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DOCTOR OF PHILOSOPHY

Topological groupoids, measures and representations

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# TOPOLOGICAL GROUPOIDS, MEASURES AND REPRESENTATIONS.

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#### ERRATUM

The argument of line 15 to line 18, page 99, is incorrect and should be replaced by the following:- " next observe that  $\mu'_o(E) = \int_X \chi \omega_x(E \cap G(x, x_o)) d\mu'_1 = \int_X \omega_x(E \cap G(x, x_o)) d\mu'_1$ where  $\mu'_1 = \mu_1|_Y$ . Finally, we note that  $\int \omega_x d\mu'_1$ is equivalent to  $\int \omega_x d\mu''$  if, and only if,  $\mu'_1$ is equivalent to  $\mu''_1$ ." The conclusion of Theorem 4.2.11 now follows.



## TOPOLOGICAL GROUPOIDS, MEASURES AND REPRESENTATIONS.

by

Anthony Karel Seda.

A thesis submitted to the University of Wales in support of the application for the degree of Doctor of Philosophy.

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May 1974.



## TO MY PARENTS.

#### DECLARATION.

Except where otherwise indicated, this thesis contains the results of the candidate's own investigations and all sources of information have been acknowledged. None of these results are being concurrently submitted for any other degree.

## The Director of Studies.

The Candidate.

#### ACKNOWLEDGMENTS.

The candidate wishes to express his sincere appreciation and gratitude to Professor R. Brown, his Director and Supervisor of Studies, for the constant help received during the course of this work, for encouragement and advice and for his constant readiness to discuss a problem.

Thanks are also due to the referee of my papers "Haar Measures for Groupoids" and "Convolution Algebras for Groupoids" for suggestions and comments which led to the following results of Chapter 4. Specifically, his comments led to Theorem 4.4.11, the technique used to obtain 4.4.14 and the example of Section 4.5.10. I would also like to thank Drs. R. Harte, F. Holland and T. Porter for their helpful discussions during the latter part of the preparation of the manuscript.

The candidate is also indebted to the Science Research Council for a maintenance grant during the course of the first two years (1970-1972) of this work.

Finally, I would like to thank Mrs. C. I. Price for her excellent job of typing.

#### SUMMARY

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This thesis is concerned with topological groupoids, that is, with categories in which each morphism is an isomorphism topologised in such a way that the algebraic operations are compatible with the topology. Three main areas are examined, they are: topological aspects, measure theoretic considerations and, thirdly, representations of groupoids. In the first of these, it is shown that the base space of the universal bundle of J. Milnor is a classifying space for certain topological groupoids. The second aspect concerns a notion of invariant measure for groupoids which generalises that of a group. It is shown that such measures always exist on a locally compact Hausdorff topological groupoid, and a classification is given with suitable restrictions. Convolution algebras are then constructed and applications to differential geometry and transformation groups are considered. Finally, a theory of unitary representations of locally compact Hausdorff topological groupoids is presented. Amongst the results obtained, is a version for groupoids of the classical Peter-Weyl theory for groups.

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- Operations on Representations.

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#### INTRODUCTION

In 1926, nearly twenty years before categories appeared, H. Brandt in " Uber ein Verallgemeinerung des Gruppenbegriffes " (Math. Ann. 96, (1926), 360-366) gave a formal definition of the term groupoid as used in this thesis. This work attracted little attention however, and groupoids lay dormant in the literature for a comparatively long time. Their revival and much of the present day interest centered on them is due chiefly to Ch. Ehresmann in his fundamental work on local structures and differential geometry, see for example " Gattungen von lokalen Strukturen ", Jahresbericht Deutsch. Math. Ver., Bd. 60 (1957) 49-77. And for a summary, see " Categories et structures ", Dunod, Paris, 1965.

More recently, G.W. Mackey [1], [2] has opened up a new line of attack on many problems in ergodic theory. It is well known that a transitive group action determines and is determined up to equivalence by, a conjugacy class of subgroups, namely, the isotropy subgroups at the points of the G-space. For a non transitive action there are no such subgroups, but Mackey has been able to associate with an ergodic action an object which, in the general case, does what the closed subgroup does in the special case. This object is termed a virtual subgroup. These considerations led him to the concepts of ergodic groupoid and virtual group. This work is guided mainly by the analogy between group actions and representations. On the algebraic and topological sides, work of P.J.Higgins and R. Brown has shown that certain well known constructions in group theory and homotopy theory can be nicely formulated in terms of groupoids and that there are advantages in doing this. In particular, we cite the proof of the theorems of Nielsen-Schreier and Kurosh given by Higgins in Higgins [1], and the version of the van Kampen theorem given by R. Brown, see Brown [1].

One aspect of groupoids which has so far received little attention is the following: the existence of an invariant measure m on any locally compact Hausdorff topological group G is a fact of fundamental importance in several important areas of modern mathematics. For example, in the study of the function spaces L'(G) and  $C_c(G)$ , the existence of m gives natural algebra structures to these spaces and, in fact, L'(G) becomes a Banachalgebra. If G is compact, then, using m, it is possible to average metrics on metrizable G-spaces to obtain invariant metrics. Similarly, when dealing with representations of G, one can construct unitary representations of G from linear ones. These facts were first put to use by Peter and Weyl in their theory of compact Lie groups. It is this area of mathematics with which this thesis is concerned and the original problem, suggested as a research topic by my supervisor, is that of generalising the above facts from groups G to groupoids G. Thus, Chapters 4 and 5 constitute the heart of the thesis, Chapters 1, 2 and 3 are prerequisite and grew out of the need for a preparatory study of topological groupoids.

The breakdown of the chapters is as follows.

Chapter 1 is introductory, it establishes notation, terminology and the basic (well known) facts we need throughout. It also

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studies the relationship between groupoids and G-spaces and shows, essentially, that the study of effective G-spaces is equivalent to that of groupoids. Specifically, we show that the category of principal G-bundles E over B with base point \* and an equivariant embedding  $i: G \rightarrow E$  is equivalent to the category of transitive groupoids  $\mathcal{G}$ with object set B together with a specific isomorphism  $G \rightarrow \mathcal{G}\{*\}$ . We note also that Theorem 1.4.11 is an improvement on the Cayley theorem of Ramsey [1], and it is interesting to contrast this theorem, regarded as a representation theorem, with Theorem 4.17 of Chapter 5.

Chapter 2 contains the point set topology that is needed for the study of Haar systems of measures in Chapter 4 and the study of representations in Chapter 5. Several interesting examples of topological groupoids are discussed and conditions are given which ensure that the groupoid,  $\tilde{G}$ , associated with a G-space, is locally trivial.

Chapter 3 studies the locally trivial groupoids of Ehresmann, and particular attention is focused on the groupoid, G(S), of admissible maps associated with a principal bundle S. This chapter is to some extent a continuation of Chapter 1 and we prove topological versions of several results of Chapter 1. There is one situation, however, which evades this study, namely the situation of G(S) for a non locally trivial principal bundle S. The reasons for this are discussed and it is for these reasons that we prefer to treat the algebraic case of Chapter 1 separately from the topological case in Chapter 3. Throughout this chapter, and elsewhere, the G-space approach to fibre bundles, due to H.Cartan as expounded by Husemoller [1], has been adopted. There are several advantages in this. Firstly, this

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approach is more general than that using charts; secondly, the introduction of charts is made only where it is necessary and, moreover, the charts take care of themselves in this theory, see Theorem 3.4.4; thirdly, this approach leads to simpler proofs. We comment here that Chapter 3 is in no way meant to be a new approach to the theory of fibre bundles. On the contrary, we adopt the view-point that G-spaces and fibre bundles are well known, but groupoids are not. Thus, we attempt to relate groupoids to bundles and not vice versa.

In Chapter 4, we study what seems to be a suitable generalisation of the concept of invariant measure for a group, see Definition 4.4.1. This definition is arrived at via the following three considerations:

1) The natural generalisation for groupoids G of left multiplication in a group is provided by the function  $L_s$  defined by  $L_s(\alpha) = s\alpha$ , where s is some fixed element of G. Since  $s\alpha$  is defined if, and only if,  $\alpha \in \operatorname{cost}_G T(s)$  and then  $L_s : \operatorname{cost}_G T(s) \longrightarrow \operatorname{cost}_G T(s)$ , to build in to the definition the maximum amount of left invariance, one should consider measures  $\mu_x$  defined on  $\operatorname{cost}_G x$ , for each object x, which are preserved by the functions  $L_s$ .

ii) If G acts on a fibre space P:S  $\longrightarrow$  X with metric space fibres  $(S_x, d_x)$ , in order to obtain invariant metrics on the fibres of S one would like to consider an expression of the type

 $d_x^*(s_1, s_2) = \int d_{\pi(\alpha)} (\alpha^{-1} \cdot s_1, \alpha^{-1} \cdot s_2) d\mu$ Again this makes sense if, and only if,  $\alpha \in \text{cost}_G x$  and so  $\mu$  must be a measure  $\mu_x$  defined on  $\text{cost}_G x$ . If  $d_x^*$  is to be isometric, the measures

 $\mu_x$  must be invariant with respect to the functions  $L_s$ , see 5.3.6. iii) If f, g  $\in C_c(G)$  one would like to define a convolution f\*g by

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a formula of the type  $f \star g(\alpha) = \int f(\beta)g(\bar{\beta}_{\alpha})d\mu(\beta)$ . This makes sense if, and only if,  $\beta \in \operatorname{cost}_{G} \Pi(\alpha)$  and, again, it follows that  $\mu$  ought to be defined on  $\operatorname{cost}_{G} \Pi(\alpha)$ . Moreover, to obtain associativity of  $\star$ , we need exactly the condition that the  $\mu_{\chi}$  be preserved by the functions  $L_{g}$ , for each  $s \in G$ , see Theorem 4.5.9. If we want to consider L'(G), then we need a measure m on G. In order to carry out technical constructions, m must be suitably related to the measures  $\mu_{\chi}$ . It turns out that conditions i) and ii) of Definition 4.4.1 are satisfactory in this respect.

It should be observed that there is a resemblance between our notion of invariant measures on a groupoid, and Mackey's notion of ergodic groupoid, see Mackey [1], [2]. However, it is the considerations above that motivate our definition and not those of ergodic theory. The introduction of measure classes, in our case, is merely an attempt to give a classification of invariant measures, and measure classes bring some order into what would otherwise be considerable chaos.

An attempt to construct suitable invariant measures for groupoids has also been made by J.J. Westman, see Westman [2]. He defines measures  $\mu_{xy}$  on G(x,y), for each pair of objects x and y, rather than on costars. His definition is, therefore, essentially different from ours. We compare Westman's system and ours in Chapter 4.

Probably the most important result of Chapter 4, as far as applications are concerned, is Theorem 4.4.18.

The final chapter, Chapter 5, considers a theory of representations of topological groupoids G in terms of actions of G on a fibre space  $P: S \longrightarrow X$ . The highlight of this chapter would appear to be Theorem 5.4.12, which shows how to extend a group action

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to one of a groupoid. We use this theorem in discussing operations on representations, notably the operation of direct sum. This theorem also makes it possible to prove embedding theorems for compact locally trivial Lie groupoids of a type considered by R.S.Palais, for compact Lie groups; in which we embed an action of such a groupoid in a Euclidean fibre bundle, rather than in Euclidean space. These theorems will be discussed elsewhere.

Following Halmos, we use the symbol **M** to indicate the end of proof of a theorem, lemma, proposition, remark etc. The working of our internal reference system is self evident and needs no further comment.

#### ABSTRACT GROUPOIDS

1.

#### Chapter 1.

### 80 Introduction

In this chapter we introduce our basic definitions and give some natural examples of groupoids. The major part of the chapter is concerned with a discussion of the general structure of groupoids in terms of admissible maps between fibres of a "bundle with structural sheaf". In the same spirit, covering morphisms of groupoids are related to morphisms of bundles with structural sheaf.

#### <u>81</u> Basic defnitions.

We start with the defnition of a groupoid, which is nothing more than a small category with inverses. We refer to Brown [1] for more details concerning our basic definitions and terminology throughout the first three sections of this chapter. The first three sections are entirely expository, but serve to establish our notation and collect together the elementary facts we need concerning groupoids.

#### 1.1.1. Definition.

A groupoid G consists of :-

a) A set ob(G), called the set of <u>objects</u> or <u>vertices</u> of G.
b) For each pair of objects x, y, a set G(x,y) of <u>elements</u> or <u>morphisms</u> of G, with <u>initial point</u> x and <u>final point</u> y.
c) A function (.): G(y,z) x G(x,y) -- G(x,z)

(β,a) → β.a

called the <u>composition in G</u> and defined for all triples (x,y,z) of objects.

These terms are subject to the axioms:-G1)  $G(x,y) \cap G(x',y') = \phi$  unless x = x' and y = y'. G2) If  $\alpha \in G(x,y)$ ,  $\beta \in G(y,z)$  and  $\gamma \in G(z,w)$ , then  $\gamma \cdot (\beta \cdot \alpha) = (\gamma \cdot \beta) \cdot \alpha$ . Thus, (.) is associative and we can write  $\gamma \cdot \beta \cdot \alpha$  without ambiguity. G3) For each object x of G there exists an element  $I_x \in G(x,x)$ with the properties:-

 $a \cdot I_x = a$  for all a with initial point x

and  $I_x \cdot a = a$  for all a with final point x.

The elements  $I_x$  are called the <u>units</u> or <u>identities</u> of G. G4) For each element  $a \in G(x,y)$  there exists an element  $a^{-1} \in G(y,x)$ satisfying  $a \cdot a^{-1} = I_y$  and  $a^{-1} \cdot a = I_x$ .

 $\alpha^{-1}$  is called an <u>inverse</u> of  $\alpha$ .

1.1.2. Remarks.

i) We shall often denote ob(G) by obG or X and speak of "the groupoid G over X".

ii) 1.1.1 a) and b) imply that a groupoid G over X is "small" in the sense that the class Mor  $G = \bigcup_{x,y \in X} G(x,y)$  of elements of G is, in fact, a set.

Note that we usually allow ourselves to confuse G with Mor(G) unless we specifically wish to refer to the elements of G.
iii) Again, 1.1.1 b) permits us to define two functions π and π' as follows:-

 $\pi$  : G  $\longrightarrow$  ob(G)

defined by  $\pi(a)$  = initial point of a,

and  $\pi'$ : G  $\longrightarrow$  ob(G)

defined by  $\pi(a)$  = final point of a.

We call  $\pi$  the <u>initial function</u> of G and  $\pi'$  the <u>final</u> <u>function</u> of G.

iv) If we introduce the set  $\mathfrak{D} \subset G \times G$ , where

 $\mathfrak{D} = \left\{ (\beta, \alpha) \in G \times G ; \pi'(\alpha) = \pi(\beta) \right\},$ 

then we can regard the composition as a function (.) :  $\mathfrak{D} \longrightarrow \mathfrak{G}$ .

(β,α) → β.α

Note that we usually write  $\beta a$  for the composite  $\beta \cdot a$  of  $\beta$  and a. v) It is straightforward to show that the element  $I_x$  is unique, for each  $x \in ob(G)$ , see Brown [1]. This fact permits us to define a function  $u : ob(G) \longrightarrow G$  called the <u>unit function</u> of G.  $x \longmapsto I_x$ 

vi) We shall always, in future, write  $G\{x\}$  for the set G(x,x) and we note that this set is, in fact, a group under the composition given in G, with identity element  $I_x$ . We call the group  $G\{x\}$  the <u>object</u> or <u>vertex group</u> at x.

For the sake of notation, we introduce the sets  $St_G x = \{ \alpha \in G ; \pi(\alpha) = x \}$  - called the "Star in G at x", and  $Cost_G x = \{ \alpha \in G ; \pi'(\alpha) = x \}$  - called the "Costar in G at x". We will also write I(G) for the set  $\{ I_x ; x \in ob(G) \}$  of identities in G.

- vii) It is straightforward to show that the inverse  $a^{-1}$  of an element a of G is unique, see Brown [1] again, and we can, therefore, define the <u>inverse function</u> inv : G  $\longrightarrow$  G of G. a  $\longmapsto$  a  $a^{-1}$
- viii) Let  $a \in G(x,y)$ , then since  $a^{-1}a = I_x$  and  $a \cdot (a^{-1}a) = a$  we sometimes refer to  $I_x$  as the <u>right identity</u> of a. Similarly,  $I_y$ is sometimes referred to as the <u>left identity</u> of a.
- ix) Instead of the notation  $a \in G(x,y)$  we shall make use of the a notations  $a : x \longrightarrow y$  or  $x \longrightarrow y$  when these appear more convenient.
- x) Using the identification u : ob(G) I(G), we can define a groupoid as a set equipped with a partial, associative multiplication which possesses identities and inverses, subject to axioms of type
   G2) .... G4). This latter definition is equivalent to 1.1.1 but will not be specifically used in this thesis; see Mackey [1], however.
- §2. Some examples and special types of groupoid.
- 1.2.1. The following example appears in Brown [1].

Let G be a group with identity element e. Then G can be regarded as a groupoid with object set  $\{e\}$  and composition just the binary operation of the group. Its morphisms, or elements, are just the elements g of G with initial and final point e, i.e. g:e --- e.

In fact, a groupoid G is a group if, and only if,  $\mathfrak{D} = G \times G$ if, and only if, ob(G) is a singleton set.

1.2.2. The following example is well known.

Let U be a family (set) of sets and for each pair  $u_1$ ,  $u_2 \in U$  let

$$G(u_{1},u_{2}) = \left\{ \text{bijections}: u_{1} \longrightarrow u_{2} \right\}$$
(So  $G(u_{1},u_{2})$  may be empty). Let  $G = \bigcup G(u_{1},u_{2})$   
 $u_{1},u_{2} \in \bigcup$ 

then G becomes a groupoid in a natural way by defining the composition (.) to be the usual composition of functions. Thus, if  $f: u_1 \rightarrow u_2$  $\varepsilon G(u_1, u_2)$  and  $g: u_2 \rightarrow u_3 \quad \varepsilon G(u_2, u_3)$ , we define  $g \cdot f: u_1 \rightarrow u_3$ by  $g \cdot f(x) = g(f(x))$ .

The identity  $I_u$  is simply the identity function u - uand the inverse of  $f: u_1 - u_2$  is the inverse function  $f^{-1}: u_2 - u_1$ . Verification of the axioms G1) .... G4) is routine. Note that G1) ensures that we distinguish between  $f: u_1 - u_2$  and  $f|_{u_1'}: u_1' - f(u_1')$ where  $u_1' \subset u_1$  and  $u_1' \in U$ .

Other examples of this type can be constructed by endowing the sets  $u \in U$  with certain structures, and requiring the functions  $f \in G$ to respect these structures. For example, if U is a set of topological spaces, we require the elements f of G to be homeomorphisms. Again, if U is a collection of rings, we require the elements f of G to be ring isomorphisms.

#### <u>1.2.3</u>. <u>Definition</u>.

A groupoid G is said to be <u>transitive</u> or <u>connected</u> (abstractly) if  $G(x,y) \neq \phi$  for each pair x,y of objects of G. Otherwise G is said to be (abstractly) <u>disconnected</u>. If  $G(x,y) = \phi$  for all x and y, with  $x \neq y$ , G is said to be (abstractly) <u>totally disconnected</u>. <u>1.2.4.</u> <u>Definition</u>.

A groupoid G is said to be <u>discrete</u> if it is totally disconnected and  $G\{x\} = \{I_x\}$  for all  $x \in ob(G)$ .

Clearly ob(G) can be identified with the discrete groupoid I(G), for any groupoid G.

1.2.5. The following example appears in Mackey [1].

Let X be any set and let  $G \subset X \times X$  be an equivalence relation on X. Thus,

i)  $(x,x) \in G$  for all  $x \in X$ .

ii) if 
$$(x_1, x_2) \in G$$
, then  $(x_2, x_1) \in G$ .

iii) if  $(x_1, x_2) \in G$  and  $(x_2, x_3) \in G$ , then  $(x_1, x_3) \in G$ .

We can turn G into a groupoid as follows. We take ob(G)to be X, and for each pair x,y of objects we define  $G(x,y) = \{(x,y)\}$ if  $(x,y) \in G_{i} = \phi$  otherwise.

Composition is defined by the rule

$$(y,z) \cdot (x,y) = (x,z)$$

The units of G are the pairs (x,x) for each  $x \in X$  and the inverse of (x,y) is the pair (y,x). Verification of the groupoid axioms is, again, routine.

Notice that in this particular groupoid G(x,y) is either empty or a singleton set, for all x and y. Such a groupoid is said to be <u>principal</u>. As observed by Mackey [2], there is a natural one to one correspondence between principal groupoids on the one hand, and sets with an equivalence relation on the other.

A rather important case occurs when the equivalence relation is trivial in the sense that  $G = X \times X$ , that is, any pair of elements of X are related. In this case G(x,y) is a singleton set for all x and y in X. A groupoid with this property is called a <u>tree</u> groupoid. The importance of tree groupoids will become apparent in the next section. <u>1.2.6.</u> Suppose G is any transitive groupoid over X and let  $x_o$  be any object of G. Since G is connected,  $G(x,y) \neq \phi$  for all  $x, y \in X$ , and, using the axiom of choice, we can construct a tree groupoid T over X, with  $T \subset G$ ; where  $\tau_{xy}$  denotes the unique element of T(x,y), for all x,y in X. In fact, T is a wide tree subgroupoid of G, see 1.3.1.

Now suppose a  $\varepsilon$  G(x,y), then it is easily seen that we can represent a in the form

$$u = \tau_{x,y} \alpha' \tau_{x,x}$$

for some unique a  $\varepsilon G\{x_{\beta}\}$ . And, with this notation, we have the relation  $(\beta \alpha)' = \beta' \alpha'$ .

Thus, G can be recovered from the tree T and one vertex group  $G[x_{\beta}]$ .

We refer to Brown [1], Chapter 6 for complete details of this.

Let G be a transitive groupoid over X. Then the sets G(x,y) are cardinal equivalent for all x,y in X. In fact, all the vertex groups are isomorphic.

Proof. See Brown [1], Chapter 6.

Proposition.

<u>1.2.7</u>.

1.2.8. The following has appeared in many places, see Mackey [1].

Let S be a set and G a group acting on the right of S, thus we have a map

•: S × G --- S

such that the relations:

i)  $(s \cdot g_1) \cdot g_2 = s \cdot g_1 g_2$ 

ii) s.e = s, e the identity of G,

hold for all  $s \in S$  and all  $g, g \in G$ , where g = denotes the product $of <math>g_1$  and  $g_2$  in G. We define a groupoid structure  $\tilde{G}$  on  $S \times G$  as follows:-

i) Take  $ob(\widetilde{G}) = S$ .

ii) For  $s_1, s_2 \in S$  define  $\widetilde{G}(s_1, s_2)$  by  $\widetilde{G}(s_1, s_2) = \{(s_1, g) \in \{s_1\} \times G; s_1 \cdot g = s_2\}$ .

iii) Define composition by :-

$$\widetilde{G}(s_{2},s_{3}) \times \widetilde{G}(s_{1},s_{2}) \longrightarrow \widetilde{G}(s_{1},s_{3})$$
$$((s_{2},g_{2}), (s_{1},g_{1})) \longmapsto (s_{1},g_{1}g_{2})$$

Thus,  $(s_2, g_2) \cdot (s_1, g_1)$  is defined if and only if,  $s_1, g_2 = s_2$  and in this case  $(s_2, g_2)(s_1, g_1) = (s_1, g_2)$ .

