x y}$.
For the second part;
$\partial_{n}^{0}\left(m+{ }_{n} m_{1}\right)=a_{n}^{0} m=c_{1}^{n-1} x \quad$.
$\partial_{n}^{1}\left(m+m_{1}\right)=a_{n}^{1} m=c_{1}^{n-1} y \quad$.
Thus $m+_{n} m_{1} \in M_{n}(x, y)$.
ii) Since $r$. $j$ is always defined, so we need to prove that $r \cdot{ }_{n} m \in M_{n}(x, y)$, that is, $\partial_{i}^{\alpha}\left(r \cdot n^{m}\right)=\varepsilon_{1}^{n-2} 0_{x y}$, $i<n$ and $(\alpha, i) *(0,1)$ and $\partial_{n}^{o}\left(r \cdot n^{m}\right)=\varepsilon_{1}^{n-1} x, \partial_{n}^{1}\left(r \cdot n^{m}\right)=c_{1}^{n-1} y$. For the first part, we get ;
$\partial_{i}^{\alpha}\left(r \cdot{ }_{n} m\right)=r \cdot{ }_{n-1} \partial_{i}^{\alpha} m=r \cdot{ }_{n-1} c_{1}^{n-2} 0_{x y}=c_{1}^{n-2} 0_{x y}$. For the second part ;

$$
\begin{array}{rlr}
\partial_{n}^{0}\left(r ._{n} m\right) & =a_{n}^{0} m & \text { by }(4.1 .5)(i) \\
& =c_{1}^{n-1} x & \\
\partial_{n}^{0}\left(r{ }_{n} m\right) & =\partial_{n}^{0} m & \text { by }(4.1 .5)(i) \\
& =c_{1}^{n-1} y &
\end{array}
$$

Thus $r$. $m \in M_{n}(x, y)$.
(ii) Since $m *_{n} m$ is defined if and only if $\partial_{n}^{0} m=\partial_{n}^{1} m$
and clearly $a_{n}^{0} m,=c_{1}^{n-1} y=a_{n}^{1} m$, then $m *_{n} m^{\prime}$ is defined. We need to show that $m *_{n} m \in M_{n}(x, z)$, that is,
$\partial_{i}^{\alpha}\left(m *_{n} m^{\prime}\right)=c_{1}^{n-2} 0_{x z}, i<n,(\alpha, i) *(0,1)$ and
$\partial_{n}^{0}\left(m *_{n} m^{\prime}\right)=\varepsilon_{1}^{n-1} x \quad, \quad \partial_{n}^{1}\left(m *_{n} m^{\prime}\right)=c_{1}^{n-1} z \quad$.
For the first part ;

$$
\begin{aligned}
\partial_{i}^{\alpha}\left(m *_{n} m^{\prime}\right) & =\partial_{i}^{\alpha} m *_{n-1} \partial_{i}^{\alpha} m^{\prime} \quad \text { by }(4.1 .4)(i) \\
& =c_{1}^{n-2} 0_{x y} *_{n-1} c_{1}^{n-2} 0_{x z}=c_{1}^{n-2} 0_{x z}
\end{aligned}
$$

For the second part;
$\partial_{n}^{0}\left(m *_{n} m^{\prime}\right)=\partial_{n}^{0} m=c_{1}^{n-1} x \quad$ and
$\partial_{n}^{1}\left(m *_{n} m^{\prime}\right)=\partial_{n}^{1} m^{\prime}=c_{1}^{n-1} z$. Thus $m *_{n} m^{\prime} \in M_{n}(x, z)$.
Now to prove that $m *_{n} m^{\prime}=\varepsilon_{1}^{n-1} 0_{x z}$ if $m \in M_{n}(x, y)$, $m^{\prime} \in M_{n}(y, z)$ and $n \geqslant 3$.
$m *_{n} m^{\prime}=\left(m *_{n} m{ }^{\prime}\right) *_{n-1}\left(c_{1} \partial_{n-1}^{1} m *_{n} c_{1} a_{n-1}^{1} m^{\prime}\right)$
$=\left(m *_{n-1} c_{1} a_{n-1}^{1} m\right) *_{n}\left(m, *_{n-1} c_{1} \partial_{n-1}^{1} m \prime\right)$ by (4.1.6)(ii).
Since $m \in M_{n}(x, y), m^{\prime} \in M_{n}(y, z)$, then
$\partial_{n-1}^{2} m=c_{1}^{n-2} 0_{x y}$ and $a_{n-1}^{1} m^{\prime}=c_{1}^{n-2} 0_{y z}$ (for $n=2$ these
equations are not true). Thus
$*_{n} m=\varepsilon_{1}^{n-1} 0_{x y} *_{n} \varepsilon_{1}^{n-1} 0_{y z}=\varepsilon_{1}^{n-1} 0_{x z}$.
iv) Given $m, m_{l} \in M_{n}(x, y)$ such that $m+j m_{l}, m+{ }_{f} m_{l}$ are defined for $j * k$, then
$m_{j} m_{1}=\left(m+_{k} 0_{x y}\right)+j\left(0_{x y}+m_{1}\right)$, where $O_{x y}=\varepsilon_{1}^{n-1} O_{x y}$ $=\left(m+j 0_{x y}\right)+_{k}\left(0_{x y}+m_{j}\right) \quad$ by $(4.1 .6)(i)$

$$
=m+{ }_{k} m_{l}
$$

Lemma 4.3.3: Let, $n \geqslant 1$ and $m \in M_{n}(x, y)$, a $\in M_{1}(w, x)$ and ${ }^{b} \in M_{1}(y, z)$, then $a_{m}, m^{b}$ as defined in theorem (4.3.1) lie in $M_{n}(w, y), M_{n}(x, z)$ respectively . This action is preserved by the map $\delta: M_{n} \longrightarrow M_{n-1}$ for $n \geqslant 2$. Further, if $m \in M_{n}(x, y)$, $\mathbb{m}_{1} \in M_{2}(w, x)$ and $m_{2} \in M_{2}(y, z)$, then

$$
\begin{aligned}
& \delta m_{1_{m}}= \begin{cases}c_{1}^{n-1} 0_{w y} & n \geqslant 3 \\
m_{1} m & n=2\end{cases} \\
& m_{m} m_{2}= \begin{cases}c_{1}^{n-1} 0_{x z} & n \geqslant 3 \\
m m_{2} & n=2\end{cases}
\end{aligned}
$$

Proof: Since a $\in M_{1}(w, x)$, then $\varepsilon_{1}^{n-1}$ a $\in M_{n}(w, x)$ and hence
$c_{1}^{n-1} a *_{n} m=a_{m} \in M_{n}(w, y)$ by lemma (4.3.2).
We prove similarly that $m^{b} \in M_{n}(x, z)$.
Let $m \in M_{n}(x, y)$ and a $\in M_{1}(w, x)$, then
$\delta\left(a_{m}\right)=\delta\left[c_{1}^{n-1} a *_{n}^{m}\right]=\delta c_{1}^{n-1} a *_{n} \delta m=c_{1}^{n-1} a *_{n} \delta m$
(since $\delta c_{1}^{n-1} a=c_{1}^{n-1} a$ ). Then
$\delta\left({ }^{( } m\right)=\varepsilon_{1}^{n-1} a *_{n} \delta_{m}=(\delta m)$. Also we can prove similarly
that $\delta\left(m^{b}\right)=(\delta m)^{b}$; thus the action is preserved by the $\operatorname{map} 8$.
For n 23 , we have

$$
\delta_{m_{2}}=e_{1}^{n-1} \delta_{m} *_{n} m=e_{1}^{n-2} m_{1} *_{n} m\left(\text { since } m_{1} \in M_{2}(w, x),\right.
$$

and $\left.\delta_{m_{1}} \in M_{1}(w, x)\right)$. Then

$$
\delta_{\mathrm{m}_{1}}=\varepsilon_{1}^{\mathrm{n}-2_{m_{1}}} *_{\mathrm{n}} m=\varepsilon_{1}^{\mathrm{n}-1} 0_{w y} \quad \text { by lemma (4.3.2)(iii). }
$$

We prove similarly that $m^{\delta m_{2}}=\varepsilon_{1}^{n-1} 0_{x z}$ for $n 23$.

For $n=2$, we have

$$
\begin{aligned}
& \delta_{\mathrm{m}_{1} \mathrm{~m}}=c_{1} \delta_{\mathrm{m}_{1}} *_{2} \mathrm{~m}=\left(\varepsilon_{1} \delta_{\mathrm{m}} *_{2} \mathrm{~m}\right) *_{1}\left(\mathrm{~m}_{1} *_{2} \varepsilon_{2} 0_{\mathrm{xy}}\right) \\
= & \left(c_{1} \delta_{m_{1}} *_{1} \mathrm{~m}_{1}\right) *_{2}\left(\mathrm{~m} *_{1} \varepsilon_{2} 0_{\mathrm{xy}}\right) \quad \text { by }(4.1 .6)(\mathrm{ii}) \\
= & m_{1} \mathrm{~m}
\end{aligned}
$$

As shown in the diagram,

$=\underbrace{w}_{0}{ }_{0}^{m p} y$
We can prove similarly that $m^{\delta m_{2}}=m m_{2}$. This is the
Complete proof of theorem (4.3.1) .
It is clear that the construction of theorem (4.3.1) gives
a functor

$$
y:(\underline{\omega-A l g}) \rightarrow-\cdots(\underline{\text { Cos }})
$$

## THE EQUIVALENCE BETWEEN n-TUPLE ALGEBROIDS AND CROSSED COMPLEXES FOR $n=3$ AND 4

## 0. Introduction:

In this chapter we define w-algebroids, and n-tuple algebroids, with connections, and we prove that there exists an equivalence between the category of n-tuple algebroids (with connections) and the category of n-truncated crossed complexes (over algebroids) for the cases $n=3$ and 4 . Moreover we give a conjecture for the general form of the operation of our folding operation on compositions, which if true would give the equivalence of the categories of W-algebroids and crossed complexes.

## 1. $\omega$-ALGEBROIDS WITH CONNECTIONS:

In chapter IV section 1 we gave the definition of Walgebroid without connections. In this section we add extra structure to that definition to get an w-algebroid with Connections ; namely
Definition 5.1.1: Let A be an w-algebroid (without connections) . We say that $A$ is an w-algebroid with Connections (or simply an w-algebroid) if it has for $n \geqslant 2$ additional structure maps $\Gamma_{i}, \Gamma_{i}^{\prime}: A_{n-1} \rightarrow A_{n}(i=1, \ldots, n-1)$ satisfying the following relations (5.1.2)

$$
\Gamma_{i} \Gamma_{j}= \begin{cases}\Gamma_{j+1} \Gamma_{i} & (i<j)  \tag{5.1.2}\\ \Gamma_{j} \Gamma_{i-1} & (i>j)\end{cases}
$$

$$
a_{i}^{\alpha} \Gamma_{j}= \begin{cases}r_{j-1} \partial_{i}^{\alpha} & (i<j)  \tag{5.1.2}\\ \Gamma_{j} a_{i-1}^{\alpha} & (i>j+1)\end{cases}
$$

$$
\partial_{i}^{\alpha} \Gamma_{j}^{\prime}= \begin{cases}\Gamma_{j-1}^{\prime} a_{i}^{\alpha} & (i<j)  \tag{5.1.2}\\ \Gamma_{j}^{\prime} \partial_{i-1}^{\alpha} & (i \Delta j+1)\end{cases}
$$

$$
\Gamma_{i} \Gamma_{j}^{\prime}= \begin{cases}\Gamma_{j+1}^{\prime} \Gamma_{i} & (i \angle j)  \tag{5.1.2}\\ \Gamma_{j}^{\prime} \Gamma_{i-1} & (i>j+1)\end{cases}
$$

(5.1.3) If $a, b \in A_{n}$ and $a+j b$ is defined, then

$$
\begin{aligned}
& \Gamma_{i}\left(a+{ }_{j} b\right)= \begin{cases}\Gamma_{i} a+{ }_{j+1} \Gamma_{i} b & (i<j) \\
\Gamma_{i} a+{ }_{j} \Gamma_{i} b & (i \Delta j)\end{cases} \\
& \Gamma_{i}^{\prime}\left(a+{ }_{j} b\right)= \begin{cases}\Gamma_{i}^{\prime} a+{ }_{j+1} \Gamma_{i}^{\prime} b & (i<j) \\
\Gamma_{i}^{\prime} a+{ }_{j} \Gamma_{i}^{b} & (i>j)\end{cases}
\end{aligned}
$$

(5.1.4) If $a, b \in A_{n}$ and $a * j b$ is defined, then

$$
\begin{align*}
& \Gamma_{i}^{\prime} \Gamma_{j}^{\prime}= \begin{cases}\Gamma_{j+1}^{\prime} \Gamma_{i}^{\prime} & (i\langle j) \\
\Gamma_{j}^{\prime} \Gamma_{i-1}^{\prime} & (i>j)\end{cases}  \tag{5.1.2}\\
& \Gamma_{i} \varepsilon_{j}= \begin{cases}c_{j+1} \Gamma_{i} & (i<j) \\
c_{j} \Gamma_{i-1} & (i>j) \\
c_{j}^{2} & (i=j)\end{cases}  \tag{5.1.2}\\
& \Gamma_{i}^{\prime} \epsilon_{j}= \begin{cases}c_{j+1} \Gamma_{i}^{\prime} & (i \angle j) \\
c_{j} \Gamma_{i-1}^{\prime} & (i>j) \\
c_{j}^{2} & (i=j)\end{cases} \\
& \left.\begin{array}{l}
\partial_{j}^{0} \Gamma_{j}=\partial_{j+1}^{o} r_{j}=i d \\
\partial_{j}^{1} r_{j}=\partial_{j+1}^{1} r_{j}=c_{j} a_{j}^{1}
\end{array}\right\} \\
& \left.\begin{array}{l}
a_{j}^{i} \Gamma_{j}^{\prime}=a_{j+1}^{1} \Gamma_{j}^{\prime}=i d \\
a_{j}^{0} \Gamma_{j}^{\prime}=a_{j+1}^{0} \Gamma_{j}^{\prime}=c_{j} a_{j}^{0}
\end{array}\right\}
\end{align*}
$$

$$
\begin{align*}
& \Gamma_{i}\left(a *_{j} b\right)= \begin{cases}\Gamma_{i} a *_{j+1} \Gamma_{i} b & (i<j) \\
\Gamma_{i} a *_{j} \Gamma_{i} b & (i>j)\end{cases}  \tag{5.1.4}\\
& \Gamma_{i}^{\prime}\left(a *_{j} b\right)= \begin{cases}\Gamma_{i}^{\prime} a *_{j+1} \Gamma_{i}^{\prime} b & (i<j) \\
\Gamma_{i}^{\prime} a *_{j} \Gamma_{i}^{\prime} b & (i>j)\end{cases} \tag{5.1.4}
\end{align*}
$$

$\Gamma_{j}^{\prime} a *_{j+1} \Gamma_{j}^{a}=\varepsilon_{j}^{a}, r_{j}^{\prime a} *_{j} \Gamma_{j}^{a}=\dot{c}_{j+1}^{a}$ (5.1.4)(iii).
(5.1.5) If $a \in A_{n}$ and $r \in R$, then

$$
\begin{align*}
& \Gamma_{i}\left(r \cdot_{j} a\right)= \begin{cases}r \cdot{ }_{j+1} \Gamma_{i}^{a} & (i<j) \\
r \cdot{ }_{j} \Gamma_{i}^{a} & (i>j)\end{cases}  \tag{5.1.5}\\
& r_{i}^{\prime}\left(r \cdot{ }_{j} a\right)= \begin{cases}r \cdot{ }_{j+1} \Gamma_{i}^{\prime a} & (i<j) \\
r \cdot{ }_{j} \Gamma_{i}^{\prime} & (i>j)\end{cases} \tag{5.1.5}
\end{align*}
$$

[Note that the case $i=j$ in $5.1 .3,5.1 .4,5.1 .5$ are covered by the rules in (5.1.6)(i ,ii) and (5.1.7) below].
(5.1.6) (i) Let $a, b \in A_{n}$ with $\partial_{i}^{\alpha} a=\partial_{i}^{\alpha}$, then

$$
\Gamma_{i}^{1}\left(\partial_{i+1}^{0} a+{ }_{i} \partial_{i+1}^{0} b\right) *_{i+1}\left(a+{ }_{i} b\right) *_{i+1} \Gamma_{i}\left(\partial_{i+1}^{1} a+{ }_{i} \partial_{i+1}^{1} b\right)=
$$

$$
\left(\Gamma_{i}^{\prime} \partial_{i+1}^{0} a *_{i+1} a *_{i+1} \Gamma_{i} a_{i+1}^{1} a\right)+_{i+1}\left(\Gamma_{i}^{\prime} \partial_{i+1}^{0} b *_{i+1} b *_{i+1} \Gamma_{i} a_{i+1}^{1} b\right)
$$

(5.1.6) (ii) Let $a \in A_{n}$ and $r \in R$, then
$r_{i} \partial_{i+1}^{0}\left(r r_{i} a\right) *_{i+1}\left(r r_{i} a\right) *_{i+1} r_{i} a_{i+1}^{1}\left(r r_{i} a\right)=$
${ }^{r}{ }_{i+1}\left(\Gamma_{i}^{\prime} a_{i+1}^{0} a *_{i+1} a *_{i+1} \Gamma_{i} a_{i+1}^{1} a\right)$.
(5.1.7) Let $a, b \in A_{n}$ with $\partial_{j}^{1} a=\partial_{j}^{0} b$. Then

$$
\begin{aligned}
& r_{j}\left(a *_{j} b\right)=\left\{\begin{array}{ll}
\Gamma_{j} a & c_{j} b \\
c_{j+1} b & \Gamma_{j} b
\end{array}\right\} \\
& \Gamma_{j}^{\prime}\left(a *_{j} b\right)=\left\{\begin{array}{ll}
\Gamma_{j}^{\prime} & c_{j+1} a \\
c_{j} a & \Gamma^{\prime} j^{b}
\end{array}\right\}
\end{aligned}
$$



Note that for all $n \geqslant 2$ and $1 \leqslant i \leqslant n-1$, the pair ( $A_{n}, A_{n-1}$ ) With the two algebroid structures in direction $i$ and $i+1$ forms a double algebroid with connections as in chapter II .

It is clear that we can define an $\omega$-subalgebroid in a similar way to that in chapter IV.
Definition 5.1.8: A morphism between $\omega$-algebroids ( $f: \underline{A} \rightarrow$ B) is a morphism of algebroids (without connections) preserving the connections . We denote the resulting category of Walgebroids by ( $\omega$-Alg) .
Definition 5.1.9: An m-tuple algebroid $A$ is an m-truncated cubical complex with connections having $n$ algebroid structures in dimension $n(n \leqslant m)$, and satisfying all the laws for an Walgebroid in so far they make sense . Thus 2-tuple algebroids are exactly the double algebroids of chapter II.

We move on now to give the first stage of constructing a functor from (Crs) to ( $\underline{\omega-A l g \text { ). As is to be expected, this }}$ requires the construction of a "folding operation".

## 2. FOLDING OPERATIONS:

In this section, we introduce an operation $\Phi$ on cubes in an w-algebroid $A$ which has the effect of "folding" all faces of a $\in A_{n}$ onto the ( 0,1 )-th face, so that they can be combined to form a "word" in the "folded" faces of a This operation $\Phi$ transforms a into an element of the associated crossed complex $\gamma$ A . It is important that $\Phi a$ is constructed from a and the "shell" ${ }^{\text {a a consisting of all the faces } \partial_{j}^{\alpha} \mathrm{a}}$ of a . This will imply that a itself can be reconstructed from \$a and the shell ${ }^{\text {da }}$.

Note: The folding operation $\Phi$ has a similar effect to the folding operation in the $\omega$-groupoid context in [B-Hi-2]. However the case of $\omega$-algebroids present considerably more technical difficulty.

First, we define operation $\psi_{j}, \Phi_{j}: A_{n} \rightarrow A_{n}$ by the formulae

$$
\begin{aligned}
& \Psi_{j} a=\Gamma_{j}^{\prime} \partial_{j+1}^{0} a *_{j+1} a *_{j+1} \Gamma_{j} \partial_{j+1}^{1} a, \\
& \Phi_{j} a=\Psi_{j} a-{ }_{j+1} c_{j} \partial_{j}^{2} \psi_{j} a, \text { for } a \in A_{n} \text { and } 1 \leqslant j \leqslant n-1 .
\end{aligned}
$$

Also we define $\Phi^{\prime}=\Phi_{n-1} \ldots \Phi_{1}$. It may be checked that $\psi_{j}, \Phi_{j}$
are well defined ; the proof is essentially the same as that in (5.2.1), (5.2.2) below.

Second, we define $\Phi_{j}^{\prime}=\psi_{j}-{ }_{n} c_{j} \partial_{j}^{1} \psi_{j}$, where $\Phi_{j}^{\prime}$ will be shown to be well defined on $\Phi_{j+1}^{\prime} \cdots \phi_{n-2}^{\prime} \Phi^{\prime}$ a , for a $\epsilon A_{n}$. We define $\Phi^{\prime \prime}=\Phi_{1}^{\prime} \ldots \Phi_{n-2}^{\prime}$ and will show later that $\Phi^{\prime \prime}$ is well defined on elements $\Phi^{\prime} a$.

Finally , we define $\Phi=\Phi^{\prime \prime} \Phi^{\prime}$.
Now to give pictures for the above definitions, we shall Use the cube in dimension 3 . $l_{\text {let }}$ a $\in A_{3}$ have edges and vertices given by :


So $\Psi_{1} a$ is in the form

and hence $\Phi_{1} a$ is of the type


Thus $\Psi_{2} \Phi_{2} a$ is in the form


And so $\Phi_{2} \Phi_{1} a$ is given by


Where $t=p\left(q f-h q^{\prime}\right)-\left(r e-g r^{\prime}\right) s^{\prime}$.