(Note that one can proceed similarly with G acting on the left of S.) It follows that  $\tilde{G} = \bigcup \tilde{G}(s_1, s_2)$  is a groupoid over S. The unit  $I_s = (s,e)$  and  $(s,g)^{-1} = (s \cdot g, g^{-1})$ . To verify G2), consider  $\{(s_1, g_2), (s_2, g_2)\} \cdot (s_1, g_1)$  where  $s_1 \cdot g_1 = s_2$ and  $s_2 \cdot g_2 = s_3$ . Then we have that  $\{(s_3, g_3) \cdot (s_2, g_2)\} (s_1, g_1) = (s_2, g_2 g_3) \cdot (s_1, g_1)$  $= (s_1, g_1(g_2 g_1))$ . On the other hand,  $(s_2,g_2) \cdot (s_1,g_1)$  is defined and is equal to  $(s_1,g_2)$ , and since  $s_1 \cdot g_1 g_2 = (s_1 \cdot g_1) \cdot g_2 = s_2 \cdot g_2 = s_3$ , we see that  $(s_2,g_2) \cdot \{(s_2,g_2)(s_1,g_1)\}$  is defined and equals

 $(s_3, g_3)$   $(s_1, g_2, g_2) = (s_1, (g_2, g_2)g_3).$ 

Thus, by the associativity of the group multiplication we have

$$(s_1, (g_1g_2)g_3) = (s_1, g_1(g_2g_3)),$$
  
 $\{(s_1, g_1), (s_1, g_2)\} = \{(s_1, g_1), (s_1, g_2), (s_1,$ 

hence  $(s_3, g_3) \{ (s_2, g_2) (s_1, g_1) \} = \{ (s_3, g_3) (s_2, g_2) \} (s_1, g_1)$ and so G2) is verified.

The other groupoid axioms are easily verified and so  $\tilde{G}$  is a groupoid over S. Note that  $\tilde{G}$  is connected in the abstract sense if, and only if, G acts transitively on S. Also the vertex group  $\tilde{G}{s} = G_s$  - the stability or isotropy subgroup of G. <u>1.2.9</u>. Suppose G and H are groups and we form the set Hom(G,H) = {f: G  $\rightarrow$  H; f is a homomorphism } We construct a groupoid GH over the set Hom(G,H) as follows:-

i) ob(GH) = Hom(G,H).

ii) For f,g 
$$\varepsilon$$
 ob(GH), we define GH(f,g) by  
GH(f,g) = {(a,f,g)  $\varepsilon$  H × {f} × {g}; for all  
 $\beta \varepsilon G, g(\beta) = \alpha f(\beta) \alpha^{-1}$ }.

iii). Define the composition by

 $GH(g,h) \times GH(f,g) \longrightarrow GH(f,h)$   $((\forall,g,h), (a,f,g)) \longmapsto (\forall a,f,h).$ Since  $(\forall a)f(\beta)(\forall a)^{-1} = \forall a f(\beta) a^{-1} x^{-1}$   $= \forall g(\beta) x^{-1}$   $= h(\beta),$ 

this composition is well defined and makes GH into a groupoid; where  $GH = \bigcup GH(f,g)$ . Hom(G,H)

In fact,  $I_f = (e,f,f)$ , where e = identity of H, and  $(a,f,g)^{-1} = (a^{-1},g,f)$ .

The verification of axioms G1) .... G4) is routine. This is an example of the "Functor-Categories" of Freyd and Mitchell and has

appeared elsewhere.

<u>1.2.10</u>. Another good example of a groupoid is provided by the fundamental groupoid,  $\pi X$ , of a topological space X. Since we have no need of this groupoid, we omit the details of its construction and refer the reader to Brown [1].

83. Further definitions and properties of groupoids.

1.3.1. Definition.

· Let G be a groupoid over X and let H be a groupoid over Y. Then H is called a <u>subgroupoid</u> of G if

- a)  $Y \subseteq X$ .
- b) for all  $x, y \in Y$  we have  $H(x, y) \subseteq G(x, y)$ , so that  $H \subseteq G$ .
- c) composition of elements in H coincides with the restriction to H of the composition in G.
- d) For each  $y \in Y$  the identity in  $H\{y\}$  coincides with the identity of  $G\{y\}$ .

H is called a <u>full</u> subgroupoid if H(x,y) = G(x,y) for all  $x,y \in Y$ , and H is called <u>wide</u> if Y = X.

For example, the tree groupoid constructed in 1.2.6 is a wide subgroupoid of G, and if G is any groupoid over X and  $x \in X$ , then  $G\{x\}$  is a full subgroupoid of G.

Having defined subgroupoids, we could go on and define "normal" subgroupoids and ultimately, the notion of "quotient" groupoid. However, we do not make use of either of these concepts but details can be found in Brown [1] and Higgins [1].

1.3.2. Products of groupoids.

Let  $\{G_a\}_I$  be a family of groupoids indexed by the set I, and let  $X_a = ob(G_a)$ . We form the product G of the  $\{G_a\}_I$  as follows:-We take

 $X = \prod_{a \in I} X_a \text{ as our object set}$ 

and set  $G = \prod_{\alpha} G_{\alpha}$ , as a set.  $\alpha \in I$ 

We define a composition in G as follows. Let  $(x_{q_{i}}), (y_{q_{i}})$ 

and  $(z_{\alpha}) \in X$  then, by definition of G, we have  $G((x_{\alpha}), (y_{\alpha})) = \{(g_{\alpha}) \in \prod_{\alpha \in I} G_{\alpha}; g_{\alpha} \in G_{\alpha}(x_{\alpha}, y_{\alpha})\}$ 

So we define composition by:

$$G((y_{\alpha}),(z_{\alpha})) \times G((x_{\alpha}),(y_{\alpha})) \longrightarrow G((x_{\alpha}),(z_{\alpha})),$$
$$((g'_{\alpha}),(g_{\alpha})) \longmapsto (g'_{\alpha}g_{\alpha}),$$

where  $g'_{a}g_{a}$  is the composite of  $g'_{a}$  and  $g_{a}$  in  $G_{a}$ .

With this law of composition, it is easy to verify that G is a groupoid over X with

$$I_{(x_{\alpha})} = (I_{x_{\alpha}})$$
 and  $(g_{\alpha})^{-1} = (g_{\alpha}^{-1}).$ 

It is also clear that G is connected abstractly if, and only if,  $G_a$  is connected abstractly for each  $a \in I$ . Again we refer to Brown [1] for details.

#### 1.3.3. Components in a groupoid

Let G be any groupoid over X and let  $x \in X$ . Let  $C_{x_o}$  be the full subgroupoid of G on all objects  $y \in X$  such that  $G(x_o, y) \neq \phi$ . Then, if x,  $y \in ob(C_{x_o})$ ,  $G(x,y) \neq \phi$ , since it contains the composite  $\beta \alpha$  for some  $\alpha \in G(x, x_o)$  and some  $\beta \in G(x_o, y)$ . Thus,  $C_{x_o}$  is transitive and is clearly the maximal transitive subgroupoid of G with x as one of its objects.  $C_{x_o}$  is, therefore, called the <u>component</u> of G containing (or determined by)  $x_o$ .

<u>1.3.4</u>. The relation  $x \sim y$  if and only if,  $G(x,y) \neq \phi$  is an equivalence relation on X, and its equivalence classes are precisely the object sets of the components of G.

#### 1.3.5. Morphisms of Groupoids

Let G and H be two groupoids. A morphism (or homemorphism)  $f: G \longrightarrow H$  assigns to each object x of G an object f(x) of H, and to each element  $a \in G(x,y)$  an element  $f(a) \in H(f(x),f(y))$  such that:-M1). If  $I_x \in G\{x\}$  is the identity at x in G, then  $f(I_x) = I_{f(x)}$  - the identity of H at f(x). M2). If  $a: x \longrightarrow y$  and  $\beta: y \longrightarrow z$  are elements in G, then  $f(\beta a) = f(\beta) \cdot f(a)$ . Thus, a morphism consists of a pair of maps (f, obf), where obf denotes the map induced by f on object sets, thus obf :  $ob(G) \longrightarrow ob(H)$ Note that it is a consequence of M1) and M2) that if

(f, obf): G  $\longrightarrow$  H is a morphism, then  $f(a^{-1}) = f(a)^{-1}$ , for :-

$$f(aa^{-1}) = f(I_{\pi'}(a)) = I_{f(\pi'}(a)),$$

 $f(a).f(a^{-1}) = I_{f(\pi'(a))}$ .

thus

Similarly

 $f(a^{-1}) \cdot f(a) = I_{f(\pi(a))} \cdot f(a^{-1}) = f(a)^{-1}$ .

Whence

A morphism of groupoids is clearly a functor of the underlying categories. Since we have an identity morphism  $I_G : G \longrightarrow G$ , for any groupoid G, we can define the notion of isomorphism of groupoids: <u>1.3.6</u>. <u>Definition</u>.

Groupoids G and H are said to be <u>isomorphic</u> if there is a morphism  $f: G \longrightarrow H$  and a morphism  $g: H \longrightarrow G$  such that  $gf = I_G$ and  $fg = I_H$ . We denote g by  $f^{-1}$  usually, and we call such an f an <u>isomorphism</u>.

Note that if  $f_1: G_1 \longrightarrow G_2$  and  $f_2: G_2 \longrightarrow G_3$  are morphisms, then  $f_2f_1: G_1 \longrightarrow G_3$  is a morphism. However, unlike the case of groups, the image Im(f) of a morphism of groupoids  $f: G \longrightarrow H$ is not necessarily a subgroupoid of H, see Brown[2]. Nevertheless, Ker  $f = \{ \alpha \in G ; f(\alpha) \in I(H) \}$  - the <u>Kernel</u> of  $f_1$  - is a subgroupoid of G.

Let  $f: G \longrightarrow H$  be a morphism of Groupoids and recall that  $St_{G}x = \pi^{-1}(x)$  (see 1.1.2 vi). Let  $x \in ob(G)$  and suppose  $a \in St_{G}x$ , then  $\pi(f(a)) = f(x)$  and so  $f(a) \in St_{H}f(x)$ . Thus, f induces a map  $St_{G}f: St_{G}x \longrightarrow St_{H}f(x)$ , for each object x of G.

<u>1.3.7</u>. <u>Definition</u>. (see Brown [1] and Higgins [1]). We say a morphism f : G — H of groupoids is

a) star injective

b) star surjective or a fibration

c) star bijective or a covering morphism

if, for each  $x \in ob(G)$ ,  $st_G^f$  is injective, surjective or bijective respectively.

## S4. The structure of groupoids.

In this section we formulate the notions of principal bundle S with structural sheaf, admissible map between its fibres and the associated groupoid  $\mathcal{G}(S)$  of admissible maps. The notion of bundle with structural sheaf is not essentially new and there are similar definitions to be found in many places in the literature, see Hirzebruch [1] for example; they are, however, topological versions. Also, there are definitions of "admissible map" between the fibres of a principal coordinate bundle, see Steenrod [1]; but these definitions involve coordinate functions in their description, and will not, therefore, generalise, especially to the non locally trivial topological case.

Our definitions are general and using them we obtain the new results Theorems 1.4.11 and 1.5.10. We shall, however, relate our definitions to the existing ones in Chapter 3, where we also give a general definition for fibre bundles. Unfortunately, we are unable to prove analogues of 1.4.11 and 1.5.10 for non-locally trivial topological groupoids due to the lack, at present, of a suitable topology for G(S). For this reason, we consider the algebraic case and the locally trivial topological case separately.

There are several consequences of this study. Firstly, it reveals the essentially G-space theoretic nature of groupoids; secondly we can give topological versions of our results in Chapter 3 (for locally trivial topological groupoids) and this leads to a homotopy classification. As another consequence, we recover the topology of Danesh-Naruei [1] for the fundamental groupoid  $\pi X$ , of a space X. 1.4.1. Suppose H is a group and S a right H-set.

Thus, we have an action  $S \times H \xrightarrow{\bullet} S$ 

 $(s,h) \mapsto s.h$ 

of H on the right of S. Let  $s \cdot H = \{s \cdot h; h \in H\}$  - the <u>orbit</u> of s

under H, let X = S/H be the set of orbits and let  $r : S \longrightarrow X$ 

be the canonical surjection of S onto X. We shall denote by  $\alpha(S)$ the triple (S, ., r) consisting of the set S together with the action '.' of H on S and the natural surjection r. We make the following definition, which, of course, is not new:-

#### <u>1.4.2</u>. Definition.

An <u>H-morphism</u>  $g: S \longrightarrow S'$  of right H sets S and S', is a map  $g: S \longrightarrow S'$  such that g(s.h) = g(s).h for all  $s \in S$  and  $h \in H$ .

#### <u>1.4.3</u>. Definition.

A map P : S --- B of S onto B will be called an <u>H-bundle</u> if the following conditions are satisfied:

i) There is an action . of the group H on the right of the set S. ii)  $\alpha(S)$  and (S, ., P) are isomorphic in the sense that there is a bijection  $f: S/_{H} \longrightarrow B$  such that the diagram:-

> $S \xrightarrow{I} S$  $F \downarrow \qquad \downarrow P$  $S/H \xrightarrow{B} B$

is commutative, where I is the identity map.  $P: S \longrightarrow B$  will be called a <u>principal H-bundle</u> if the action of H on S is <u>effective</u> in the sense that if there exists s, h and h' with s.h = s.h', then h = h'; H is called the <u>structure group</u> of the bundle. As usual, we call S the <u>total space</u>, B the <u>base space</u> and P the <u>projection</u>.

We remark that our notation a(S) and our terminology is borrowed from Husemoller [1]. Of course we are not assuming that S is a topological space at present, nor are we assuming continuity of any of the functions defined above. Following Husemoller, we shall, in future, denote an H-bundle P: S — B by (S, P, B). <u>1.4.4.</u> Definition.

By a <u>sheaf of groups</u> over a set Y, we mean a map  $\sigma : \Sigma \longrightarrow Y$ of  $\Sigma$  onto Y such that  $\Sigma_y = \overline{\sigma'(y)}$  has a group structure, for each  $y \in Y$ . We shall denote this by  $(\Sigma, \sigma, Y)$ .

We now formulate our notion of "bundle with structural sheaf" in :-

1.4.5. Definition.

A map P : S  $\rightarrow$  B will be called a (<u>principal</u>) <u>bundle with</u> <u>structural sheaf</u> if :-

i) There is given a partition  $\{B_y\}_{y \in Y}$  of B and a sheaf of groups  $\sigma : \mathbb{Z} \longrightarrow Y$ .

ii)  $P: P(B_y) \longrightarrow B_y$  is a (principal)  $\geq_y$  bundle for each  $y \in Y$ . Thus, S is the disjoint union of the sets  $P'(B_y)$ .

For notational convenience, we denote  $\overline{P'}(B_y)$  by  $S_y$  and often refer simply to the bundle S, or , more precisely, (S, P, B) with sheaf  $\Sigma \cdot \overline{\Sigma}$  is called the <u>structure sheaf</u> of S and we call the bundles P :  $S_y \longrightarrow B_y$  the <u>component (principal) bundles</u>. The set  $\overline{P'}(b)$ ,  $b \in B$ , is called the <u>fibre</u> of S over b.

Notice that the notion of "sheaf of groups" is exactly that of "totally disconnected groupoid", see 1.2.3., but is used here for reasons of terminology.

The concepts of "sheaf of groups" and "bundle with structural sheaf" are, of course , not new and similar definitions can be found in many places in the literature. However, the definitions which occur elsewhere are usually "topologised" and , at present , we are not concerned with topological aspects.

Observe that if (S, P, B) is a bundle with sheaf  $\geq$  and beB, then beB for some unique yeY and there is an induced action

 $P'(b) \times Z_y \longrightarrow P'(b)$ 

of  $\Sigma_y$  on P<sup>-1</sup>(b). With this in mind we formulate:-<u>1.4.6</u>. <u>Definition</u>.

Let (S, P, B) be a principal bundle with structural sheaf  $(\Sigma, \sigma, Y)$ . Then a map h between fibres of  $S_y$ ,  $y \in Y$ , is called <u>admissible</u> if h is a  $\mathcal{Z}_y$  morphism in the sense of 1.4.2. <u>1.4.7</u>. <u>Theorem</u>.

Let (S, P, B) be a principal bundle with structural sheaf  $(z, \sigma, Y)$ . Then:-

a) Any admissible map is a bijection.

b) The inverse of an admissible map is admissible.

c) The composite of two admissible maps is admissible.

- d) The identity  $I: P^{-1}(b) \longrightarrow P^{-1}(b)$  is admissible.
- e) Given b<sub>1</sub>, b<sub>2</sub>  $\in$  B<sub>y</sub> for some y, and s<sub>1</sub>  $\in$  P<sup>-1</sup>(b<sub>1</sub>) and

 $s_2 \in P^{-1}(b_2)$ , there is a unique admissible map  $\gamma : P^{-1}(b_1) \longrightarrow P^{-1}(b_2)$ such that  $\gamma(s_1) = s_2$ .

### Proof.

a) Suppose b,  $b_2 \in B_y$  and that

$$\eta : P^{-1}(b_1) \longrightarrow P^{-1}(b_2)$$

is admissible.

Suppose  $s_1 \neq s_2$  in  $P^{-1}(b_1)$ , then, by definition of orbit, there exists  $\omega \in \mathbb{Z}_y$  such that  $s_1 \cdot \omega = s_2 \cdot So \ \gamma(s_2) = \gamma(s_1 \cdot \omega) = \gamma(s_1) \cdot \omega$ Now, since  $s_1 \neq s_2$ ,  $\omega \neq identity$  in  $\mathbb{Z}_y$ ,

hence  $\eta(s_1) \cdot \omega \neq \eta(s_1)$  by effectiveness of the  $\Sigma_y$  action, thus  $\eta(s_1) \neq \eta(s_2)$  and so  $\eta$  is injective.

Now let  $q \in P^{-1}(b_2)$  and let  $s \in P^{-1}(b_1)$ . Set  $q' = h(s) \in P^{-1}(b_2)$ and, again using the definition of orbit, let  $\omega \in \Sigma_y$  be such that

Then  $\gamma(s \cdot \omega) = \gamma(s) \cdot \omega = q' \cdot \omega = q$  and so  $\gamma$  is surjective and, hence, bijective.

The proofs of b), c) and d) are straightforward and will be omitted.

e) To prove e), suppose  $b_1, b_2 \in B_y$  and  $s_1 \in P^{-1}(b_1)$ ,  $s_2 \in P^{-1}(b_2)$ . Now, given any  $s \in P^{-1}(b_1)$ , there exists  $\omega \in \sum_y$  such that  $s_1 \cdot \omega = s$ and  $\omega$  is unique with this property. Define  $\gamma : P^{-1}(b_1) \longrightarrow P^{-1}(b_2)$ as follows

i)  $\lambda(s_1) = s_2$ ii) if  $s = s_1 \cdot \omega$ , define  $\lambda(s)$  by  $\lambda(s) = s_2 \cdot \omega$ . We show  $\lambda$  is admissible.

Now, 
$$\mu(s.\omega') = \mu((s, .\omega).\omega')$$
  
=  $\mu(s, .\omega\omega')$   
=  $s_2. \omega\omega' = (s_2.\omega).\omega'$   
=  $\mu(s). \omega'$ 

So  $\chi$  is admissible, and it is clear that  $\chi$  is unique with respect to the property  $\chi(s_1) = s_2 \cdot \blacksquare$ <u>Note</u> that e) shows that any admissible map  $\eta : P^{-1}(b_1) \longrightarrow P^{-1}(b_2)$ is uniquely determined by a pair  $(s_1, s_2) \in P^{-1}(b_1) \times P^{-1}(b_2)$  such that  $\chi(s_1) = s_2$ .

Theorem 1.4.7. allows us to consider the groupoid G(S) of admissible maps between fibres of S, see 1.2.2, where S is a principal bundle with structural sheaf. If  $G(S)(B_y)$  denotes the full subgroupoid of G(S) over  $B_y$ , then  $G(S)(B_y)$  is connected abstractly and its vertex groups are isomorphic with  $\sum_{y}$  (by effectiveness). 1.4.8.

Now let G be any groupoid over X, and let  $Y \subset X$  be a set of objects such that each connected component  $C_y$  of G is determined by a unique element y of Y, (i.e. Y is a "section" of the set of equivalence classes of 1.3.4). For each  $y \in Y$  let  $B_y = \{x \in X;$  $G(x,y) \neq \phi\}$ , so that  $C_y = G(B_y)$  - the full subgroupoid of G over  $B_y$ . Set  $\Sigma_y = C\{y\}$  for each  $y \in Y$  and define  $\Sigma = \bigcup_{y \in Y} \sum_y$ , then there is a natural surjection  $\sigma: \Sigma \longrightarrow Y$ , where  $\sigma^{-1}(y) = \sum_y$  for each  $y \in Y$ , and  $(\Sigma, \sigma, Y)$  is a sheaf of groups over Y. Define S by  $S = \bigcup_{y \in Y} St_G y$ 

and let  $\pi': S \longrightarrow X$  be the final map,

so  $\pi'(St_{G}y) = B_{y}$  and :-

1.4.9. Proposition.

 $(St_{g}y, \pi', B_{y})$  is a principal  $\sum_{y}$  - bundle for each  $y \in Y$ . <u>Proof</u>. There is a natural action of  $\sum_{y} = G\{y\}$  on the right of  $St_{G}y$  define by :-

$$\operatorname{St}_{G}^{y} \times \mathbb{Z}_{y} \longrightarrow \operatorname{St}_{G}^{y}$$
,  
( $\beta$ ,a)  $\longmapsto \beta$ a

where  $\beta a$  denotes the composite of  $\beta$  and a in G. This is in fact an action, by the associativity of the composition in G, and is, moreover, effective since  $\beta a = \beta a' \implies a = a'$ .

If  $\beta : y \longrightarrow x$  in  $\operatorname{St}_{G}y$ , then  $\beta \cdot \sum_{y} = G(y,x)$  (see 1.2.7) and so the orbit set  $\operatorname{St}_{G}y / \sum_{y} = \{\{G(y,x)\}\}; x \in B_{y}\}$ . Define

$$f: \operatorname{St}_{G} y / \mathbb{Z}_{y} \longrightarrow \mathbb{B}_{y}$$
$$\{G(y, x)\} \longmapsto x$$

then the diagram



commutes since  $fr(\beta) = f(\beta \cdot G\{y\}) = \pi'(\beta)$ . Thus,  $St_Gy$  is a principal  $\sum_{y}$  bundle by Definition 1.4.3.

It is now clear that  $(S. \pi', X)$  is a principal bundle with structural sheaf  $(\Sigma, \sigma, Y)$  as in 1.4.5. Now form the groupoid G(S) of admissible maps with  $S = (S, \pi', X)$  as in 1.4.8, we have :-<u>1.4.10</u>. Lemma.

Let  $b_1$ ,  $b_2 \in B_y$  and let  $\gamma : \pi^{-1}(b_1) \longrightarrow \pi^{-1}(b_2)$  be admissible, then there exists unique  $\lambda \in G(b_1, b_2)$  such that  $\gamma(\alpha) = \lambda \alpha$ for all  $\alpha \in \pi^{\prime -1}(b_1)$ . Proof.

First note that if  $\gamma(\alpha) = \lambda \alpha$  for all  $\alpha \in \pi'^{-1}(b_1)$  and some  $\lambda \in G(b_1, b_2)$ , then  $\gamma$  is admissible. For if  $\alpha' \in \Sigma_y$ , then  $\gamma(\alpha \cdot \alpha') = \lambda(\alpha \cdot \alpha') = \lambda(\alpha) \alpha' = \gamma(\alpha) \cdot \alpha'$ .

Hence  $\eta$  is admissible.