Now we have $\Psi_{1} \Phi_{2} \Phi_{1}$ a of the type


Thus $\Phi_{\mathrm{a}}=\Phi_{1}^{\prime} \Phi_{2} \Phi_{1} \mathrm{a}$ is in the form


This shows that the vertices and the edges of $\Phi$ a are Appropriate to an element of $\gamma$ A. We will prove later that \$a does belong to $\gamma \underline{A}$.
The laws of the previous sections imply various laws for the ${ }^{0}$ operations $\psi_{j}, \Phi_{j}$.
Lemma 5.2.1:

$$
\begin{aligned}
& \partial_{i}^{\alpha} \psi_{j}= \begin{cases}\Psi_{j-1} \partial_{i}^{\alpha} & (i<j) \\
\Psi_{j} \partial_{i}^{\alpha} & (i>j+1)\end{cases} \\
& \partial_{j+1}^{\alpha} \Psi_{j}=c_{j} \partial_{j}^{\alpha} \partial_{j+1}^{\alpha}\left(=c_{j} \partial_{j}^{\alpha} \partial_{j}^{\alpha}\right) \quad \text { (5.2.1)(ii) } \\
& a_{j}^{0} \psi_{j}^{a}=\partial_{j}^{o}{ }^{a}{ }_{j} a_{j+1}^{1} \text { a for } a \in A_{n}(5.2 .1)(i i i) \\
& \partial_{j}^{1} \psi_{j} a=a_{j+1}^{0} a *_{j} \partial_{j}^{1} a \quad \text { for } a \in A_{n}(5.2 .1)(i v)
\end{aligned}
$$

## Proof:

(5.2.1)(i): For $i<j$, let a $\in A_{n}$. Then
$\partial_{i}^{\alpha} \psi_{j}^{a}=\partial_{i}^{\alpha}\left[\Gamma_{j}^{\prime} \partial_{j+1}^{0} a *_{j+1} a *_{j+1} \Gamma_{j} \partial_{j+1}^{2} a\right]$
$=\partial_{i}^{\alpha} \Gamma_{j} \partial_{j+1}^{0} a \quad *_{j} \partial_{i}^{\alpha} \quad *_{j} \partial_{i}^{\alpha} \Gamma_{j} \partial_{j+1}^{1} a \quad$ by (4.1.4)(i)
$=\Gamma_{j-1}^{\prime} \partial_{j}^{0} \partial_{i}^{\alpha} a *_{j} \partial_{i}^{\alpha}{ }_{a} *_{j} \Gamma_{j-1} \partial_{j}^{1} \partial_{i}^{\alpha} a=\psi_{j-1} \partial_{i}^{\alpha} a$
by (5.1.2)(vii,viii) and (4.1.1)(i).
(5.2.1)(i): For $i>j+1$, let a $\in A_{n}$. Then

$$
\partial_{i}^{\alpha} \psi_{j} a=\partial_{i}^{\alpha}\left[\Gamma_{j}^{\prime} \partial_{j+1}^{0} a *_{j+1} a *_{j+1} r_{j} \partial_{j+1}^{1} a\right]
$$

$$
=\partial_{i}^{\alpha} \Gamma_{j}^{\prime} \partial_{j+1}^{0} a *_{j+1} \partial_{i}^{\alpha} a *_{j+1} \partial_{i}^{\alpha} \Gamma_{j} \partial_{j+1}^{1} a \quad \text { by (4.1.4)(i) }
$$

$=\Gamma_{j}^{\prime} \partial_{j+1}^{0} \partial_{i}^{\alpha}{ }_{a} *_{j+1} \partial_{i}^{\alpha}{ }_{a} *_{j+1} \Gamma_{j} a_{j+1}^{1} \partial_{i}^{\alpha}{ }_{a}=\psi_{j} \partial_{i}^{\alpha}{ }_{a}$
by (5.1.2)(vii,viii) and (4.1.1)(i) .
(5.2.1)(ii): Let a $\in A_{n}$. Then
$\partial_{j+1}^{\alpha} \psi_{j} a=\partial_{j+1}^{\alpha}\left[\Gamma_{j}^{\prime} \partial_{j+1}^{0} a *{ }_{j+1} a *_{j+1} \Gamma_{j} a_{j+1}^{1} a\right]$.
If $\alpha=0$, we get

$$
\begin{aligned}
\partial_{j+1}^{0} \psi_{j}^{a} & =a_{j+1}^{0} \Gamma_{j}^{\prime} \partial_{j+1}^{0} \\
& =c_{j} \partial_{j}^{0} \partial_{j+1}^{0} a
\end{aligned} \quad \text { by the algebroid axiom } \quad \text { (5.1.2)(vi). }
$$

If $\alpha=1$, we get
$\partial_{j+1}^{2} \psi_{j} a=a_{j+1}^{1} \Gamma_{j} a_{j+1}^{1} a$ by the algebroid axioms

$$
=c_{j} \partial_{j}^{1} \partial_{j+1}^{1} a \quad \text { by (5.1.2)(v). }
$$

Thus $\partial_{j+1}^{\alpha} \Psi_{j}=c_{j} \partial_{j}^{\alpha} \partial_{j+1}^{\alpha}$.
(5.2.1)(iii):

$$
\partial_{j}^{0} \psi_{j}^{a}=\partial_{j}^{0}\left[\Gamma_{j}^{\prime} \partial_{j+1}^{0} a *_{j+1}^{a} *_{j+1} \Gamma_{j} \partial_{j+1}^{1} a\right]
$$

$=\partial{ }_{j}^{0} \Gamma_{j} \partial_{j+1}^{0} a *{ }_{j} \partial_{j}^{0} a *_{j} \partial_{j}^{0} \Gamma_{j} \partial_{j+1}^{1} a$
$=c_{j} \partial_{j}^{0} \partial_{j+1}^{0} a *{ }_{j}^{0} a *_{j} \partial_{j+1}^{1} a$
by (5.1.2)(v ,vi)
$=\partial_{j}^{0}{ }^{0}{ }_{j} \partial_{j+1}^{1} a \quad$ (since $c_{j} \partial_{j}^{0} \partial_{j+1}^{0}$ a is an identity). (5.2.1)(iv):

$$
\begin{aligned}
& \partial_{j}^{1} \Psi_{j} a=\partial_{j}^{1}\left[\Gamma_{j}^{\prime} \partial_{j+1}^{0} a *_{j+1} a *_{j+1} r_{j} \partial_{j+1}^{1} a\right] \\
& =\partial_{j}^{1} \Gamma_{j}^{\prime} \partial_{j+1}^{0} a *_{j} \partial_{j}^{1} a *_{j} \partial_{j}^{1} r_{j} \partial_{j+1}^{1} a \quad \text { by (4.1.4)(i) } \\
& =\partial_{j+1}^{0} a *_{j} \partial_{j}^{1} a *_{j} c_{j} \partial_{j}^{1} \partial_{j+1}^{1} a \quad \text { (5.1.2) (v, vi) } \\
& =\partial_{j+1}^{0} a *_{j} \partial_{j}^{1} a \quad \text { (since } c_{j} \partial_{j}^{1} \partial_{j+1}^{1} a \text { is an identity) }
\end{aligned}
$$

Corollary 5.2.2:

$$
\begin{array}{ll}
\partial_{i}^{\alpha} \Phi_{j}= \begin{cases}\Phi_{j-1} \partial_{i}^{\alpha} & (i<j) \\
\Phi_{j} \partial_{i}^{\alpha} & (i>j+1)\end{cases} \\
\partial_{j+1}^{\alpha} \Phi_{j}=\varepsilon_{j} \partial_{j}^{\alpha} \partial_{j+1}^{\alpha} & \left(=\varepsilon_{j} \partial_{j}^{\alpha} \partial_{j}^{\alpha}\right) \tag{5.2.2}
\end{array}
$$

and for a $\in A_{n}$, we have

$$
\begin{align*}
& \partial_{j}^{0} \Phi_{j} a=\left(\partial_{j}^{0} *_{j} \partial_{j+1}^{1} a\right)-{ }_{j}\left(\partial_{j+1}^{0} a *_{j} \partial_{j}^{1} a\right)(5.2 .2)(i i i),  \tag{5.2.2}\\
& \partial_{j}^{1} \Phi_{j} a=c_{j+1}\left(c_{1}^{j-1}\left(\partial_{1}^{0}\right)^{j+1} a_{1}, \varepsilon_{1}^{j-1}\left(\partial_{1}^{1}\right)^{j+1} a\right)(5.2 .2)(i v), \\
& \partial_{j+1}^{\alpha} \Phi_{j} \ldots \Phi_{1}=c_{1}^{j}\left(\partial_{1}^{\alpha}\right)^{j+1} \tag{5.2.2}
\end{align*}
$$

Proof: (5.2.2)(i): Let a $\in A_{n}$. Then for $i<j$

$$
\begin{aligned}
& \partial_{i}^{\alpha} \Phi_{j} a=a_{i}^{\alpha}\left[\psi_{j} a-{ }_{j+1} c_{j} \partial_{j}^{1} \psi_{j} a\right] \\
& =\partial_{i}^{\alpha} \psi_{j} a-{ }_{j} \partial_{i}^{\alpha_{i}}{ }_{j} \partial_{j}^{1} \Psi_{j}^{a} \quad b y(4.1 .3)(i) \\
& =\psi_{j-1} \partial_{i}^{\alpha}{ }^{\alpha}-c_{j-1} \partial_{j-1}^{1} \psi_{j-1} \partial_{i}^{\alpha} \quad b y(5.2 .1)(i) \text { and (4.1.1)(i,iii) } \\
& =\Phi_{j-1} \partial_{i}^{\alpha} \text {. }
\end{aligned}
$$

We can prove similarly that $\partial_{i}^{\alpha} \Phi_{j}=\Phi_{j} \partial_{i}^{\alpha}$, for $i>j+1$.
(5.2.2)(ii): This is immediate from the algebroid axioms and (5.2.1)(ii).
(5.2.2)(iii):

$$
\begin{align*}
& \partial_{j}^{0} \Phi_{j} a=\partial_{j}^{0}\left(\psi_{j} a-{ }_{j+1} c_{j} \partial_{j}^{1} \psi_{j}^{a}\right) \\
& =\partial_{j}^{0} \psi_{j}^{a}-{ }_{j} \partial_{j}^{0} c_{j} \partial_{j}^{1} \psi_{j}^{a} \quad \text { by (4.1.3)(i) }  \tag{4.1.3}\\
& =\partial_{j}^{0} \psi_{j}^{a}-{ }_{j} \partial_{j}^{1} \psi_{j}^{a} \quad \text { by (4.1.1)(iii) }  \tag{4.1.1}\\
& =\left(\partial_{j}^{0} a *_{j} \partial_{j+1}^{1} a\right)-{ }_{j}\left(\partial_{j+1}^{0} a *_{j} a_{j}^{1} a\right) \quad b y(5.2 .1)(i i i, i v)
\end{align*}
$$ (5.2.2)(iv):

$\partial_{j}^{2} \Phi_{j} a=\partial_{j}^{1}\left(\psi_{j}^{a}-{ }_{j+1} c_{j} a_{j}^{1} \psi_{j}^{a}\right)$
$=\partial_{j}^{1} \psi_{j}{ }^{a}-{ }_{j+1} a_{j}^{1} c_{j} \partial_{j}^{1} \psi_{j}^{a} \quad$ by (4.1.3)(i)
$=\partial_{j}^{1} \psi_{j}{ }^{a}-{ }_{j+1}{ }_{j}^{1} \psi_{j}{ }^{a} \quad$ by (4.1.1)(iii)
$=\zeta_{j+1}\left(\varepsilon_{1}^{j-1}\left(\partial_{1}^{0}\right)^{j+1} a, \varepsilon_{1}^{j-1}\left(\partial_{1}^{1}\right)^{j+1} a\right)$.
(5.2.2)(v):
$\partial_{j+1}^{\alpha} \Phi_{j} \Phi_{j-1} \ldots \ldots \Phi_{1}=c_{j} \partial_{j}^{\alpha} \partial_{j}^{\alpha} \Phi_{j-1} \ldots \Phi_{1} \quad b y(5.2 .2)(i i)$
$=c_{j} \partial_{j}^{\alpha}\left[c_{j-1} \partial_{j-1}^{\alpha} \partial_{j-1}^{\alpha} \Phi_{j-2} \cdots \Phi_{1}\right]$ by (5.2.2)(ii)
$=\epsilon_{j-1} \varepsilon_{j-1} \partial_{j-1}^{\alpha} \partial_{j-1}^{\alpha}\left[\partial_{j-1}^{\alpha} \Phi_{j-2} \ldots \Phi_{1}\right]$ by (4.1.1)(i,ii,iii) .
Thus by induction, we get

$$
\partial_{j+1}^{\alpha} \Phi_{j} \ldots \ldots \Phi_{1}=c_{1}^{j}\left(\partial_{1}^{\alpha}\right)^{j+1} .
$$

Lemma 5.2.3:

$$
\psi_{j} c_{i}= \begin{cases}c_{i} \psi_{j-1} & (i<j)  \tag{5.2.3}\\ c_{i} \psi_{j} & (i>j+1)\end{cases}
$$

$$
\begin{equation*}
\psi_{j} c_{j}=\psi_{j} c_{j+1}=c_{j} \tag{5.2.3}
\end{equation*}
$$

The proof of the above lemma is clear by using a similar argument to that in lemma (5.2.1).

## corollary 5.2.4:

$$
\Phi_{j} c_{i}= \begin{cases}c_{i} \Phi_{j-1} & (i<j)  \tag{5.2.4}\\ c_{i} \Phi_{j} & (i>j+1)\end{cases}
$$

If $a \in A_{n-1}$, then
$\Phi_{j} c_{j} a=\Phi_{j} c_{j+1} a=\varsigma_{j+1}\left(c_{1}^{j-1}\left(\partial_{1}^{0}\right)^{j+1} a, c_{1}^{j-1}\left(\partial_{1}^{2}\right)^{j+1} a\right)$ (5.2.4)(ii)
$\Phi_{n-1} \cdots \ldots \Phi_{j} \varepsilon_{j} a=s_{n}\left(\varepsilon_{1}^{n-2}\left(\partial_{1}^{0}\right)^{n} a, \varepsilon_{1}^{n-2}\left(\partial_{1}^{1}\right)^{n} a\right) \quad$ (5.2.4)(iii)
Proof:
(5.2.4)(i): For $i<j$, let a $\in A_{n-1}$, then
$\phi_{j} c_{i} a=\psi_{j} c_{i}{ }^{a}-{ }_{j+1} c_{j} \partial_{j}^{1} \psi_{j} c_{i} a$
$=c_{i} \psi_{j-1} a-{ }_{j+1} c_{j} c_{j-1} a_{j-1}^{1} \psi_{j-1} a \quad b y(5.2 .3)(i),(4.1 .1)(i i, i i i)$
$=c_{i} \Phi_{j-1} a \quad$ by (4.1.3) (ii).
We can prove similarly that $\Phi_{j} c_{i}=c_{i} \Phi_{j}$, for $i>j+1$. (5.2.4) (ii):
$\phi_{j} c_{j} a=\psi_{j} c_{j}^{a-}{ }_{j+1} c_{j} \partial_{j}^{1} \psi_{j} c_{j}{ }^{a}$

$$
\begin{array}{ll}
=c_{j}^{a-j+1} c_{j} \partial_{j}^{1} c_{j}^{a} & \text { by }(5.2 .3)(i i) \\
=c_{j}^{a-}-j+1 \\
c_{j}^{a} & \text { by }(4.1 .1)(i i i) \\
=c_{j+1}\left(c_{1}^{j-1}\left(\partial_{1}^{0}\right)^{j+1} a, c_{1}^{j-1}\left(a_{1}^{1}\right)^{j+1} a\right)
\end{array}
$$

By a similar argument one can prove that
${ }_{j} c_{j+1} a=\varsigma_{j+1}\left(c_{1}^{j-1}\left(\partial_{1}^{0}\right)^{j+1} a, c_{1}^{j-1}\left(\partial_{1}^{1}\right)^{j+1} a\right)$.
(5.2.4)(iii): This is clear by using (5.2.4)(ii).

Lemma 5.2.5:

$$
\begin{aligned}
& \psi_{j} \Gamma_{i}=\left\{\begin{array}{lll}
\Gamma_{i} \psi_{j-1} & (i<j) \\
\Gamma_{i} \psi_{j} & (i>j+1)
\end{array}\right. \\
& \psi_{j} \Gamma_{j}=c_{j} \\
& \psi_{j} \Gamma_{j+1}=r_{j}^{\prime} *_{j+1} \Gamma_{j+1} *_{j+1} \Gamma_{j} c_{j+1} \partial_{j+1}^{1} \quad(5.2 .5)(i)(i i i)
\end{aligned}
$$

Proof:
(5.2.5)(i) Let a $\in A_{n-1}$. Then for $i<j$, we get
$\psi_{j} \Gamma_{i} a=\Gamma_{j}^{\prime} \partial_{j+1}^{0} \Gamma_{i} a *_{j+1} \Gamma_{i} a *_{j+1} \Gamma_{j} a_{j+1}^{1} \Gamma_{i} a$
$=\Gamma_{j}^{\prime} \Gamma_{i} \partial_{j}^{0}{ }^{a} *_{j+1} \Gamma_{i} a *_{j+1} \Gamma_{j} \Gamma_{i} \partial_{j}^{1} a \quad b y(5.1 .2)(v i i, v i i i)$
$=\Gamma_{i} \Gamma_{j-1}^{\prime} \partial_{j}^{0}{ }^{a} *_{j+1} \Gamma_{i} a *_{j+1} \Gamma_{i} \Gamma_{j+1} \partial_{j}^{1}{ }^{\mathrm{a}}$ by (5.1.2) (ix)
$=\Gamma_{i}\left(\Gamma_{j-1}^{\prime} \partial_{j}^{0} a *{ }_{j} a *_{j} \Gamma_{j-1} a_{j}^{1} a\right)$
$=r_{i} \psi_{j-1} a$.
We can prove similarly that $\psi_{j} \Gamma_{i}=\Gamma_{i} \psi_{j}$, if $i \Delta j+1$.
(5.2.5) (ii): Let $a \in A_{n-1}$. Then
$\psi_{j} \Gamma_{j}{ }^{a}=r_{j}^{\prime} \partial_{j+1}^{0} r_{j}^{a} *_{j+1} \Gamma_{j}^{a} *_{j+1} \Gamma_{j} \partial_{j+1}^{1} \Gamma_{j}{ }^{a}$.
$=\Gamma_{j}^{\prime} a *_{j+1} \Gamma_{j}{ }^{a} *_{j+1} \Gamma_{j} c_{j} a_{j}^{1} a \quad b y(5.1 .4)(i i i)$

$=c_{j}$ a (since $c_{j} c_{j} a_{j}^{1 a}$ is an identity).
(5.2.5)(iii): Let a $\in A_{n-1}$. Then
$\psi_{j} r_{j+1} a=r_{j}^{\prime} \partial_{j+1}^{0} r_{j+1} a *_{j+1} r_{j+1} a *_{j+1} \Gamma_{j} \partial_{j+1}^{2} r_{j+1} a$
$=\Gamma_{j}{ }^{a} *_{j+1} \Gamma_{j+1} a *_{j+1} \Gamma_{j} c_{j+1} a_{j+1}^{1} a$ by (5.1.2)(v).