Now suppose  $\lambda : \pi'^{-1}(b_1) \longrightarrow \pi'^{-1}(b_2)$  is admissible, let  $\beta_1 \in \pi'^{-1}(b_1)$  set  $\lambda = \lambda(\beta_1)$  and define  $\lambda$  by  $\lambda = \chi \beta_1^{-1}$ . Then  $\lambda(\alpha) = \lambda \alpha$  for all  $\alpha \in \pi'^{-1}(b_1)$ , using 1.2.6, and  $\lambda$  is unique by the effectiveness of the group actions.

These results lead finally to :-

#### <u>1.4.11</u>. Theorem.

Any groupoid G is isomorphic to a groupoid of admissible maps between fibres of a principal bundle with structural sheaf.

## Proof.

Let G be any groupoid over X and let  $(S, \pi', X)$  be the principal bundle with structural sheaf  $(\Sigma, \sigma, Y)$  as constructed in 1.4.8. Form the groupoid G(S) and define

> $\Gamma : G \longrightarrow \mathcal{G}(S)$ by  $\Gamma(\lambda) = \gamma_{\lambda}$  on elements

and ob  $\Gamma$  = identity on X,

the where  $\chi_{\lambda}$  is/admissible map determined by  $\lambda$  in the sense of Lemma 1.4.10. By 1.4.10, we have immediately that  $\Gamma$  is bijective and, moreover, the diagrams

both commute.

Now  $\Gamma(\lambda_{2}\lambda_{1}) = \lambda_{\lambda_{2}\lambda_{1}}$ , where  $\lambda_{\lambda_{2}\lambda_{1}}(\alpha) = \lambda_{2}\lambda_{1}\alpha$   $= \lambda_{2}(\lambda_{1}(\alpha)) = \lambda_{\lambda_{2}}(\lambda_{\lambda_{1}}(\alpha))$ , consequently  $\Gamma(\lambda_{2}\lambda_{1}) = \Gamma(\lambda_{2})\Gamma(\lambda_{1})$ . Also  $\Gamma(I_{x}) = \lambda_{I_{x}}$  and  $\lambda_{I_{x}}(\alpha) = I_{x}\alpha = \alpha$ , so that  $\lambda_{I_{x}}$  is the identity admissible map. The result now follows for it is an easily proved general fact that the inverse of a bijective homomorphism of groupoids is itself a homomorphism of groupoids and, hence, an isomorphism.

The result we have just established can be regarded as the first part of a general representation theorem for abstract groupoids. The second, and last, step will be taken in the next section, where we discuss the "uniqueness" of the principal bundle determining a given groupoid in the sense of 1.4.11.

### 85. Morphisms of Principal Bundles.

This section will be devoted to the notions of "morphism" and "isomorphism" between principal bundles with structural sheaf. We will show that if (S, P, B) and (S', P', B') both determine a given groupoid G, then (S, P, B) and (S', P', B') are isomorphic. This result is important in the classification of locally trivial groupoids, which is carried out in Chapter 3.

## <u>1.5.1</u>. <u>Definition</u>.

Suppose  $(\Sigma, \sigma, Y)$  and  $(\Sigma, \sigma', Y')$  are two sheaves of groups. The <u>homomorphism</u>  $(\Sigma, \sigma, Y) \longrightarrow (\Sigma, \sigma', Y')$  is a pair of maps  $K: \Sigma \longrightarrow \Sigma'$  and  $\overline{K}: Y \longrightarrow Y'$  such that :

i)  $\Sigma \xrightarrow{K} \Sigma'$   $\sigma \downarrow \qquad \int \sigma'$   $\gamma \xrightarrow{\overline{K}} \gamma'$ i) The restriction  $\kappa_{\gamma} = \kappa \downarrow : \Sigma_{\gamma} \rightarrow \Sigma$ 

commutes.

ii) The restriction  $K_y = K | : \Sigma_y \to \Sigma'$  is a homomorphism of groups.

One defines <u>isomorphism</u> of sheaves in the obvious way, and in fact a homomorphism  $(\kappa, \bar{\kappa})$  is an isomorphism if and only if,  $\kappa$  and  $\bar{\kappa}$  are bijections.

The class of sheaves forms a category in a natural way with morphisms the homomorphisms of sheaves.

We need the following notion of homomorphism of bundles :-

#### <u>1.5.2</u>. Definition.

Let (S, P, B) and (S, P, B) be (principal) bundles with structural sheaves  $(\Sigma, \sigma, Y)$  and  $(\Sigma, \sigma', Y')$  respectively. Then a pair  $(f, \overline{f})$  of maps is a morphism or bundle map  $(S, P, B) \longrightarrow (S', P', B')$ with respect to a homomorphism  $(\kappa, \bar{\kappa}) : (\Sigma, \sigma, Y) \longrightarrow (\Sigma, \sigma', Y')$ of sheaves if :-

i)  $f: S \longrightarrow S'$  is a bijection,  $\overline{f}: B \longrightarrow B'$  is a surjection and the diagram :-S E S'

commutor

	PI = IP'	
ii)	$ \begin{array}{cccc}  & \mathcal{B} & \xrightarrow{\mathcal{B}} & \mathcal{B} \\ \overline{f}(B_{y}) & \subseteq & B'_{\overline{K}(y)} & \text{for all } y \varepsilon \end{array} $	Y, so that $f: S_y \longrightarrow S'_{\vec{x}}(y)$
iii)	$S_y \times Z_y \xrightarrow{\bullet} S_y$	commutes for each $y \in Y$ .
•	fxKy	
	$S'_{\bar{\kappa}(y)} \times Z_{\bar{\kappa}(y)} \longrightarrow S_{\bar{\kappa}(y)}$	

In the special case when the two sheaves involved both collapse to a single group, that is, in the case of bundles with structural group, 1.5.2 reduces to the appropriate notion of morphism for such bundles. Note that ii) now becomes redundent of course. Similarly, our notion of isomorphism 1.5.4 will include this special case.

#### 1.5.3. Remark.

Suppose we have a bundle map  $(f,\overline{f})$  as in the previous definition. Let  $y \in Y$  and  $x \in B_y$  and set  $f_x = f|_{p-1}(x) : p^{-1}(x)$ -  $P'^{-1}(f(x))$ . Then  $f_x$  is 1-1 since f is bijective. If (S, P, B) and (S', P', B') are both principal bundles, then the commutativity of iii) together with the effectiveness of the actions of  $\Sigma_{y}$  and  $\Sigma_{\vec{k}(y)}$  implies that  $K_{y}$  is injective, for each  $y \in Y$ .

The class of bundles with structure sheaf forms a category in a natural way, its morphisms being the bundle maps as defined in 1.5.2. 1.5.4. Definition.

By a bundle isomorphism  $(f, \bar{f})$  of (S, P, B) and (S', P', B').

we mean a bundle map  $(f,\overline{f})$  with respect to an isomorphism  $(\kappa,\overline{\kappa})$  of sheaves, such that  $\overline{f}$  is bijective and  $\overline{f}(B_y) = B'_{\overline{\kappa}(y)}$  for all  $y \in Y$ . Two bundles (S, P, B) and (S', P', B') will be called <u>isomorphic</u> if there is an isomorphism  $(f,\overline{f})$ :  $(S, P, B) \longrightarrow (S', P', B')$ . 1.5.5.

The following observations 1.5.6, 7 and 8 are key facts in proving the Theorem 1.5.10.

Suppose S is an effective right H-space and we form the H-bundle (S, r, X). Let  $x_o \in X$ , then  $r^{-1}(x_o) = H_o \subset S$  is the orbit of any one  $s_o \in H_o$ , that is,  $H_o = s_o \cdot H \cdot H_o$  inherits a (non-canonical) group structure from H as follows, define

 $(s_0 \cdot h_1) (s_0 \cdot h_2) = s_0 \cdot h_1 h_2$ . This law of composition turns  $H_0$  into a group with identity  $s_1$  and

$$e : H \longrightarrow H$$
$$h \longmapsto s_{\bullet} \cdot h$$

is an isomorphism of groups.

Define an action of H on S by

 $S \times H_{a} \longrightarrow S$ (s.s.h)  $\mapsto$  s.h.

It is easily seen that this action is effective, so we can form the principal H<sub>o</sub> bundle  $(S, r_o, X_o)$ , where  $X_o = S/H_o$  and  $r_o$  the canonical surjection associated with the H<sub>o</sub> action.

<u>1.5.6</u>. <u>Proposition</u> (S, r, X) and (S, r,  $X_o$ ) are isomorphic bundles. <u>Proof</u>.

Define  $f : S \longrightarrow S$  to be the identity

$$\begin{array}{ccc} \text{ind} \quad \overline{\mathbf{f}} : X \longrightarrow X_{o} \\ \text{by} \quad \text{s.H} \longmapsto \text{s.H}_{o} \end{array}$$

Then  $\overline{f}$  is a bijection and the diagram

$$\begin{array}{c} S \xrightarrow{f} S \\ r \downarrow & \downarrow r \\ X \xrightarrow{f} X_{o} \end{array}$$

commutes.
Finally, the diagram

$$S \times H \longrightarrow S$$
  
 $f \times e \downarrow \qquad f$   
 $S \times H_0 \longrightarrow S$ 

also commutes since :

$$f(s) \cdot e(h) = s \cdot (s \cdot h) = s \cdot h$$
  
=  $f(s \cdot h)$ 

Thus (S, r, X) and (S, r, X) are isomorphic.

1.5.5 and 1.5.6 show that any H-bundle is isomorphic to a bundle in which the structure group is embedded, as a fibre, in the total space. We shall call  $x_o$  the <u>base point</u> of S.

<u>1.5.7</u>. Now suppose, as in 1.5.5, that

is an effective action of the group  $H_o$  on S, and that  $H_o \subseteq S$ . This action induces an effective transitive action on any orbit of  $H_o$ , in particular there is the induced action

in which H acts effectively and transitively on the right of H .

Let  $rm : H_{o} \times H_{o} \longrightarrow H_{o}$  denote the action of  $H_{o}$  on the right of  $H_{o}$  obtained by right multiplication; this is an effective transitive action. It is well known that (.) and r.m. are equivalent, so there is a group isomorphism  $\Theta: H_{o} \longrightarrow H_{o}$  and a bijection  $\phi: H_{o} \longrightarrow H_{o}$  such that

$$\begin{array}{cccc} H_{\circ} \times H_{\circ} & \stackrel{\bullet}{\longrightarrow} & H_{\circ} \\ \phi \times \phi & & & \downarrow \phi \\ H_{\circ} \times H_{\circ} & \stackrel{\Gamma. m.}{\longrightarrow} & H_{\circ} \end{array}$$

is a commutative diagram.

Let X be the orbit set for the action (.) and let  $r: S \longrightarrow X$  be the usual map. We can choose a transversal  $\{s_x \in r^{-1}(x); x \in X\}$  of S, that is, a cross section of r, which is such that  $s_x$  is the identity of H, where  $H = r^{-1}(x)$ . Then we have

 $S = \bigcup_{\mathbf{x} \in \mathbf{Y}} S_{\mathbf{x}} \cdot \mathbf{H}_{\mathbf{x}}$ Define  $S' = \bigcup_{x \in X} s_x \times H$  - which is a disjoint union - where we identify  $s_x \times H_{o}$  and H . The action r.m. extends to an action, which we still denote r.m. where r.m. : S'x H \_\_\_ S'  $(s_x \times h, h_z) \mapsto s_x \times h, h_z$ . Also, we can extend  $\phi$  orbit-wise, that is, we define  $\phi_x : S_x \cdot H_o \longrightarrow S_x \times H_o$  $(s_r,h) \longrightarrow s_r \times \phi(h)$ to obtain a map, which we still denote  $\phi$  , where  $\phi = \bigcup_{x \in X} \phi_x : s \longrightarrow s'.$ Then  $\phi$  is bijective and S  $\xrightarrow{\phi}$  S' is, clearly, commutative  $\begin{array}{c} \downarrow \\ \downarrow \\ \\ \times \\ \xrightarrow{\mathbf{I}} \\ \\ \end{array} \\ \end{array}$  where I is the identity on X. Consider S×H ---- S Øx0 S'× H, r.m. S'  $(s_{x},h_{z},h_{z}) \in S \times H_{o}$ , then :let  $\phi(\bullet)(s_{y} \cdot h_{y}, h_{z}) = \phi(s_{y} \cdot h_{y} \cdot h_{z})$ =  $s_x \neq (h_h)$  $\mathbf{r} \cdot \mathbf{m} \cdot (\phi \times \Theta)(\mathbf{s}_{\mathbf{x}} \cdot \mathbf{h}_{\mathbf{y}}, \mathbf{h}_{\mathbf{z}}) = \mathbf{r} \cdot \mathbf{m} \cdot (\mathbf{s}_{\mathbf{x}} \times \phi(\mathbf{h}_{\mathbf{y}}), \Theta(\mathbf{h}_{\mathbf{z}}))$ also =  $s_x \times \phi(h_1) \otimes (h_2)$ 

Thus, the above diagram commutes and so we have an isomorphism  $(S, r, X) \longrightarrow (S', r', X)$  in the sense of 1.5.4. Thus, any H-bundle (S, P, B) is isomorphic to a bundle (S', P', B') in which the structure group is embedded, as a fibre, in S', and the induced right action of the structure group on itself is that of right multiplication. We shall call (S', P', B') a <u>regular</u> bundle and say that (S', P', B') is a

=  $s_x \times \phi(h_1 \cdot h_2)$ .

regular representative of (S, P, B) .

Let (S, P, B) be any principal bundle with structural sheaf  $(\Sigma, \sigma, Y)$ . By working over each component  $B_y$ , we can obtain a principal bundle (S', P', B'), with sheaf  $(\Sigma', \sigma', Y)$ , in which each of the component principal bundles is regular in the sense of 1.5.7; and such that (S, P, B) and (S', P', B') are isomorphic. (S', P', B') is called <u>regular</u> and we say it is a <u>regular representative</u> of (S, P, B). 1.5.9. Lemma. Suppose (S, P, B) and (S', P', B') are principal bundles with structural sheaves. Then an isomorphism  $(\Gamma, \overline{\Gamma})$ :  $g(S) \longrightarrow g(S')$  of groupoids induces a bundle isomorphism  $\overline{\Gamma}: \operatorname{St}_{g(S)}^{X} \longrightarrow \operatorname{St}_{g(S')}^{\overline{\Gamma}}(x)$ , for each  $x \in B$ .

Let  $x \in B$ . Since  $\Gamma$  is an isomorphism the restriction  $\Gamma |_{St} = \tilde{\Gamma} : St \xrightarrow{X} \longrightarrow St \xrightarrow{\tilde{\Gamma}} (X)$  is a bijection, and  $St_{g(S)} \xrightarrow{St} \xrightarrow{\tilde{\Gamma}} St_{g(S')} \xrightarrow{\tilde{\Gamma}} St_{g(S')} \xrightarrow{\tilde{\Gamma}} \cdots \xrightarrow{\tilde{\Gamma}} St_{g(S')} \xrightarrow{\tilde{\Gamma}} \cdots \xrightarrow{\tilde{\Gamma}} St_{g(S')} \xrightarrow{\tilde{\Gamma}} \cdots \xrightarrow{\tilde{\Gamma}}$ 

Let  $\omega$ :  $g(S)\{x\} \rightarrow G(S')\{\overline{\Gamma}(x)\}$  be the restriction of  $\Gamma$ ,  $\omega$  is an isomorphism of groups and

$$\begin{array}{c|c} st_{g(s)}^{x} \times g(s) \{x\} & \longrightarrow & st_{g(s)}^{x} \\ \hline \vec{n} \times \omega & & & \downarrow \vec{r} \\ St_{g(s')}^{\overline{r}(x)} \times g(s') \{\overline{r}x\} & \longrightarrow & st_{g(s')}^{\overline{r}(x)} \end{array}$$

commutes, since

$$\tilde{\Gamma}(\mathfrak{h},\mathfrak{a}) = \Gamma(\mathfrak{h}\mathfrak{a}) = \Gamma(\mathfrak{h}), \Gamma(\mathfrak{a})$$

whereas

 $\tilde{\Gamma}(\eta).\omega(\alpha) = \tilde{\Gamma}(\eta).\Gamma(\alpha) = \Gamma(\eta)\Gamma(\alpha).$ 

Thus,  $\tilde{\Gamma}$ :  $\operatorname{St}_{q(S)}^{\times} \longrightarrow \operatorname{St}_{q(S')}^{\tilde{\Gamma}(X)}$  is a bundle isomorphism with respect to the isomorphism  $\omega$  of groups. Recall that Theorem 1.4.11 showed that any groupoid G can be regarded as a groupoid of admissible maps, G(S), for some bundle (S, P, B). We now complete this classification by proving :-

## 1.5.10. Theorem.

Suppose (S, P, B) and (S', P', B') are principal bundles with structural sheaves. Then the groupoids G(S) and G(S') are isomorphic if and only if, (S, P, B) and (S', P', B') are isomorphic bundles.

Let (S, P, B) and (S', P', B') have sheaves  $(\Sigma, \sigma, Y)$ and  $(\Sigma', \sigma', Y')$  respectively, and suppose  $(f, \overline{f}) : (S, P, B) \longrightarrow (S', P', B')$ is an isomorphism with respect to the isomorphism  $(\kappa, \overline{\kappa}) : (\Sigma, \sigma, Y)$  $\longrightarrow (\Sigma', \sigma', Y')$  of sheaves.

We show there is an isomorphism

$$\Gamma: g(s) \longrightarrow g(s')$$

of groupoids.

Let  ${B_y}_{y}$  and  ${B_y}_{y'}$  denote the partitions associated with B and B' as in 1.4.5, so that

> $\overline{f}(B_y) = B'_{\overline{K}}(y)$ . Form G(S) and G(S') as usual and define

$$r : g(s) \rightarrow g(s')$$

i) on object sets B and B'

ob  $\Gamma = \overline{f} : B \longrightarrow B'$ . ii) If  $x_1, x_2 \in B_y$  and  $\gamma \in G(S)(x_1, x_2)$ , we define  $\gamma' \in G(S')(\overline{f}(x_1), \overline{f}(x_2))$  as follows. Since f is a fibrewise bijection, we can define  $\gamma'$  by

 $b' = f.b.f^{-1}$ 

with f restricted to appropriate fibres. The assignment  $\chi \mapsto \chi'$  now defines  $\Gamma$  on elements.

One easily shows that  $\chi'$  is admissible, so that  $\Gamma$  is well defined, and also that  $\Gamma$  is an isomorphism of groupoids. This demonstrates the sufficiency of the conclusion.

Conversely, suppose we have an isomorphism

$$\Gamma : g(s) \longrightarrow g(s')$$

 $\overline{\mathbf{f}}$  : B  $\longrightarrow$  B'

whose induced map on object sets is

Then  $\Gamma$  induces a bijection  $\overline{K}$ :  $Y \longrightarrow Y'$  which is such that  $\overline{f}(B_y) = B'_{\overline{K}(y)}$ .

We can suppose, by the sufficiency which we have proved, that (S, P, B) and (S', P', B') are regular in the sense of 1.5.8. So, if (S, P, B) has sheaf  $(\Sigma, \sigma, Y)$  and (S', P', B') has sheaf  $(\Sigma', \sigma', Y')$ , there are distinguished points  $x_y \in B_y$ , for each  $y \in Y$ , and  $x'_y \in B'_y$ , for each  $y \in Y'$ , such that :

i) 
$$\Sigma_y = P^{-1}(x_y)$$
 and  $\Sigma'_y = P'^{-1}(x'_y)$  for  $y \in Y$  and  $y \in Y'$   
ii)  $\overline{f}(x_y) = x'_{\overline{\kappa}(y)}$  for all  $y \in Y$ 

iii) the induced actions

$$P^{-1}(x_y) \times Z_y \longrightarrow P^{-1}(x_y)$$
 and  $P^{\prime -1}(x_y^{\prime}) \times Z_y^{\prime} \longrightarrow P^{\prime -1}(x_y^{\prime})$ 

are those of right multiplication.

Now  $\Gamma$  induces an isomorphism

$$y: G(S_y) \longrightarrow G(S_{\vec{k}}(y))$$
 of transitive groupoids,

whose induced map on objects is

$$\overline{\mathbf{f}}_{\mathbf{y}} = \overline{\mathbf{f}}|_{\mathbf{B}_{\mathbf{y}}} : \mathbf{B}_{\mathbf{y}} \longrightarrow \mathbf{B}'_{\mathbf{K}(\mathbf{y})},$$

and  $\overline{f}_{\mu}$  is a bijection for each  $y \in Y$ .

I,

I,

There are natural indentifications

$$z_{y} \rightarrow g(s) \{x_{y}\}$$

$$g \rightarrow h_{g}$$

$$z_{y} \rightarrow g(s') \{x_{y'}\}$$

$$g' \rightarrow h$$

and

for all  $y \in Y$  and  $y \in Y'$ , where  $\gamma_g(s) = g \cdot s = gs$  and  $\gamma_{g'}(s') = g' \cdot s' = g's' \cdot I_y$  and  $I'_y$  are group isomorphisms. Since  $\Gamma_y$ :  $G(S) \{x_y\} \rightarrow G(S') \{x'_{\bar{K}}(y)\}$ is a group isomorphism, we can define a group isomorphism

 $\kappa_{y}: \Sigma_{y} \longrightarrow \Sigma'_{\vec{\kappa}(y)}$ , for each  $y \in Y$ , by requiring

26.

to commute.

$$\sum_{\vec{k}(y)}' \frac{\mathbf{I}_{\vec{k}}(y)}{\mathbf{G}(s')\{x'_{\vec{k}(y)}\}}$$

 $\Sigma_y \xrightarrow{I_y} Q(s)\{x_y\}$ 

The collection  $\{\kappa_y\}$  defines a bijective map

 $\Sigma \xrightarrow{K} \Sigma'$ 

$$\mathsf{K}: \Sigma \longrightarrow \Sigma$$

and

commutes,

so that 
$$(K, \overline{K})$$
:  $(\Sigma, \sigma, Y) \longrightarrow (\Sigma, \sigma', Y')$ 

is an isomorphism of sheaves.

The next step is to construct an isomorphism  $(f_y, \overline{f_y}) : S_y \longrightarrow S'_{\tilde{\kappa}(y)}$  of principal bundles, and then "extend" over S, as usual. To do this, we construct a bundle isomorphism

$$\phi_{y} : S_{y} \longrightarrow St_{\mathcal{G}}(S)^{x_{y}},$$

the construction of which, when carried out for  $S'_{\vec{\kappa}}(y)$ , leads also to a bundle isomorphism

$$\varphi'_{\bar{\kappa}(y)}: s'_{\bar{\kappa}(y)} \longrightarrow st_{\bar{g}}(s') x'_{\bar{\kappa}(y)}$$

We then obtain  $f_v$  by requiring

$$S_{y} \xrightarrow{f_{y}} S'_{\bar{\kappa}(y)}$$
to commute,  

$$\begin{array}{c} \varphi_{y} \\ \varphi_{y} \\ St_{g(S)} \times_{y} \xrightarrow{\tilde{\Gamma}} St_{g(S')} \times \\ g(S') \times \\ \bar{\kappa}(y) \end{array}$$

where  $\tilde{\Gamma}$  denotes the bundle isomorphism of Lemma 1.5.9.

Let  $e_y$  be the identity of  $\mathbb{Z}_y$ , for each  $y \in Y$ , so that  $e_y \in S_y \cap P^{-1}(x_y)$ . Define  $\varphi_y : S_y \longrightarrow St_{\mathcal{G}(S)} \times_y$  by  $\varphi_y(s) =$  unique admissible map  $\eta$  such that  $\eta(e_y) = s$ . (See Theorem 1.4.7 e). Then  $\varphi_y$  is a bijection, by 1.4.7, and

$$S_{y} \xrightarrow{\beta_{y}} St_{g(S)} \times_{y}$$

$$P = \prod_{B_{y}} \frac{\pi}{B_{y}} B_{y}$$

commutes

Next consider the diagram :-

S, x Zy  $\phi_{y} \times I_{y}$ Øy  $St_{q(s)}^{x_y} \times G(s) \{x_y\} \xrightarrow{\bullet} St_{q(s)}^{x_y}$ 

Let  $(s, \omega) \in S_y \times Z_y$ , then  $\phi_y(.)(s, \omega) = \phi_y(s, \omega)$  and  $\phi_y(s, \omega)(e_y) = s \cdot \omega$ . On the other hand,  $(.)(\phi_y \times I_y)(s, \omega) = \phi_y(s) \cdot I_y(\omega)$  $= \phi_y(s) \gamma_{\omega}$ .

However, by our choice of regular representatives we have :

$$\phi_{y}(s) \gamma_{\omega}(e_{y}) = \phi_{y}(s)(\omega e_{y}) = \phi_{y}(s)(e_{y}\omega)$$
$$= \phi_{y}(s)(e_{y}\omega) = \phi_{y}(s)(e_{y})\omega = s.\omega$$

Hence the above diagram commutes, and so  $\oint_y$  is a bundle isomorphism for each  $y \in Y$ . Hence we obtain bundle isomorphisms  $(f_y, \overline{f}_y)$  for each  $y \in Y$  and the collection  $\{f_y\}$  defines a bijection  $f : S \longrightarrow S'$ .

It is clear that  $(f,\overline{f})$ :  $(S, P, B) \longrightarrow (S', P', B')$  is an isomorphism of bundles with respect to the isomorphism  $(\kappa, \overline{\kappa})$  of sheaves.

This completes the proof of the theorem.

### <u>1.5.11</u>. Remarks.

i) In Chapter 3 we will show that topological versions of the results of §4 and §5 can be proved for locally trivial groupoids. This will lead, with very little extra effort, to a homotopy classification of locally trivial groupoids.

ii) Theorems 1.4.11 and 1.5.10 allow us to formulate bundle theoretic concepts in terms of groupoid - theoretic ones, and conversely. For example, the notion of normal subgroupoid and quotient groupoid lead to notions of "normal sub-bundle" and "quotient bundle".

#### §6. Covering Morphisms.

In this section, we shall discuss covering morphisms in terms of the concepts and results of §4 and §5. For simplicity, we shall

suppose throughout this section that all groupoids are connected in the abstract sense, although the results established here hold generally. This can be seen by working over transitive components and using sheaves of groups, rather than single groups. We establish a result, for covering morphisms, analogous to 1.2.6 for groupoids.

<u>1.6.1</u>. Let  $\widetilde{G}$  be a transitive groupoid over  $\widetilde{X}$ , let G be a transitive groupoid over X and suppose that P :  $\widetilde{G} \longrightarrow G$  is a covering morphism (see 1.3.7). Let  $\overline{P}$  denote the induced map ob P :  $\widetilde{X} \longrightarrow X$  on objects. We can choose base points  $\widetilde{x}_{\circ} \in \widetilde{X}$  and  $x_{\circ} \in X$  with  $\overline{P}(\widetilde{x}_{\circ}) = x_{\circ}$  (the sheaves involved here each consists of precisely one group), and then we have a commutative diagram



where  $\tilde{\Gamma}$  and  $\Gamma$  are the isomorphisms defined in 1.4.11. Note that the bottom map is actually  $\Gamma P \tilde{\Gamma}^{-1}$ , of course, but we denote it by P without causing confusion; it is a covering morphism of transitive groupoids.

st <sub>G</sub> <sup>x</sup> <sub>o</sub> <u>r</u> <sub>o</sub> st <sub>G</sub> <sub>v</sub> <sub>o</sub>

π' μ \_ = μπ'

Since P is a covering morphism, we have a bijective map  $P_{g} = \operatorname{St} P : \operatorname{St} \tilde{x}_{g} \longrightarrow \operatorname{St} x_{g}$   $G_{g}$ 

which is such that :

a)

commutes.

Ъ)

 $\tilde{P}_{o} = P_{o} \Big|_{\tilde{G}\{\tilde{x}_{o}\}} : \tilde{G}\{\tilde{x}_{o}\} \longrightarrow G\{x_{o}\}$  is an injective group

homomorphism, and also

c)  $\operatorname{St}_{\widetilde{G}} \widetilde{x}_{\circ} \times \widetilde{G} \{ \widetilde{x}_{\circ} \} \xrightarrow{\bullet} \operatorname{St}_{\widetilde{G}} \widetilde{x}_{\circ}$  $\begin{array}{c} P_{\circ} \times \widetilde{P}_{\circ} \\ St \\ G \end{array} \xrightarrow{\bullet} St \\ C \end{array} \xrightarrow{\bullet} \operatorname{St}_{\widetilde{G}} x$ 

commutes for :  $P_{o}(\beta \cdot a) = P_{o}(\beta a) = P_{o}(\beta)P_{o}(a) = P_{o}(\beta)\tilde{P}_{o}(a) = P_{o}(\beta)\tilde{P}_{o}(a)$ , for all  $(\beta, a) \in St_{\tilde{a}} \times \tilde{G}\{\tilde{x}_{o}\}$ . Thus,  $(P_{o}, \tilde{P})$  is a bundle map  $St_{\tilde{G}} \xrightarrow{\tilde{x}} \longrightarrow St_{\tilde{G}}$  with respect to  $\tilde{P}_{o}$  as in Definition 1.5.2.

<u>1.6.2.</u> Lemma.

Suppose  $\widetilde{G}$  and G are connected groupoids over  $\widetilde{X}$  and X respectively and we have a pair of maps

 $P_{o} : St_{\widetilde{G}} \xrightarrow{\widetilde{x}} \longrightarrow St_{\widetilde{G}} \xrightarrow{\widetilde{x}}_{o}$ and  $\overline{P} : \widetilde{X} \longrightarrow X$  satisfying :i)  $\widetilde{P}_{o} = P_{o} \Big|_{\widetilde{G}\{\widetilde{x}_{o}\}} : \widetilde{G}\{\widetilde{x}_{o}\} \longrightarrow G\{x_{o}\}$  and is a group homomorphism. ii)  $(P_{o}, \overline{P}) : (St_{\widetilde{X}}, \pi', \widetilde{X}) \longrightarrow (St_{X}, \pi', X)$  is a bundle map with respect to  $\widetilde{P}_{o}$ .

Then P<sub>o</sub> can be extended to a covering morphism  $P: \tilde{G} \longrightarrow G$  with  $ob P = \overline{P}$ .

Proof.

First note that the hypotheses immediately yield that  $\overline{P}(\tilde{x}_{o}) = x_{o}$ ,  $\tilde{P}_{o}$  is injective and  $\overline{P}$  is a surjection.

Choose a wide tree subgroupoid  $\widetilde{T}$  in  $\widetilde{G}$  and let  $\mathcal{T}_{\widetilde{X}}$  denote the unique element of  $\widetilde{T}(\widetilde{x},\widetilde{x})$ . Define  $\mathcal{T}_{\overline{P}(\widetilde{X})}$  by  $\mathcal{T}_{\overline{P}(\widetilde{X})} = P_o(\mathcal{T}_{\widetilde{X}})$  $\varepsilon G(x,\overline{P}(\widetilde{X}))$ . Now by 1.2.6, any element  $\widetilde{\alpha} \in \widetilde{G}(\widetilde{x},\widetilde{y})$  can be represented as

$$\widetilde{\alpha} = \tau_{\widetilde{y}} \widetilde{\alpha}_{c} \tau_{\widetilde{x}}^{-1}$$

for some unique element  $\tilde{\alpha}_{o}$  of  $\tilde{G}\{\tilde{x}_{o}\}$ . Now define  $P: \tilde{G} \longrightarrow G$  by  $P(\tilde{\alpha}) = P(\tau_{\tilde{y}}) \tilde{P}(\tilde{\alpha}_{o}) P_{o}(\tau_{\tilde{x}})^{-1}$  on elements, and define P on objects by  $ob P = \overline{P}$ . We claim that  $(P,\overline{P})$  is a covering morphism of groupoids.

Let  $\tilde{\beta} = \tau_{\tilde{z}} \tilde{\beta}_{\tilde{y}} \tau_{\tilde{y}}^{-1} \epsilon \tilde{G}(\tilde{y}, \tilde{z})$  then we have  $\tilde{\beta}\tilde{a} = \tau_{\tilde{z}} \tilde{\beta}_{\tilde{a}} \tilde{a}_{\tilde{z}} \tau_{\tilde{x}}^{-1}$ and so  $P(\tilde{\beta}\tilde{a}) = P_{\sigma}(\tau_{\tilde{z}})\tilde{P}_{\sigma}(\tilde{\beta}_{\sigma}\tilde{a}_{\sigma})P_{\sigma}(\tau_{\tilde{x}})^{-1}$ . Since  $\tilde{P}_{\sigma}$  is a homomorphism of groups, we have

 $P(\tilde{\beta}\tilde{a}) = P_{o}(\tau_{\tilde{z}})\tilde{P}_{o}(\tilde{\beta})\tilde{P}_{o}(\tilde{a}) P_{o}(\tau_{\tilde{x}})^{-1}$ 

Also,  $P(\tilde{\beta})P(\tilde{a})$  is defined in G, by hypothesis ii), and

$$P(\tilde{\beta})P(\tilde{\alpha}) = \left\{ P_{o}(\tau_{\tilde{z}})\tilde{P}_{o}(\tilde{\beta}_{o})P_{o}(\tau_{\tilde{y}})^{-1} \right\} \left\{ P_{o}(\tau_{\tilde{y}})\tilde{P}_{o}(\tilde{\alpha}_{o})P_{o}(\tau_{\tilde{x}})^{-1} \right\}$$

$$= P_{o}(\tau_{\tilde{z}})\tilde{P}_{o}(\tilde{\beta}_{o})\tilde{P}_{o}(\tilde{\alpha}_{o})P_{o}(\tau_{\tilde{x}})^{-1} .$$
Thus, we have  $P(\tilde{\beta}\tilde{\alpha}) = P(\tilde{\beta}) P(\tilde{\alpha}) .$  If  $\tilde{\alpha} = I_{\tilde{x}} = \tau_{\tilde{x}} I_{\tilde{x}} \tau_{\tilde{x}}^{-1} ,$  then
$$P(\tilde{\alpha}) = P_{o}(\tau_{\tilde{x}})\tilde{P}_{o}(I_{\tilde{x}_{o}})P_{o}(\tau_{\tilde{x}})^{-1}$$

$$= P_{o}(\tau_{\tilde{x}}) I_{x} P_{o}(\tau_{\tilde{x}})^{-1} .$$
Hence,  $P(I_{\tilde{x}}) = \tau_{\tilde{P}(\tilde{x})} I_{x} \tau_{o}^{-1} P(\tilde{x})$  and so we have  $P(I_{\tilde{x}}) = I_{\tilde{P}(\tilde{x})} .$  Thus, P
is a morphism of goupoids.

Next, let  $\tilde{x} \in \tilde{X}$  and let  $x = \overline{P}(\tilde{x})$  and consider  $\operatorname{St}_{\widetilde{G}} P$  :  $\operatorname{St}_{\widetilde{G}} \tilde{x} \longrightarrow \operatorname{St}_{\widetilde{G}} x$ . Now  $\mathcal{T}_{\widetilde{X}}$  induces a map  $\mathcal{T}_{\widetilde{X}}^{*}$  :  $\operatorname{St}_{\widetilde{G}} \tilde{x} \longrightarrow \operatorname{St}_{\widetilde{G}} \tilde{x}_{\widetilde{s}}$  defined by  $\mathcal{T}_{\widetilde{X}}^{*}(\tilde{\alpha}) = \tilde{\alpha} \mathcal{T}_{\widetilde{X}}$  and, similarly,  $\mathcal{T}_{X}$ induces a map  $\operatorname{St}_{\widetilde{G}} x \longrightarrow \operatorname{St}_{\widetilde{G}} x_{\widetilde{s}}$ , and these maps are bijective. Since Pis a morphism, the diagram



is commutative, whence  $\operatorname{St}_{\widetilde{G}} P$  is bijective for each  $\widetilde{x} \in \widetilde{X}$ . Thus, P is a covering morphism of groupoids.

<u>1.6.3</u>. We next show that the extension carried out in 1.6.2. is independent of the choice of the tree  $\widetilde{T}$  in  $\widetilde{G}$ .

With the hypothesis of Lemma 1.6.2. holding, let  $\widetilde{T}_1$  and  $\widetilde{T}_2$  be any two wide tree subgroupoids of  $\widetilde{G}$ , and let  $\mathcal{T}_{\widetilde{X}} \in \widetilde{T}_1(\widetilde{x}_2, \widetilde{x})$ ,

 $T_{\widetilde{X}}^2 \in \widetilde{T}_2(\widetilde{x}, \widetilde{x})$ . Let  $P_1$  and  $P_2$  denote the respective extensions of  $P_0$  by  $\widetilde{T}_1$  and  $\widetilde{T}_2$  as in 1.6.2. Now any element  $\widetilde{\alpha} \in \widetilde{G}(\widetilde{x}, \widetilde{y})$  can be represented either as

$$\tilde{a} = \tau_{\tilde{y}} \tilde{a} \tau_{\tilde{x}}^{-1}$$

in terms of  $\widetilde{T}_{i}$  and  $\widetilde{G}\{\widetilde{x}_{c}\}$  or as  $\widetilde{\alpha} = \mathcal{T}_{\widetilde{Y}}^{2} \widetilde{\alpha}_{z} \mathcal{T}_{\widetilde{Y}}^{2-1}$  31 •

in terms of  $\widetilde{T}_{2}$  and  $\widetilde{G}\{\widetilde{x}_{s}\}$ . Thus, we have :

Let  $\mathcal{T}_{\widetilde{X}}^{'-1} \mathcal{T}_{\widetilde{X}}^{2} = \widetilde{\beta} \in \widetilde{G}\{\widetilde{x}_{\circ}\}$ , so that  $\mathcal{T}_{\widetilde{X}}^{'} = \mathcal{T}_{\widetilde{X}}^{'} \widetilde{\beta}$ . Since  $P_{o}$  is a bundle map, we have  $P_{o}(\mathcal{T}_{\widetilde{X}}^{'}) = P_{o}(\mathcal{T}_{\widetilde{X}}^{'}) P_{o}(\widetilde{\beta}) = P_{o}(\mathcal{T}_{\widetilde{X}}^{'}) \widetilde{P}_{o}(\mathcal{T}_{\widetilde{X}}^{'-1} \mathcal{T}_{\widetilde{X}}^{'})$ , whence  $\widetilde{P}_{o}(\mathcal{T}_{\widetilde{X}}^{'-1} \mathcal{T}_{\widetilde{X}}^{'}) = P_{o}(\mathcal{T}_{\widetilde{X}}^{'})^{-1} P_{o}(\mathcal{T}_{\widetilde{X}}^{'})^{-1-----*}$ 

We now obtain

$$P_{z}(\tilde{a}) = P_{o}(\tau_{\tilde{y}}^{2}) \tilde{P}_{o}(\tilde{a}_{z}) P_{o}(\tau_{\tilde{x}}^{2})^{-1}$$

$$= P_{o}(\tau_{\tilde{y}}^{2}) P_{o}(\tau_{\tilde{y}}^{2-1}\tau_{\tilde{y}}^{i}) \tilde{P}_{o}(\tilde{a}_{z}) P_{o}(\tau_{\tilde{x}}^{i-1}\tau_{\tilde{x}}^{2}) P_{o}(\tau_{\tilde{x}}^{2})^{-1}$$

$$= P_{o}(\tau_{\tilde{y}}^{i}) \tilde{P}_{o}(\tilde{a}_{z}) P_{o}(\tau_{\tilde{x}}^{i})^{-1}$$

by use of \* . Hence,  $P_{z}(\tilde{a}) = P_{1}(\tilde{a})$  and so  $P_{1} = P_{2}$ . Lemma 1.6.2 and 1.6.3 yield the following analogue of 1.2.6 : 1.6.4. Theorem.

Let P :  $\widetilde{G} \longrightarrow G$  be a covering morphism of transitive groupoids  $\widetilde{G}$  and G, and let  $\widetilde{x}$  be any object of  $\widetilde{G}$ . Then P is uniquely determined by  $\operatorname{St}_{\widetilde{G}}^{\operatorname{P}}$  :  $\operatorname{St}_{\widetilde{G}}^{\operatorname{x}} \longrightarrow \operatorname{St}_{\operatorname{G}}^{\operatorname{ob}}\operatorname{P}^{\operatorname{x}}$  and any wide tree subgroupoid of  $\widetilde{G}$ .

Proof.

Let  $P_o = St_{\widetilde{G}}P$  :  $St_{\widetilde{G}}\tilde{x} \longrightarrow St_{\widetilde{G}} \operatorname{ob} P\tilde{x}$ , let  $\overline{P} = \operatorname{ob} P$ and let  $\widetilde{T}$  be any wide tree subgroupoid of  $\widetilde{G}$  with  $\mathcal{T}_{\widetilde{y}} \in \widetilde{T}(\widetilde{x}, \widetilde{y})$ . By 1.6.1,  $P_o$  is a bundle map with respect to  $\widetilde{P}_o = P_o |_{\widetilde{G}\{\widetilde{x}\}}$ , and so by 1.6.2 and 1.6.3  $P_o$  can be extended uniquely by  $\widetilde{T}$  to a covering morphism  $P_i$  :  $\widetilde{G} \longrightarrow G$  with  $\operatorname{ob} P_i = \overline{P} = \operatorname{ob} P$ . If  $\widetilde{a}$  is represented as  $\widetilde{a} = \mathcal{T}_{\widetilde{Z}} \widetilde{a}_o \mathcal{T}_{\widetilde{y}}^{-1}$ , then

$$P_{i}(\tilde{\alpha}) = P_{i}(\tau_{\tilde{z}}) P_{i}(\tilde{\alpha}) P_{i}(\tau_{\tilde{y}})^{-1}$$
$$= P(\tau_{\tilde{z}}) P(\tilde{\alpha}) P(\tau_{\tilde{y}})^{-1}$$
$$= P(\tau_{\tilde{z}}) P(\tilde{\alpha}) P(\tau_{\tilde{y}})^{-1} = P(\tilde{\alpha})$$

Thus P = P and the theorem is proved.

# 1.6.5. Remark.

and

It is possible to formulate a notion of "equivalence of bundle maps", and it follows then that

 $\operatorname{St}_{\widetilde{G}} P$  :  $\operatorname{St}_{\widetilde{G}} \widetilde{x} \longrightarrow \operatorname{St}_{G} \operatorname{ob} P \widetilde{x}$  $\operatorname{St}_{\widetilde{G}} P$  :  $\operatorname{St}_{\widetilde{G}} \widetilde{y} \longrightarrow \operatorname{St}_{G} \operatorname{ob} P \widetilde{y}$ 

are "equivalent", for each pair  $\tilde{x}, \tilde{y} \in \tilde{X}$ . And so a covering morphism determines uniquely, and is determined by, an "equivalence class" of bundle maps, see 3.6.5.

80. Introduction.

Chapter 2.

In this chapter, following Ehresmann [1], we introduce the concept of "topological groupoid" and derive some basic properties which will be needed in later chapters.

Some of the examples of §2 of Chapter 1 have natural topologies which are compatible with the groupoid structure, and turn these groupoids into topological groupoids.

The results of §4 in Chapter 1 will also be discussed briefly in this chapter, but a thorough discussion of §5 of Chapter 1 will be presented in Chapter 3 for the case of "locally trivial" groupoids.

§1. Topological groupoids.

Our basic definition is :-

2.1.1. Definition. (see Ehresmann [1] for example)

A <u>topological groupoid</u> G is a groupoid G in which the sets Mor(G) and ob(G) are topological spaces, and the following functions are continuous :-

i) Composition :  $\mathfrak{D} \longrightarrow Mor(G)$  and inverse :  $Mor(G) \longrightarrow Mor(G)$ . ii)  $\pi$  and  $\pi'$  :  $Mor(G) \longrightarrow ob(G)$ 

iii)  $u : ob(G) \longrightarrow Mor(G)$ , where  $u(x) = I_x$ .

2.1.2. Remarks.

Recall that we allow ourselves to confuse Mor(G) with G
 with G (see 1.1.2 Remark ii), and so in future we will regard G
 itself as a topological space.

ii)

It is to be understood that  $\mathfrak{P}$  always has the relative topology inherited from the product topology on  $G \times G$ .

iii) Let  $I(G) \subset G$  have the relative topology as a subspace of G, then Axiom ii) of 2.1.1. implies that  $\pi|_{I(G)}$  :  $I(G) \longrightarrow ob(G)$ is continuous, and iii) implies that  $u : ob(G) \longrightarrow I(G)$  is continuous. Thus, the natural identification  $ob(G) \xrightarrow{u} I(G)$  is a homeomorphism.

The sets  $cost_{\alpha}x$ ,  $star_{\alpha}x$  and  $G\{x\}$ , for  $x \in ob(G)$ , all iv) inherit subspace topologies, and in this topology G{x} is a topological group. It is to be understood, as in the case of I(G) also, that these spaces always have the subspace topology of G.

We shall say that G is a Hausdorff, compact, locally compact. normal etc. topological groupoid, provided that both G and ob(G) are Hausdorff, compact, locally compact, normal etc. Of course. the appropriate condition on ob(G) will often be a consequence of that on G, for example, compactness of ob(G) follows from that of G as does the condition of being Hausdorff.

The connectedness of G and ob(G) as topological spaces is. in general, not related to the connectedness of G in the abstract For example, any group with the discrete topology is dissense. connected as a topological space, but is an abstractly connected topological groupoid. On the other hand, any groupoid G with the indiscrete topology on G and ob(G) is a topological groupoid. which is connected as a topological space. It is not, of course, necessarily connected in the abstract sense. However, we prove in a corollary to the result 2.4.3 below that if G is locally trivial and ob(G) is connected, then G is abstractly connected.

Axiom i) of the Definition 2.1.1 immediately yields that inverse : G ---- G is a homeomorphism.

There are other interesting structures one can place on a viii) groupoid G with respect to which the algebraic operations of the groupoid are compatible. For example, Borel structures in which the topological spaces are replaced by Borel spaces in Definition 2.1.1 . and the continuous functions there are replaced by Borel measurable Thus, we obtain Borel groupoids and these will be considered ones. in Chapter 4. Likewise, we can replace the topological spaces by Cr differentiable manifolds and replace the continuous functions by C<sup>r</sup> differentiable ones. Thus we obtain C<sup>r</sup> Lie groupoids, and these

vi)

v)

vii)

have been studied by Ehresmann.

Before moving on to consider examples of topological groupoids, we shall prove some simple, but useful, results using little more than the pefinition 2.1.1.

#### 2.1.3. Proposition.

a) Let G be a topological groupoid in which ob(G) is a T, space, then for all x, y  $\varepsilon$  ob(G) the sets i) G(x,y) ii) G{x} iii) St<sub>G</sub>x and iv) cost<sub>G</sub>x are all closed in G. If ob(G)is a Hausdorff space, then v)  $\bigcup_{x \in ob(G)} G{x}$  is closed in G.

b) Let G be any topological groupoid over X. If  $U \subset X$  is open (resp. closed), then G(U) - the full subgroupoid of G on U - is open (resp. closed) in G.