Corollary 5.2.6:

$$
\Phi_{j} r_{i}= \begin{cases}\Gamma_{i} \Phi_{j-1} & (i<j)  \tag{5.2.6}\\ r_{i} \Phi_{j} & (i>j+1)\end{cases}
$$

For a $\in A_{n-1}$, we have

$$
\begin{gather*}
\Phi_{j} r_{j}^{a}=\varsigma_{j+1}\left(\varepsilon_{1}^{j-1}\left(\partial_{1}^{0}\right)^{j+1} a, \varepsilon_{1}^{j-1}\left(\partial_{1}^{1}\right)^{j+1} a\right) \quad \text { (5.2.6)(ii) } \\
\Phi_{j} r_{j+1}=\left(r_{j}^{\prime} *_{j+1} r_{j+1} *_{j+1} \Gamma_{j} c_{j+1} \partial_{j+1}^{1}\right)-{ }_{j+1} \\
\left(c_{j} *_{j+1} r_{j+1} c_{j} a_{j}^{1}\right) \quad(5.2 .6)(i i i) . \tag{5.2.6}
\end{gather*}
$$

Proof:
(5.2.6)(i): For $i<j$ and a $\in A_{n-1}$, we get

$$
\begin{aligned}
& \Phi_{j} \Gamma_{i} a=\psi_{j} \Gamma_{i} a-{ }_{j+1} \varepsilon_{j} \partial_{j}^{1} \psi_{j} \Gamma_{i}^{a} \\
& =\Gamma_{i} \psi_{j-1} a-{ }_{j+1} c_{j}{ }^{1}{ }_{j} \Gamma_{i} \psi_{j-1} a \quad \text { by (5.2.5)(i) } \\
& =\Gamma_{i} \psi_{j-1} a-{ }_{j+1} \varepsilon_{j} \Gamma_{i} \partial_{j-1}^{1} \psi_{j-1} a \quad \text { by (5.1.2)(vii) } \\
& =\Gamma_{i} \psi_{j-1} a-_{j+1} r_{i} c_{j-1} \partial_{j-1}^{1} \psi_{j-1} a \quad b y(5.1 .2)(i i i) \\
& =\Gamma_{i}\left(\psi_{j-1} a-c_{j-1} \partial_{j-1}^{1} \psi_{j-1} a\right) \quad b y(5.1 .3)(i i i) \\
& =\Gamma_{i} \Phi_{j-1}{ }^{\mathbf{a}} \text {. }
\end{aligned}
$$

We can prove similarly that $\Phi_{j} \Gamma_{i}=\Gamma_{i} \Phi_{j}$, for $i>j+1$. (5.2.6)(ii): Let a $\in A_{n-1}$. Then

$$
\begin{aligned}
\Phi_{j} \Gamma_{j} a & =\psi_{j} \Gamma_{j}^{a}-{ }_{j+1} c_{j} a_{j}^{1} \psi_{j} \Gamma_{j}^{a} \\
& =c_{j} a-{ }_{j+1} c_{j} a_{j}^{1} c_{j}^{a} \quad \text { by }(5.2 .5)(i i) \\
& =c_{j}^{a}-{ }_{j+1} c_{j}^{a} \quad b y(4.1 .1)(i i i) \\
& =c_{j+1}\left(c_{1}^{j-1}\left(\partial_{1}^{0}\right)^{j+1} a, c_{1}^{j-1}\left(a_{1}^{1}\right)^{j+1} a\right)
\end{aligned}
$$

(5.2.6)(iii): Let a $\in A_{n-1}$. Then

$$
\begin{aligned}
& \Phi_{j} \Gamma_{j+1} a=\Psi_{j} \Gamma_{j+1} a-{ }_{j+1} \varepsilon_{j}{ }^{a}{ }_{j} \Psi_{j} \Gamma_{j+1} a \\
& =\left(\Gamma_{j}^{\prime} a *_{j+1} \Gamma_{j+1} a *_{j+1} \Gamma_{j} c_{j+1} \partial_{j+1}^{1} a\right)-_{j+1}\left(\varepsilon_{j} \partial_{j}^{1} \Gamma_{j}^{\prime} a *_{j+1}\right. \\
& \left.c_{j} \partial_{j}^{1} \Gamma_{j+1} a *{ }_{j+1} c_{j} \partial_{j}^{1} \Gamma_{j} c_{j+1} \partial_{j+1}^{1} a\right) \quad b y(5.2 .5)(i, i i) \\
& =\left(\Gamma_{j}^{\prime} a *_{j+1} \Gamma_{j+1} a *_{j+1} \Gamma_{j} c_{j+1} \partial_{j+1}^{1} a\right)-_{j+1}\left(c_{j} \partial_{j}^{1} \Gamma_{j}^{\prime}{ }^{a} *_{j+1}\right. \\
& \left.c_{j} a_{j}^{1} r_{j+1} a *{ }_{j+1} c_{j} a_{j}^{1} r_{j} c_{j+1} a_{j+1}^{1} a\right) \text { by (4.1.4)(i),(5.1.4)(i) } \\
& =\left(\Gamma_{j}^{\prime} a *_{j+1} \Gamma_{j+1} a *_{j+1} \Gamma_{j} c_{j+1} \partial_{j+1}^{1} a\right)-{ }_{j+1}\left(\varepsilon_{j} a *_{j+1} \Gamma_{j+1} c_{j} \partial_{j}^{1} a\right. \\
& \left.*_{j+1} c_{j}{ }^{3}\left(\partial_{j}^{1}\right)^{2} a\right) \text { by (5.1.2)(iii, v, vi, vii) and (4.1.l)(ii,iii) } \\
& =\left(\Gamma_{j}^{\prime} a *_{j+1} \Gamma_{j+1}^{a} *_{j+1} \Gamma_{j} \varepsilon_{j+1} \partial_{j+1}^{1} a\right)-{ }_{j+1}\left(\varepsilon_{j}^{a} *_{j+1} \Gamma_{j+1} \varepsilon_{j} \partial_{j}^{1} a\right) \\
& \text { ( since } \left.r_{j+1} c_{j} \partial_{j}^{1} a *_{j+1} c_{j}{ }^{3}\left(\partial_{j}^{1}\right)^{2} a=r_{j+1} c_{j} \partial_{j}^{1} a\right) \text {. }
\end{aligned}
$$

Lemma 5.2.7:

$$
\begin{array}{cc}
\Psi_{j} \Gamma_{i}^{\prime}= \begin{cases}\Gamma_{i}^{\prime} \Psi_{j-1} & (i<j) \\
\Gamma_{i}^{\prime} \Psi_{j} & (i \Delta j+1)\end{cases} \\
\Psi_{j} \Gamma_{j}^{\prime}=c_{j} & (5.2 .7)(i)  \tag{5.2.7}\\
\Psi_{j} \Gamma_{j+1}^{\prime}=\Gamma_{j}^{\prime} c_{j+1} \partial_{j+1}^{0} *_{j+1} \Gamma_{j+1}^{\prime}{ }^{*}{ }_{j+1} \Gamma_{j} \quad(5.2 .7)(i i)
\end{array}
$$

## Proof:

(5.2.7)(i): Let a $\in A_{n-1}$. Then for $i \angle j$
$\psi_{j} \Gamma_{i}^{\prime} a=\Gamma_{j}^{\prime} a_{j+1}^{0} \Gamma_{i}^{\prime} a *_{j+1} \Gamma_{i}^{\prime} a *_{j+1} \Gamma_{j} a_{j+1}^{1} \Gamma_{i}^{\prime} a$
$=\Gamma_{j}^{\prime} \Gamma_{i}^{\prime} \partial_{j}^{0} a *_{j+2} \Gamma_{i}^{\prime} a *_{j+1} \Gamma_{j} \Gamma_{i}^{\prime} a_{j}^{1} a \quad$ by ${ }^{\prime}(5.1 .2)(v i i)$
$=\Gamma_{i}^{\prime} \Gamma_{j-1}^{\prime} \partial_{j}^{0} a *_{j+1} \Gamma_{i}^{\prime} a *_{j+1} \Gamma_{i}^{\prime} \Gamma_{j-1} a_{j}^{1} a \quad b y(5.1 .2)(i i, i x)$
$=\Gamma_{i}^{\prime}\left(\Gamma_{j-1}^{\prime} \partial_{j}^{0} a *_{j} a *_{j} \Gamma_{j-1} \partial_{j}^{1} a\right)$ by (5.1.4)(iv)
$=\Gamma_{i}^{\prime} \Psi_{j-1}{ }^{a}$.

We can prove similarly that $\Psi_{j} \Gamma_{i}^{\prime}=\Gamma_{i}^{\prime} \psi_{j}$, for $i>j+1$. (5.2.7)(ii): Let a $\in A_{n-1}$. Then

$$
\begin{aligned}
& \Psi_{j} \Gamma_{j}^{\prime}{ }^{a}=\Gamma_{j}^{\prime} \partial_{j+1}^{0} \Gamma_{j}^{\prime}{ }^{a} *_{j+1} \Gamma_{j}^{\prime}{ }^{a}{ }^{*}{ }_{j+1} \Gamma_{j} a_{j+1}^{1} \Gamma_{j}^{\prime}{ }^{a} \\
& =\Gamma_{j}^{\prime} c_{j} \partial_{j}^{0} a *_{j+1} \Gamma_{j}^{\prime} a *_{j+1} \Gamma_{j}^{a} \quad \text { by (5.1.2)(vi) } \\
& =r_{j}^{\prime} c_{j} a_{j}^{0}{ }^{a} *_{j+1} c_{j}{ }^{a} \quad \text { by (5.1.4)(vi) } \\
& =\varepsilon_{j} c_{j} \partial_{j}^{0}{ }^{a} *_{j+1} \varepsilon_{j} a \quad b y(5.1 .2)(i v) \\
& =\boldsymbol{c}_{j}{ }^{\mathrm{a}} \text {. }
\end{aligned}
$$

(5.2.7)(iii): Let a $\in A_{n-1}$. Then

Corollary 5.2.8:

$$
\begin{gather*}
\Phi_{j} \Gamma_{i}^{\prime}=\left\{\begin{array}{ll}
\Gamma_{i}^{\prime} \Phi_{j-1} & (i<j) \\
\Gamma_{i}^{\prime} \Phi_{j} & (i>j+1)
\end{array} \quad(5.2 .8)(i)\right.  \tag{5.2.8}\\
\Phi_{j} \Gamma_{j}^{\prime}=\varsigma_{j+1}\left(c_{1}^{j-1}\left(\partial_{1}^{0}\right){ }^{j+1} a, \varepsilon_{1}^{j-1}\left(\partial_{1}^{1}\right)^{j+1} a\right)  \tag{5.2.8}\\
\Phi_{j} \Gamma_{j+1}^{\prime}=\left(\Gamma_{j}^{\prime} c_{j+1} \partial_{j+1}^{0} *_{j+1} \Gamma_{j+1}^{\prime} *_{j+1} \Gamma_{j}\right)-_{j+1}\left(c_{j} \Gamma_{j}^{\prime} \partial_{j}^{1} *_{j+1}\right. \\
\left.c_{j}^{2} \partial_{j}^{1}\right)
\end{gather*}
$$

Proof: The proofs of (i), (ii) are similar to that of corollary (5.2.6) .
(5.2.8)(iii): Let a $\in \mathbb{A}_{n-1}$. Then

$$
\begin{aligned}
& \Phi_{j} \Gamma_{j+1}^{\prime} a=\psi_{j} \Gamma_{j+1}^{\prime} a-{ }_{j+1} c_{j} \partial_{j}^{1} \psi_{j} \Gamma_{j+1}^{\prime} a \\
& =\left(\Gamma_{j}^{\prime} c_{j+1} \partial_{j+1}^{0} a *_{j+1} \Gamma_{j+1}^{\prime} a *_{j+1} \Gamma_{j} a\right)-{ }_{j+1} c_{j} \partial_{j}^{1}\left(\Gamma_{j}^{\prime} c_{j+1} a_{j+1}^{0} a\right. \\
& \left.*_{j+1} \Gamma_{j+1}^{\prime} a *_{j+1} \Gamma_{j} a\right)
\end{aligned}
$$

## Proposition 5.2.9:

i) If $a, b \in A_{n}$ with $\partial_{j}^{\alpha}=\partial_{j}^{\alpha} b$, where $\alpha=0,1$, then

$$
\psi_{i}\left(a+{ }_{j} b\right)= \begin{cases}\psi_{i} a+{ }_{j} \psi_{i} b & (j \neq i) \\ \Psi_{i} a+{ }_{i+1} \psi_{i} b & (j=i)\end{cases}
$$

ii) If $a, b \in A_{n}$ with $\partial_{j}^{1} a=\partial_{j}^{0} b$, then

iii) If a $\in A_{n}$ and $r \in R$ then

$$
\Psi_{i}(r \cdot j a)= \begin{cases}r \cdot j \psi_{i} a & j \neq i \\ r \cdot j+1 & \psi_{i} a\end{cases}
$$

## Proof:

i) Let $j \geq i+1$. Then

$$
\begin{aligned}
& \Psi_{i}\left(a+{ }_{j} b\right)=\Gamma_{i}^{\prime} \partial_{i+1}^{0}\left(a+{ }_{j} b\right) *_{i+1}\left(a+{ }_{j} b\right) *_{i+1} \Gamma_{i} \partial_{i+1}^{1}\left(a+{ }_{j} b\right) \\
& =\left(\Gamma_{i}^{\prime} a_{i+1}^{0} a+{ }_{j} \Gamma_{i}^{\prime} \partial_{i+1}^{0} b\right) *_{i+1}\left(a+{ }_{j} b\right) *_{i+1}\left(\Gamma_{i} a_{i+1}^{1} a+{ }_{j} \Gamma_{i} a_{i+1}^{1} b\right) \\
& b y(4.1,3)(i), \text { and }(5.1 .3)(i, i i)
\end{aligned}
$$

$$
=\left(\Gamma_{i}^{i} \partial_{i+1}^{0} a{ }_{i+1} a *{ }_{i+1} \Gamma_{i} \partial_{i+1}^{i} a\right)+_{j}\left(\Gamma_{i}^{i} \partial_{i+1}^{0} b * i+1 \quad b * i+1\right.
$$

$$
\begin{aligned}
& \left.*_{j+1} c_{j} \partial_{j}^{2} \Gamma_{j+1} a_{j+1} c_{j} \partial_{j}^{1} \Gamma_{j} a_{j}\right) \quad \text { by }(4.1 .4)(i, i i)
\end{aligned}
$$

$$
\begin{aligned}
& \left.c_{j} \Gamma_{j}^{, \partial_{j}^{2} a}{ }_{j+1} c_{j}^{2} a_{j}^{2} a\right) \quad \text { by }(5.1 .2)(v, v i, v i i) \\
& =\left(\Gamma_{j}^{1} c_{j+1} \partial_{j+1}^{0} a_{j+1} \Gamma_{j+1}^{\prime} a_{j+1} \sum_{j+1}^{a}\right)_{j+1}\left(c_{j} \Gamma_{j}^{\prime} \partial_{j}^{1} a_{j}^{*}{ }_{j+1}\right. \\
& \left.c_{j}^{2} \partial_{j}^{2} a\right)
\end{aligned}
$$

$\left.\Gamma_{i} \partial_{i+1}{ }^{2}\right)$
$=\Psi_{i}{ }^{+}{ }_{j} \psi_{i} b$.
We can prove similarly that $\Psi_{i}\left(a+{ }_{j} b\right)=\Psi_{i} a+{ }_{j} \Psi_{i} b$ for $j<i$.
For $\mathbf{j}=\mathrm{i}+\mathrm{l}$, the result is clear by using distributivity and the hypothesis that $\partial_{i+1}^{\alpha} a=\partial_{i+1}^{\alpha} b$.

Now to prove the second part of (i), for $\mathbf{j}=\mathbf{i}$, we refer to the case $n=2$. Thus by proposition (3.2.3)

$$
\psi_{j}\left(a+{ }_{j} b\right)=\psi_{j} a+{ }_{j+1} \psi_{j} b .
$$

ii) Let $j<i$. Then we have
$\Psi_{i}\left(a *_{j} b\right)=\Gamma_{i}^{\prime} a_{i+1}^{0}\left(a *_{j} b\right) *_{i+1}\left(a *_{j} b\right) *_{i+1} \Gamma_{i} a_{i+1}^{1}\left(a *_{j} b\right)$
$=\left(\Gamma_{i}^{\prime} \partial_{i+1}^{0} a *_{j} \Gamma_{i}^{\prime} \partial_{i+1}^{0} b\right) *_{i+1}\left(a *_{j} b\right) *_{i+1}\left(\Gamma_{i} a_{i+1}^{1} a *_{j} \Gamma_{i} a_{i+1}^{1} b\right)$ by (4.1.4)(i) and (5.1.4)(i,ii)
$=\left(\Gamma_{i}^{\prime} \partial_{i+1}^{0} a *_{i+1} a *_{i+1} \Gamma_{i} a_{i+1}^{1} a\right) *_{j}\left(\Gamma_{i}^{\prime} \partial_{i+1}^{0} b *_{i+1} b *_{i+1} \Gamma_{i} a_{i+1}^{1} b\right)$ by (4.1.6)(ii)
$=\Psi_{i}{ }^{a}{ }^{*}{ }_{j} \Psi_{i} \mathbf{b}$.
We can prove similarly that $\Psi_{i}\left(a *_{j} b\right)=\Psi_{i} a *_{j} \Psi_{i} b$, for $j>i+1$.

The equalities for $\mathbf{j = i , i + 1}$ follow from proposition (3.2.3) since $\left(A_{n}, A_{n-1}\right)$ is a double algebroid for direction $i, i+1$. iii) For $j>i+1$, then we have

$$
\psi_{i}\left(r \cdot \cdot_{j} a\right)=\Gamma_{i}^{\prime} \partial_{i+1}^{0}\left(r \cdot{ }_{j} a\right) *_{i+1}\left(r \cdot{ }_{j} a\right) *_{i+1} \Gamma_{i} a_{i+1}^{1}\left(r \cdot j_{j} a\right)
$$

$$
=\Gamma_{i}^{\prime}\left(r \cdot{ }_{j-1} \partial_{i+1}^{0} a\right) *_{i+1}\left(r \cdot j_{j} a\right) *_{i+1} \Gamma_{i}\left(r_{j-1} \partial_{j+1}^{1} a\right)
$$

by (4.1.5)(i)
$=\left(r \cdot j_{j} \Gamma_{i}^{\prime} a_{i+1}^{0} a\right) *_{i+1}\left(r \cdot{ }_{j} a\right) *_{i+1}\left(r \cdot{ }_{j} r_{i} a_{i+1}^{1} a\right)$ by (5.1.5)(i ,ii)

$$
\begin{aligned}
& =r \cdot j\left[\Gamma_{i}^{\prime} \partial_{i+1}^{0} a *_{i+1} a *_{i+1} \Gamma_{i} \partial_{i+1}^{1} a\right] \text { by (4.1.5) (iii) } \\
& =r \cdot{ }_{j} \Psi_{i}^{a} .
\end{aligned}
$$

We can prove similarly that $\Psi_{i}(r \cdot j a)=r \cdot j \Psi_{i} a, f o r$ $\mathbf{j}<\mathbf{i}$ and $\mathbf{j}=\mathbf{i}+1$.

Finally, for $j=i$, we refer again to the case $n=2$, then

$$
\Psi_{i}\left(r \cdot_{i} a\right)=r \cdot{ }_{i+1} \Psi_{i}^{a}
$$

## Corollary 5.2.10:

i) If $a, b \in A_{n}$ with $\partial_{j}^{\alpha} a=\partial_{j}^{\alpha}$, then

$$
\Phi_{i}\left(a+{ }_{j} b\right)= \begin{cases}\Phi_{i} a+{ }_{j} \Phi_{i} b & \text { if } j \neq i \\ \Phi_{i} a+{ }_{j+1} \Phi_{i} b & \text { if } j=i\end{cases}
$$

ii) If $a, b \in A_{n}$ with $a_{j}^{1} a=\partial_{j}^{0} b$, then

iii) If a $\in A_{n}$ and $r \in R$. Then

$$
\Phi_{i}(r \cdot j a)= \begin{cases}r \cdot{ }_{j} \Phi_{i} a & (j \neq i) \\ r{ }_{i+1} \Phi_{i}^{a} & (j=i)\end{cases}
$$

Proof: i) For $j \neq i$, let $j>i+1$. Then

$$
\begin{aligned}
& \Phi_{i}\left(a+{ }_{j} b\right)=\psi_{i}\left(a+{ }_{j} b\right)-{ }_{i+1} c_{i} a_{i}^{1} \psi_{i}\left(a+{ }_{j} b\right) \\
& =\left(\Psi_{i} a+{ }_{j} \Psi_{i} b\right)-{ }_{i+1} c_{i} \partial_{i}^{1}\left(\Psi_{i} a+{ }_{j} \Psi_{i} b\right) \quad b y(5.2 .9)(i) \\
& =\left(\Psi_{i} a+{ }_{j} \psi_{i} b\right)-_{i+1}\left(\varepsilon_{i} \partial_{i}^{1} \Psi_{i} a+{ }_{j} c_{i} \partial_{i}^{1} \psi_{i} b\right) \quad b y(4,1.3)(i, i i) \\
& =\left(\Psi_{i} a-{ }_{i+1} c_{i} \partial_{i}^{1} \psi_{i} a\right)+{ }_{j}\left(\psi_{i} b-_{i+1} c_{i} a_{i}^{1} \Psi_{i} b\right) \text { by (4.1.6)(i) } \\
& =\Phi_{i} \mathrm{a}+{ }_{j} \Phi_{i} \mathrm{~b} \quad \text {. }
\end{aligned}
$$

We can prove similarly that $\Phi_{i}\left(a+{ }_{j} b\right)=\Phi_{i} a+_{j} \Phi_{i} b$, for $j<i, j=i+1$. Finally for $j=i$ we refer to the case $n=2$, thus $\Phi_{i}\left(a+_{i} b\right)=\Phi_{i} a{ }_{i+1} \Phi_{i} b$. Also one can prove the above equation by using (5.2.9)(i) and (4.1.3)(i ,ii). ii) For $j * i, i+1$, let $j>i+1$. Then
$\Phi_{i}\left(a *_{j} b\right)=\psi_{i}\left(a *_{j} b\right)-_{i+1} c_{i} a_{i}^{1} \psi_{i}\left(a *_{j} b\right)$
$=\left(\Psi_{i} a *_{j} \Psi_{i} b\right)-{ }_{i+1} c_{i} a_{i}^{1}\left(\Psi_{i} a *_{j} \Psi_{i} b\right) \quad b y(5.2 .9)(i i)$
$=\left(\Psi_{i} a *_{j} \Psi_{i} b\right)-_{i+1}\left(c_{i} \partial_{i}^{1} \Psi_{i} a *_{j} c_{i} \partial_{i}^{1} \Psi_{i} b\right)$ by (4.1.3)(i ,ii)
$=\left(\Psi_{i} a-{ }_{i+1} c_{i} \partial_{i}^{1} \Psi_{i} a\right) *_{j}\left(\Psi_{i} b-_{i+1} \varepsilon_{i} \partial_{i}^{1} \Psi_{i} b\right) \quad b y(4.1 .6)(i i i)$
$=\Phi_{i} a *{ }_{j} \Phi_{i} b \quad$.
We can prove similarly that $\Phi_{i}\left(a *_{j} b\right)=\Phi_{i} a_{j}{ }_{j} \Phi_{i} b$,for $\mathbf{j}<\mathbf{i}$.
Again the equalities for $\mathbf{j}=\mathbf{i}, \mathbf{j}=\mathbf{i}+1$ follow from
proposition (3.2.3) since ( $A_{n}, A_{n-1}$ ) is a double algebroid for direction $i, i+1$.
iii) For $j * i$ let $j>i+1$. Then
$\Phi_{i}\left(r \cdot{ }_{j} a\right)=\psi_{j}\left(r \cdot{ }_{j} a\right)-_{i+1} c_{i} \partial_{i}^{1} \psi_{i}\left(r \cdot{ }_{j} a\right)$
$=\left(r \cdot{ }_{j} \Psi_{i}^{a}\right)-{ }_{i+1} c_{i} \partial_{i}^{1}\left(r \cdot{ }_{j} \Psi_{i}^{a}\right)$
$=\left(r \cdot j \Psi_{i}^{a}\right)-_{i+1}\left(r \cdot{ }_{j} \varepsilon_{i} \partial_{i}^{1} \Psi_{i} a\right) \quad$ by (4.1.5)(i ,ii)
$=r \cdot j\left[\Psi_{i}^{a-}{ }_{i+1} c_{i} \partial_{i}^{1} \psi_{i}^{a}\right] \quad$ by distributivity
$=r \cdot{ }_{j}{ }^{\mathrm{a}}{ }^{\mathrm{a}}$.
We can prove similarly that $\Phi_{i}\left(r \cdot j^{a}\right)=r \cdot{ }_{j} \Phi_{i} a$, for $j<i$ and $\mathbf{j}=\mathbf{i + 1}$.

Finally, if $j=i$ we refer to the case $n=2$, then $\Phi_{i}\left(r \cdot_{i} a\right)=r \cdot{ }_{i+1} \Phi_{i} a \cdot$ Also one can prove the above equation by using (5.2.9)(iii) and (4.1.5)(i,ii).

In the following corollary we give the general formulae for $\Phi(a+j b)$ and $\Phi(r \cdot j a)$ for $a, b \in A_{n}, r \in R$ such that $a+j b$ is defined. We delay giving the formulae for $\Phi(a \neq j b)$.

Corollary 5.2.11:
i) If $a, b \in A_{n}$ with $\partial_{j}^{\alpha} a=a_{j}^{\alpha} b$, where $\alpha=0,1$; then

$$
\Phi^{\prime}(a+j b)=\Phi^{\prime} a+n \Phi^{\prime} b
$$

ii) If $a \in A_{n}$ and $r \in R$, then

$$
\Phi^{\prime}\left(r \cdot j^{a}\right)=r \cdot{ }_{n} \Phi^{\prime} a
$$

Proof:

$$
\text { i) } \begin{aligned}
& \Phi^{\prime}\left(a+{ }_{j} b\right)=\Phi_{n-1} \cdots \Phi_{1}\left(a+_{j} b\right) \\
= & \Phi_{n-1} \cdots \Phi_{j}\left(\Phi_{j-1} \ldots \Phi_{1} a+{ }_{j} \Phi_{j-1} \ldots \Phi_{1} b\right) \quad b y(5.2 .10)(i) \\
= & \Phi_{n-1} \cdots \Phi_{j+1}\left(\Phi_{j} \ldots \Phi_{1} a+{ }_{j+1} \Phi_{j} \ldots \Phi_{1} b\right) \text { by (5.2.10)(i) } \\
= & \left(\Phi_{n-1} \ldots \Phi_{1} a\right)+_{n}\left(\Phi_{n-1} \ldots \Phi_{1} b\right) \quad b y \text { induction and (5.2.10)(i) } \\
= & \Phi^{\prime} a+{ }_{n} \Phi^{\prime} b .
\end{aligned}
$$

$$
\begin{aligned}
\text { ii) } \Phi^{\prime}(r \cdot j a)= & \Phi_{n-1} \cdots \Phi_{1}(r \cdot j a) \\
=\Phi_{n-1} \cdots \Phi_{j}\left(r \cdot{ }_{j+1} \Phi_{j-1} \cdots \Phi_{1} a\right) & \text { by }(5.2 .10)(i i i) \\
=\Phi_{n-1} \cdots \Phi_{j+1}\left(r \omega_{j} \Phi_{j} \cdots \Phi_{1} a\right) & \text { by }(5.2 .10)(i i i)
\end{aligned}
$$

Thus by induction, we get
$\Phi^{\prime}\left(r \cdot j^{a}\right)=r \cdot{ }^{\prime \prime} a \quad$
Now we apply the operation $\Phi^{\prime \prime}$, but before that we show that $\Phi_{j}^{\prime}$ is well defined on the required elements, namely

Remark 5.2.12: For $1\left\langle j\left\langle n-2, \Phi_{j}^{\prime}\right.\right.$ is well defined on
$\Phi_{j+1}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a$, for $a \in A_{n}$.
Proof: By its definition $\Phi_{j}^{\prime}$ is well defined on $\Phi_{j+1}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime}$ a if and only if $\partial_{n}^{\alpha_{\psi}}{ }_{j} \Phi_{j+1}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a=\partial_{n}^{\alpha_{j}} \partial_{j}^{1} \psi_{j} \Phi_{j+1}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a$, $\alpha=0,1$.

First the left hand side $=$
$\partial_{n}^{\alpha_{j}} \Phi_{j+1}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a=\psi_{j} a_{n}^{\alpha_{\Phi}}{ }_{j+1} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a \quad$ by (5.2.1) (i).
Let $\Phi_{j+2}^{\prime} \cdots \Phi_{n-2}^{\prime} \Phi^{\prime} a=b$, then
L.H.S $=\psi_{j} \partial_{n}^{\alpha_{n}}\left[\Psi_{j+1} b-_{n} c_{j+1} a_{j+1}^{1} \Psi_{j+1} b\right]=\psi_{j} \partial_{n}^{\alpha_{j+1}}{ }_{j}$
by the algebroid axiom.
Thus by repeating this procedure $(n-j-l)$ times, we get

$$
\begin{aligned}
& \partial_{n}^{\alpha_{j}} \Psi_{j+1}^{\prime} \Phi_{n-2}^{\prime} \Phi^{\prime} a=\Psi_{j} \Psi_{j+1} \ldots \Psi_{n-2} \partial_{n}^{\alpha_{\Phi}^{\prime} a} \\
& =\Psi_{j} \ldots_{n-2}\left[\left(c_{1}\right)^{n-1}\left(\partial_{1}^{\alpha_{1}}\right)^{n} a\right] \quad \text { by }(5.2 .2)(v) \\
& =\left(c_{1}\right)^{n-1}\left(\partial_{1}^{\alpha_{1}}\right)^{n} a \quad \text { by }(5.2 .3)(i i)
\end{aligned}
$$

On the other hand,
R.H.S $=\partial_{n}^{\alpha_{j}} \partial_{j}^{1} \psi_{j} \Phi_{j+1}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a=c_{j} \partial_{j}^{1} \partial_{n}^{\alpha_{j}} \Psi_{j} \Phi_{j+1}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a$

$$
\text { by }(4.1 .1)(i, i i i)
$$

$=c_{j} \partial_{j}^{1}\left[\left(c_{1}\right)^{n-1}\left(a_{1}^{\alpha}\right)^{n} a\right)$ by the above argument
$=\left(c_{1}\right)^{n-1}\left(\partial_{1}^{\alpha}\right)^{n} a \quad$ by $(4.1 .1)(i i i)$.
Thus $\Phi_{j}^{\prime}$ is well defined for all $1 \leqslant j \leqslant n-2$.
Lemma 5.2.13: Let $u, v \in A_{n}$ be such that $u+_{n} v$ is defined and $\Phi^{\prime \prime}$ is defined on $u+_{n} v$. Then $\Phi^{\prime \prime}\left(u+_{n} v\right)=\Phi^{\prime \prime} u+_{n} \Phi^{\prime \prime} v$.

Proof: $\Phi^{\prime \prime}\left(u+_{n} v\right)=\Phi_{1}^{\prime} \ldots \Phi_{n-2}^{\prime}\left(u+_{n} v\right)$
$=\Phi_{1}^{\prime} \ldots \Phi_{n-3}^{\prime}\left[\psi_{n-2}\left(u+_{n} v\right)-_{n} c_{n-2} a_{n-2}^{1} \psi_{n-2}\left(u+_{n} v\right)\right]$

$$
=\Phi_{1}^{\prime} \ldots \Phi_{n-3}^{\prime}\left[\left(\psi_{n-2} u+_{n} \psi_{n-2} v\right)-{ }_{n} c_{n-2} \partial_{n-2}^{1}\left(\psi_{n-2} u+_{n} \psi_{n-2}\right)\right]
$$

by (5.2.9)(i)
$=\Phi_{1}^{\prime} \ldots \Phi_{n-3}^{\prime}\left[\left(\Psi_{n-2}^{u}-c_{n-2} a_{n-2}^{1} \psi_{n-2}^{u}\right)+_{n}\left(\Psi_{n-2}^{v}-{ }_{n} c_{n-2} \partial_{n-2}^{1}\right.\right.$

$$
\left.\psi_{n-2} v\right) \quad \text { by }(4.1 .3)(i, i i)
$$

$=\Phi_{1}^{\prime} \ldots \Phi_{n-3}^{\prime}\left[\Phi_{n-2}^{\prime} u{ }_{n} \Phi_{n-2}^{\prime}{ }^{v}\right]$. Thus by induction we get $\Phi "\left(u+_{n} v\right)=\Phi " u+_{n} \Phi " v$.

## Corollary 5.2.14:

i) If $a, b \in A_{n}$ with $\partial_{j}^{\alpha} a=\partial_{j}^{\alpha} b$, then

$$
\Phi\left(a+{ }_{j} b\right)=\Phi a+_{n} \Phi b .
$$

ii) If a $\epsilon A_{n}$ and $r \in R$, then

$$
\Phi\left(r \cdot{ }_{j} a\right)=r \cdot n \Phi a
$$

## Proof:

ii) $\Phi\left(r \cdot j^{a}\right)=\Phi^{\prime \prime} \Phi^{\prime}\left(r \cdot j^{a}\right)=\Phi^{\prime \prime}\left(r \cdot \Phi^{\prime \prime} a\right)$ by (5.2.11) (iii)

$$
=\Phi_{2}^{\prime} \ldots \Phi_{n-2}^{\prime}\left(r \cdot_{n} \Phi^{\prime} a\right)
$$

$$
\begin{aligned}
& \text { i) } \Phi(a+j b)=\Phi^{\prime \prime} \Phi^{\prime}(a+j b)=\Phi^{\prime \prime}\left(\Phi^{\prime} a+{ }_{n} \Phi^{\prime} b\right) \text { by (5.2.11)(i) } \\
& =\Phi_{1}^{\prime} \ldots \Phi_{n-2}^{\prime}\left(\Phi^{\prime} a{ }_{n} \Phi^{\prime} b\right) \\
& =\Phi_{1}^{\prime} \ldots \Phi_{n-3}^{\prime}\left[\psi_{n-2}\left(\Phi^{\prime} a+_{n} \Phi^{\prime} b\right)-c_{n-2} c_{n-2}^{1} \psi_{n-2}\left(\Phi^{\prime} a+_{n} \Phi^{\prime} b\right)\right] \\
& =\Phi_{1}^{\prime} \ldots \Phi_{n-3}^{\prime}\left[\left(\Psi_{n-2} \Phi^{\prime} a+_{n} \Psi_{n-2} \Phi^{\prime} b\right)-_{n} c_{n-2} a_{n-2}^{1}\left(\Psi_{n-2} \Phi^{\prime} a+{ }_{n}\right.\right. \\
& \left.\left.\Psi_{n-2} \Phi^{\prime} b\right)\right] \quad \text { by (5.2.9)(i) } \\
& =\Phi_{1}^{\prime} \ldots \Phi_{n-3}^{\prime}\left[\left(\psi_{n-2} \Phi^{\prime} a-{ }_{n} c_{n-2} a_{n-2}^{1} \psi_{n-2} \Phi^{\prime} a\right)+{ }_{n}\left(\psi_{n-2} \Phi^{\prime} b-n\right.\right. \\
& \left.c_{n-2} a_{n-2}^{2} \psi_{n-2} \Phi^{\prime} b\right) \quad \text { by (4.1.3)(i,ii) } \\
& =\Phi_{1}^{\prime} \cdots \Phi_{n-3}^{\prime}\left(\Phi_{n-2}^{\prime} \Phi^{\prime} a+_{n} \Phi_{n-2}^{\prime} \Phi^{\prime} b\right)=\Phi^{\prime \prime} \Phi^{\prime} a+_{n} \Phi^{\prime \prime} \Phi^{\prime} b \text { by induction } \\
& =\Phi a+{ }_{n} \Phi b \text {. }
\end{aligned}
$$

$$
\begin{aligned}
& =\Phi_{1}^{\prime} \ldots \Phi_{n-3}^{\prime}\left[\Psi_{n-2}\left(r \cdot n \Phi^{\prime} a\right)-{ }_{n} c_{n-2} \partial_{n-2}^{1} \Psi_{n-2}\left(r \cdot n \Phi^{\prime} a\right)\right] \\
& =\Phi_{1}^{\prime} \ldots \Phi_{n-3}^{\prime}\left[\left(r \cdot{ }_{n} \Psi_{n-2} \Phi^{\prime} a\right)-_{n}\left(r \cdot{ }_{n} c_{n-2} \partial_{n-2}^{1} \Psi_{n-2} \Phi^{\prime} a\right)\right] \\
& \text { by (5.2.9)(iii) and (4.1.5)(i,ii) } \\
& =\Phi_{1}^{\prime} \ldots \Phi_{n-3}^{\prime}\left[r \cdot{ }_{n}\left(\Psi_{n-2} \Phi^{\prime} a-{ }_{n} c_{n-2} a_{n-2}^{1} \Psi_{n-2} \Phi^{\prime} a\right)\right]
\end{aligned}
$$

by distributivity
$=\Phi_{2}^{\prime} \ldots \Phi_{n-9}^{\prime}\left[r \cdot{ }_{n} \Phi_{n-2}^{\prime} \Phi^{\prime} a\right]$.
Thus by induction, we get $\Phi\left(r \cdot j^{a}\right)=r \cdot{ }_{n} \Phi a$.
Recall from chapter IV $\$ 1$ that the function
$5_{j}: A_{n-1} \times A_{n-1} \rightarrow A_{n}$ gives the zero from $u$ to $v$ in $A_{n-1}$
for the $j$-th algebroid structure of $A_{n}$. Also we write $\beta_{0} a=\left(a_{1}^{0}\right)^{n_{a}}, \beta_{1} a=\left(a_{1}^{1}\right)^{n_{a}}$, the first and last vertices respectively of the element a $\in A_{n}$.

Lemma 5.2.15: If $(\alpha, i) \not(0,1), i<n-1$ and a $\in A_{n}$, then $\partial_{i+1}^{\alpha} \Phi_{i}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a=\zeta_{n-1}\left(\left(c_{1}\right)^{n-2} \beta_{0} a,\left(c_{1}\right)^{n-2} \beta_{1} a\right)$.

Proof: Let $\Phi_{i+1}^{\prime} \cdots \Phi_{n-2}^{\prime} \Phi^{\prime} a=b$, then

$$
\begin{aligned}
& \partial_{i+1}^{\alpha} \Phi_{i}^{\prime} \cdots \Phi_{n-2}^{\prime} \Phi^{\prime} a=\partial_{i+1}^{\alpha} \Phi_{i}^{\prime} b=\partial_{i+1}^{\alpha}\left[\Psi_{i} b-{ }_{n} c_{i} \partial_{i}^{1} \psi_{i} b\right] \\
& =\partial_{i+1}^{\alpha} \Psi_{i} b-_{n-1} \partial_{i+1}^{\alpha} c_{i} \partial_{i}^{1} \psi_{i}^{b} \quad b y(4.1 .3)(i) \\
& =c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} b-n-1 c_{i} \partial_{i}^{2} c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} \quad b y(5.2 .1)(i) \text { and (4.1.1)(i,iii) } \\
& =c_{i} \partial_{i}^{\alpha_{j}} \alpha_{i}^{\alpha}-{ }_{n-1} c_{i} a_{i}^{\alpha_{j} \alpha_{b}} \quad \text { by (4.1.1)(iii) } \\
& =\zeta_{n-1}\left(\partial_{n-1}^{o} c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} b, \partial_{n-1}^{1} c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} b\right) \quad .
\end{aligned}
$$

Claim 5.2.16:
$\partial_{n-1}^{0} c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} b=c_{1}^{n-2} \beta_{0} a$ and $a_{n-1}^{1} c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} b=c_{1}^{n-2} \beta_{1} a$.

The proof of the above claim is indicated in Appendix Il .
Thus $\partial_{i+1}^{\alpha} \Phi_{i}^{\prime} \cdots \Phi_{n-2}^{\prime} \Phi^{\prime} a=\zeta_{n-1}\left(c_{1}{ }^{n-2} \beta_{0} a, c_{1}{ }^{n-2} \beta_{1} a\right)$.
Proposition 5.2.17: If $(\alpha, i) \notin(0,1)$ and a $\in A_{n}$, then
i) $\partial_{i}^{\alpha_{\Phi}}=s_{n-1}\left(\left(c_{1}\right)^{n-2} \beta_{0} a,\left(c_{1}\right)^{n-2} \beta_{1} a\right)$, for $i<n$,
ii) $\partial_{n}^{\alpha} \Phi a=\left(c_{1}\right)^{n-1}\left(\partial_{1}^{\alpha}\right)^{n}$ a.

Proof: i) $\partial_{i}^{\alpha} \Phi a=\partial_{i}^{\alpha \prime \Phi^{\prime} \Phi^{\prime} a}=\partial_{i}^{\alpha} \Phi_{i}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a$.
Let $\Phi_{2}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a=b_{1}$, then
$\partial_{i}^{\alpha} \Phi a=\partial_{i}^{\alpha_{i}^{\prime} b_{1}}=\partial_{i}^{\alpha}\left[\psi_{2} b_{1}-{ }_{n} c_{1} \partial_{2}^{2} \Psi_{1} b_{1}\right]$
$=\partial_{i}^{\alpha} \Psi_{1} b_{1}-_{n-1} \partial_{i}^{\alpha} c_{1} \partial_{2}^{1} \psi_{1} b 1 \quad$ by (4.1.3)(i)
$=\psi_{1} \partial_{i}^{\alpha} b_{1}-_{n-1} c_{1} \partial_{1}^{1} \psi_{1} \partial_{i}^{\alpha} b_{1}$ by (5.2.1)(i) and (4.1.1)(i ,iii)
Let $\Phi_{3}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a=b_{2}$, then
$\partial_{i}^{\alpha} \Phi a=\Psi_{1} \partial_{i}^{\alpha_{\Phi}^{\prime} b_{2}}-_{n-1} c_{1} \partial_{1}^{1} \Psi_{1} \partial_{i}^{\alpha_{\Phi}^{\prime}} b_{2}$
$=\Psi_{1} \partial_{i}^{\alpha}\left[\psi_{2} b_{2}-{ }_{n} c_{2} \partial_{2}^{1} \psi_{2}^{b} b_{2}\right]-{ }_{n-1} c_{1} \partial_{1}^{1} \psi_{1} \partial_{i}^{\alpha}\left[\psi_{2} b_{2}-{ }_{n} c_{2} \partial_{2}^{1} \psi_{2} b_{2}\right]$
$=\Psi_{1}\left[\psi_{2} \partial_{i}^{\alpha} b_{2}-_{n-1} c_{2} \partial_{2}^{1} \psi_{2} \partial_{i}^{\alpha_{2}}\right]-{ }_{n-1} c_{1} \partial_{1}^{1} \psi_{1}\left[\psi_{2} a_{i}^{\alpha_{b}}-_{n-1}\right.$
$\left.c_{2} \partial_{2}^{1} \Psi_{2} \partial_{i}^{\alpha_{2}}\right]$ by (4.1.3)(i),(5.2.1)(i) and (4.1.1)(i ,iii)
$=\left[\Psi_{1} \Psi_{2} \partial_{i}^{\alpha} b_{2}-_{n-1} \Psi_{1} c_{2} \partial_{2}^{1} \psi_{2} \partial_{i}^{\alpha} b_{2}\right]-_{n-1}\left[c_{1} \partial_{1}^{1} \psi_{1} \psi_{2} \partial_{i}^{\alpha} b_{2}-_{n-1}\right.$ $\left.c_{1} \partial_{1}^{1} \Psi_{1} c_{1} \partial_{2}^{1} \psi_{2} \partial_{i}^{\alpha_{2}}\right]$ by (5.2.9)(i) and (4.1.3)(i ,ii)
$=\left[\Psi_{1} \psi_{2} \partial_{i}^{\alpha} b_{2}-_{n-1} c_{1} \partial_{2}^{1} \psi_{2} \partial_{i}^{\alpha_{2}}\right]-_{n-1}\left[c_{1} a_{1}^{1} \Psi_{1} \Psi_{2} \partial_{i}^{\alpha_{b}}-_{n-1}\right.$ $\left.c_{1} a_{2}^{1} \Psi_{2} \partial_{i}^{\alpha_{2}}\right]$ by (5.2.3)(ii) and (4.1.1)(iii)
$=\Psi_{1} \Psi_{2} \partial_{i}^{\alpha_{b}}-_{n-1} c_{1} \partial_{1}^{1} \Psi_{1} \psi_{2} \partial_{i}^{\alpha_{b}}$.
Thus by repeating this procedure, we have ;
$\partial_{i}^{\alpha} \Phi_{a}=\psi_{1} \ldots \psi_{i-2} \partial_{i}^{\alpha} b_{i-2}-_{n-1} c_{1} \partial_{1}^{2} \psi_{1} \ldots \psi_{i-2} \partial_{i}^{\alpha} b_{i-2}$,
where $b_{i-2}=\Phi_{i-1}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a$. Thus
$\partial_{i}^{\alpha} \Phi_{a}=\Psi_{1} \ldots \Psi_{i-2}\left[\zeta_{n-1}\left(\left(\varepsilon_{1}\right)^{n-2} \beta_{0} a,\left(\varepsilon_{1}\right)^{n-2} \beta_{1} a\right)\right]--_{n-1}$
$c_{1} \partial_{1}^{1} \Psi_{1} \ldots \psi_{i-2}\left[\delta_{n-1}\left(\left(c_{1}\right)^{n-2} \beta_{0} a,\left(c_{1}\right)^{n-2} \beta_{1} a\right)\right]$
by lemma (5.2.15)
$=\Sigma_{n-1}\left(\left(\varepsilon_{1}\right)^{n-2} \beta_{0} a,\left(\varepsilon_{1}\right)^{n-2} \beta_{1} a\right)$.
ii) $\partial_{n}^{\alpha^{\Phi} a}=\partial_{n}^{\alpha^{\prime \prime} \Phi^{\prime} a}=\partial_{n}^{\alpha_{1}}{ }_{1}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a$
$=\partial_{n}^{\alpha}\left[\Psi_{1} \Phi_{2}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a-n c_{1} \partial_{1}^{1} \Psi_{2} \Phi_{2}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a\right]$
$=\partial_{n}^{\alpha} \Psi_{1} \Phi_{2}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a \quad$ by the algebroid axiom
$=\Psi_{2} \partial_{n}^{\alpha_{2}}{ }_{2}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a \quad b y(5.2 .1)(i)$
$=\Psi_{1} \ldots \Psi_{n-2} a_{n}^{\alpha} \Phi^{\prime} a \quad$ by the induction
$=\psi_{1} \ldots \psi_{n-2} a_{n}^{\alpha_{\Phi_{n-1}}} \ldots . \Phi_{1} a=\Psi_{1} \ldots \psi_{n-2}\left(\left(c_{1}\right)^{n-1}\left(\partial_{1}^{\alpha}\right)^{n} a\right)$ by (5.2.2)(v)
$=\left(c_{1}\right)^{n-1}\left(\partial_{1}^{\alpha}\right)^{n}$ a by (4.4.3)(ii),(4.1.1)(ii) and induction. $\square$ Thus \$a is an element in the associated crossed complex yA.