#### Proof. a)

Let  $x \in ob(G)$  be any object of G. Since we can display  $St_G x$  as  $\pi^{-1}(x)$ , the continuity of  $\pi$  and the hypothesis ob(G) be  $T_i$  imply that  $St_G x$  is closed in G, since  $\{x\}$  is closed in ob(G). This proves iii) and iv) is similar since  $cost_G x = \pi'^{-1}(x)$ . Since  $G(x,y) = \pi^{-1}(x) \wedge \pi'^{-1}(y)$ , it is now immediate that G(x,y) is closed in G, which is i); of course ii) now follows with x = y.

To prove v), we have

$$\pi : G \longrightarrow ob(G) \text{ and}$$
$$\pi' : G \longrightarrow ob(G)$$

are continuous functions into a Hausforff space. If  $a \in \bigcup_{x \in ob} (G)$ then  $a \in G\{y\}$  for some y and  $\pi(a) = y = \pi'(a)$ . On the other hand, if  $\pi(a) = \pi'(a) = y$  for  $a \in G$ , then  $a \in G\{y\}$  and so  $a \in \bigcup_{x \in ob} (G)$ Thus,  $\bigcup_{x \in ob} (G)$  is the set of points on which  $\pi$  and  $\pi'$  coincide,  $x \in ob(G)$ and so  $\bigcup_{x \in ob} (G)$ 

For any set  $U \subset X$ , we have

 $G(U) = \pi^{-1}(U) \, , \quad \pi'^{-1}(U) \, ,$ and so the result follows from the continuity of  $\pi$  and  $\pi'$ .

It is a consequence of this result, that, for  $T_{i}$  object space, the subspaces  $St_{G}x$ ,  $cost_{G}x$  and  $G\{x\}$  are compact resp. locally compact, for each object x, if G is compact resp. locally compact. This remark will be important later.

In general, however,  $\operatorname{St}_{G} x$  is not open in G and so G is not generally a disjoint union or sum of the spaces  $\operatorname{St}_{G} x$ . (or of the spaces  $\operatorname{Cost}_{G} x$ ). Likewise, G(x,y) is not generally open and so  $\operatorname{St}_{G} x$ is itself not generally a sum of the G(x,y), for  $y \in \operatorname{ob}(G)$ .

2.1.4. Proposition.

Ъ)

a Hausdorff space
a) Let G be a topological groupoid over / X. Then the set of
units I(G) is closed in G if and only if, G is Hausdorff.
b) If G is a Hausdorff groupoid, then D is closed in G × G.
Proof.

a) First suppose that I(G) is closed in G. Let  $\Theta : \mathfrak{D} \longrightarrow G$ denote the composition function and let  $\varphi$ :  $G \longrightarrow G$  denote the inverse function. Then  $\mathfrak{S}^{-1}(I(G)) = \{(\alpha, \alpha^{-1}) \in G \times G\}$  is closed in  $G \times G$ , by b), since  $\Theta$  is continuous. Further, the map  $I \times \varphi$ :  $G \times G \longrightarrow G \times G$ defined by  $I \times \varphi(\alpha, \beta) = (\alpha, \beta^{-1})$  is continuous and so

$$(\mathbf{I} \times \phi)^{-1} \left( \begin{smallmatrix} -1 \\ \odot \end{smallmatrix}^{-1} (\mathbf{I}(\mathbf{G})) = (\mathbf{I} \times \phi)^{-1} \left\{ \begin{smallmatrix} (a, a^{-1}) \in \mathbf{G} \times \mathbf{G} \right\}$$
$$= \left\{ (a, a) \in \mathbf{G} \times \mathbf{G} \right\}$$
$$= \Delta (\mathbf{G}) - \text{the diagonal of } \mathbf{G}$$

is closed in G. By the well known fact that a space Y is Hausdorff if, and only if,  $\Delta(Y)$  is closed in Y × Y, we conclude that G is a Hausdorff space, and hence a Hausdorff groupoid by Remark v) of 2.1.2. Conversely, suppose G is a Hausdorff groupoid, then G × G is a Hausdorff space. Let X = ob(G) and define

 $f: \bigcup_{\mathbf{x} \in \mathbf{X}} G \times G$ 

 $a \mapsto (a,a^2)$ 

and  $g : \bigcup_{x \in X} G \times G$ 

$$a \mapsto (a^2, a)$$
.

Since the composite of f with the projection on either factor is continuous, we have f is continuous. Similarly, g is continuous. Now, clearly  $f(\alpha) = g(\alpha)$  if  $\alpha = I_x$  for some  $x \in X$ , on the other hand, if  $f(\alpha) = g(\alpha)$ , then  $\alpha = \alpha^2$  and so  $\alpha = I_x$  for some  $x \in X$ . Thus, the set of points on which f and g coincide is I(G) and so I(G) is closed in  $\bigcup_{x \in X} G\{x\}$ . However,  $\bigcup_{x \in X} G\{x\}$  is closed in G  $x \in X$ .

since G is Hausdorff, by 2.1.3 v). Thus, I(G) is closed in G and so the proof of part a) is complete.

b) By definition of  $\mathfrak{P}$  ,  $\mathfrak{P}$  is the pull-back

 $\begin{array}{c} \mathfrak{D} \longrightarrow \mathbf{G} \\ \downarrow \qquad \qquad \downarrow \mathbf{\pi}' \\ \mathbf{G} \xrightarrow{\mathbf{\pi}} \times \end{array}$ 

where X = ob(G). So if G is a Hausdorff groupoid, X is Hausdorff and so  $\mathcal{D}$  is closed in  $G \times G$ .

We have a topological version of 1.2.7 :-

#### 2.1.5. Proposition.

Let G be a topological groupoid over X and suppose G is transitive. Then :

a) G(x,y) and G(x',y') are homeomorphic for any objects
x, y, x' and y' of G. In particular, the vertex groups G{x}
and G{y} are isomorphic topological groups for any x, y e ob(G).
b) For all objects x and y of G, St<sub>G</sub>x and St<sub>G</sub>y are homeomorphic and a similar statement holds for costars.
<u>Proof</u>.

a) Choose a wide tree subgoupoid T in G and let  $\mathcal{T}_{xy} \in T(x,y)$ . As shown in 1.2.7,  $a \vdash \tau_{vv}' a \tau_{xx}'$ 

is bijective. From the continuity of the composition in G, it follows that  $\phi$  is continuous and also that  $\phi^{-1}$  is continuous. Hence,  $\phi$  is a homeomorphism.

In particular,  $\phi$  :  $G\{x\} \longrightarrow G\{y\}$  $\alpha \longmapsto T_{xy} \alpha T_{yx}$ 

is an isomorphism of topological groups.

With the notation of a), the map

 $a \mapsto \tau_{xy} a$ 

 $\phi$  : St<sub>G</sub>x --- St<sub>G</sub>y

is a homeomorphism since the composition in G is continuous.

Note that we have proved rather more than is stated in 2.1.5 in so much that the homeomorphisms  $\phi$  are determined by G .

§2. Examples of topological groupoids.

In this section, we discuss natural topologies for some of the examples of §2 Chapter 1, and show that, with these topologies, the examples of §2 are topological groupoids.

2.2.1.

Ъ)

Any topological group, for example the real line, is a topological groupoid with one object. More generally, suppose  $\{G_a; a \in I\}$  is a family of topological groups. Give I the discrete topology and give  $G = \bigcup_{a \in I} G_a$  the sum topology. Then G is a groupoid over I, called a facil the "sum" of the groups  $G_a$ , with composition defined in terms of that given in each  $G_a$ . In fact,

 $\mathcal{D} = \left\{ (g,h) \in G \times G ; gh \text{ is defined} \right\}$ =  $\bigcup_{a \in I} G_a \times G_a \subset G \times G , and then$  $a \in I$ 

composition :  $\mathfrak{D} \longrightarrow G$  is defined by  $(g,h) \longmapsto gh$ .

If  $e_{\alpha} \in G_{\alpha}$  is the identity of  $G_{\alpha}$ , for  $\alpha \in I$ , then  $I_{\alpha} = e_{\alpha}$ . We show next that G is a topological groupoid :-

Now  $G_{a}$  is open in G and so  $G_{a} \times G_{a}$  is open in  $G \times G$ , for each  $a \in I$ . So  $\bigoplus$  is open in  $G \times G$  and, by definition of the sum topology,  $U \subset G$  is open if, and only if,  $U \cap G_{a}$  is open in  $G_{a}$ 

(and hence in G) for each  $a \in I$ . Thus :-

$$(\operatorname{comp})^{-1}(U) = (\operatorname{comp})^{-1}(\bigcup \cup \cup \cap G_{\alpha})$$
  
$$= \bigcup \operatorname{comp}^{-1}(\bigcup \cap G_{\alpha})$$
  
$$= \varepsilon I$$

which is open in  $G \times G$  and hence, in  $\mathfrak{D}$ . Thus, the continuity of the composition is established and likewise that of the other functions. Clearly, the "sum"  $G = \bigcup_{\alpha} G_{\alpha}$  of a family of topological groupoids  $G_{\alpha}$ becomes a topological groupoid in precisely the same way.

2.2.2.

Let X be an n-dimensional differentiable manifold and let TX denote the tangent bundle to X. Then TX can be regarded as a fibre bundle with fibre  $\mathbb{R}^n$  and structure group  $GL(\mathbb{R}^n)$  - the general linear group. Let G(X) denote the groupoid of admissible (i.e. linear bijections) maps between tangent planes of TX. Then G(X) can be made into a topological groupoid (in fact a Lie groupoid) in a natural way. The details of this are a special case of a general construction given in Chapter 3 and will, therefore, be omitted.

This example is a particular example of the groupoids constructed in 1.2.2.

2.2.3.

Let G be any topological groupoid and H a subgroupoid of G as defined in 1.3.1. Then H becomes a topological subgroupoid of G if it is given the subspace topology. This follows from the fact that the restriction of a continuous function to a subspace is continuous. 2.2.4.

Let X be any topological space and let T be a tree groupoid

on X with  $\tau_{xy} \in T(x,y)$ . If we topologise T by requiring the function

 $\phi : T \longrightarrow X \times X$   $\tau_{yy} \longmapsto (x,y)$ 

to be a homeomorphism, then T becomes a topological tree groupoid over X. To verify this, we note that essentially  $\mathfrak{D} = \{(y,z), (x,y) \in (X \times X) \times (X \times X)\}$ and that the continuity of the composition function is equivalent to the continuity of

 $\Theta : \mathfrak{D} \longrightarrow X \times X$  $((y,z), (x,y)) \longmapsto (x,z)$ 

The diagram



commutes, where  $P_1$  and  $P_2$  denote projections, and so  $P_1 \Theta$  is continuous. Likewise,  $P_2 \Theta$  is continuous and so  $\Theta$  is continuous. Hence, the composition function in T is continuous. A similar argument establishes the continuity of the inverse function, that of  $\pi$ and  $\pi'$  and also that of u.

Since any principal groupoid G on X, as defined in 1.2.5, can be regarded as a subgroupoid of T, it follows from 2.2.3 that G is a topological groupoid in the subspace topology.

We note that the product topology on T (i.e. that induced by  $\phi$  ) is the coarsest topology on T making T a topological groupoid. This follows from the fact that the product topology on  $X \times X$  is initial with respect to the projections.

2.2.5.

Let S be a topological space and let G be a topological group acting continuously on the right of S, that is, we have a continuous action :  $S \times G \xrightarrow{\bullet} S$ . We show that the groupoid  $\widetilde{G}$  of

1.2.8 is a topological groupoid over S as follows. As a set,  $\tilde{G}$  is  $\cdot S \times G$ , and so we give  $\tilde{G}$  the product topology of  $S \times G$ . Since  $S \times G \longrightarrow S$ , we have a natural map

 $(S \times G) \times G \xrightarrow{\circ} S \times G$  $((s_1, g_1), g_2) \longmapsto (s_1, g_1 g_2)$ 

and this is continuous since  $P_1(0)((s,g_1),g_2) = s$ , which is the projection on the first factor, and  $P_2(0)((s,g_1),g_2) = g_1g_2$ , which is the composite  $(S \times G) \times G \xrightarrow{\text{identification}} S \times (G \times G) \xrightarrow{P_2} G \times G \xrightarrow{\text{multp}^n} G$ and so  $P_2(0)$  is continuous, thus 0 is continuous. Now  $\mathfrak{D} = \{((s_2,g_2),(s_1,g_1)) \in \widetilde{G} \times \widetilde{G}; s_1 \cdot g_1 = s_2\},$ and composition  $: \mathfrak{D} \longrightarrow \widetilde{G}$  is defined by  $(s_2,g_2) \cdot (s_1,g_1) = (s_1,g_1g_2),$ 

which is the composite

 $\begin{array}{cccc} & & & \\$ 

and so the composition function is continuous. Continuity of the remaining functions is easily checked and so  $\tilde{G}$  is a topological groupoid over S. This result is, in fact, a particular case of the equivalence of the category of "topological actions" with that of "topological coverings", see Hardy [1].

#### 2.2.6.

For some results on topologising the example of 1.2.10, that is, the fundamental groupoid of X, we refer the reader to Danesh-Naruei [1]. We note, however, that if X is arcwise connected and arcwise locally connected, then X admits a universal covering,  $(\tilde{X}, P)$ , which is a locally trivial principal bundle over X with discrete fibre as shown by Steenrod [1]. Since  $\pi X$  is isomorphic to  $G(\tilde{X})$ , we can topolgise  $\pi X$ using the results of Chapter 3. We give more details of this in Chapter 3. §3. Morphisms of topological groupoids.

Having defined "topological groupoid", it is natural to formulate a definition of "morphism of topological groupoid", this we do in :-

#### <u>2.3.1</u>. <u>Definition</u>.

By a <u>homomorphism</u> (or just <u>morphism</u>)  $f : G \longrightarrow H$  of topological groupoids, we mean a morphism of abstract groupoids which is continuous on both objects and elements.

The topological groupoids and morphisms clearly form a category, and we have a forgetful functor from this category into the category of abstract groupoids and homomorphisms.

Next we show that a product of topological groupoids can readily be made into a topological groupoid in:-

2.3.2.

Let  $\{G_a\}$  be a family of topological groupoids with  $a \in I$ 

 $X_a = ob(G_a)$ . Form the product groupoid  $TG_a = G$  over  $TT_a = X_{a \in I}^a$ as in 1.3.2, and give G and X the usual product topology. We show

G is a topological groupoid over X, and to do this we have to verify the continuity of the usual functions. As usual, let

$$\begin{aligned} & \begin{split} & \begin{split} & \end{split} \\ & \end{split} & \end{split} \\ & \end{split} & \end{split} \\ & \end{split} & \end{split} & \end{split} \\ & \end{split} & \end{split} & \end{split} & \end{split} & \end{split} \\ & \end{split} \\ & \end{split} \\ & \end{split} & \ends$$

this function is clearly continuous for each  $\beta \in I$  .

Let

 $\Theta_{\beta} : \mathfrak{D} \longrightarrow \mathfrak{G}_{\beta},$   $((\mathfrak{g}_{\alpha}), (\mathfrak{h}_{\alpha})) \longmapsto \mathfrak{G}_{\beta}\mathfrak{h}_{\beta}$ 

then  $\vartheta_{\beta}$  is the composite of  $\not_{\beta}$  and the composition function in  $G_{\beta}$ is, therefore, continuous. Since we have  $P_{\beta}(\text{composition in } G) = \vartheta_{\beta}$ , where  $P_{\beta}$  denotes projection on the  $\beta$ -factor, we have that composition in G is continuous.

A similar argument establishes the continuity of the inverse

function G ----> G .

$$\pi((g_{a})) = (\pi_{a}(g_{a})) ,$$

where  $\pi_{_{\mathbf{G}}}$  denotes the initial function of  $\mathtt{G}_{_{\mathbf{G}}}$  , and so  $\pi = \prod_{\alpha} \pi_{\alpha} : \prod_{\alpha} G_{\alpha} \longrightarrow \prod_{\alpha} X_{\alpha}$  is a continuous function. Similarly,  $\pi' = \prod_{\alpha} \pi'_{\alpha}$  and  $u = \prod_{\alpha} u_{\alpha}$  are both continuous and so,  $\alpha \in I$ by Definition 2.1.1, G is a topological groupoid over X . Note that the projection  $P_{\beta}$ :  $\Pi \ G_{\alpha} \longrightarrow G_{\beta}$  is a  $a \in I$ morphism of topological groupoids. It is a consequence of this, that T  $G_a$  of topological groupoids  $\{G_a\}_{a, \in T}$  has the the product usual universal property of products. Explicitly, suppose  $f_{\alpha}$ : G  $\longrightarrow$  G<sub>\alpha</sub> is a morphism of topological groupoids for each  $\alpha \in I$ . Then there is a unique morphism  $f : G \longrightarrow \Pi G_{\alpha}$ , of topological groupoids, such that  $P_a f = f_a$  for each  $a \in I$ . The following proposition, which has an immediate generalisation to a family  $\{G_{a}\}_{a \in I}$  of groupoids together with a family  $\{f_{a}\}_{a \in I}$ of morphisms, where  $f_a$ :  $G_a \longrightarrow H$ , is an example of the "Initial Structures" of Bourbaki.

2.3.3. Proposition.

Let  $f : G \longrightarrow H$  be a morphism of groupoids of G onto H, where H is a topological groupoid. Then f induces a topology on G compatible with the groupoid structure of G and f is then a morphism of topological groupoids.

#### Proof.

Define a set  $U \subset G$  to be open if, and only if,  $U = f^{-1}(V)$ for some open set  $V \subset H$ , and, if  $\overline{f} = ob(f)$ , define  $\overline{U} \subset ob(G)$  to be open if, and only if,  $\overline{U} = \overline{f}^{-1}(\overline{V})$  for some open set  $\overline{V} \subset ob(H)$ . This defines a topology on G and on ob(G), and f and  $\overline{f}$  being surjective are both continuous and open maps.

Since the diagram below is commutative:-



where  $\theta_{\rm G}$  and  $\theta_{\rm H}$  denote the composition functions of G and H respectively, it follows that  $\theta_{\rm G}$  is continuous, for if U  $\subset$  G is open, U = f<sup>-1</sup>(V) for some open V  $\subset$  H, and then  $\theta_{\rm G}^{-1}({\rm U}) = \theta_{\rm G}^{-1}({\rm f}^{-1}({\rm V}))$ = (f  $\times$  f)<sup>-1</sup>( $\theta_{\rm H}^{-1}({\rm V})$ ),

which is open in  $\mathfrak{D}_{G}$ . The continuity of the inverse function is proved similarly and the commutativity of the diagrams :

#### <u>S4</u>. <u>Locally trivial topological groupoids</u>.

In this section, we introduce Ehresmann's notion of locally trivial topological groupoids, derive some of their basic properties and give some examples of these groupoids. Locally trivial topological groupoids are, essentially, topological groupoids for which the initial and final maps have local sections, and the existence of such sections leads to a local product structure on the elements.

<u>2.4.1</u>. <u>Definition</u>. (See Ehresmann [1] for example).

Let G be a topological groupoid over X. Then G is said to be <u>locally trivial</u> if there is an indexed family  $\{U_{\alpha}, x_{\alpha}, \lambda_{\alpha}\}_{\alpha \in I}$  where:i)  $\{U_{\alpha}\}_{\alpha \in I}$  is an open cover of X and  $x_{\alpha} \in U_{\alpha}$  for each  $\alpha \in I$ . ii)  $\lambda_{\alpha}$  is a continuous function mapping  $U_{\alpha}$  into G, for each  $a \in I$ , and  $\lambda_{a}(x) \in G(x, x_{a})$  for all  $x \in U_{a}$ . Thus,  $\lambda_{a}$  is a local section of the map  $\pi : \operatorname{cost}_{G} x_{a} \longrightarrow X$ .

We call the family  $\{U_{\alpha}, x_{\alpha}, \lambda_{\alpha}\}_{\alpha \in I}$  a <u>local trivialisation</u> of G, and call the cover  $\{U_{\alpha}\}$  a <u>trivialising cover</u>. We call the sets  $U_{\alpha}$  <u>trivialising neighbourhoods</u> and say G <u>trivialises over</u>  $U_{\alpha}$ .

We shall say that G is <u>globally trivial</u> if there is a distinguished point  $x_o \in X$  and a continuous function  $\lambda : X \longrightarrow G$ such that  $\lambda(x) \in G(x,x_o)$  for all  $x \in X$ . Thus, if G is locally trivial and  $U_a$  is a trivialising neighbourhood in X, then  $G(U_a)$  the full subgroupoid of G over  $U_a$  - is globally trivial.

If G has a trivialising cover  $\{U_{\alpha}, x_{\alpha}, \lambda_{\alpha}\}_{I}$ , we can always assume that  $\lambda_{\alpha}(x_{\alpha}) = I_{x_{\alpha}}$  for each  $\alpha \in I$ , for, if  $\mu_{\alpha}$ :  $U_{\alpha} \longrightarrow G$ is continuous and  $\mu_{\alpha}(x) \in G(x, x_{\alpha})$  for all  $x \in U_{\alpha}$ , then

 $\lambda_{\alpha}$  :  $U_{\alpha} \longrightarrow G$ 

defined by  $\lambda_{\alpha}(x) = \mu_{\alpha}(x_{\alpha})^{-1}\mu_{\alpha}(x)$  is clearly continuous,  $\lambda_{\alpha}(x) \in G(x, x_{\alpha})$ and  $\lambda_{\alpha}(x_{\alpha}) = I_{x_{\alpha}}$ . We shall, in future, always suppose that  $\lambda_{\alpha}(x_{\alpha}) = I_{x_{\alpha}}$ , for all  $\alpha \in I$ .

We also remark that from a trivialising cover  $\{U_{\alpha}, x_{\alpha}, \lambda_{\alpha}\}$  we can, merely by composing  $\lambda_{\alpha}$  with the inverse function, obtain a "trivialising cover"  $\{U_{\alpha}, x_{\alpha}, \mu_{\alpha}\}$  with  $\mu_{\alpha}(x) \in G(x_{\alpha}, x)$  for all  $x \in U_{\alpha}$ . The converse is also true. It will be a matter of convenience whether we choose  $\lambda_{\alpha}(x) \in G(x, x_{\alpha})$  or  $\lambda_{\alpha}(x) \in G(x_{\alpha}, x)$ , although we normally employ the former sense.

Finally, we remark :-

2.4.2.

Suppose  $\{U_{a}, x_{a}, \lambda_{a}\}_{T}$  is a trivialising cover for G so that

is continuous and  $\lambda_{\alpha}(x) \in G(x, x_{\alpha})$  for all  $x \in U_{\alpha}$ . Let  $x_{\beta}$  be any point of  $U_{\alpha}$ , let  $\tau_{\alpha\beta} \in G(x_{\alpha}, x_{\beta})$  (which is non empty since  $\lambda_{\alpha}(x_{\beta})^{-1} \in G(x_{\alpha}, x_{\beta})$ ), let  $U_{\beta} = U_{\alpha}$  and define  $\mu_{\beta}$  :  $U_{\beta} \longrightarrow G$  by  $\mu_{\beta}(x) = \tau_{\alpha\beta} \lambda_{\alpha}(x)$ .

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Then  $\mu_{\beta}$  is continuous,  $\mu_{\beta}(x) \in G(x, x_{\beta})$  for all  $x \in U_{\beta}$  and  $\{U_{\beta}, x_{\beta}, \mu_{\beta}\}_{I}$  is another local trivialisation of G. So we see that within  $U_{\alpha}$  the choice of  $x_{\alpha}$  is arbitrary.

Next, suppose that  $\mathcal{U} = \{U_j; j \in J\}$  is an open base for the topology of X, so that each  $U_\alpha$  can be written  $U_\alpha = \bigcup_{\substack{j \in J_\alpha \\ j \in J_\alpha}} J_j$  for some subset  $J_\alpha \subseteq J$ . By choosing  $x_j \in U_j$  for  $j \in J$  and restricting  $\lambda_\alpha$ , we obtain a trivialisation  $\{U_j, x_j, \lambda_j\}_J$  for G where the  $U_j$  are basic open sets. This fact will be needed later on.