Corollary 5.2.18: $\Phi a=a$ if and only if a is an element in $y \mathbb{A}$. In particular $\boldsymbol{\Phi}^{\mathbf{2} b}=\mathbf{\Phi}$ b for all $b \in \mathbb{A}$.

Proof: Let a $\in Y A$, then a $\in M_{n}(x, y)$ for some $x, y \in M_{0}$. By (4.4.14) we get $\Psi_{j} a=a$, then $\Phi_{j} a=a$ and hence $\Phi^{\prime} a=a$, thus $\Phi$ a $=$ a . The converse is trivial.

In the previous discussion we investigated the folding operation $\Phi$ in the general case except for the formulae for $\Phi(a * j b)$. We give now the formulae for $\Phi(a * j b)$ in the 3 and 4 dimensional cases.

Also we shall prove that there exist an equivalence between the category $(\underline{C r s})^{3}$ of 3 -truncated crossed complexes and the category $(\underline{\omega-A l g})^{3}$ of 3 -tuple algebroids . We start first with the formulae for $\Phi\left(a{ }^{\prime}{ }_{j} b\right)$ namely;

Proposition 5.2.19: If $a, b \in A_{n}$ and $a *_{j} b$ is defined then, for $n=3$ (resp.4) and $1<j<3$ (resp. $1 \leqslant j \leqslant 4$ )
$\Phi\left(a *_{j} b\right)=u_{j}(\Phi b) \quad{ }_{n}(\Phi a)^{v}{ }_{j} b$
Where $u_{j} a=a_{1}^{0} \ldots \partial_{j-1}^{0} \quad \partial_{j+1}^{0} \ldots \partial_{n}^{0} a$ and $v_{j} b=a_{1}^{2} \ldots a_{j-1}^{2} a_{j+1}^{1} \ldots a_{n}^{1} b \quad$.

The proof of the above proposition is indicated in Appendix $I V$.

Proposition 5.2.20:

1) For $n=3$, let $a \in A_{2}$. Then
i) $\Phi c_{i} a=0$ in dimension 3 for $1 \leqslant i \leqslant 3$,
ii) $\Phi \Gamma_{j} a=0$ in dimension 3 for $1 \leqslant j \leqslant 2$,
iii) $\phi \Gamma_{j}^{\prime} a=0$ in dimension 3 for $1 \leqslant j \leqslant 2$.
2) For $n=4$, let $a \in A_{3}$. Then
i) $\Phi \varepsilon_{i} a=0$ in dimension 4 for $1 \leqslant i \leqslant 4$,
ii) $\phi \Gamma_{j} a=0$ in dimension 4 for $1 \leqslant j \leqslant 3$,
iii) $\phi \Gamma_{j}^{\prime} a=0$ in dimension 4 for $1 \leqslant j \leqslant 3$.

The proof is given in appendix $V$.

Now we move to define an extra structure on an w-algebroid A , which we call "thin structure".

Definition 5.2.20: An element a $\in A_{n}$ (for $n=3,4$ is called thin if and only if $\Phi a=0$ in dimension $n$.

For all $n$, the collection of thin elements of $A_{n}$ is closed under the operations $+j, \cdot j, 1 \leqslant j \leqslant n$, and this is also been proved for $\boldsymbol{*}_{j}$ if $n \leqslant 4$.

## 3. COSKELETON OF $\omega$-ALGEBROIDS:

If one ignores the elements of dimension higher than $n$ in an $\omega$-algebroid, one obtains an n-tuple algebroid $A^{n}$. R.Brown and P.J.Higgins [ $B-H i-2]$ have constructed the skeleton and the coskeleton in the $\omega$-groupoid case. We will follow the notations and terminology of [B-Hi-2] to costruct a coskeleton in an $\omega$-algebroid.

We start to construct the coskeleton in terms of "shells" as follows :

In a cubical complex $K$, an r-shell means a family $a=\left(a_{i}^{\alpha}\right)$
of $r$-cubes $(i=1, \ldots, r+1, \alpha=0,1)$ satisfying
$\partial_{j}^{\beta} a_{i}^{\alpha}=\partial_{i-1}^{\alpha} a_{j}^{\beta}$ for $1 \leqslant j<i \leqslant r+1$ and $\alpha, \beta \in\{0,1\} \quad$ (5.3.1)(i)
In particular the faces $a_{i}^{\alpha} b$ of any ( $r+1$ )-cube form an r-shell $\underline{\partial} \mathrm{b}$. We denote $b y \mathrm{KK}_{\mathrm{r}}$, the set of all r-shells of $K$ (c.f. Duskin's "Simplicial kernel" [D-l]).

Let $K=\left(K_{n}, \ldots, K_{o}\right)$ be an $n$-truncated cubical complex. Then $K^{\prime}=\left(\square K_{n}, K_{n}, \ldots, K_{0}\right)$ will denote the $(n+1)$-truncated cubical complex in which, for any a $\epsilon \mathrm{K}_{\mathrm{n}}, \partial_{i}^{\alpha}$ is defined to
be $a_{i}^{\alpha}$ and for any $b \in K_{n}, c_{j} b$ is defined to be the $n$-shell c , where

$$
c_{i}^{\alpha}= \begin{cases}c_{j-1} \partial_{i}^{\alpha} & (i<j)  \tag{5.3.1}\\ c_{j} \partial_{i-1}^{\alpha} b & (i>j) \\ b & (i=j)\end{cases}
$$

If $K$ has connections, we can also define $\Gamma_{j} b=d, \Gamma_{j}^{\prime} b=e$ where

$$
\begin{align*}
& d_{i}^{\alpha}=\left\{\begin{array}{lll}
\Gamma_{j-1} \partial_{i}^{\alpha} & (i<j) & d_{j}^{0}=d_{j+1}^{0}=b \\
\Gamma_{j} \partial_{i-1}^{\alpha} b & (i>j+1) & d_{j}^{1}=d_{j+1}^{1}=c_{j} a_{j}^{0} b
\end{array}\right.  \tag{5.3.1}\\
& e_{i}^{\alpha}=\left\{\begin{array}{lll}
\Gamma_{j-1}^{\prime} \partial_{i}^{\alpha} b & (i<j) & e_{j}^{0}=e_{j+1}^{0}=c_{j} \partial_{j}^{0} b \\
\Gamma_{j}^{\prime} \partial_{i-1}^{\alpha} b & (i>j+1) & e_{j}^{1}=e_{j+1}^{1}=b
\end{array}\right. \tag{5.3.1}
\end{align*}
$$

In this way $K$ ' becomes an ( $n+1$ )-truncated cubical complex with connections .

Now we replace $K$ by an $n$-tuple algebroid $A$. We define $+j$, $*_{j}, \cdot j$ in $\square A_{n}$ as follows :

For $+_{j}$, let $a, b \in \square A_{n}$ with $a_{j}^{\alpha} a=\partial_{j}^{\alpha} b$. Define $a+{ }_{j} b=f$ where

$$
f_{i}^{\alpha}= \begin{cases}a_{i}^{\alpha}+{ }_{j-1} b_{i}^{\alpha} & (i<j) \\ a_{i}^{\alpha}+{ }_{j} b_{i}^{\alpha} & (i>j) \\ a_{i}^{\alpha}\left(=b_{i}^{\alpha}\right) & (i=j)\end{cases}
$$

For $*_{j}$, let $a, b \in A_{n}$ with $\partial_{j}^{1} a=\partial_{j}^{0} b$. Define $a *_{j} b=g$ where

$$
\begin{equation*}
g_{j}^{0}=a_{j}^{0} \quad, g_{j}^{1}=b_{j}^{1} \tag{5.3.1}
\end{equation*}
$$

$$
g_{i}^{\alpha}=\left\{\begin{array}{lll}
a_{i}^{\alpha} & *_{j-1} b_{i}^{\alpha} & (i<j) \\
a_{i}^{\alpha} & { }_{j} b_{i}^{\alpha} & (i>j)
\end{array}\right.
$$

Finally, for $\cdot j$, let $a \in A_{n}$ and $r \in R$, then we define $r \cdot j a=h$, where

$$
h_{i}^{\alpha}=\left\{\begin{array}{ll}
r \cdot j-1 a_{i}^{\alpha} & (i<j) \\
r \cdot j_{i}^{a_{i}^{\alpha}} & (i \Delta j) \\
a_{i}^{\alpha} & (i=j)
\end{array} \quad(5.3 .1)(v i i i)\right.
$$

Proposition 5.3.2: The above structure $A^{\prime}=\left(\square A_{n}, \ldots, A_{0}\right)$ is an ( $n+1$ )-truncated $\omega$-algebroid.

The proof of the above proposition is given in Appendix VI. Proposition 5.3.3: If $A^{n}=\left(A_{n}, \ldots . ., A_{0}\right)$ is an n-tuple algebroid, then the $\omega$-algebroid $\bar{A}^{n}$ with

$$
\bar{A}_{m}^{n}= \begin{cases}A_{m}^{n} & \text { for } m \leqslant n \\ q^{m-n} A_{n}^{n} & \text { for } m>n\end{cases}
$$

and operations defined as above, is the $n$-coskeleton of $A^{n}$. Proof: If $\underline{B}$ is any $\omega$-algebroid and $f_{k}: B_{k} \rightarrow A_{k}$ are defined for $k=0,1, \ldots, n$, that is

so as to form a morphism of n-tuple algebroids from n-truncated $B$ to $n$-truncated $A$, then there is a unique extension to morphism of $\omega$-algebroids $f: \underline{B} \rightarrow \bar{A}^{n}$ defined inductively $b y$, for $b \in \underline{B}, f_{m} b=c$, where $c_{i}^{\alpha}=f_{m-1} a_{i}^{\alpha} b$ (m $>n$ ) . This shows that $\bar{A}^{n} \cong \operatorname{Cosk}^{n} A^{n}$.

We apply now the folding operations $\Psi_{j}, \Phi_{j}, \Phi^{\prime}{ }_{j}$, $\Phi$ in the $\omega$-algebroid $\operatorname{Cosk}^{n} \underline{A}^{n}$, where $A^{n}=\left(A_{n}, \ldots, A_{0}\right)$. Given an n-shell $a=\left(a_{i}^{\alpha}\right) \in \square A_{n}$, we obtain $n$-shells $\Psi_{j} a, \Phi_{j} a, \Phi_{j}^{\prime} a^{a}$ and $\Phi$ a . By proposition (5.2.16) $\Phi a \in \gamma \square A^{n}$, that is, all faces of фa are zero except the faces $(0,1),(\alpha, n)$. If $B$ is a given $\omega$-algebroid, adjointness gives a canonical morphism $f: \underline{B} \rightarrow \operatorname{Cosk}^{n} \underline{B} \cong \operatorname{Cosk}^{n}(n-t r u n c a t e d \underline{B})$ with $f_{r+1} b=\underline{a}$ for $b \in B_{r+1}$. Since $f$ preserves the folding operations, so for any $b \in B_{n}$ and $n \geqslant 3$, we get

$$
\begin{equation*}
\Phi \underline{a} b=\underline{\partial} \Phi b \tag{5.3.4}
\end{equation*}
$$

Proposition 5.3.5: Let $A$ be an $\omega$-algebroid and let $\underline{M}=y \underline{A}$ be its associated crossed complex . Let a $\in \square_{n-1}$ and $\varepsilon \in M_{n}(u, v)$ where $u=\beta_{0} a, v=\beta_{1} a$. Then a neccessary and sufficient condition for the existence of $b \in A_{n}$ such that $\underline{\partial} b=a$ and $\Phi b=\varepsilon$ is that $\delta \varepsilon=\delta \Phi \underline{\partial} a$. Further if $b$ exists, it is unique.

Proof: The essential point is that the folding $\Phi$ a of a is constructed from a by applying operations defined on the shell da of $a$. Further, each of the individual components of $\Phi$ is reversible, given full information on the shell of the element to which it is applied, For example, an element b may be reconstructed from $\Psi_{i} b$ and $\partial b$ - the proof of this essentially the same as in the 2-dimensional case. Then
$\Phi_{i} b=\Psi_{i} b-{ }_{i+1} c_{i} a_{i}^{1} \Psi_{i} b$ and so $b$ may be reconstructed from $\Phi_{i} b$ and $\partial b$. A similar remark applies to $\Phi_{i}^{\prime} b$, when defined . $\square$

It is clear from proposition (5.3.2) that ( $D A_{2}, A_{2}, A_{1}, A_{0}$ ) is a 3-tuple algebroid, whenever ( $A_{2}, A_{1}, A_{0}$ ) is a 2-tuple algebroid.

In the next section we prove that there exists an equivalence between the category ( $\mathbf{C r s}^{3}$ of 3 -truncated crossed complexes and the category ( $\omega$ - Alg $)^{3}$ of 3-truncated algebroids.

## 4. THE EQUIVALENCE OF CATEGORIES:

In this section we start to construct a 3 -tuple algebroid from a 3 -truncated crossed complex by using the folding operation . Then we prove the equivalence of the above categories .

Let $\underline{M}^{3}=\left(M_{3}, M_{2}, M_{1}, M_{0}\right)$ be a 3 -truncated crossed complex and let $A_{0}=M_{0}, A_{1}=M_{1}$. Then $A_{2}=\gamma \lambda M_{2}$, constructed as in chapter III . Define

$$
A_{3}=\left\{(\underline{a}, \varepsilon): \underline{a} \in \square_{2}, \varepsilon \in M_{3} \text { such that } \delta \varepsilon=\partial_{1}^{0} \Phi_{\mathrm{a}}\right\} \text {. }
$$

define the maps $\varepsilon_{j}, \partial_{j}^{\alpha}, \Gamma_{i}, \Gamma_{i}$, for $\alpha=0,1$,
$j=1,2,3$ and $i=1,2$ in the following way :
Let $a \in A_{2}$, define $\varepsilon_{j} a=\left(\varepsilon_{j} a, 0\right)$, where $\varepsilon_{j} a$ is defined by
 dimension 3 , see proposition (5.2.19)) . Define $\partial_{j}^{\alpha}$ : $A_{3} \rightarrow A_{2}$ by, if $(\underline{a}, \varepsilon) \in A_{3}$, then it is clear that $\partial_{j}^{\alpha}(\underline{a}, \varepsilon)=a_{j}^{\alpha}$. Finally, define $\Gamma_{j}(\underline{a}, \xi)\left(\Gamma_{j} \underline{a}, 0\right)$ and $\Gamma_{j}^{\prime}(\underline{a}, \varepsilon)=\left(\Gamma_{j} \underline{a}, 0\right)$. It is clear that $\Gamma_{j} a, \Gamma_{j}^{\prime} a \in A$ by proposition (5.2.19).

We define now the appropriate algebraic structure on $A_{3}$ namely ; additions , compositions and scalar multiplications as follows :

First for $(\underline{a}, \varepsilon),(\underline{b}, n) \in A_{3}$ with $\partial_{j}^{\alpha} \underline{\underline{a}}=a_{j}^{\alpha} \underline{b}$, we define
$(\underline{a}, \underline{\varepsilon})+j(\underline{b}, \pi)=[(\underline{a}+j \underline{b}), \varepsilon+\pi]$ for $j=1,2,3$.
Note that this definition make sense. Thus we have
$a_{1}^{0} \Phi\left(\underline{a}+{ }_{j} \underline{b}\right)=\partial_{1}^{0}\left[\Phi \underline{a}+{ }_{3} \Phi \underline{b}\right]=a_{1}^{0} \Phi \underline{a}+{ }_{3} \partial_{1}^{0} \Phi \underline{b}=\delta \xi+\delta n$
$=\delta(\varepsilon+\pi)$.
Second, for $(\underline{a}, \varepsilon),(\underline{b}, n) \in A_{3}$ with $\partial_{j}^{1} \underline{a}=a_{j}^{0} \underline{b}$, then we define
$(\underline{a}, \underline{\varepsilon}) *_{j}(\underline{b}, \eta)=\left[\left(\underline{a} *_{j} \underline{b}\right), u_{j} \underline{\theta}_{\eta}+\varepsilon^{\mathbf{v}} \mathbf{j} \underline{b}\right]$.
We must verify the appropriate boundary condition, namely
$a_{1}^{0} \Phi\left(\underline{a} *_{1} \underline{b}\right)=a_{1}^{0}\left[\left(\Phi_{\underline{a}}\right)^{v} \underline{b}{ }_{3} u_{1} \underline{\underline{a}}(\Phi \underline{b})\right]$ by (5.2.18)
$=\partial_{1}^{0}\left[(\Phi \underline{a})^{v_{1} \underline{b}}\right]+{ }_{3} a_{1}^{0}\left[u_{1} \underline{a}(\Phi \underline{b})\right]=\left(\partial_{1}^{0} \Phi_{\underline{a}} *_{3} a_{1}^{0} c_{1}^{2} a_{2}^{1} a_{3}^{1} \underline{b}\right)+{ }_{3}$

$$
\left(\partial_{1}^{0} c_{1}^{2} a_{2}^{1} \partial_{3}^{1} \underline{a} *_{3} \partial_{1}^{0} \Phi \underline{b}\right)=\left(\delta \varepsilon * c_{1} \partial_{2}^{1} \partial_{3}^{1} \underline{b}\right)+\left(c_{1} \partial_{2}^{1} \partial_{3}^{1} \underline{a} * \delta n\right)
$$

$=\delta\left[\left(\varepsilon * \varepsilon_{1} \partial_{2}^{1} \partial_{3}^{1} \underline{b}\right)+\left(\varepsilon_{1} \partial_{2}^{1} \partial_{3}^{1} \underline{a} * n\right)\right]=\delta\left[\varepsilon^{v} \underline{b}+u_{1} \underline{\underline{a}} n\right]$.
We can verify similarly that $(\underline{a}, \varepsilon) \quad *_{j}(\underline{b}, n)$ is well defined for $\mathbf{j}=2,3$.

> Finally for $(\underline{a}, \varepsilon) \in A_{3}$ and $r \in R$, we define $r \cdot j(\underline{a}, \varepsilon)=(r \cdot j \underline{a}, r \cdot \varepsilon) \cdot$ Again it is easy to
show that this definition make sense. Thus we are ready to give the first result of this section, namely ;

Proposition 5.4.1: The above structure is a 3-tuple algebroid.

The proof of this proposition is similar to that of proposition (3.3.1)

Thus any 3 -truncated crossed complex gives rise to a 3-tuple algebroid. This constructiondefines a functor $\lambda$ from the category ( $\underline{(r s})^{3}$ of 3 -truncated crossed complexes to the category $(\underline{\omega-A l g})^{3}$ of 3 -tuple algebroids, that is $\lambda:(\underline{\text { Crs }})^{3} \rightarrow(\underline{\omega-A l g})^{3}$.

Now we move on to prove the equivalence between these two categories .

Theorem 5.4.2: The functors $\gamma$, $\lambda$ defined previously form an adjoint equivalence

$$
y:(\underline{\omega-A l g})^{3} \cdots(\underline{\operatorname{Crs}})^{3}: \lambda
$$

The proof again is similar argument to that of theorem (3.4.1).

Note that one can prove there exist an equivalence between the category ( WAlg $^{4}$ ( of 4-tuple algebroids and the category (Crs) ${ }^{4}$ of 4-truncated crossed complexes by using similar arguments and the formulae of the folding operation for the composition which was given in (5.2.18) .