The following result, though simple, is quite important. 2.4.3. Proposition.

Let G be a locally trivial topological groupoid over X, let  $x_o \in X$  and let  $C_{x_o}$  be the transitive component of G determined by  $x_o$  in the sense of 1.3.3. Then the object set of  $C_{x_o}$  is both open and closed in X.

Proof.

Let  $Y_{x_o} = ob(C_{x_o})$ , let  $\{U_a, x_a, \lambda_a\}_I$  be a trivialising cover of X, and suppose  $Y_{x_o} \cap U_a \neq \phi$ . Let  $y \in Y_{x_o} \cap U_a$ and let  $x \in U_a$ , so that  $\lambda_a(x) \in G(x, x_a)$  and  $\lambda_a(y) \in G(y, x_a)$ , whence  $\lambda_a(x)^{-1} \lambda_a(y) \in G(y, x)$  and so  $x \in Y_{x_o}$ . Thus, if  $Y_{x_o} \cap U_a \neq \phi$ we must have  $U_a \subseteq Y_{x_o}$ . However,  $\{U_a\}_I$  certainly covers  $Y_{x_o}$  and, consequently,  $Y_{x_o}$  is a union of some of the  $U_a$  and is, therefore, open. Since the object sets  $Y_{x_o}$  of the components  $C_{x_o}$  partition X, they must be closed in X also.

#### Corollary 1.

If X is connected and G is a locally trivial groupoid over X, then G is transitive.

If G is a locally trivial groupoid over X, then G is the topological sum of its transitive components. <u>Proof</u>.

By 2.4.3., the object sets of the transitive components of G are both open and closed in X, so, by 2.1.3. b), each transitive component is both open and closed in G. Thus, G is the topological sum of the transitive components.

Notice that the converse of corollary 1 is false, as the following example shows. Let 9 be the tree groupoid on  $\{0,1\}$ , let  $\{0,1\}$  have the discrete topology and let  $\{0,1\} \times \{0,1\}$  have the product of the discrete topologies, which is, of course, discrete. Now let  $\phi$ :  $9 \longrightarrow \{0,1\} \times \{0,1\}$  be the natural bijection as in 2.2.4 and use  $\phi$  to topologise 9. Then 9 is a topological tree groupoid as shown in 2.2.4 and is abstractly connected. It is, moreover, globally trivial, for we can distinguish  $0 \in \{0,1\}$  and define

> $\lambda : \{0,1\} \longrightarrow \mathcal{G} ,$ by  $\lambda (0) = I_0$  $\lambda (1) = i^{-1} \in \mathcal{G}(I,0) ;$

then  $\lambda$  is continuous since {0,1} is discrete. However, {0,1} is not connected of course.

Proposition 2.4.3 also yields :

2.4.4. Proposition.

Let G be a topological groupoid over a connected space X .

Then G is locally trivial if, and only if, there is a distinguished point  $x_{o} \in X$ , an open cover  $\{U_{a}\}_{I}$  of X and continuous functions  $\mu_{a}: U_{a} \longrightarrow G$  such that  $\mu_{a}(x) \in G(x, x_{o})$  for all  $x \in U_{a}$ . <u>Proof</u>.

For the necessity, suppose  $\{U_a, x_a, \lambda_a\}_I$  is a trivialising cover of G then, since G is transitive by 2.4.3, we can distinguish  $x_o \in X$  and choose elements  $\tau_a \in G(x_a, x_o)$ . Now define  $\mu_a : U_a \longrightarrow G$ by  $\mu_a(x) = \tau_a \lambda_a(x)$  to obtain the conclusion.

Conversely, given an open cover  $\{U_{\alpha}\}_{I}$  of X and continuous functions  $\mu_{\alpha}: U_{\alpha} \longrightarrow G$  such that  $\mu_{\alpha}(x) \in G(x, x_{o})$  for all  $x \in U_{\alpha}$ then, since G must be transitive, we can choose  $x_{\alpha} \in U_{\alpha}$  and  $T_{\alpha} \in G(x_{o}, x_{\alpha})$ , for all  $\alpha \in I$ . Now define  $\lambda_{\alpha}: U_{\alpha} \longrightarrow G$  by  $\lambda_{\alpha}(x) = T_{\alpha} \mu_{\alpha}(x)$  to obtain a local trivialisation  $\{U_{\alpha}, x_{\alpha}, \lambda_{\alpha}\}_{I}$  for G.

For connected spaces X , or more generally for transitive groupoids G , we can regard 2.4.4 as an equivalent definition of local triviality.

- 2.4.5. Proposition.
- a) A finite product  $\prod_{i=1}^{n} G_i$  of topological groupoids is locally i=1 trivial if, and only if, each of the  $G_i$  is locally trivial.
- b) A product  $\mathbb{M}G_i$  of a family  $\{G_i; i \in I\}$  of topological groupoids is locally trivial if each  $G_i$  is locally trivial, and all but a finite number are globally trivial.

c) If a product ΠG<sub>i</sub> of a family {G<sub>i</sub>; i ε I} of topological groupoids is locally trivial, then each G<sub>i</sub> is locally trivial.
 <u>Proof</u>.

a) Suppose  $G_{i}$ , i = 1, 2, ..., n, is locally trivial and  $\begin{cases} U_{a_{i}}, x_{a_{i}}, \lambda_{a_{i}} \\ i & a_{i} & e A_{i} \end{cases}$ is a local trivialisation of  $G_{i}$ , 1,2,..., n.

It is clear that  $\left\{ \begin{array}{c} U_{a_1} \times U_{a_2} \times \cdots \times U_{a_n}, (x_{a_1}, x_{a_2}, \cdots, x_{a_n}), \lambda_{a_1} \times \lambda_{a_2} \times \cdots \times \lambda_{a_n} \right\}$ is a local trivialisation of  $\prod_{i=1}^{n} G_i$  indexed by  $A_i \times A_2 \times \cdots \times A_n$ . The converse of a) will follow from c). Let  $G_{i_1}, G_{i_2}, \dots, G_{i_n}$  be locally trivial for b)  $i_i, i_2, \dots, i_n \in I$  and suppose  $G_i$  is globally trivial for  $i \neq i_1, i_2, \dots, i_n$ . Let  $X_i = ob(G_i)$ . Let  $\{U_{\alpha_i}, x_{\alpha_i}, \lambda_{\alpha_i}\}$  $i_j = i_j A_i$ be a local trivialisation of  $G_{i}$ , for j = 1, 2, ..., n, and let  $\{X_i, x_i, \lambda_i\}$  be a global trivialisation of  $G_i$  for  $i \neq i_1, i_2, \dots, i_n$ . Then.  $\left\{ \begin{array}{ccc} U_{a_{i}} \times & U_{a_{i}} \times & \cdots \times & U_{a_{i}} \times & \prod & X_{i}, & (x_{i}), \\ & & & & & \\ & & & & \\ & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & &$ where  $x_i \in \{x_{a_i}\}$  for  $i = i_j$  and  $j = 1, 2, \dots, n$ , is a local trivialisation of  $TTG_i$  indexed by  $A_i \times A_i \times \cdots \times A_i$ . Let  $ob(G_i) = X_i$  for all  $i \in I$  and suppose c)  $\{U_{a}, (x_{i})_{a}, \lambda_{a}\}$  is a local trivialisation of  $TG_{i}$ . By 2.4.2, we can suppose that the sets U are basic open sets, so that U is of the form  $U_i \times U_i \times \cdots \times U_i \times \Pi X_i$ , where  $U_i$  is open in  $i \neq i_1, \dots, i_n$ .  $X_{i_i}$ ,  $j = 1, \dots, n$ . Now  $\lambda_a : U_a \longrightarrow TIG_i$  is continuous and is such that  $\lambda_{a}((y_{i})) \in G((y_{i}), (x_{i})_{a})$  for all  $(y_{i}) \in U_{a}$ , where  $G = \pi G_{i}$ . Choose an index  $j \in I$  and let  $P_j$  denote the projection on the j<sup>th</sup> factor, then

$$T_{a} = P_{j}(U_{a})$$

is an open set in  $X_j$  containing  $x_a = P_j((x_i)_a)$ . The choice of  $U_a$ 

as a basic open set permits us to define a local section

 $S_{a} : V_{a} \longrightarrow U_{a}$ of  $P_{j}$  by  $S_{a}(y) = (y_{i})$ , with  $y_{i} = P_{i}((x_{i})_{a})$  for  $i \neq j$  and  $y_{j} = y$ . Finally, we define

 $\mu_{a} : V_{a} \longrightarrow G_{j}$ 

by " $\mu_{a}$  is the composite  $P_{j}\lambda_{a}S_{a}$ ", where  $P_{j}$  in this composite denotes the projection  $TTG_{j} \longrightarrow G_{j}$ . Since the sets  $V_{a}$  cover  $X_{j}$ ,  $\{V_{a}, x_{a}, \mu_{a}\}$  is a local trivialisation of  $G_{j}$ . This completes the proof of the proposition.

We also have an obvious result concerning "sums".

## 2.4.6.

A sum  $\bigcup_{\alpha \in I} G_{\alpha}$  of topological groupoids is locally trivial if, and only if each  $G_{\alpha}$  is locally trivial.

We shall need this result later in Chapter 3.

2.4.7. Some examples of locally trivial groupoids.

i) Any topological group is a globally trivial topological groupoid, and so is the groupoid consisting of the union of a family

 $\{G_a; a \in I\}$  of topological groups topologised as in 2.2.1.

- ii) If X is a differentiable manifold, and G(X) is the groupoid of 2.2.2, then G(X) is a locally trivial groupoid, but not generally a globally trivial one. See Chapter 3 for details.
- iii) If T is a tree groupoid over X topologised as in 2.2.4, then T is globally trivial since the projection  $P_1 : X \times X \longrightarrow X$ has a section s, determined by a point  $x \in X$ , in the sense that  $s_1(x) = (x, x_0)$ .
- iv) If X is an arcwise connected arcwise locally connected space, then the fundamental groupoid  $\pi X$  is locally trivial. For details see Danesh-Naruei [1], although this fact also follows from our

results of Chapter 3 and the fact that  $\pi X$  is isomorphic with  $G(\tilde{X})$ , where  $\tilde{X}$  is the universal covering space of X. §5. Principal "H-spaces" and topological groupoids.

Suppose S is a topological space and H a topological group acting continuously on the right of S, that is, we have a continuous action  $S \times H \xrightarrow{\cdot} S$  of H on S. Suppose H acts effectively, as defined in Chapter 1, and let  $S^*$  be defined by :-

 $S^* = \{(s,s,h) \in S \times S; h \in H\} \subset S \times S.$ 

The effectiveness of the action of H means that there is a function  $t : S^* \longrightarrow H$  defined by

and called the <u>translation</u> function. Give  $S^*$  the subspace topology of  $S \times S$ ; following Husemoller [1] we make :

2.5.1. Definition.

S is a <u>principal</u> H-space if the translation function t : S — H is continuous.

We have :-

2.5.2. Proposition.

Let G be a transitive topological groupoid over X and let  $x_o \in X$ . Then  $St_G x_o$  is a principal  $G\{x_o\}$  space. Proof.

As shown in Chapter 1, there is a natural effective action

$$\operatorname{St}_{G} x_{\circ} \times G\{x_{\circ}\} \longrightarrow \operatorname{St}_{G} x_{\circ}$$

induced by the composition in G. Since the composition is continuous, this action is continuous.

Let  $S = St_G x_{\circ}$ , then  $S^* = \{(\beta, \beta \alpha); \alpha \in G\{x_{\circ}\}\}$  and t :  $S^* \longrightarrow G\{x_{\circ}\}$  is defined by  $t((\beta, \beta')) = \beta^{-1}\beta'$ , which is continuous since the composition and inverse maps are continuous. We say that an action of H on S is <u>locally transitive</u> if there is a covering  $\{U_i\}$  of S by open sets  $U_i$  such that for any pair s, s'  $\in U_i$ , there exists h  $\in$  H with the property s.h = s.

## 2.5.3. Proposition.

Let G be a topological group acting effectively and continuously on the right of the space S, and let  $\widetilde{G}$  be the groupoid of 2.2.5. Then  $\widetilde{G}$  is locally trivial if, and only if, S is a principal G-space which is locally transitive.

#### Proof.

For the necessity, let  $\{U_{i}, s_{i}, \lambda_{i}\}$  be a local trivialisation of  $\tilde{G}$ . Then, given s,  $s' \in U_{i}$ , we have  $\lambda_{i}(s')^{-1}\lambda_{i}(s) \in \tilde{G}(s, s')$  and so S is locally transitive with respect to the covering  $\{U_{i}\}$ . The sets  $S_{ij}^{*}$  $= U_{i} \times U_{j} \cap S^{*}$  form an open covering of  $S^{*}$ . Suppose  $S_{ij}^{*} \neq \phi$ , then  $\tilde{G}(s_{i}, s_{j})$  $\neq \phi$ , let  $h \in \tilde{G}(s_{i}, s_{j})$ . The function  $t : S_{ij}^{*} \rightarrow G$  can be resolved into the sequence  $S_{ij}^{*} \longrightarrow \tilde{G} \times \tilde{G} \xrightarrow{\text{composition}} \tilde{G} \xrightarrow{P_{2}} G$  $(s, s') \longmapsto (\lambda_{i}(s), \lambda_{j}(s')) \longmapsto \lambda_{j}(s')^{-1} h \lambda_{i}(s) \longmapsto t(s, s')$ and so t is continuous and S is a principal G-space.

Conversely, suppose S is locally transitive with respect to the covering  $\{U_i\}$  of S and t is continuous. For each index i choose  $s_i \in U_i$  and define  $\lambda_i$ :  $U_i \longrightarrow \widetilde{G}$  by  $\lambda_i(s) = (s,t(s,s_i))$ , to obtain the local trivialisation  $\{U_i, s_i, \lambda_i\}$  of  $\widetilde{G}$ .

Notice that if G acts effectively, then  $\widetilde{G}$  is principal as defined in 1.2.5.

2.5.4. Remark. Suppose we have a topological group H acting continuously on the right of a space S. Following Husemoller [1], we give the orbit set X the quotient topology by the natural surjection r :  $S \longrightarrow X$ ; then r is continuous and, in fact, open. We define an Hmorphism g of two H-spaces S and S', as in 1.4.2 except that we now ask for g to be continuous. We then call a continuous surjection  $P: S \longrightarrow B$  an H-bundle, if there is a continuous H-space structure on S and a homeomorphism  $f: B \longrightarrow S/H$  such that the diagram



is commutative, where I is the identity map  $S \longrightarrow S$ . If (S, P, B) is an H-bundle, then it follows that P must be open and, hence, an identification map. Finally, we call an H-bundle (S, P, B) a principal H-bundle if the H-space structure on S is effective and the translation function t :  $S^* \longrightarrow H$  is continuous.

Now suppose G is any transitive topological groupoid over X and let  $x_{c} \in X$ . By 2.5.2,  $St_{G}x_{c}$  is a principal  $G\{x_{c}\}$  space, and  $\pi'$ :  $St_{G}x_{c} \longrightarrow X$  is a continuous surjection. However,  $\pi'$ :  $St_{G}x_{c} \longrightarrow X$  is not generally a principal  $G\{x_{c}\}$  bundle, since  $\pi'$  is not generally a quotient map. An example to show this can be given as follows. Let G be a group acting transitively on a set S which contains at least 2 elements (so that G cannot be the trivial group),  $S_{3}$  acting on  $\{1,2,3\}$  - say. Give G the discrete topology and S the indiscrete topology, then G is a topological group and any action of G on S is continuous. Now form the topological groupoid  $\widetilde{G}$  over S and let  $s_{c} \in S$ . Clearly,  $St_{G}s_{c} = \{s_{c}\} \times G$  and so  $St_{G}s_{c}$  receives the discrete topology as a subspace of  $S \times G$ . It follows, now, that  $\pi'$ :  $St_{G}s_{c} \longrightarrow S$  is continuous but is neither open nor a quotient map.

By relaxing the condition that P be a quotient map, in our previous definitions, we do obtain that  $(St_Gx_o, \pi', X)$  is a "principal  $G\{x_o\}$  bundle" in an Obvious weaker sense than that above.

Chapt	er 3.	LOCALLY	TRIVIAL	TOPOLOGICAL	GROUPOIDS	
80.	Introdu	ction.	e street			•••

This chapter is concerned with a fairly detailed study of certain aspects of locally trivial topological groupoids. The main purpose of this study is to clarify the structure of locally trivial topological groupoids in preparation for establishing the results of the later chapters. For example, one construction of a "Haar system" of measures, in Chapter 4, depends on the local product structure of a locally trivial topological groupoid. Furthermore, "admissible maps" arise naturally in dealing with "representations" of groupoids and "actions" of groupoids on fibre spaces. A secondary purpose is to use our results to obtain a homotopy classification of certain locally trivial groupoids.

Yet another purpose of this chapter is to clarify, complete and extend the work of Ehresmann and Danesh-Naruei [1] on locally trivial topological groupoids. In particular, Definition 3.3.1 and its associated results are new, as are the results of Sections 4, 5 and 6. In fact, it appears that much of Ehresmann's work, though used by others, has not been set down in detail. However, Danesh-Naruei [1] presents a detailed account of the results of §1 and §2, and these sections are almost entirely contained in his work, though they are formulated in a different way. In fact, we only include in §1 and §2 those results we need later on.

Whilst it is not our intention to develop a comprehensive theory of fibre bundles with structural sheaf, we need to investigate certain concepts within such a theory. Our definition of bundle morphism is more general than that of Husemoller [1], as indeed are certain associated results; this is true even in the particular case of bundles with structural group, to the extent that we need to work with respect to a given group homomorphism. Our basic reference to standard fibre bundle theory is Husemoller [1], although we shall occasionally refer to

Steenrod [1] .

## <u>§1.</u> <u>Cocycles and Transition functions</u>.

The following proposition is a convenient and obvious reformulation of local triviality.

#### 3.1.1. Proposition.

Let G be a transitive topological groupoid over X. Then G is locally trivial if, and only if, there is a distinguished point  $x_o \in X$ , an open cover  $\{U_i\}_{i \in I}$  of X and continuous functions  $\lambda_i : U_i \longrightarrow G$  such that  $\lambda_i(x) \in G(x, x_o)$  for each  $x \in U_i$ . <u>Proof</u>.

This is, more or less, the same argument as used in proving 2.4.4.

The system  $\{U_i, \lambda_i, x_o; i \in I\}$  will be called a <u>local trivia</u>-<u>lisation of G based at x</u>.

3.1.2.

Now suppose G and  $\{U_{i}, \lambda_{i}, x_{o}; i \in I\}$  are as in 3.1.1 and define  $g_{ji} : U_{i} \cap U_{j} \longrightarrow G\{x_{o}\}$  by  $g_{ji}(x) = \lambda_{j}(x)\lambda_{i}(x)^{-1}$ , for those pairs i,j of indices such that  $U_{i} \cap U_{j} \neq \emptyset$ . The continuity of the composition, inverse and the  $\lambda_{i}$  implies the continuity of the  $g_{ji}$ , and it is readily seen that the system  $\{g_{ji}; j, i \in I\}$  is a system of transition functions in the sense of Husemoller [1]; that is, the functions  $g_{ji}$  satisfy the cocycle condition  $g_{\kappa i}(x) = g_{\kappa j}(x)g_{ji}(x)$ , for all i, j,  $\kappa \in I$  and x an element of  $U_{i} \cap U_{j} \cap U_{\kappa}$ . We call the system  $\{g_{ji}; i, j \in I\}$  the "system of transition functions determined by the local trivialisation  $\{U_{i}, \lambda_{i}, x_{o}; i \in I\}$  of G".

There is a notion of equivalence of two systems  $\{g_{ji}\}$  and  $\{g_{ji}\}$  of transition functions, see Husemoller [1].

§2. Atlases and locally trivial topological groupoids.

Locally trivial topological groupoids have a local product

structure as far as their topology is concerned; this is shown in the (sketch) proof of the following theorem of Ehresmann. (Some details of the proof will be needed later).

3.2.1. Theorem. (see Danesh-Naruei [1]).

Let G be a transitive locally trivial topological groupoid over X and let  $x_o \in X$ . Then  $\pi'$ :  $St_G x_o \longrightarrow X$  is a locally trivial bundle in the sense of Chapter 2, Husemoller [1]. <u>Proof</u>.

Let  $\{U_i, \lambda_i, x_i; i \in I\}$  be a local trivialisation for G based at x, and define

 $\phi_i$ :  $U_i \times G\{x_o\} \longrightarrow \pi'^{-1}(U_i) \cap St_{G}x_o$ 

by  $\phi_i(x,a) = \lambda_i(x)^{-1} a$ . The diagram



commutes, where  $P_i$  is the projection on the first factor, and it is easily seen that  $\phi_i$  is a homeomorphism. Thus the theorem follows.

If  $(B \times G\{x_o\}, P, B)$  denotes the product princiapl  $G\{x_o\}$ bundle over B, then we regard  $B \times G\{x_o\}$  as a right (principal)  $G\{x_o\}$  - space where

 $(B \times G\{x_{\circ}\}) \times G\{x_{\circ}\} \xrightarrow{\bullet} B \times G\{x_{\circ}\}$ is defined by  $(b,a) \cdot \beta = (b,a\beta)$ . Then, if  $\phi_{i} : U_{i} \times G\{x_{\circ}\} \xrightarrow{\bullet} \pi^{\bullet-1}(U_{i}) \cap St_{G}x_{\circ}$  is defined as in 3.2.1, it follows that  $\phi_{i}$  is a  $G\{x_{\circ}\} - map$ . That is to say,  $\phi_{i}((x,a) \cdot \beta) = \phi_{i}(x,a) \cdot \beta$ . Thus,  $(\phi_{i}, U_{i})$  is a chart for  $St_{G}x_{\circ}$  over  $U_{i}$  and so a local trivialisation  $\{U_{i}, \lambda_{i}, x_{\circ}; i \in I\}$  of G induces, in a natural way, an atlas  $\{U_{i}, \phi_{i}\}_{i \in I}$  of charts for  $St_{G}x_{\circ}$ .

Note also that the system  $\left\{ \textbf{g}_{ji} \right\}$  of transition functions
associated with this atlas is given by  $g_{ji}(x) = \lambda_j(x) \lambda_i(x)^{-1}$ , and thus coincides with the system of 3.1.2.

Recall that we showed in Chapter 2, that, if G is any transitive topological groupoid over X,  $St_Gx_o$  need not be a principal  $G\{x_o\}$  bundle as defined by Husemoller [1]. However, we now have: 3.2.2. Corollary.

If G is a transitive locally trivial topological groupoid over X and  $x \in X$ , then  $St_G x_o$  is a locally trivial principal  $G\{x_o\}$  - bundle over X.

Proof.

The commutativity of the diagram



in 3.2.1, means that  $\pi'$  is locally open (since P, is) and, hence, is an open map. Consequently,  $\pi'$  is a quotient map.

3.2.3.

Let G be a transitive locally trivial topological groupoid and let  $\{U_{i}, \lambda_{i}, x_{o}\}$  and  $\{U_{i}, \lambda'_{i}, x_{o}\}$  be two local trivialisations for G both based at  $x_{o}$ ; by considering common refinements, we can suppose they both have the same coordinate neighbourhoods. If  $g_{ji}$ and  $g'_{ji}$  denote their respective transition functions (3.1.2), then we have  $g_{ji}(x) = \lambda_{j}(x)\lambda_{i}(x)^{-1}$  and  $g'_{ji}(x) = \lambda'_{j}(x)\lambda'_{i}(x)^{-1}$ . Now define  $\mu_{i}: U_{i} \longrightarrow G\{x_{o}\}$  by  $\mu_{i}(x) = \lambda_{i}(x)\lambda'_{i}(x)^{-1}$ , then  $\mu_{i}$  is continuous for each i and it is readily seen that we have the relation  $g'_{ji}(x) = \mu_{j}(x)^{-1}g_{ji}(x)\mu_{i}(x)$ 

holding for each pair of indeces i, j and all  $x \in U_{j} \cap U_{j}$ . Thus, by 2.6 of Husemoller [1] any two local trivialisations of G give equivalent systems of transition functions and, hence, give isomorphic

locally trivial principal bundle structures to Stax.

We also have a theorem concerning costars which is analogous to 3.2.1.

3.2.4. Theorem.

Let G be a transitive topological groupoid over X and let  $x_o \in X$ . Then  $cost_G x_o$  is a left principal  $G\{x_o\}$  - space. If G is locally trivial, then  $\pi : cost_G x_o \longrightarrow X$  is a locally trivial left principal  $G\{x_o\}$  - bundle.

#### Proof.

The first assertion is proved in exactly the same fashion as we proved that  $St_Gx_o$  is a principal  $G\{x_o\}$ -space in Chapter 2, making the obvious necessary changes in our definitions.

For the second assertion; using the choice of local trivialisation as in the proof of 3.2.1 we define

 $\phi_{i}: G\{x_{o}\} \times U_{i} \longrightarrow \pi^{-1}(U_{i}) \land \operatorname{cost}_{G} x_{o} \text{ by } \phi_{i}(a, x) = a \lambda_{i}(x)$ 

One easily checks that  $\phi_i$  is a fibre preserving  $G\{x_o\}$ -homeomorphism and so the result follows.

Notice that the transition functions  $\{h_{ji}\}$  - say - associated with the atlas  $\{U_i, \phi_i\}$  are given by  $h_{ji}(x) = g_{ji}(x)^{-1}$ , where  $g_{ji}(x)$ is defined as in 3.1.2. Thus, two different local trivialisation yield equivalent systems of transition functions and, hence, isomorphic locally trivial principal bundle structures on cost  $g_{i}^{x}$  .

Finally, we state the following theorem of Ehresmann. 3.2.5. Theorem. (See Danesh-Naruei [1]).

Let G be a transitive locally trivial topological groupoid over X and let  $x_o \in X$ . Then G can be given the structure of a coordinate bundle (see Steenrod [1]) over X × X with projection  $(\pi,\pi'): G \longrightarrow X \times X$ , fibre  $G\{x_o\}$  and group  $G\{x_o\} \times G\{x_o\}$ acting (possibly ineffectively) by left and right (inverse) multiplication.

#### Proof.

We omit the details which are similar to those of the previous Theorems 3.2.1 and 3.2.4. Details can, in fact, be found in Danesh-Naruei [1].

#### §3. Fibre Bundles and Admissible Maps.

In this section, we will introduce our notion of admissible map between fibres of a fibre bundle, which will generalise the definition of Chapter 1, and obtain a groupoid of admissible maps. This definition will also generalise that of Steenrod [1], which is the one used by Ehresmann, (see also Westman [1]), in that our definition does not need local triviality. In the case of a locally trivial fibre bundle, the groupoid of admissible maps has a natural topology (Ehresmann) which makes it a locally trivial topological groupoid, see Westman [1].

Let (S, P, B) be a principal H-bundle as in Husemoller [1] (or Chapter 2) and suppose  $H \times F \xrightarrow{\bullet} F$  is a continuous action of the group H on the left of a space F. Then there is a natural continuous action  $(S \times F) \times H \xrightarrow{\bullet} S \times F$  defined by  $(s,f) \cdot h = (s \cdot h, h^{-1}, f)$  of H on the right of  $S \times F$ . Now, following Husemoller [1], we set  $S_F = (S \times F)/H$  topologised as a quotient with respect to the canonical surjection  $r : S \times F \longrightarrow S_F$ . Finally, define  $P_F : S_F \longrightarrow B$  by  $P_F((s,f) \cdot H) = P(s)$  to obtain "the fibre bundle  $(S_F, P_F, B)$  over B with fibre F, group H and associated principal H-bundle (S, P, B)".

For a point  $b \in B$ , the fibre  $P_F^{-1}(b)$  over b is given by

$$P_{F}^{-1}(b) = \{(s,f), H; p(s) = b, f \in F\}$$

and it is easily seen that

$$P_{F}^{-1}(b) = \{(s_{o}, f), H; f \in F\}$$

where s<sub>o</sub> is some fixed element of  $P^{-1}(b)$ . Furthermore, given such a fixed element s<sub>o</sub>  $\epsilon P^{-1}(b)$ , we have a homeomorphism

$$K(s_{o},b): F \longrightarrow P_{F}^{-1}(b)$$

define by  $\kappa(s_{s_{1}},b)(f) = (s_{s_{1}},f)\cdot H$ . Now, given a map  $\gamma : P_{F}^{-1}(b_{1}) \longrightarrow P_{F}^{-1}(b_{2})$  and elements  $s_{1} \in P^{-1}(b_{1})$  and  $s_{2} \in P^{-1}(b_{2})$ , we have the diagram

$$P_{F}^{-1}(b_{1}) \xrightarrow{h} P_{F}^{-1}(b_{2})$$

$$((s_{1},b_{1})) \xrightarrow{h} F$$

$$F \xrightarrow{h} F$$

where h is the map induced on the fibre making the diagram commute. Due to the presence of both left and right actions of the group H, a suitable, canonical definition of admissible map in the spirit of 1.4.6 does not seem possible. We propose the

<u>3.3.1</u>. Definition.

A map  $\gamma : P_F^{-1}(b_1) \longrightarrow P_F^{-1}(b_2)$  is said to be <u>admissible</u> if, given any pair of elements  $s_1 \in P^{-1}(b_1)$ ,  $s_2 \in P^{-1}(b_2)$ , there exists an element  $h(s_1, s_2) \in H$  such that the induced map  $\overline{\gamma} : F \longrightarrow F$  is defined by  $\overline{\gamma}(f) = h(s_1, s_2)$  of for all  $f \in F$ . Thus,  $\overline{\gamma}$  corresponds to the operation of the element  $h(s_1, s_2)$  on the left of F. <u>3.3.2</u>.

We shall first show that if  $\gamma : P_F^{-1}(b_1) \longrightarrow P_F^{-1}(b_2)$ satisfies the Definition 3.3.1 for one choice of  $s_1, s_2$ , then it does so with respect to any other choices.

So suppose  $\kappa(s_2, b_2)^{-1} \ \ \ \kappa(s_1, b_1) : F \longrightarrow F$  corresponds to left action of the element  $g \in H$ , and let  $s'_i \in P^{-1}(b_i)$  and  $s'_2 \in P^{-1}(b_2)$  be any other choice of elements of S. If we let  $\omega_i$ be the unique element of H such that  $s'_i = s_i \cdot \omega_i$ , then we have, by definition of  $\kappa(s'_i, b_i)$ , that  $\kappa(s'_i, b_1)(f) = (s'_1, f) \cdot H$ 

$$= (s_1 \cdot \omega_1, f) \cdot H$$
  
=  $((s_1, \omega_1, f) \cdot \omega_1) \cdot H$   
=  $(s_1, \omega_1, f) \cdot H$   
=  $(s_1, \omega_1, f) \cdot H$   
=  $K(s_1, b_1)(\omega_1, f)$ .

Similarly, if  $\omega_2 \in H$  is the unique element of H such that  $s'_2 = s_2 \cdot \omega_2$ , then we have  $K(s'_2, b_2)(f) = K(s_2, b_2)(\omega_2 \cdot f)$ . Thus,

$$\begin{split} \kappa (s'_{2}, b_{2})^{-1} \mathcal{H} & \kappa (s'_{1}, b_{1})(f) = \kappa (s'_{2}, b_{2})^{-1} \mathcal{H}((s_{1}, \omega_{1}, f), H) \\ & = \kappa (s'_{2}, b_{2})^{-1} \mathcal{H} & \kappa (s_{1}, b_{1})(\omega_{1}, f) \\ & = \kappa (s_{2}, b_{2})^{-1} \mathcal{H} & \kappa (s_{1}, b_{1})(\omega_{2}, \omega_{1}, f) \\ & = g_{*}(\omega_{2}^{-1}\omega_{1}, f) \\ & = (g_{*}\omega_{1}^{-1}\omega_{1}) \cdot f \end{split}$$

And so  $K(s'_{2}, b_{2})^{-1} \succ K(s'_{1}, b_{1})$  also corresponds to left action of an element of H.

We show next that , ignoring the topology, the Definition 3.3.1 does generalise our definition of Chapter 1. To do this, we first make the following observation. Suppose (S, P, B) is any principal H-bundle and we let H act on itself by left multiplication, then we can form the fibre bundle  $(S_H, P_H, B)$  with group and fibre H and associated principal H-bundle (S, P, B). There is a natural identification of (S, P, B) and  $(S_H, P_H, B)$  defined by

> $S \longrightarrow S_H$ s  $\longmapsto$  (s, t(s,s)).H = (s,e).H

where t denotes the translation function (see §5 of Chapter 2) and e denotes the identity of H. If we fix  $s_o \in P^{-1}(b)$ , this map can be regarded as the map  $s \longleftrightarrow (s_o, t(s_o, s))$ . H and then, under this identification, the map  $\ltimes(s_o, b)$  becomes

$$(s_{o},b) : H \longrightarrow P^{-1}(b)$$

$$(s_{o},b)(h) = s_{o} \cdot h \cdot b$$

defined by

We shall show that  $\gamma : P^{-1}(b_1) \longrightarrow P^{-1}(b_2)$  is admissible in the sense of 1.4.6 if, and only if,  $\kappa(s_2, b_2)^{-1} \gamma \kappa(s_1, b_1) : H \longrightarrow H$ corresponds to left multiplication by an element of H, for each choice of  $s_1 \in P^{-1}(b_1)$  and  $s_2 \in P^{-1}(b_2)$ . Suppose first, then, that  $\gamma$  is admissible in the sense of 1.4.6, that  $\gamma(s_1) = \overline{s}$  and let  $\omega$  be the unique element of H such that  $s_2 \cdot \omega = \overline{s}$ .

Now

$$\bar{\eta}(h) = \kappa(s_2, b_2)^{-1} \eta \kappa(s_1, b_1)(h)$$

$$= \kappa(s_2, b_2)^{-1} \eta(s_1, h)$$

$$= \kappa(s_2, b_2)^{-1} (\eta(s_1), h)$$

$$= \kappa(s_2, b_2)^{-1} (s_1, h)$$

$$= \kappa(s_2, b_2)^{-1} (s_1, h)$$

$$= \kappa(s_2, b_2)^{-1} (s_2, w), h$$

$$= \kappa(s_2, b_2)^{-1} (s_2, w)$$

Thus,  $\bar{h}(h) = \omega \cdot h$  where  $\omega$  is an element of H uniquely determined by the choice of s, and s<sub>2</sub>.

Conversly, if  $\kappa(s_2, b_2)^{-1} \prec \kappa(s_1, b_1)(h) = \omega \cdot h$  for some  $\omega$ , then we have:

$$\psi(\mathbf{s} \cdot \mathbf{h}) = \kappa(\mathbf{s}_{2}, \mathbf{b}_{2}) \cdot \omega \cdot \kappa(\mathbf{s}_{1}, \mathbf{b}_{1})^{-1} (\mathbf{s} \cdot \mathbf{h})$$
$$= \kappa(\mathbf{s}_{2}, \mathbf{b}_{2}) \cdot \omega \cdot \mathbf{h}'$$

where h' is such that  $s \cdot h = s_1 \cdot h'$ . Thus,  $h_1(s \cdot h) = s_2 \cdot \omega h'$ . Now  $h_1(s) \cdot h = K(s_2, b_2) \omega K(s_1, b_1)^{-1}(s) \cdot h$   $= K(s_2, b_2) \omega (h' h^{-1}) \cdot h$   $= (s_2 \cdot \omega h' h^{-1}) \cdot h$  $= s_2 \cdot \omega h'$ .

And, hence, h is admissible in the sense of 1.4.6. Thus our Definition 3.3.1 is a generalisation of Definition 1.4.6.

3.3.3. Proposition.

Let  $(S_F, P_F, B)$  be the fibre bundle with fibre F, group H and associated principal H-bundle (S, P, B). Then:

a) Any admissible map is a bijection.

b) The inverse of an admissible map is admissible.

c) The composition of two admissible maps is admissible.

d) The identity map on any fibre is admissible.

Proof.

The proof of a), b) and d) is straightforward. To prove c), we proceed in the obvious way and make use of 3.3.2.

Proposition 3.3.3 allows us to form the groupoid  $G(S_F)$  of admissible maps between the fibres of  $S_F$  and whose object set is B.

We shall say that a group H acts <u>faithfully</u> on the left of a space F if the relation  $h \cdot f = f$ , for all  $f \in F$ , implies h is the identity of H. This condition is weaker than the condition that H acts effectively as discussed in Chapters 1 and 2.

We now prove that the vertex group  $G(S_F)\{b\}$ , for any  $b \in B$ , is isomorphic with H if, and only if, H acts faithfully on F. To prove this, consider  $\kappa(s,b)^{-1} \downarrow \kappa(s,b)$  for some choice of  $s \in P^{-1}(b)$ and  $\downarrow \in G(S_F)\{b\}$ . Then the function  $H \longrightarrow G(S_F)\{b\}$  defined by  $h \longmapsto \kappa(s,b) h \kappa(s,b)^{-1}$  is easily seen to be a homomorphism of groups, and is surjective by 3.3.1. Suppose that  $\kappa(s,b) h \kappa(s,b)^{-1} =$  $\kappa(s,b) h' \kappa(s,b)^{-1}$ , then we have the relation

$$h \cdot K(s,b)^{-1}(f) = h' K(s,b)^{-1}(f)$$

holding for all  $f \in P_F^{-1}(b)$ . Hence,  $h'^{-1}h \cdot \kappa(s,b)^{-1}(f) = \kappa(s,b)^{-1}(f)$ for all  $f \in P_F^{-1}(b)$ . Since  $\kappa(s,b)$  is a homeomorphism, this last relation implies h = h' if, and only if, H acts faithfully and so we have the required conclusion.

Now  $S_F$ , B and H are topological spaces and so it is natural to ask if there is a canonical topology on  $G(S_F)$  determined by those of  $S_F$ , B and H, in which  $G(S_F)$  is a topological groupoid. It is not clear if this is the case in general; however, in the case of a locally trivial fibre bundle  $S_F$ , the problem has a natural solution (due to Ehresmann) and we present the construction later on in this section.

We remark that the results of this section can be formulated, and carried out, with respect to a sheaf of groups, rather than a single group. However, we content ourselves with a single group and observe that this suffices in the important case of local triviality; this remark is a consequence of 2.4.3 and its corollaries, see §4. In fact, the appropriate generalisations of the results of §2 are given in §4 of this chapter. In the case of a fibre bundle  $(S_F, P_F, B)$  with associated principal H-bundle (S, P, B), the groupoids  $G(S_F)$  and G(S) are both transitive. Also, there is a fibration  $\mathcal{C} : G(S) \longrightarrow G(S_F)$ defined as follows. Let  $S^{\bigstar} \subset S$  be such that  $S^{\backsim} \cap P^{-1}(b) = \{s_b\}$ - the singleton set containing  $s_b$  - for each  $b \in B$ . Given  $\flat \in G(S)(b_1, b_2)$  there exists unique  $\omega \in H$  such that  $\flat (s_{b_1}) = s_{b_2} \cdot \omega$ , define  $\mathcal{C}$  by  $\mathcal{C}(\flat) = \mathsf{K}(s_2, b_2) \cdot \omega \cdot \mathsf{K}(s_1, b_1)^{-1}$  on elements, and let  $\mathcal{C}$ be the identity map on objects. Thus, the construction of  $G(S_F)$  can be described in terms of fibrations. If H acts faithfully, then  $\mathcal{C}$ is an isomorphism of groupoids, and conversely.

Our next task is to relate admissible maps and charts in the case when  $S_F$  is locally trivial and we are given an atlas for  $S_F$ . 3.3.4.

Suppose  $(S_F, P_F, B)$  is the usual fibre bundle with fibre F, group H and associated principal H-bundle (S, F, B). Suppose  $S_F$ is locally trivial and that  $\{U_i, \phi_i; i \in I\}$  is an atlas for  $S_F$ . Thus,  $\phi_i : \Theta(U_i) \longrightarrow P_F^{-1}(U_i)$  is a fibre bundle isomorphism over  $U_i$ , where, for a space B,  $\Theta(B) = (B \times F, P, B)$  is the product fibre bundle over B with fibre F. Also, by 2.2. of Chapter 5 of Husemoller [1], given two charts  $\phi_i, \phi_2 : \Theta(U) \longrightarrow P_F^{-1}(U)$  there is a unique continuous function  $g: U \longrightarrow H$  such that  $\phi_i(b,f) = \phi_2(b,g(b),f)$  for all  $(b,f) \in U \times F$ . Consequently, if  $(U_i, \phi_i)$  and  $(U_j, \phi_j)$  are two charts such that  $U_i \cap U_j \neq \phi$ , there is a unique continuous function  $\mathcal{E}_{ji}: U_i \cap U_j \longrightarrow H$  such that  $\phi_i(b,f) = \phi_j(b,\mathcal{E}_{ji}(b),f)$  for all  $(b,f) \in (U_i \cap U_j) \times F$ . In fact, if the homeomorphism  $\phi_{i,x}: F \longrightarrow P_F^{-1}(x)$  is defined by  $\phi_{i,x}(f) = \phi_i(x,f)$ , then  $\mathcal{E}_{ji}(x) = \phi_{j,x}^{-1}, \phi_{i,x}$ .

With this notation we now prove:

3.3.5. Theorem. Let  $\{U_i, \phi_i; i \in I\}$  be an atlas for  $S_F$ . Then a map

We shall prove that for any  $x \in B$  and coordinate neighbourhood  $U_i$  containing x,  $\phi_{i,x} = \kappa(s,x)$  for some choice of  $s \in P^{-1}(x)$ . The result then follows by 3.3.2.

With the notation of Husemoller [1], let  $\mu = (S, P, B)$  and let  $\ell = \mu[F] = (S_F, P_F, B)$ . For any subset  $A \subset B$ ,  $\mu[F]|_A$  and  $(\mu|_A)[F]$  are canonically A-isomorphic, see Husemoller [1] page 46, and so it suffices to consider  $\ell$  as the trivial bundle and a chart  $\phi : \theta(B) \longrightarrow \ell$ .

If  $f = (B \times H, P, B)$  denotes the product principal H-bundle, then  $\Theta(B) = f[F]$  as a fibre bundle and, if f[F] = (Z, q, B), we have a natural identification  $(g, I_B) : (Z, q, B) \longrightarrow (B \times F, P, B)$ where  $g(b,h,f) \cdot H = (b,h \cdot f)$ , see Husemoller [1], page 46. Thus, under this identification, we are considering a fibre bundle isomorphism  $\phi : f[F] \longrightarrow e$ , which we still denote by  $\phi$  without causing confusion. By definition of a fibre bundle isomorphism,  $\phi$  is the quotient of the H-morphism  $u \times I_F : B \times H \times F \longrightarrow S \times F$  "by the orbits", where  $u : f \longrightarrow \mu$  is a principal bundle isomorphism. Thus  $\phi : Z \longrightarrow S_F$  is defined by  $\phi((b,h,f) \cdot H) = (u(b,h),f) \cdot H$ .

Finally, if we fix  $b \in B$ , set h = the identity e of H and put s = u(b,e), then  $s \in P^{-1}(b)$  is fixed and  $\varphi$  restricts to a map  $\varphi_b : F \longrightarrow P_F^{-1}(b)$  defined by

 $\phi_{b}(f) = \phi((b,e,f) \cdot H)$  $= (u(b,e),f) \cdot H$  $= \kappa(s,b)(f) \cdot H$ 

Thus  $\phi_b = \kappa(s,b)$  and the required conclusion follows.

As already mentioned in the introduction to this section, Theorem 3.3.5 is the definition of admissible map which already exists in the literature. (see Danesh-Naruei [1]).

This theorem makes it possible to topologise  $G(S_F)$  in the case  $S_F$  is locally trivial, and we now give a sketch of this construction in:

<u>3.3.6.</u> <u>Topologising  $G(S_F)$ .</u>

Let  $S_F$  be the fibre bundle with fibre F, group H and associated principal H-bundle (S, P, B) as usual, and suppose  $\{U_i, \not >_i; i \in I\}$  is an atlas for  $S_F$ . We shall now suppose that H acts faithfully on F and will retain this hypothesis in future unless otherwise stated.

Let 
$$G(S_F)(U_i, U_j) = \bigcup_{\substack{x \in U \\ y \in U_i}} G(S_F)(x, y)$$

and define

$$\begin{split} & \mathcal{L}_{ij}: \mathcal{G}(S_F)(U_i,U_j) \longrightarrow U_i \times U_j \times H \\ & \mathcal{L}_{ij}(\varsigma) = (\pi(\varsigma), \pi'(\varsigma), \varsigma'_{ij}), \text{ where } \varsigma'_{ij} \text{ is } \end{split}$$

defined by

by

$$\xi'_{ij} = \phi_{j,\pi'(\xi)}^{-1} \cdot \xi \cdot \phi_{i,\pi(\xi)}$$

The maps  $\gamma_{ij}$  are bijective, for all i and j, since H acts faithfully, and we topologise  $G(S_F)$  by taking them as homeomorphisms, and taking the sets  $G(S_F)(U_i, U_j)$  as a sub-base. It turns out that, with this topology,  $G(S_F)$  is a locally trivial topological groupoid over B. In fact, a local trivialisation  $\{U_i, x_i, \lambda_i\}$  can be defined for  $G(S_F)$  as follows. For any coordinate neighbourhood  $U_i$  of  $S_F$  we choose  $x_i \in U_i$  and define  $\lambda_i : U_i \longrightarrow G(S_F)$  by  $\lambda_i(x) = \gamma_{ii}^{-1}(x, x_i, e)$ ,

where e is the identity of H .

We shall not give any details of this, but we refer to Danesh-Naruei [1] or the papers of Ehresmann, see also Westman [1]. Of course,  $G_{\lambda}(S_{F})$  is "locally a product" as a topological space.

66**.** <sup>.</sup>

# <u>3.3.7.</u> Topologising $\pi X$ .

Let X be a topological space with the local conditions of §14, Steenrod [1] and let  $P: \widetilde{X} \longrightarrow X$  be the universal covering of X. Since  $\tilde{X}$  is a locally trivial principal bundle with group  $\pi_1(X)$  - the fundamental group of X , see Steenrod [1] , we can topologise  $G(\widetilde{X})$  using 3.3.6. However, an inspection of  $\pi(X)$  shows that we can identify  $\pi(X)$ and  $\operatorname{G}(\widetilde{\operatorname{X}})$  . Thus, we can topologise the fundamental groupoid  $\pi(\operatorname{X})$  in such a way that  $\pi(X)$  is a locally trivial topological groupoid over X. This topology has been discussed from a different point of view in Danesh-Naruei [1] .

### <u>84</u>.

## Isomorphism Theorems.

In this section, we shall prove topological versions of the Isomorphism Theorems 1.4.11 and 1.5.10 for the case of locally trivial topological groupoids. Whilst Ehresmann has pointers in the direction we now follow, he seems to have no account of the following results and certainly has given no details.