Now we end this chapter with a conjecture for the higher dimension ; namely for the n-dimensional case .

First suppose that, if $n \geqslant 2,1 \leqslant j \leqslant n$ and $a, b \in A_{n}$ are such that $a * j b$ is defined, then
$\Phi\left(a *_{j} b\right)=u_{j}^{a}(\Phi b) \quad{ }_{n}(\Phi a)^{v_{j}}{ }^{b} \quad$.
Then there exist an equivalence between the category (Crs) of


In the next chapter we suggest possible further work in the same area to link the above ideas with homological and homotopical algebra.

## CONJECTURED RESULTS ON ALGEBROIDS, $\omega$-ALGEBROIDS

## AND CROSSED COMPLEXES

In this chapter we recall briefly various results on groupoids , $\omega$-groupoids , crossed complexes (over groupoids) and we conjecture that these results carry over to algebroids , $\omega$-algebroids and crossed complexes (over algebroids) . We also conjecture a relation between the ideas of this thesis and algebraic geometry.

First we recall some results in the groupoid case, namely ;

1) R.Brown [Br-4] has explored the notion of a fibration of groupoids : i.e. a morphism $\Phi: A \rightarrow B$ of groupoids such that whenever $x \in A_{0}$ and $b \in B_{1}$ with $\delta O_{b}=\Phi(x)$, there exists a $\in A_{1}$ such that $\Phi(a)=b$ and $\delta^{\circ} a=x$. Also he proved that if one starts in the category of groups, then certain constructions lead naturally to fibrations of a groupoids .
2) J.Howie [Ho-l] extended the notion of a fibration of groupoids to that of a fibration of crossed complexes (over groupoids) . Simply a fibration in the category of crossed complexes (over groupoids) is a morphism $\psi: M \rightarrow N$ such that each groupoid morphism $\psi_{n}: M_{n} \rightarrow N_{n}(n \geqslant l)$ is a fibration of groupoids .
3) The homotopy addition lemma is given in [B-Hi-2]. This is
precisely the formulae of the (0,l)-th face for the element ${ }^{\text {( }}$ a where $\Phi$ is a folding operation in an $\omega$-groupoid $G$ and $a \in G_{n}$.
4) The results presented by J.H.C.Whitehead [Wh-2] are generalised in [B-Hi-4] . Namely the functor $\Delta^{\prime}$ : (free crossed complexes) $\rightarrow-\rightarrow$ (chain complexes with operators) is generalised to a functor $\Delta$ : (crossed complexes) $\rightarrow$ (chain complexes with operators) . Moreover in [B-Hi-4] a right adjoint for the functor $\Delta$ is constructed.
5) The equivalence between the category of $\infty$-groupoids and the category of crossed complexes (over groupoids) is proved in [B-Hi-6] by using the equivalence of the category of $\omega$-groupoids with the category of crossed complexes (over groupoids) . More precisely, they have proved that for any $\omega$-groupoid there exists an o-groupoid containing the associated crossed complex.
6) The notion of a tensor product $A$ B of crossed complexes (over groupoids) $A, B$ and internal hom functor $\operatorname{CRS}(A, B)$ is given in [B-Hi-7] . The category of crossed complexes is given the structure of a symmetric monoidal closed category . The crossed complex $\operatorname{CRS}(A, B)$ is in dimension 0 the set of all morphisms A $\rightarrow-$ B . In dimension $m \geqslant 1$, it consists of m-fold homotopies $h: A \rightarrow B$ over morphisms $f: A \rightarrow B$.

The material given above has been done in the groupoid case. It is reasonable to conjecture that it will carry over to the algebroid case

Now we move on to conjecture other possible result to link the ideas of algebroids with the algebraic geometry .

In this thesis we have only dealt with the case of associative algebroids and not commutative algebroids . The notion of commutative algebroids ought to be definable, and ought to be relevant to algebraic geometry.

Finally, the notions of seaparable, central algebroids have been defined categorically in [Mi-1,2,3] . Namely let $A$ be an R-algebroid. A is called separable if $A$, considered as its own hom functor: is projective as
$A^{e}=A \otimes A^{0 P}$ - module. It is central if the map $R \rightarrow \operatorname{Hom}_{A} e(A, A)$ is an isomorphism. Possibly this notion can be extended to higher dimensions.

## APPENDIX I

## Verification of Theorem (3.1.7) and Lemma (3.1.8):

i) The definition of $\theta_{1}, \theta_{2}$ :

Let $a, b, c, d \in D_{1}$ with $c d=a b$ and $\propto$ has boundary given by


Since $\theta_{1}(a \underset{b}{c} d)=\left(\varepsilon_{1} c *_{2} \Gamma^{\prime} d\right) *_{1}\left(\Gamma a *_{2} \varepsilon_{2} b\right)$ and $\varepsilon_{1} c, \Gamma^{\prime} d$,
[a, $c_{1} b$ have boundaries given by

and then ( $\left.c_{1} c *_{2} \Gamma^{\prime} d\right)$, ( $\left[a *_{2} \varepsilon_{1} b\right.$ ) have boundaries in the form


Thus $\left(\varepsilon_{1} c *_{2} \Gamma^{\prime} d\right) *_{1}\left(\Gamma a *_{2} \varepsilon_{1} b\right)$ is defined (since $\left.c d=a b\right)$; namely


Similarly for the definition of $\theta_{2}$.
ii) Lemma 3.1.8:
$\theta_{1}(a \underset{b}{c} d)=\left(c_{1} c *_{2} \Gamma^{\prime} d\right) *_{1}\left(\Gamma a *_{2} c_{1} b\right)$ which is
diagrammatically given by




## APPENDIX II

## Verification of proposition (3.2.3) diagrammatically:

i) Let $\alpha, \beta \in D$ be given by

thus $\alpha+{ }_{1} \beta$ is in the form

$\Phi\left(\alpha+{ }_{1} \beta\right)=\left[\Gamma^{\prime}\left(a+a_{1}\right) *_{2}\left(\alpha+_{1} \beta\right) *_{2}\left[\left(d+d_{1}\right)\right]-2 c_{1}\left(a+a_{1}\right) b\right.$,


On the other hand ;


ii) Similarly for $\Phi(\alpha+2 \beta)=\Phi \alpha+{ }_{2} \Phi \beta$.
iii) For $\Phi\left(\alpha *_{1} \beta\right)=\left(\Phi \alpha *_{2} \varepsilon_{1} \partial_{2}^{1} \beta\right)+_{2}\left(\varepsilon_{1} \partial_{1}^{0} \alpha *_{2} \Phi \beta\right)$.

Let $\alpha, \beta$ be given by

and so $\alpha *_{1} \beta$ is in the form


Now $\Phi\left(\alpha *_{1} \beta\right)=\left(\Gamma^{\prime} a a^{\prime} *_{2}\left(\alpha *_{1} \beta\right) *_{2} \Gamma d d^{\prime}\right)-_{2} \varepsilon_{1} a a^{\prime} e$, which is diagrammatically pictured as



$=$

iv) For $\Phi\left(\alpha *_{2} \beta\right)=\left(c_{1} \partial_{1}^{0} \alpha *_{2} \Phi \beta\right)+_{2}\left(\Phi \alpha *_{2} c_{1} \partial_{1}^{2} \beta\right)$.

Let $\alpha, \beta$ be given by

then $\alpha *_{2} \beta$ is in the form

$$
{ }_{y}^{\mathrm{a} \underbrace{}_{\mathrm{b}} \mathrm{c}_{2} \beta} \mathrm{cc}^{\prime} \mathrm{e}
$$

Then $\Phi\left(\alpha *_{2} \beta\right)=\left(\Gamma \cdot a *_{2}\left(\alpha *_{2} \beta\right) *_{2} \Gamma d\right)-\varepsilon_{2} \varepsilon_{1} a b b$ which is in the form

$=$

 $=$

$=$

$=\left(\varepsilon_{1} c *_{2} \Phi \beta\right)+_{2}\left(\Phi \alpha *_{2} \varepsilon_{1} b^{\prime}\right)$.
v) The rules $\Phi(r \cdot 1 \propto)=r \cdot 2 \Phi \alpha$ and $\Phi(r \cdot 2 \alpha)=r \cdot 2 \Phi \alpha$ are clear.

## The proof of claim (5.2.16):

We give a complete proof for the first part of (5.2.16). Recall that $b=\Phi_{i+1}^{\prime} \cdots \Phi_{n-2}^{\prime} \Phi^{\prime} a$.
$\partial_{n-1}^{0} c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} b=a_{n-1}^{0} c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} \Phi_{i+1}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a$
$=c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} a_{n}^{\circ} \Phi_{i+1}^{\prime} \cdots \Phi_{n-2}^{\prime} \Phi^{\prime} a \quad b y(4.1 .1)(i, i i i)$.
Let $\Phi_{i+2}^{\prime} \cdots \phi_{n-2}^{\prime} \Phi^{\prime} a=c \cdot$ Then
$\partial_{n-1}^{0} c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} b=c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} \partial_{n}^{o} \Phi_{i+1}^{\prime} c=c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} \partial_{n}^{0}\left[\Psi_{i+1} c-n c_{i+1} \partial_{i+1}^{1} \psi_{i+1} c\right.$
$=c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} \partial_{n}^{0} \psi_{i+1} c^{i} \quad$ by (4.1.3)(i)
$=\varepsilon_{i} a_{i}^{\alpha} \partial_{i}^{\alpha} \psi_{i+1} \partial_{n}^{0} c \quad$ by (5.2.1)(i)
$=c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} \psi_{i+1} \partial_{n}^{0} \Phi_{i+2}^{\prime} \ldots \Phi_{n-2}^{\prime} \Phi^{\prime} a$. Thus by repeating this
procedure, we get
$\partial_{n-1}^{0} c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} b=c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} \psi_{i+1} \ldots \ldots \psi_{n-2} \partial_{n}^{0} \Phi^{\prime} a$.
Now we look at $\partial_{n}^{0} \Phi^{\prime} a=a_{n}^{0} \phi_{n-1} \ldots \Phi_{1} a$
$=c_{n-1} \partial_{n-1}^{0} \partial_{n-1}^{0} \Phi_{n-2} \cdots \Phi_{1} a \quad$ by (5.2.2)(ii).
Thus by repeating this procedure, we get
$\partial_{n}^{0} \Phi^{\prime} a=c_{n-1} \partial_{n-1}^{0} c_{n-2} \partial_{n-2}^{0} \ldots \ldots c_{1} \partial_{1}^{0} \partial_{1}^{0} a$
$=\left(c_{1}\right)^{n-1}\left(\partial_{1}^{0}\right)^{n}$.
Thus $\partial_{n-1}^{0} c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} b=c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha} \psi_{i+1} \ldots \psi_{n-2}\left(c_{i}\right)^{n-1}\left(\partial_{1}^{0}\right)^{n} a$.

$$
\begin{array}{ll}
=c_{i} \partial_{i}^{\alpha} \partial_{i}^{\alpha}\left(c_{1}\right)^{n-1}\left(\partial_{1}^{0}\right)^{n} a & \quad\left(\text { since } \psi_{i} c_{i+1}=c_{i}\right) \\
=\left(c_{1}\right)^{n-2}\left(\partial_{1}^{0}\right)^{n_{a}} & \text { by }(4.1 .1)(i i, i i i) \\
=\left(c_{1}\right)^{n-2} \beta_{0}^{a} \quad
\end{array}
$$

We can prove similarly that
$a_{n-1}^{1} c_{i} a_{i}^{\alpha} \partial_{i}^{\alpha}{ }_{b}=\left(c_{1}\right)^{n-2} \beta_{1} a$

We give a complete proof for the case $n=3$. We start with $j=1$. Then

$$
\Phi\left(a *_{1} b\right)=\Phi_{1}^{\prime} \Phi_{2} \Phi_{1}\left(a *_{1} b\right)
$$

$$
=\Phi_{1}^{\prime} \Phi_{2}\left[\left(\Phi_{1} a *_{2} c_{1} \partial_{2}^{1} b\right)+_{2}\left(c_{1} \partial_{2}^{0} a *_{2} \Phi_{2} b\right)\right] \text { by (5.2.10)(ii) }
$$

$$
=\Phi_{1}^{\prime} \Phi_{2}\left(\Phi_{1} *_{2} c_{1} \partial_{2}^{1} b\right)+{ }_{3} \Phi_{1}^{\prime} \Phi_{2}\left(c_{2} \partial_{2}^{0} a *_{2} \Phi_{2} b\right)
$$

$$
\text { by }(5.2 .10)(i) \text { and }(5.2 .13)
$$

$$
=\Phi_{1}^{\prime}\left[\left(\Phi^{\prime} a *_{3} c_{2} a_{3}^{1} c_{1} a_{2}^{1} b\right)+{ }_{3}\left(c_{2} \partial_{3}^{0} \Phi_{1} a *_{3} \Phi_{2} c_{2} a_{2}^{1} b\right)\right]+{ }_{3} \Phi_{1}^{\prime}\left[\left(\Phi_{2} c_{1} a_{2}^{0} a\right.\right.
$$

$$
\left.\left.*_{3} c_{2} \partial_{3}^{1} \Phi_{2} b\right)+{ }_{3}\left(c_{2} \partial_{3}^{0} c_{1} \partial_{2}^{0} a *_{3} \Phi^{\prime} b\right)\right] \text { by (5.2.10)(ii) }
$$

$$
=\Phi_{1}^{\prime}\left(\Phi^{\prime} a *_{3} c_{1}^{2}\left(a_{2}^{1}\right)^{2} b\right)+{ }_{3} \Phi_{1}^{\prime}\left(c_{2} a_{3}^{0} \Phi_{1} a *_{3} c_{1} \Phi_{2} a_{2}^{1} b\right)+{ }_{3}
$$

$$
\Phi_{1}^{\prime}\left(c_{1} \Phi_{2} \partial_{2}^{0} a *_{3} c_{2} \partial_{3}^{1} \Phi_{1} b\right)+\Phi_{1}^{\prime}\left(c_{1}^{2}\left(a_{2}^{0}\right)^{2} a *_{3} \Phi^{\prime} b\right)
$$

$$
\text { by }(5.2 .13),(4.1 .1)(\mathrm{ii}, \mathrm{iii}) \text { and }(5.2 .4)(\mathrm{i}) \text {. }
$$

$$
\text { Let } A_{1}=\Phi_{1}^{\prime}\left(\Phi^{\prime} a *_{3} c_{1}^{2}\left(a_{2}^{1}\right)^{2} b\right)
$$

$$
=\Psi_{1}\left(\Phi^{\prime} a *_{3} \varepsilon_{1}^{2}\left(\partial_{2}^{1}\right)^{2} b\right)-c_{3} c_{1} a_{1}^{1} \psi_{1}\left(\Phi^{\prime} a *_{3} \varepsilon_{1}^{2}\left(\partial_{2}^{1}\right)^{2} b\right)
$$

$$
=\left(\Psi_{1} \Phi^{\prime} a *_{3} \Psi_{1}\left(c_{1}\right)^{2}\left(a_{2}^{1}\right)^{2} b\right)--_{3}\left(c_{1} a_{1}^{1} \Psi_{1} \Phi^{\prime} a *_{3} c_{1} a_{1}^{1} \Psi_{1}\left(c_{1}\right)^{2}\left(a_{2}^{1}\right)^{2} b\right)
$$

$$
\text { by }(5.2 .9)(i i) \text { and }(4.1 .4)(i, i i)
$$

$$
=\left(\Psi_{1} \Phi^{\prime} a *_{3}\left(c_{1}\right)^{2}\left(a_{2}^{1}\right)^{2} b\right)-{ }_{3}\left(c_{1} \partial_{1}^{1} \Psi_{1} \Phi^{\prime} a *_{3} \cdot\left(c_{1}\right)^{2}\left(a_{2}^{1}\right)^{2} b\right)
$$

$$
\text { by }(5.2 .3)(i i) \text { and }(4.1 .1)(i i i)
$$

$$
=\left(\psi_{1} \Phi^{\prime} a-c_{1} \partial_{1}^{1} \Psi_{2} \Phi^{\prime} a\right) *_{3}\left(c_{1}\right)^{2}\left(\partial_{2}^{1}\right)^{2} b \text { by distributivity }
$$

$$
=\Phi a *_{3}\left(c_{1}\right)^{2}\left(\partial_{2}^{1}\right)^{2} b \text {. Since } a_{2}^{1} a_{2}^{1}=a_{2}^{1} a_{3}^{1} \text {, then }
$$

$$
A_{1}=(\Phi a)^{v_{1} b}
$$

We can prove similarly that

$$
\begin{aligned}
& A_{4}=\Phi_{1}^{\prime}\left[\left(c_{1}\right)^{2}\left(a_{2}^{0}\right)^{2} a *_{3} \Phi^{\prime} b\right]=\left(c_{1}\right)^{2}\left(a_{2}^{0}\right)^{2} a *_{3} \Phi b=u_{1}(\Phi b) . \\
& \text { Let } A_{2}=\Phi_{1}^{\prime}\left(\varepsilon_{2} a_{3}^{0} \Phi_{1} a *_{3} \varepsilon_{1} \Phi_{2} a_{2}^{1} b\right) \\
& =\psi_{1}\left(\varepsilon_{2} \partial_{3}^{0} \Phi_{1} a *_{3} c_{1} \Phi_{2} \partial_{2}^{1} b\right)-{ }_{3} c_{1} \partial_{1}^{1} \psi_{1}\left(c_{2} \partial_{3}^{0} \Phi_{1} a *_{3} \varepsilon_{1} \Phi_{2} \partial_{2}^{1} b\right) \\
& =\left(\Psi_{1} c_{2} \partial_{3}^{0} \Phi_{1} a *_{3} \Psi_{1} \varepsilon_{1} \Phi_{2} \partial_{2}^{1} b\right)-3\left(c_{1} \partial_{1}^{1} \Psi_{1} c_{2} \partial_{3}^{0} \Phi_{1} a *_{3} c_{1} \partial_{1}^{1} \Psi_{1} c_{1} \Phi_{2} \partial_{2}^{1} b\right) \\
& \text { by (5.2.9)(ii) and (4.1.4)(i,ii) } \\
& =\left(c_{1} \partial_{3}^{0} \Phi_{1} a *_{3} c_{1} \Phi_{2} \partial_{2}^{1} b\right)-{ }_{3}\left(c_{1} \partial_{3}^{0} \Phi_{1} a *_{3} c_{2} \Phi_{2} \partial_{2}^{1} b\right) \\
& \text { by (5.2.3)(ii) and (4.1.1)(iii). }
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
& A_{3}=\Phi_{1}^{\prime}\left(c_{1} \Phi_{2} \partial_{2}^{0} a *_{3} c_{2} \partial_{3}^{1} \Phi_{1} b\right) \\
& =\Psi_{1}\left(c_{1} \Phi_{2} \partial_{2}^{0} a *_{3} c_{2} \partial_{3}^{1} \Phi_{1} b\right)-{ }_{3} c_{1} a_{1}^{1} \Psi_{1}\left(c_{2} \Phi_{2} a_{2}^{0} a *_{3} c_{2} a_{3}^{1} \Phi_{2} b\right) \\
& =\left(\Psi_{1} c_{1} \Phi_{2} \partial_{2}^{0} a *_{3} \Psi_{1} c_{2} \partial_{3}^{1} \Phi_{1} b\right)-{ }_{3}\left(c_{1} \partial_{1}^{1} \Psi_{1} c_{1} \Phi_{2} a_{2}^{0} a *_{3} c_{1} \partial_{1}^{1} \psi_{2} c_{2} \partial_{3}^{1} \Phi_{1} b\right) \\
& \text { by (5.2.9)(ii) and (4.1.4)(i,ii) } \\
& =\left(c_{1} \Phi_{2} a_{2}^{0} a *_{3} c_{1} a_{3}^{1} \Phi_{1} b\right)-c_{3}\left(c_{2} \Phi_{2} a_{2}^{0} a *_{3} c_{1} \partial_{3}^{1} \Phi_{1} b\right) \\
& \text { by (5.2.3)(ii) and (4.1.1)(iii). }
\end{aligned}
$$

Thus $A_{2}+{ }_{3} A_{3}=\left[\left(c_{1} a_{3}^{0} \Phi_{1} a *_{3} c_{1} \Phi_{2} a_{2}^{1} b\right)+{ }_{3}\left(c_{1} \Phi_{2} a_{2}^{0} a *_{3}\right.\right.$

$$
\begin{aligned}
& \left.\left.c_{1} \partial_{3}^{1} \Phi_{1} b\right)\right]-3\left[\left(c_{1} \partial_{3}^{0} \Phi_{1} a *_{3} c_{1} \Phi_{2} \partial_{2}^{1} b\right)+{ }_{3}\left(c_{1} \Phi_{2} \partial_{2}^{0} a *_{3} c_{1} \partial_{3}^{1} \Phi_{1} b\right)\right] \\
& =\zeta_{3}\left\{\partial_{3}^{0}\left[\left(c_{1} \partial_{3}^{0} \Phi_{2} a *_{3} c_{1} \Phi_{2} \partial_{2}^{1} b\right)+{ }_{3}\left(c_{1} \Phi_{1} \partial_{2}^{0} a *_{3} c_{1} \partial_{3}^{1} \Phi_{1} b\right)\right],\right. \\
& \left.\partial_{3}^{1}\left[\left(c_{1} \partial_{3}^{0} \Phi_{1} a *_{3} c_{1} \Phi_{2} \partial_{2}^{1} b\right)+{ }_{3}\left(c_{1} \Phi_{2} \partial_{2}^{0} a *_{3} c_{1} \partial_{3}^{1} \Phi_{1} b\right)\right]\right\} .
\end{aligned}
$$

First we compute
$\partial_{3}^{0}\left[\left(c_{1} \partial_{3}^{0} \Phi_{1} a *_{3} c_{1} \Phi_{2} \partial_{2}^{1} b\right)+{ }_{3}\left(c_{1} \Phi_{2} \partial_{2}^{0} a *_{3} c_{1} \partial_{3}^{1} \Phi_{1} b\right)\right]$
$=\partial_{3}^{0} c_{1} \partial_{3}^{0} \Phi_{1}$ a by (4.1.3)(i) and (4.1.4)(i)

| $=c_{1} \partial_{2}^{0} \partial_{2}^{0} \Phi_{1} a$ | by $(4.1 .1)(i, i i i)$ |
| :--- | :--- |
| $=c_{1} \partial_{2}^{0} c_{1} \partial_{1}^{0} \partial_{1}^{0} a$ | by (5.2.2)(ii) |
| $=\left(c_{1}\right)^{2}\left(a_{1}^{0}\right)^{3} a$ | by (4.1.1)(iii). |

Second, we compute
$a_{3}^{1}\left[\left(\varepsilon_{1} a_{3}^{0} \Phi_{1} a *_{3} c_{1} \Phi_{2} a_{2}^{1} b\right)+{ }_{3}\left(\varepsilon_{1} \Phi_{2} \partial_{2}^{0} a *_{3} \varepsilon_{1} a_{3}^{1} \Phi_{1} b\right)\right]$
$=a_{3}^{1} c_{1} \partial_{3}^{1} \Phi_{1} b \quad$ by (4.1.3)(i) and (4.1.4)(i)
$=c_{1} a_{2}^{1} a_{2}^{1} \Phi_{1} b \quad$ by (4.1.1)(i,iii)
$=c_{1} a_{2}^{1} c_{1} \partial_{1}^{1} \partial_{1}^{1} b \quad$ by (5.2.2)(ii)
$=\left(c_{1}\right)^{2}\left(a_{1}^{1}\right)^{3} b, \quad b y(4.1 .1)(i i i)$.
Thus $A_{2}+{ }_{3} A_{3}=5_{3}\left[\left(c_{1}\right)^{2}\left(a_{1}^{0}\right)^{3} a,\left(c_{1}\right)^{2}\left(a_{1}^{1}\right)^{3} b\right]$.
Therefore $\Phi\left(a *_{1} b\right)=A_{1}+{ }_{3} A_{4}=u_{1}{ }^{a}(\Phi a)+{ }_{3}(\Phi b)^{v_{1} b}$.
Second for $j=2$, we get
$\Phi\left(\mathrm{a} *_{2} \mathrm{~b}\right)=\Phi_{1}^{\prime} \Phi_{2} \Phi_{1}\left(\mathrm{a} *_{2} \mathrm{~b}\right)$
$=\Phi_{1}^{\prime} \Phi_{2}\left[\left(c_{1} a_{1}^{0} a *_{2} \Phi_{1} b\right)+{ }_{2}\left(\Phi_{1} a *_{2} c_{1} a_{1}^{1} b\right)\right]$ by (5.2.10)(ii)
$=\Phi_{1}^{\prime} \Phi_{2}\left(c_{1} a_{1}^{0} a *_{2} \Phi_{1} b\right)+\Phi_{1}^{\prime} \Phi_{2}\left(\Phi_{1} a *_{2} c_{1} a_{1}^{1} b\right)$ by (5.2.10)(i) and (5.2.13)
$=\Phi_{1}^{\prime}\left[\left(\Phi_{2} c_{2} a_{1}^{0} a *_{3} c_{2} a_{3}^{1} \Phi_{1} b\right)+{ }_{3}\left(c_{2} a_{3}^{0} c_{1} a_{1}^{0} a *_{2} \Phi^{\prime} b\right)\right]+{ }_{3}$ $\Phi_{1}^{\prime}\left[\left(\Phi^{\prime} a *_{3} c_{2} a_{3}^{1} c_{1} \partial_{1}^{1} b\right)+{ }_{3}\left(\varepsilon_{2} \partial_{3}^{0} \Phi_{1} a *_{3} \Phi_{2} c_{1} a_{1}^{1} b\right)\right]$ by (5.2.10)(ii)
$=\Phi_{1}^{\prime}\left(c_{1} \Phi_{1} a_{1}^{o} a *_{3} c_{2} a_{3}^{1} \Phi_{1} b\right)+{ }_{3} \Phi_{1}^{\prime}\left(c_{1}^{2} \partial_{2}^{o} \partial_{1}^{o} a *_{3} \Phi^{\prime} b\right)+{ }_{3}$
$\Phi_{1}^{\prime}\left(\Phi^{\prime} a *_{3} c_{1}^{2} a_{2}^{2} a_{1}^{1} b\right)+{ }_{3} \Phi_{1}^{\prime}\left(c_{2} \partial_{3}^{0} \Phi_{1} a *_{3} c_{1} \Phi_{1} a_{1}^{1} b\right)$ by (5.2.13) , (5.2.4)(i) and (4.1.1)(i,ii,iii).

$$
\begin{aligned}
& \text { Let } A_{1}=\Phi_{1}^{\prime}\left(c_{1} \Phi_{1} \partial_{1}^{0} a *_{3} c_{2} \partial_{3}^{1} \Phi_{1} b\right) \\
& =\psi_{1}\left(c_{2} \Phi_{1} \partial_{1}^{0} a *_{3} \varepsilon_{2} \partial_{3}^{1} \Phi_{1} b\right)-c_{3} c_{1}^{1} \Psi_{1}\left(c_{1} \Phi_{1} \partial_{1}^{0} a *_{3} c_{2} \partial_{3}^{1} \Phi_{2} b\right) \\
& =\left(\Psi_{2} c_{1} \Phi_{1} \partial_{1}^{0} a *_{3} \Psi_{1} c_{2} a_{3}^{1} \Phi_{2} b\right)-c_{3}\left(c_{1} a_{1}^{1} \Psi_{1} c_{2} \Phi_{1} \partial_{1}^{0} a *_{3} c_{2} \partial_{3}^{1} \Phi_{1} b\right) \\
& \text { by (5.2.9)(ii) and (4.1.4)(i,ii) } \\
& =\left(c_{1} \Phi_{1} a_{1}^{0} a *_{3} c_{1} \partial_{3}^{1} \Phi_{1} b\right)-c_{3}\left(c_{1} \Phi_{1} a_{1}^{0} a *_{3} c_{1} \partial_{3}^{1} \Phi_{1} b\right) \\
& \text { by (5.2.3)(ii) and (4.1.1)(iii). }
\end{aligned}
$$

On the other hand;
Let $A_{4}=\Phi_{1}^{\prime}\left(c_{2} \partial_{3}^{0} \Phi_{1} a *_{3} c_{1} \Phi_{1} \partial_{1}^{1} b\right)$
$=\left(c_{1} \partial_{3}^{0} \Phi_{1} a *_{3} c_{1} \Phi_{1} \partial_{1}^{1} b\right)-c_{3}\left(c_{1} \partial_{3}^{0} \Phi_{1} a *_{3} c_{1} \Phi_{1} \partial_{1}^{1} b\right)$
by similar way as above.
By using similar argument as above, we get

$$
\begin{aligned}
& A_{1}+{ }_{3} A_{4}=5_{3}\left[\left(c_{1}\right)^{2}\left(\partial_{1}^{0}\right)^{3} a,\left(\varepsilon_{1}\right)^{2}\left(\partial_{1}^{1}\right)^{3} b\right] \\
& \text { Let } A_{2}=\Phi_{1}^{\prime}\left[\left(c_{1}\right)^{2} \partial_{2}^{0} a_{1}^{0} a \quad *_{3} \Phi^{\prime} b\right) \\
& =\Psi_{1}\left[\left(c_{1}\right)^{2} a_{1}^{0} a_{3}^{0} a *_{3} \Phi^{\prime} b\right]-3 c_{1} a_{1}^{1} \Psi_{1}\left[\left(c_{1}\right)^{2} a_{1}^{0} a_{3}^{0} a *_{3} \Phi^{\prime} b\right] \\
& \text { by (4.1.1)(i) } \\
& =\left[\Psi_{1}\left(c_{1}\right)^{2} a_{1}^{0} a_{3}^{0} a *_{3} \psi_{1} \Phi^{\prime} b\right]-{ }_{3}\left[c_{1} \partial_{1}^{1} \Psi_{1}\left(c_{1}\right)^{2} a_{1}^{0} \partial_{3}^{0} a *_{3} c_{1} \partial_{1}^{1} \Psi_{1} \Phi^{\prime} b\right] \\
& \text { by (5.2.9)(ii) and (4.1.4)(i,ii) } \\
& =\left[\left(c_{1}\right)^{2} \partial_{1}^{0} \partial_{3}^{0} a *_{3} \Psi_{1} \Phi^{\prime} b\right]-3\left[\left(c_{1}\right)^{2} a_{1}^{0} \partial^{0} a *_{3} c_{1} a_{1}^{1} \Psi_{1} \Phi^{\prime} b\right] \\
& \text { by (5.2.3)(ii) and (4.1.1)(iii) } \\
& =c_{1}^{2} a_{1}^{0} \partial_{3}^{0} a *_{3}\left(\Psi_{1} \Phi^{\prime} b-c_{3} c_{1}^{1} \Psi_{1} \Phi^{\prime} b\right) \text { by distributivity } \\
& =c_{1}^{2} \partial_{1}^{0} \partial_{3}^{0}{ }^{0} *_{3} \Phi b=u_{2}^{a}(\Phi b) \text {. }
\end{aligned}
$$

We can prove similarly that
$A_{3}=\Phi a *_{3} c_{1}^{2} \partial_{1}^{1} \partial_{3}^{1} b=(\Phi a)^{v_{2} b}$.

Thus $\Phi\left(a *_{2} b\right)=u^{a}(\Phi b)+_{3}(\Phi a)^{v_{3}}{ }^{b}$.
Finally for $\mathbf{j}=3$, we get

$$
\begin{array}{r}
\Phi\left(a *_{3} b\right)=\Phi_{1}^{\prime} \Phi_{2} \Phi_{1}\left(a *_{3} b\right)=\Phi_{1}^{\prime} \Phi_{2}\left(\Phi_{1} a *_{3} \Phi_{1} b\right) \\
b y(5.2 .10)(i i) \\
=\Phi_{1}^{\prime}\left[\left(c_{2} a_{2}^{0} \Phi_{1} a *_{3} \Phi^{\prime} b\right)+{ }_{3}\left(\Phi^{\prime} a *_{3} c_{2} \partial_{2}^{1} \Phi_{1} b\right)\right] \\
b y(5.2 .10)(i i) \\
=\Phi_{1}^{\prime}\left(c_{2} \partial_{2}^{0} \Phi_{1} a *_{3} \Phi^{\prime} b\right)+{ }_{3}\left(\Phi^{\prime} a *_{3} c_{2} a_{2}^{1} \Phi_{2} b\right) \text { by (5.2.13) } \\
=\Phi_{1}^{\prime}\left(c_{2} c_{1} \partial_{1}^{0} a_{1}^{0} a *_{3} \Phi^{\prime} b\right)+{ }_{3} \Phi_{1}^{\prime}\left(\Phi^{\prime} a *_{3} c_{2} c_{1} a_{1}^{1} a_{1}^{1} b\right) \\
b y(5.2 .2)(i i) .
\end{array}
$$

Let $A_{1}=\Phi_{1}^{\prime}\left(c_{2} c_{1} \partial_{1}^{0} \partial_{1}^{0} a *_{3} \Phi^{\prime} b\right)=\Phi_{1}^{\prime}\left(c_{1}^{2} \partial_{1}^{0} \partial_{2}^{0} a *_{3} \Phi^{\prime} b\right)$ by (4.1.1)(i ,ii)
$=\Psi_{1}\left[\left(c_{1}\right)^{2} a_{1}^{0} \partial_{2}^{0} a *_{3} \Phi^{\prime} b\right]-\varepsilon_{1} a_{1}^{1} \Psi_{1}\left[\left(c_{1}\right)^{2} \partial_{1}^{0} \partial_{2}^{0} a *_{3} \Phi^{\prime} b\right]$
$=\left(\Psi_{1} c_{1}^{2} \partial_{1}^{0} \partial_{2}^{0} a *_{3} \Psi_{1} \Phi^{\prime} b\right)-c_{3}\left(c_{1} \partial_{1}^{1} \Psi_{1} c_{1}^{2} \partial_{1}^{0} \partial_{2}^{0} a *_{3} c_{1} \partial_{1}^{1} \Psi_{1} \Phi^{\prime} b\right)$
by (5.2.9)(ii) and (4.1.4)(i ,ii)
$=\left[\left(c_{1}\right)^{2} \partial_{1}^{0} a_{2}^{0} a *_{3} \psi_{2} \Phi^{\prime} b\right]-{ }_{3}\left[\left(c_{1}\right)^{2} a_{1}^{0} a_{2}^{0} a *_{3} c_{1} a_{1}^{1} \psi_{1} \Phi^{\prime} b\right]$ by (5.2.3)(ii) and (4.1.1)(iii)
$=c_{1}{ }^{2} \partial_{1}^{0} \partial_{2}^{0} a *_{3} \Phi b \quad$ by distributivity
$=u_{3}{ }^{a}(\Phi b)$. We can prove similarly that $A_{2}=(\Phi a)^{v_{3} b}$.
Thus $\Phi\left(a *_{3} b\right)=u_{3}(\Phi b)+_{3}(\Phi a)^{v_{3} b}$.
This completes the proof for $n=3$. A similar direct computational proof for the case $n=4$ has been written out, but it is too lenghty to include here.

## APPENDIX V

## The proof of proposition (5.2.20):

We will prove the first case only and the second case is similar . For the first case, we start with :
i) For $1 \leqslant i \leqslant 3, \Phi c_{i} a=0$ in dimension 3 , it is immediately by using (5.2.4)(i,ii).
ii) For $\Phi \Gamma_{j}{ }^{a}=0$ in dimension 3 , we start first with $j=1$, then

$$
\begin{aligned}
& \Phi \Gamma_{1} a=\Phi_{1}^{\prime} \Phi_{2} \Phi_{1} \Gamma_{1} a=\Phi_{1}^{\prime} \Phi_{2}\left[\zeta_{2}\left(\left(\partial_{1}^{0}\right)^{2} a,\left(\partial_{1}^{1}\right)^{2} a\right)\right] \text { by }(5.2 .6)(i i) \\
& =0 \text { in dimension } 3 .
\end{aligned}
$$

$$
\text { Second, let } j=2 \text {, then }
$$

$$
\begin{gathered}
\Phi \Gamma_{2} a=\Phi_{1}^{\prime} \Phi_{2} \Phi_{1} \Gamma_{2} a=\Phi_{1}^{\prime} \Phi_{2}\left[\left(\Gamma_{1}^{\prime} a *_{2} \Gamma_{2} a *_{2} \Gamma_{1} c_{2} a_{2}^{1} a\right)-2\right. \\
\left.\left(c_{1} a *_{2} \Gamma_{2} c_{1} a_{1}^{1} a\right)\right] \quad \text { by }(5.2 .6)(i i i) \\
=\Phi_{1}^{\prime} \Phi_{2}\left[\left(\Gamma_{1}^{\prime} a *_{2} \Gamma_{2} a\right) *_{2} c_{3} \Gamma_{1} a_{2}^{1} a\right]-\Phi_{2}^{\prime} \Phi_{2}\left(c_{1} a *_{2} c_{1} \Gamma_{1} a_{1}^{1} a\right) \\
b y(5.2 .9)(i),(5.2 .13) \text { and }(5.1 .2)(i i i) \quad .
\end{gathered}
$$

$$
\text { Let } A_{1}=\Phi_{2}^{\prime} \Phi_{2}\left[\left(\Gamma_{1}^{\prime} a *_{2} \Gamma_{2} a\right) *_{2} c_{3} \Gamma_{1} \partial_{2}^{1} a\right]
$$

$$
=\Phi_{1}^{\prime}\left\{\left[\Phi_{2}\left(\Gamma_{1}^{\prime} a *_{2} \Gamma_{2} a\right) *_{3} c_{2} \partial_{3}^{1} c_{3} \Gamma_{1} a_{2}^{1} a\right]+{ }_{3}\left[\left(c_{2} \partial_{3}^{0}\left(\Gamma_{1}^{\prime} a *_{2} \Gamma_{2} a\right)\right.\right.\right.
$$

$$
\left.\left.*_{3} \Phi_{2} c_{3} \Gamma_{1} \partial_{2}^{1} a\right]\right\} \quad \text { by }(5.2 .10)(\mathrm{ii})
$$

$$
=\Phi_{1}^{\prime}\left[\Phi_{2}\left(\Gamma_{1}^{\prime} a *_{2} \Gamma_{2} a\right) *_{3} c_{2} \Gamma_{1} \partial_{2}^{1} a\right] \text { by (5.2.4)(ii),(4.1.1)(iiii). }
$$

$$
\text { Now we look at } \Phi_{2}\left(\Gamma_{1}^{\prime} a *_{2} \Gamma_{2} a\right) \text { which is equal }
$$

$$
=\left(\Phi_{2} \Gamma_{1}^{\prime} a *_{3} \varepsilon_{2} a_{3}^{1} r_{2} a\right)+{ }_{3}\left(\varepsilon_{2} \partial_{3}^{0} \Gamma_{1}^{\prime} a *_{3} \Phi_{2} \Gamma_{2} a\right)
$$

$$
\begin{aligned}
& =\Phi_{2} \Gamma_{1}^{\prime} a *_{3} c_{2} \partial_{3}^{1} \Gamma_{2} a \quad b y(5.2 .6)(i i) . \\
& \text { Thus } A_{1}=\Phi_{1}^{\prime}\left[\left(\Phi_{2} \Gamma_{1}^{\prime} a *_{3} \varepsilon_{2} a_{3}^{1} \Gamma_{2} a\right) *_{3} \varepsilon_{2} \Gamma_{1} a_{2}^{1} a\right] \\
& =\Phi_{1}^{\prime}\left[\Gamma_{1}^{\prime} \Phi_{1} a *_{3} c_{2} \partial_{3}^{1} \Gamma_{2} a *_{3} c_{2} \Gamma_{1} \partial_{2}^{1} a\right] \text { by (5.2.8)(i) } \\
& =\left(\Psi_{1} \Gamma_{1}^{\prime} \Phi_{1} a *_{3} \Psi_{2} c_{2} \partial_{3}^{1} \Gamma_{2} a *_{3} \Psi_{1} c_{2} \Gamma_{1} a_{2}^{1} a\right)-{ }_{3}\left(\varepsilon_{1} \partial_{1}^{1} \Psi_{1} \Gamma_{1}^{\prime} \Phi_{1} a *_{3}\right. \\
& \left.\varepsilon_{1} a_{1}^{1} \Psi_{1} \varepsilon_{2} \partial_{3}^{1} \Gamma_{2} a *_{3} \varepsilon_{1} \partial_{1}^{1} \Psi_{1} \varepsilon_{2} \Gamma_{1} \partial_{2}^{1} a\right) \text { by (5.2.9)(ii),(4.1.4)(i,ii) } \\
& =\left(\varepsilon_{1} \Phi_{1} a *_{3} c_{1} \partial_{3}^{1} r_{2} a *_{3} \varepsilon_{1} \Gamma_{1} a_{2}^{1} a\right)-{ }_{3}\left(\varepsilon_{2} \Phi_{1} a *_{3} c_{2} \partial_{3}^{1} r_{2} a *_{3}\right. \\
& \left.c_{1} \Gamma_{1} \partial_{2}^{1} a\right) \text { by (5.2.7)(ii),(5.2.3)(ii) and (4.1.1)(iii). }
\end{aligned}
$$

We look first at $\partial_{3}^{0}\left(\varepsilon_{1} \Phi_{1} a * c_{1} \partial_{3}^{1} \Gamma_{2} a * c_{1} \Gamma_{1} \partial_{2}^{1} a\right)$
$=\partial_{3}^{0} c_{1} \Phi_{1} a \quad$ by (4.1.4)(i)
$=c_{1} \partial_{2}^{0} \Phi_{1} a \quad$ by (4.1.1)(iii)
$=\left(c_{1}\right)^{2}\left(\partial_{1}^{0}\right)^{2}$ a by (5.2.2)(ii).

Second, $\partial_{3}^{1}\left(c_{1} \Phi_{1} a *_{3} c_{1} \partial_{3}^{1} \Gamma_{2} a *_{3} c_{1} \Gamma_{1} \partial_{2}^{1} a\right)$
$=a_{3}^{1} c_{1} \Gamma_{1} a_{2}^{1} a \quad$ by (4.1.4)(i)
$=c_{1} \Gamma_{1} \partial_{1}^{1} \partial_{1}^{1} a \quad$ by (4.1.1)(i,iii) and (5.1.2)(vii)
$=\left(c_{1}\right)^{2}\left(\partial_{1}^{2}\right)^{2} a \quad$ by (5.1.2)(iii). Thus
$A_{1}=c_{3}\left(c_{1}^{2}\left(\partial_{1}^{0}\right)^{2} a, c_{1}^{2}\left(\partial_{1}^{1}\right)^{2} a\right)$, that is $A_{1}=0$ in
dimension 3 .

$$
\text { Now we look at } A_{2}=\Phi_{1}^{\prime} \Phi_{2}\left(c_{2} a *_{2} c_{1} \Gamma_{1}^{\prime} \partial_{1}^{1} a\right)
$$

$=\Phi_{1}^{\prime} \Phi_{2} c_{1}\left(a *_{2} \Gamma_{1}^{\prime} \partial_{1}^{1} a\right) \quad b y(4.1 .4)(i i)$

$$
\begin{aligned}
= & \Phi_{1}^{\prime} c_{1} \Phi_{1}\left(a *_{2} \Gamma_{1}^{\prime} \partial_{1}^{1} a\right) \\
= & \Psi_{1} c_{1} \Phi_{1}\left(a *_{2} \Gamma_{1}^{\prime} \partial_{1}^{1} a\right)-c_{3} c_{1} a_{1}^{1} \Psi_{1} c_{1} \Phi_{1}(a, 2.4)(i) \\
= & c_{1} \Phi_{1}\left(a \Gamma_{2}^{\prime} \partial_{1}^{1} a\right) \\
& \quad b y(5.2 .3)(i i) \text { and }(4.1 .1)(i i i)
\end{aligned}
$$

We first compute $\partial_{3}^{0} \varepsilon_{1} \Phi_{1}\left(a *_{2} \Gamma_{1}^{\prime} \partial_{1}^{1} a\right)$
$=\varepsilon_{1} \partial_{2}^{0} \Phi_{1}\left(a *_{2} \Gamma_{1}^{\prime} \partial_{2}^{1} a\right)$ by (4.1.1)(iii)
$=\varepsilon_{1} \varepsilon_{1} a_{1}^{0} a_{1}^{0}\left(a *_{2} \Gamma_{1}^{\prime} a_{1}^{1} a\right)$ by (5.2.2)(ii)
$=\left(c_{1}\right)^{2}\left(\partial_{1}^{0}\right)^{2} a$ by (4.1.4)(i,ii).

Second we compute $\partial_{3}^{1} c_{1} \Phi_{1}\left(a *_{2} \Gamma_{1}^{\prime} \partial_{1}^{1} a\right)$
$=c_{1} c_{1} \partial_{1}^{1} \partial_{1}^{2}\left(a *_{2} \Gamma_{1}^{\prime} \partial_{1}^{1} a\right)$ by (4.1.1)(iii) and (5.2.2)(ii)
$=\left(c_{1}\right)^{2}\left(a_{1}^{1}\right)^{2} a \quad b y(4.1 .4)(i, i i)$.
Thus $\Phi \Gamma_{2} a=\zeta_{3}\left(c_{1}^{2}\left(\partial_{1}^{0}\right)^{2} a, c_{1}^{2}\left(\partial_{1}^{1}\right)^{2} a\right)$, and hence $\Phi \Gamma_{j} a=0$ in dimension 3 for $\mathbf{l}$ < $\mathbf{j} \mathbf{~} 2$.

We can prove similarly that $\Phi \Gamma_{j}^{\prime} a=0$ in dimension 3
for $1 \leqslant j \leqslant 2$.

## APPENDIX VI

The proof of proposition (5.3.2):

Recall that an w-algebroid $A$ is a cubical complex with connections and satisfy the axioms (4.1.3), (4.1.4), $(4.1 .5),(4.1 .6),(5.1 .3),(5.1 .4)$ and (5.1.5) .

It is clear that $A^{\prime}=\left(\square A_{n}, A_{n}, \ldots, A_{0}\right)$ is a cubical complex with connections . Thus it is enough to verify the axioms of an $\omega$-algebroid ; namely
(4.1.3)(i): Let $a, b \in A_{n}$ such that $a+j b$ is defined. Then $a_{i}^{\alpha}\left(a+{ }_{j} b\right)=a_{i}^{\alpha} f=f_{i}^{\alpha}=\left\{\begin{array}{ll}a_{i}^{\alpha}+{ }_{j-1} b_{i}^{\alpha} & (i<j) \\ a_{i}^{\alpha}+{ }_{j} b_{i}^{\alpha} & (i>j)\end{array} \quad\right.$ by $(5.3 .1)(v)$
$=\left\{\begin{array}{ll}\partial_{i}^{\alpha} a+{ }_{j-1} \partial_{i}^{\alpha} b \\ \partial_{i}^{\alpha} a+{ }_{j} \partial_{i}^{\alpha} & (i<j) \\ (i \Delta j)\end{array} \quad\right.$ by definition of $\partial_{i}^{\alpha} \quad$. (4.1.3)(ii): Let $a, b \in A_{n}$ such that $a+j b$ is defined. Then for $k<i<j$, we have
$\partial_{k}^{\alpha}\left[c_{i}\left(a+{ }_{j} b\right)\right]=c_{i-1} a_{k}^{\alpha}\left(a+{ }_{j} b\right)$
by (4.1.1)(iii)
$=c_{i-1}\left[\partial_{k}^{\alpha} a+{ }_{j-1} \partial_{k}^{\alpha} b\right]=c_{i-1} \partial_{k}^{\alpha} a+{ }_{j} c_{i-1} \partial_{k}^{\alpha}{ }_{b}$ (since $\partial_{k}^{\alpha} a, \partial_{k}^{\alpha}{ }^{\alpha}$ are elements in $A_{n-1}$ ) and by (4.1.3)(ii)
$=\partial_{k}^{\alpha} c_{i}{ }^{a}+{ }_{j} \partial_{k}^{\alpha} c_{i} a$
$=\partial_{k}^{\alpha}\left[c_{i} a+{ }_{j+1} c_{i} b\right]$. Thus $c_{i}\left(a+{ }_{j} b\right)=c_{i} a+{ }_{j+1} c_{i} b$. We can use a similar way to show that $\varepsilon_{i}\left(a+{ }_{j} b\right)=c_{i} a+{ }_{j} c_{i} b$, for $i>j$.
(4.1.4)(i): Let $a, b \in A_{n}$ such that $a *_{j} b$ is defined. Then $\partial_{i}^{\alpha}\left(a *_{j} b\right)=\partial_{i}^{\alpha} g=g_{i}^{\alpha}=\left\{\begin{array}{ll}a_{i}^{\alpha} *_{j-1} b_{i}^{\alpha} & (i<j) \\ a_{i}^{\alpha} *_{j} b_{i}^{\alpha} & (i>j)\end{array}\right.$ by (5.3.1)(vii)
$=\left\{\begin{array}{l}\partial_{i}^{\alpha}{ }^{\alpha} *_{j-1} \partial_{i}^{\alpha}{ }_{b}^{\alpha} \\ \partial_{i}^{\alpha} *_{j} \partial_{i}^{\alpha}{ }^{\alpha}\end{array}\right.$
(4.1.4)(ii): Let $a, b \in A_{n}$ such that $a *_{j} b$ is defined. Then for $k<i<j$, we get
$a_{k}^{\alpha}\left[c_{i}\left(a *_{j} b\right)\right]=\varepsilon_{i-1} a_{k}^{\alpha}\left(a *_{j} b\right) \quad$ by (4.1.1)(iii)
$=c_{i-1}\left[\partial_{k}^{\alpha} a *_{j-1} \partial_{k}^{\alpha} b\right]$ by (4.1.4) (ii) (since $\partial_{k}^{\alpha}, \partial_{k}^{\alpha}$ are elements in $A_{n-1}$ )
$=\partial_{k}^{\alpha} c_{i}{ }^{a}{ }_{j}{ }_{j} \partial_{k}^{\alpha} c_{i} b$ by (4.1.1)(i)
$=a_{k}^{\alpha}\left[c_{i} a *_{j+1} c_{i} b\right]$ by (4.1.4)(i).
Thus $c_{i}\left(a *_{j} b\right)=c_{i} a *_{j+1} c_{i} b$. We can prove similarly that $c_{i}\left(a *_{j} b\right)=c_{i} a *_{j} c_{i} b$, for $i \Delta j$.
(4.1.4)(iii): Let a $\in \mathrm{A}_{\mathrm{n}}$. Then for $\mathrm{k}<\mathrm{j}$, we get
$\partial_{k}^{\alpha}\left(\varepsilon_{j} \partial_{j}^{0}{ }^{a} *_{j} a\right)=\partial_{k}^{\alpha} c_{j} \partial_{j}^{0}{ }^{a} *_{j-1} \partial_{k}^{\alpha}{ }^{a}$
by (4.1.4)(i)
$=c_{j-1} \partial_{j-1}^{0} \partial_{k}^{\alpha} *_{j-1} \partial_{k}^{\alpha} \quad$ by (4.1.1)(i ,iii)
$=\partial_{k}^{\alpha}$ by (4.1.4)(iii) (since $\partial_{k}^{\alpha}{ }^{\alpha} \in A_{n}$ ):
Thus $c_{j} \partial_{j}^{0}{ }^{a} *_{j} a=a$. We can prove similarly that
$a=a{ }^{\prime}{ }_{j} c_{j} \partial_{j}^{1} a$.
(4.1.5)(i): Let a $\in \square A_{n}$ and $r \in R$. Then
$\partial_{i}^{\alpha}(r \cdot j a)=\partial_{i}^{\alpha} h=h_{i}^{\alpha}= \begin{cases}r \cdot{ }_{j-1} a_{i}^{\alpha} & (i<j) \\ r \cdot{ }_{j}^{a_{i}^{\alpha}} & (i>j) \text { by (5.3.1)(viii) } \\ a_{i}^{\alpha} & (i=j)\end{cases}$

(4.1.5)(ii): Let a $\in A_{n}$ and $r \in R$. Then for $k<i<j$, we get
$\partial_{k}^{\alpha}\left[c_{i}\left(r \cdot j^{a}\right)\right]=\varepsilon_{i-1} \partial_{k}^{\alpha}\left(r \cdot j^{a}\right) \quad$ by (4.1.l)(iii)
$=c_{i-1}\left(r \cdot j-1 \partial_{k}^{\alpha} a\right) \quad$ by (4.1.5)(i)

$=r \cdot{ }_{j} \partial_{k}^{\alpha} c_{i}$ a by (4.1.1) (iii)
$=\partial_{k}^{\alpha}\left(r \cdot{ }_{j+1} c_{i}^{a}\right.$ ) by (4.1.5)(i).
Thus $c_{i}(r \cdot j a)=r \cdot{ }_{j+1} c_{i} a$. Similarly we can prove the other parts.
(4.1.5)(iii): Let $a, b \in A_{n}$ and $r \in R$ such that $a{ }^{\prime}{ }_{j} b$ is defined. Then, for $k<i$ and $i=j$
$a_{k}^{\alpha}\left[r \cdot{ }_{i}\left(a *_{i} b\right)\right]=r \cdot{ }_{i-1} a_{k}^{\alpha}\left(a *_{i} b\right) \quad b y(4.1 .5)(i)$
$=r \cdot{ }_{i-1}\left(\partial_{k}^{\alpha} a *_{i-1} \partial_{k}^{\alpha} b\right) \quad$ by (4.1.4)(i)
$=\left(r ._{i-1} \partial_{k}^{\alpha} a\right) *_{i-1} \partial_{k}^{\alpha}$ by (4.1.5)(iii) (since $\partial_{k}^{\alpha}, \partial_{k}^{\alpha}{ }_{b}^{\alpha}$ are elements in $A_{n}$ )
$=\partial_{k}^{\alpha}\left(r ._{i}^{a)} *_{i-1} \partial_{k}^{\alpha} \quad\right.$ by (4.1.5)(i)
$=\partial_{k}^{\alpha}\left[\left(r \cdot{ }_{i} a\right) *_{i} b\right] \quad b y(4.1 .4)(i)$. Thus
 that $r ._{i}\left(a * *_{i}\right)=a *{ }_{i}\left(r \cdot{ }_{i} b\right)$.

Now, for $i \neq j, \operatorname{let} k<\min \{i, j\}$, then
$a_{k}^{\alpha}\left[r \cdot{ }_{i}\left(a *_{j} b\right)\right]=r \cdot{ }_{i-1} a_{k}^{\alpha}\left(a *_{j} b\right) \quad b y(4.1 .5)(i)$
$=r \operatorname{ci-1}\left(a_{k}^{\alpha} a *_{j-1} \partial_{k}^{\alpha} b\right) \quad b y(4.1 .4)(i)$
$=\left(r \cdot{ }_{i-1} \partial_{k}^{\alpha} a\right) *_{j-1}\left(r \cdot{ }_{i-1} \partial_{k}^{\alpha_{b}}\right) \quad b y(4.1 .5)(i i i)$ (since $\partial_{k}^{\alpha} a, \partial_{k}^{\alpha}$ are elements in $A_{n}$ )
$=\partial_{k}^{\alpha}(r$.i $a) *{ }_{j-1} \partial_{k}^{\alpha}\left(r \cdot{ }_{i} b\right) \quad b y(4.1 .5)(i)$
$=a_{k}^{\alpha}\left[\left(r \cdot i^{a}\right) *_{j}\left(r \cdot i^{b}\right)\right] \quad b y(4.1 .4)(i) \quad$ Thus
$r \cdot{ }_{i}\left(a *_{j} b\right)=\left(r \cdot i^{a}\right) *_{j}\left(r \cdot{ }_{i} b\right) \quad$.
(4.1.5)(iv): Let a $\in A_{n}$ and $r, s \in R$. Then for $k<i, j$ we get $\partial_{k}^{\alpha}\left[r \cdot{ }_{i}\left(s \cdot{ }_{j} a\right)\right]=r \cdot{ }_{i-1} \partial_{k}^{\alpha}\left(s \cdot{ }_{j} a\right)=r \cdot{ }_{i-1}\left(s \cdot j-1 \partial_{k}^{\alpha} a\right)$ by (4.1.5) (i)
$=s \cdot{ }_{j-1}\left(r \cdot i-1 \partial_{k}^{\alpha} a\right)$ by (4.1.5)(iv) (since $\left.\partial_{k}^{\alpha} a \in A_{n}\right)$
$=s \cdot j-1 a_{k}^{\alpha}\left(r \cdot{ }_{i} a\right)=\partial_{k}^{\alpha}\left[s \cdot j_{j}\left(r \cdot i^{a}\right)\right]$ by (4.1.5)(i).
Thus $r \cdot{ }_{i}\left(s \cdot j^{a}\right)=s \cdot j^{\left(r \cdot i^{a}\right)} \cdot$
(4.1.6)(i): Let $a, b, c, d \in \square A_{n} \operatorname{such}$ that $(a+i b)+j(c+i d)$, $(a+j c)+_{i}(b+j d)$ are defined. Then for $k<i, j$ we get
$\partial_{k}^{\alpha}\left[\left(a+{ }_{i} b\right)+_{j}\left(c+{ }_{i} d\right)\right]=a_{k}^{\alpha}\left(a+{ }_{i} b\right)+{ }_{j-1} \partial_{k}^{\alpha}\left(c+{ }_{i} d\right)$
$=\left(\partial_{k}^{\alpha} a+{ }_{i-1} \partial_{k}^{\alpha} b\right)+_{j-1}\left(\partial_{k}^{\alpha} c+_{i-1} \partial_{k}^{\alpha} d\right) \quad$ by (4.1.3)(i)
$=\left(\partial_{k}^{\alpha} a+{ }_{j-1} \partial_{k}^{\alpha} c\right)+_{i-1}\left(\partial_{k}^{\alpha}+_{j-1} \partial_{k}^{\alpha} d\right)$ by (4.1.6)(i) (since
$\partial_{k}^{\alpha}{ }^{\alpha}, \partial_{k}^{\alpha}{ }_{b}, \partial_{k}^{\alpha} c, \partial_{k}^{\alpha} d$ are in $\left.A_{n}\right)$

$$
\begin{aligned}
=\partial_{k}^{\alpha}\left(a+{ }_{j} c\right)+{ }_{i} \partial_{k}^{\alpha}\left(b+{ }_{j} d\right)= & \partial_{k}^{\alpha}\left[\left(a+{ }_{j} c\right)+{ }_{i}\left(b+{ }_{j} d\right)\right] \\
& b y(4.1 .3)(i) .
\end{aligned}
$$

Thus $\left(a+{ }_{i} b\right){ }_{j}\left(c+_{i} d\right)=\left(a+{ }_{j} c\right){ }_{i}\left(b+{ }_{j} d\right)$.
The proof of (4.1.6)(ii ,iii) are similar to that of (4.1.6)(i).
(5.1.3)(i): Let $a, b \in A_{n}$ such that $a+j b$ is defined. Then for $\mathrm{k}<\mathrm{i}<\mathrm{j}$, we get

$$
\begin{array}{ll}
\partial_{k}^{\alpha} \Gamma_{i}\left(a+{ }_{j} b\right)=\Gamma_{i-1} \partial_{k}^{\alpha}\left(a+{ }_{j} b\right) & b y(5.1 .2)(v i i) \\
=\Gamma_{i-1}\left(\partial_{k}^{\alpha}{ }^{\alpha}{ }_{j-1} \partial_{k}^{\alpha} b\right) & \text { by }(4.1 .3)(i) \tag{4.1.3}
\end{array}
$$

$=\Gamma_{i-1} \partial_{k}^{\alpha}{ }^{a}+{ }_{j} \Gamma_{i-1} \partial_{k}^{\alpha}$ by (5.1.3)(i) (since $\left.\partial_{k}^{\alpha}, \partial_{k}^{\alpha} \in A_{n-1}\right)$
$=\partial_{k}^{\alpha} \Gamma_{i} a+{ }_{j} \partial_{k}^{\alpha} \Gamma_{i} b \quad$ by (5.1.2)(vii)
$=a_{k}^{\alpha}\left[\Gamma_{i} a+{ }_{j+1} \Gamma_{i} b\right] \quad$ by (4.1.3)(i).
Thus $\Gamma_{i}(a+j b)=\Gamma_{i}{ }^{a}+{ }_{j+1} \Gamma_{i} b$. For $k>i>j$, we can use similar argument to prove that $\Gamma_{i}\left(a+{ }_{j} b\right)=\Gamma_{i} a+{ }_{j} \Gamma_{i} b$.

The proof of (5.1.3)(ii) is similar to that of (5.1.3)(i) .
(5.1.4)(i): Let $a, b \in A_{n}$ such that $a *_{j} b$ is defined. Then for $k<i<j$, we have
$a_{k}^{\alpha}\left[r_{i}\left(a *_{j} b\right)\right]=r_{i-1} \partial_{k}^{\alpha}\left(a *_{j} b\right) \quad b y(5.1 .2)(v i i)$
$=\Gamma_{i-1}\left[\partial_{k}^{\alpha}{ }^{\alpha} *_{j-1} \partial_{k}^{\alpha}\right] \quad$ by (4.1.4)(i)
$=\Gamma_{i-1} \partial_{k}^{\alpha} a *_{j} \Gamma_{i-1} \partial_{k}^{\alpha}$ by (5.1.4)(i) (since $\partial_{k}^{\alpha}, \partial_{k}^{\alpha} \in A_{n-1}$ )
$=\partial_{k}^{\alpha}\left[\Gamma_{i}{ }^{a} *_{j+1} \Gamma_{i} b\right]$ by (4.1.4)(i).

Thus $\Gamma_{i}\left(a *_{j} b\right)=\Gamma_{i} a *_{j+1} \Gamma_{i} b$. We can prove similarly that $\Gamma_{i}\left(a *_{j} b\right)=\Gamma_{i} a *_{j} \Gamma_{i} b \quad$ for $i>j$.

The proof of (5.1.4)(ii) is similar to that of (5.1.4)(i). (5.1.4)(iii): Let a $\in A_{n}$. Then for $k<j$, we get
$\partial_{k}^{\alpha}\left[\Gamma_{j}^{\prime}{ }^{a} *_{j+1} \Gamma_{j} a\right]=\partial_{k}^{\alpha} \Gamma_{j}^{\prime} a *_{j} \partial_{k}^{\alpha} \Gamma_{j} a \quad$ by (4.1.4)(i)
$=\Gamma_{j-1}^{\prime} \quad \partial_{k}^{\alpha}{ }^{\alpha} *_{j} \Gamma_{j-1} \partial_{k}^{\alpha}{ }_{a}$
by (5.1.2)(vii)
$=c_{j-1} \partial_{k}^{\alpha} a \quad b y(4.1 .1)(i i i) . T h u s \Gamma_{j}^{\prime} a *_{j+1} \Gamma_{j} a=c_{j} a$. Similarly one can prove that the second part of (5.1.4)(iii). (5.1.5)(i): Let a $\in A_{n}$ and $r \in R$. Then for $k<i<j$ we get
$\partial_{k}^{\alpha}\left[\Gamma_{i}\left(r \cdot j_{j} a\right)\right]=\Gamma_{i-1} \partial_{k}^{\alpha}\left(r \cdot{ }_{j} a\right) \quad$ by (5.1.2)(vii)
$=\Gamma_{i-1}\left(r ._{j-1} \partial_{k}^{\alpha} a\right) \quad$ by $(4.1 .5)(i)$
$=r \cdot r_{i-1} \partial_{k}^{\alpha} a \quad b y(5.1 .5)(i)$ and (since $\partial_{k}^{\alpha} a \in A_{n-1}$ )
$=r \cdot j \partial_{k}^{\alpha} \Gamma_{i}^{a}$ by (5.1.2)(vii)
$=\partial_{k}^{\alpha}\left(r \cdot{ }_{j+1} \Gamma_{i}^{a}\right)$ by (4.1.5)(i). Thus
$\Gamma_{i}\left(r \cdot{ }_{j}^{a}\right)=r \cdot{ }_{j+1} \Gamma_{i}^{a} \quad$.
We can prove similarly that $\Gamma_{i}\left(r \cdot{ }_{j} a\right)=r \cdot{ }_{j} r_{i}$, for $i \Delta j$. The proof of (5.1.5)(ii) is similar to that of (5.1.5)(i).

Thus $A^{\prime}=\left(\square A_{n}, A_{n}, \ldots, A_{0}\right)$ is an ( $n+1$ )-truncated $\omega$-algebroid .

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