We begin by recording some necessary definitions.

By a sheaf  $\sigma: \Sigma \longrightarrow Y$  of topological groups, we mean a sheaf of groups in the sense of 1.4.4 in which  $\Sigma$  and Y are topological spaces,  $\sigma$  is continuous and  $\sigma^{-1}(y)$  is a topological group, for all  $\mathbf{y} \in \mathbf{Y}$ . We shall also require the appropriate composition and inverse functions to be continuous; thus  $\geq$  is a topological groupoid over Y if the unit function u is continuous.

A continuous function P : S ---- B , of topological spaces, will be called a (principal) bundle with structure sheaf if P is a bundle with structural sheaf as in Definition 1.4.5,  $\Sigma$  is a sheaf of topological groups and P:  $S_y \longrightarrow B_y$  is a (principal)  $\geq_y$ -bundle as in Chapter 2, that is, as defined by Husemoller [1].  $P: S \longrightarrow B$ will be called <u>locally trivial</u> if each component bundle  $P_y : S_y \longrightarrow B_y$ is locally trivial and each of the sets  $B_y$  is open (and hence closed)

in B; in which case S is the topological sum of the bundles Sy .
Since, by 2.4.3 and its corollaries, any locally trivial
topological groupoid G is the topological sum of its transitive
components, it will suffice in proving our theorems to suppose that G
is transitive. In which case, it will be convenient to use the
formulation of local triviality given by 3.1.1.

Let G be any locally trivial topological groupoid over X and let  $\pi': S \longrightarrow X$  and  $\sigma: \mathbb{Z} \longrightarrow Y$  be constructed as in 1.4.8. If we give S,  $\mathbb{Z}$  and Y the appropriate subspace topologies, then  $\mathbb{Z}$  is a sheaf of topological groups (in fact, a topological groupoid), and the results of Chapter 1, those of §2 of this chapter and the above remarks show that  $\pi': S \longrightarrow X$  is a locally trivial principal bundle with structural sheaf  $\sigma: \mathbb{Z} \longrightarrow Y$ . Since each  $B_y$  is open, Y has the discrete topology in fact. Moreover,  $\mathbb{Z}$  is a topological subgroupoid of G.

## We now prove:

#### <u>3.4.1</u>. Theorem.

Let G be any locally trivial topological groupoid over X. Then G is isomorphic to G(S) for some locally trivial principal bundle with structural sheaf and base space X. In fact, we can take S to be the bundle constructed above.

#### Proof.

As already observed, we can take G to be transitive. Thus, we can take a local trivialisation  $\{U_i, \lambda_i, x_s\}$  for G, based at  $x_s$ , as in 3.1.1. We shall take S to be  $\pi': St_Gx_s \longrightarrow X$ .

By Theorem 1.4.11, we have an algebraic isomorphism  $\Gamma: G \longrightarrow G(S)$  which is defined by  $\Gamma(\alpha) = \chi_{\alpha}$ , where  $\chi_{\alpha}: \pi'^{-1}(\pi(\alpha)) \longrightarrow \pi'^{-1}(\pi'(\alpha))$  is defined by  $\chi_{\alpha}(\beta) = \alpha\beta$ . We will complete the proof of the theorem by showing that  $\Gamma$  is a homeomorphism. To do this, it suffices to consider the restriction

$$\begin{split} & \Gamma_{\mathbf{ij}} = \Gamma_{[\mathcal{G}(\mathbf{U}_{\mathbf{i}},\mathbf{U}_{\mathbf{j}})} : \mathcal{G}(\mathbf{U}_{\mathbf{i}},\mathbf{U}_{\mathbf{j}}) \longrightarrow \mathcal{G}(S)(\mathbf{U}_{\mathbf{i}}\mathbf{U}_{\mathbf{j}}) \\ \text{and show that } & \Gamma_{\mathbf{ij}} \text{ is/homeomorphism for each pair } \mathbf{i},\mathbf{j} \text{ of indices.} \\ & \text{Now S has charts } \not{\varphi_{\mathbf{i}}} : \mathbf{U}_{\mathbf{i}} \times \mathbf{G}\{\mathbf{x}_{\mathbf{o}}\} \longrightarrow \pi'^{-1}(\mathbf{U}_{\mathbf{i}}) \cap \mathbf{St}_{\mathbf{G}}\mathbf{x}_{\mathbf{o}} \\ \text{defined by } \not{\varphi_{\mathbf{i}}}(\mathbf{x},\alpha) = \lambda_{\mathbf{i}}(\mathbf{x})^{-1}\alpha \text{ , and so} \\ & \mathcal{U}_{\mathbf{ij}} : \mathcal{G}(S)(\mathbf{U}_{\mathbf{i}},\mathbf{U}_{\mathbf{j}}) \longrightarrow \mathbf{U}_{\mathbf{i}} \times \mathbf{U}_{\mathbf{j}} \times \mathcal{G}\{\mathbf{x}_{\mathbf{o}}\} \text{ is defined by} \\ & \mathcal{U}_{\mathbf{ij}}(\mathbf{h}_{\alpha}) = (\pi(\alpha),\pi'(\alpha), \phi_{\mathbf{j}}^{-1},\pi'(\alpha)\mathcal{U}_{\alpha} \xrightarrow{\mathcal{U}}_{\mathbf{i}},\pi(\alpha)) \cdot \\ \text{Thus } & \mathcal{U}_{\mathbf{ij}}(\mathbf{h}_{\alpha}) = (\mathbf{x},\mathbf{y},\lambda_{\mathbf{j}}(\mathbf{y})\alpha\lambda_{\mathbf{i}}(\mathbf{x})^{-1}) \text{ , where } \mathbf{x} = \pi(\alpha) \text{ and} \\ & \mathbf{y} = \pi'(\alpha) \text{ .} \\ & \text{Define } \quad \Theta_{\mathbf{ij}} : \mathcal{G}(\mathbf{U}_{\mathbf{i}},\mathbf{U}_{\mathbf{j}}) \longrightarrow \mathbf{U}_{\mathbf{i}} \times \mathbf{U}_{\mathbf{j}} \times \mathcal{G}\{\mathbf{x}_{\mathbf{o}}\} \\ & \text{by } \quad \Theta_{\mathbf{ij}}(\alpha) = (\pi(\alpha),\pi'(\alpha),\lambda_{\mathbf{j}}(\pi'(\alpha))\alpha\lambda_{\mathbf{i}}(\pi(\alpha)))^{-1}) \\ & \text{and, finally, define } \quad \overline{\Gamma}_{\mathbf{ij}} : \mathbf{U}_{\mathbf{i}} \times \mathbf{U}_{\mathbf{j}} \times \mathcal{G}\{\mathbf{x}_{\mathbf{o}}\} \longrightarrow \mathbf{U}_{\mathbf{i}} \times \mathbf{U}_{\mathbf{j}} \times \mathcal{G}\{\mathbf{x}_{\mathbf{o}}\} \\ & \text{to be the identity} \text{ The continuity of composition, inverse and the} \\ & \text{functions } \lambda_{\mathbf{i}} \text{ shows that } \quad \Theta_{\mathbf{ij}} \text{ is a homeomorphism}. \end{split}$$

Since the diagram: -

is commutative, it follows that  $\Gamma_{ij}$  is a homeomorphism and the theorem is established.

#### Remark.

We shall, of course, identify any effective transformation of group with the group/operators it determines, and, similarly, if G acts faithfully. It is this remark that permits us to take  $\overline{\Gamma}_{ij}$ to be the identity in proving 3.4.1, in fact it has already been used in defining the maps  $\gamma_{ij}$ .

Having obtained the topological version of 1.4.11 that we require, we now move on to the consideration of a topological version

of 1.5.10. However, we need some preliminary definitions.

A homomorphism  $(\kappa, \bar{\kappa}): \bar{\Sigma} \longrightarrow \bar{\Sigma}'$  of sheaves of topological groups is a homomorphism in the sense of Chapter 1 in which  $\kappa$ and  $\bar{\kappa}$  are continuous. With this definition we make:

## 3.4.2. Definition.

Suppose (S, P, B) and (S', P', B') are two (topological) principal bundles with structural sheaves  $\Sigma$  and  $\Sigma'$  respectively. Then a pair  $(f,\overline{f}): S \longrightarrow S'$  is a <u>bundle map</u> (or <u>morphism</u>) with respect to the homomorphism  $(\kappa, \overline{\kappa}): \overline{\Sigma} \longrightarrow \overline{\Sigma}'$ , of sheaves of topological groups, if :-

i) (f, f) is a bundle map in the sense of Definition 1.5.2.
ii) f is a homeomorphism and f is continuous.

There is, of course, a definition of bundle isomorphism. Before proceeding with the proof of our main theorem , 3.4.6, we need to investigate how a bundle map treats charts in the case of locally trivial bundles. We have the following generalisation of 1.1 page 59 Husemoller [1].

3.4.3. Lemma.

Let  $\xi = (B \times H, P, B)$  be the product principal H-bundle over B, let  $\chi = (B' \times H', P, B')$  be the product principal H'-bundle over B' and let  $\omega : H \longrightarrow H'$  be a homomorphism of groups. If  $(f,\overline{f}) : \overline{\xi} \longrightarrow \chi$  is a bundle map with respect to  $\omega$ , then  $f(b,h) = (\overline{f}(b), g(b)\omega(h))$ , for all  $(b,h) \in B \times H$ , where  $g : B \longrightarrow H'$  is continuous.

Proof.

Since  $(f,\overline{f})$  is a morphism  $f \longrightarrow h$ , we must have  $f(b,h) = (\overline{f}(b), j(b,h))$  where  $j : B \times H \longrightarrow H'$  is continuous. Now  $f((b,h_1) \cdot h_2) = f(b,h_1,h_2) = (\overline{f}(b), j(b,h_1,h_2))$ . But  $f((b,h_1) \cdot h_2) = f(b,h_1) \cdot \omega(h_2)$   $= (\overline{f}(b), j(b,h_1)) \cdot \omega(h_2)$  $= (\overline{f}(b), j(b,h_1) \cdot \omega(h_2))$ . Thus, we have the relation

 $j(b,h_1h_2) = j(b,h_1) \ \omega(h_2) \ .$ Now put  $h_1 = e$  - the identity of H, then we have  $j(b,h_2) = j(b,e) \ \omega(h_2) \ .$  Consequently,  $j(b,h) = g(b) \ \omega(h)$ , where g(b) = j(b,e) and  $g : B \longrightarrow H'$  is continuous. Now we have  $f(b,h) = (\overline{f}(b), g(b) \ \omega(h))$ 

as required. 🛤

We now prove:

3.4.4. Theorem.

Let  $\xi = (S, P, B)$  be a locally trivial principal H-bundle and let  $\chi = (S', P', B')$  be a locally trivial principal H'-bundle. Suppose  $\phi : U \times H \longrightarrow S|_U$  and  $\phi' : V \times H' \longrightarrow S'|_V$  are charts and  $(f, \overline{f}) : \xi \longrightarrow \chi$  is a bundle map with respect to the group homomorphism  $\omega : H \longrightarrow H'$ . If  $W = U \cap \overline{f}^{-1}(V) \neq \phi$ , there is a continuous function  $g : W \longrightarrow H'$  such that  $f \phi (b,h) = \phi'(\overline{f}(b), g(b)\omega(h))$ , for all  $(b,h) \in W \times H$ . In other words we have

where  $g: W \longrightarrow H$  is continuous. Proof.

Let  $(u, \overline{f})$  be defined to make the diagram

$$s|_{W} \xrightarrow{f} s'|_{V}$$
  
 $\phi \uparrow \qquad \phi'$   
 $w_{XH} \xrightarrow{u} v_{XH'}$ 

commutative. Then u is a bundle map and so by 3.4.3,  $u(b,h) = (\overline{f}(b), g(b)\omega(h))$  where  $g : \mathbb{W} \longrightarrow \mathbb{H}'$  is continuous. Thus,  $\phi'^{-1} f \phi(b,h) = (\overline{f}(b), g(b)\omega(h))$  and so we have  $f \phi(b,h) = \phi'(\overline{f}(b), g(b)\omega(h))$ .

This result has an immediate generalisation to locally trivial principal bundles with structural sheaves.

We immediately have a topological version of Lemma 1.5.9 which we present as:

<u>3.4.5</u>. Lemma.

Suppose S and S' are locally trivial principal bundles with structural sheaves and  $\Gamma: G(S) \longrightarrow G(S')$  is an isomorphism of topological groupoids. Then the restriction  $\tilde{\Gamma}: \operatorname{St}_{G}(S)^{X} \longrightarrow \operatorname{St}_{G}(S') \stackrel{\Gamma}{x}$ is an isomorphism of principal bundles, with structural sheaves, for each element x of the base space of S.

We are, at last, in a position to prove <u>3.4.6</u>. <u>Theorem</u>.

Let (S, P, B) and (S', P', B') be two locally trivial principal bundles with structural sheaves. Then the locally trivial topological groupoids G(S) and G(S') are isomorphic if, and only if, (S, P, B) and (S', P', B') are isomorphic. Proof.

First we prove the sufficiency of the conclusion, and to do this it suffices to consider the case when the sheaves involved each consists of a single group.

So suppose  $(f,\overline{f})$ :  $(S, P, B) \longrightarrow (S', P', B')$  is a bundle isomorphism with respect to the group isomorphism  $\omega : H \longrightarrow H'$ , where the groups H and H' are the groups of S and S' respectively. Define  $\Gamma : G(S) \longrightarrow G(S')$  by

i) on objects  $\Gamma = \overline{f} : B \longrightarrow B'$ .

ii) If  $x_1, x_2 \in B$  and  $\gamma \in G(S)(x_1, x_2)$ define  $\gamma' \in G(S')(\overline{f}(x_1), \overline{f}(x_2))$  by  $\gamma' = f \gamma f^{-1}$ , where the f's are appropriately restricted. The assignment  $\gamma \mapsto \gamma'$  then defines  $\Gamma$  on elements.

The proof of 1.5.10 shows that  $\Gamma$  is an isomorphism of abstract groupoids and so we have only to show that  $\Gamma$  is a homeomorphism on elements. In doing this, we can suppose that B = B' and

that  $\overline{f} = ob \Gamma$  = identity on B and we need only consider

 $\Gamma: g(s)(u_1, u_j) \longrightarrow g(s')(u_1, u_j),$ 

where  $\{U_i, \phi_i\}$  and  $\{U_i, \phi_i'\}$  are atlases for S and S' respectively. We have the diagram

where  $\tilde{\Gamma}$  is defined to make it commutative. In fact,  $\tilde{\Gamma}$  is defined by

$$\widetilde{\Gamma}(\mathbf{x},\mathbf{y},\mathbf{h}) = (\mathbf{x},\mathbf{y}, \phi_{\mathbf{j},\mathbf{y}}^{\prime-1} \mathbf{f} \phi_{\mathbf{j},\mathbf{y}} \cdot \mathbf{h} \cdot \phi_{\mathbf{i},\mathbf{x}}^{-1} \mathbf{f}^{-1} \phi_{\mathbf{i},\mathbf{x}})$$

$$\mathbf{y} \longmapsto \phi_{\mathbf{j},\mathbf{y}}^{\prime-1} \mathbf{f} \phi_{\mathbf{j},\mathbf{y}} \text{ and }$$

but the maps y

$$x \longmapsto \phi_{i,x}^{\prime-1} f \phi_{i,x}$$

are both continuous by 3.4.4 since  $(f,\overline{f})$  is a bundle isomorphism, whence  $\widetilde{\Gamma}$  is a homeomorphism and so  $\Gamma$  is a homeomorphism. This proves the sufficiency.

Conversely, suppose we have an isomorphism  $\lceil : G(S) \longrightarrow G(S')$ of topological groupoids whose induced map on object sets is  $\overline{f}$  - say. As in proving 1.5.10, we can suppose, by the sufficiency part of the theorem, that S and S' are regular in the sense of Chapter 1. Thus, there are base points  $x_o \in B$  and  $x'_o \in B'$  with  $x'_o = \overline{f}(x_o)$  and such that (S, P, B) has group  $P^{-1}(x_o)$  and (S', P', B') has group  $P'^{-1}(x'_o)$ .

We shall show that the map

$$\phi : s \longrightarrow st_{g(s)}^{x}$$

defined by  $\phi$  (5) = unique admissible map  $\gamma$  such that  $\gamma(e) = s$ , where e = identity of  $P^{-1}(x_{\phi})$ , is an isomorphism of locally trivial, principal bundles. Indeed,  $\phi$  is an isomorphism in the algebraic sense anyway (by the proof of 1.5.10) and so we have only to show that  $\phi$  is a homeomorphism. Take atlases as above and the local trivialisation of G(S) as in 3.3.6 (where  $x_i = x_{\phi}$ ). Next define  $\Theta$  :  $P^{-1}(x_{\phi}) \rightarrow G(S)[x_{\phi}]$  74•

by  $\Theta(g) = \xi_g$ , where  $\xi_g : P^{-1}(x_o) \longrightarrow P^{-1}(x_o)$  by  $\xi_g(s) = g \cdot s$ (this map was defined in the proof of 1.5.10 and was denoted "I<sub>y</sub>" there), then  $\Theta$  is an isomorphism of topological groups and so  $\varphi_{i,x_o}^{\Theta} : P^{-1}(x_o) \longrightarrow G(S)\{x_o\}$  defined by  $\varphi_{i,x_o}^{\Theta}(g) = \varphi_{i,x_o}^{\Theta}\xi_g$ is a homeorembian

is a homeomorphism.

The diagram :-

commutes, where  $\mathcal{V}_{i}(x,a) = \lambda_{i}(x)^{-1}a$ , for :

Thus, the above diagram commutes and so  $\not >$  is a homeomorphism. Similarly, we have an isomorphism

$$\phi': s' \longrightarrow st_{g}(s')^{x'_{o}}$$

of locally trivial principal bundles, and these facts, together with Lemma 3.4.5, show the existence of a bundle isomorphism  $S \longrightarrow S'$  of locally trivial principal bundles. Thus, the proof of the theorem is complete.

In the process of proving 3.4.6, we have proved a result which in itself seems worthwhile isolating. It is :

<u>3.4.7</u>. Theorem.

Let (S, P, B) be a locally trivial principal bundle with structural sheaf, let  $Y \subset B$  be a section of the equivalence classes of 1.3.4 and form the locally trivial principal bundle ( $\sqcup$  St y,  $\pi$ , B) y  $\in Y$  G(S)

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with structural sheaf. Then there is an isomorphism

 $(\phi, I)$ : (S, P, B)  $\longrightarrow$  (L) St y,  $\pi$ , B) y  $\in Y$  G(S) of principal bundles.

This result is, in fact, indicated in §8 of Steenrod [1]. Collecting all the results of this section together we have : 3.4.8. Theorem.

Let G be any locally trivial topological groupoid over X. Then :

- a) There is a locally trivial principal bundle S over X, with structural sheaf, such that  $G \cong G(S)$ .
- b) Suppose S' is another locally trivial principal bundle over
  - X with structural sheaf, then  $G(S) \cong G(S')$  if, and only if,  $S \cong S'$ .

The results of this section have many consequences two of which we state now and a third we give in Section 5 below. They will be used again in Chapter 4 and Chapter 5.

From standard bundle theory and 3.4.8 we obtain :

3.4.9. Theorem.

Let G be a transitive locally trivial topological groupoid over X. If X is contractible, then G admits a global trivialisation.

We also have immediately :

## 3.4.10. Theorem.

Any transitive locally trivial topological groupoid admits a faithful representation on some fibre bundle, as in Westman [1].

Another application of our results is given now in Section 5.

## §5. A homotopy classification.

Observe that Theorem 3.4.8 implies that there is a 1-1 correspondence between isomorphism classes of locally trivial topological groupoids on the one hand, and isomorphism classes of locally trivial principal bundles with structural sheaf on the other. Thus <u>any</u> classification of such bundles leads to a corresponding classification of groupoids, and conversely. It is this fact we exploit below.

Suppose G is any locally trivial topological groupoid over X Let  $\{B_y\}_{v \in Y}$ and suppose  $f: Z \longrightarrow X$  is a continuous function. denote the partition of X into the object sets of the transitive components of G, see Chapter 1. We can suppose  $y \in B_y$  (that is, make an identification) so that Y is a subspace of X which has the discrete topology, since each  $B_v$  is open in X . By 3.4.8, there is a locally trivial principal bundle P : S  $\longrightarrow$  X with sheaf  $\sigma: \Sigma \longrightarrow Y$ such that G(S) and G are isomorphic. Next form the induced bundle f (S) over Z which is done in the obvious way of Husemoller [1] by working over the sets  $B_v$ . Then  $f^*(S)$  is a locally trivial principal bundle over Z with sheaf  $\Sigma$ . Now form  $G(f^*(S))$ . If S, and S, are two bundles such that  $G \cong \mathcal{G}(S_1) \cong \mathcal{G}(S_2)$ , then  $S_1 \cong S_2$  by 3.4.8, hence  $f^*(S_1) \cong f^*(S_2)$ , see Husemoller [1], and so  $G(f^*(S_1)) \cong G(f^*(S_2))$  by 3.4.8 again. Thus, f induces a groupoid  $f_*(G)$  - say - over Z which is unique up to isomorphism. This construction, though formulated in an entirely different manner, is essentially the same as the construction of "induced groupoid" given by Ehresmann in Ehresmann [1] .

It will be convenient to employ the following terminology. We shall say that "a groupoid G has sheaf  $\Sigma$  " if the construction of 1.4.8 yields a sheaf isomorphic with  $\Sigma$ . We make :

3.5.1. Definition.

A locally trivial topological groupoid  $G_{\Sigma}$  over X, with sheaf  $\Sigma$ , will be called <u>universal</u> if

ii) If f and g are two continuous functions mapping Z into X such that  $f_*(G_{\Sigma})$  and  $g_*(G_{\Sigma})$  are isomorphic, then f and g are homotopic.

We have :

<u>3.5.2</u>. <u>Theorem</u>.

Let  $\sigma: \Sigma \longrightarrow Y$  be a sheaf of groups over a discrete space Y in which  $\Sigma = \bigsqcup \Sigma$  as a topological space. Then there exists a  $y \in Y$  y locally trivial topological groupoid  $G_{\Sigma}$  which is universal for locally trivial topological groupoids, with sheaf  $\Sigma$ , over paracompact spaces Z. <u>Proof</u>.

Let  $P_y : S_y \longrightarrow X_y$  denote the universal  $\Sigma_y$  bundle of Milnor (see Husemoller [1]) for each  $y \in Y$ . Let S be the disjoint union of these bundles, so that  $S = \bigsqcup S_y$ ,  $X = \bigsqcup X_y$  and  $P : S \longrightarrow X_y \in Y^y$ is the obvious map. Then  $P : S \longrightarrow X$  is a locally trivial principal bundle with sheaf  $\Sigma$ . We shall take  $G_{\Sigma}$  to be G(S).

Suppose now that G is any locally trivial topological groupoid over paracompact space Z, with sheaf  $\Sigma$ , and let  $\{B_y\}_{y \in Y}$  be the usual partition of Z associated with the sheaf  $\Sigma$ ; then  $B_y$  is paracompact for all  $y \in Y$ . The properties of  $S_y$  assert the existence of a continuous function  $f_y : B_y \longrightarrow X_y$  such that  $G(B_y) \cong (f_y)_*(G_{\Sigma}(X_y))$ . Thus, if  $f = \bigsqcup f_y : Z \longrightarrow X$ , it is immediate that we have  $f_*(G_{\Sigma}) \cong G$ .

Finally, the homotopy property follows easily from those of the functions  $f_v$  and Theorem 3.4.8.

86. Covering Morphisms of topological groupoids.

To close this chapter, we shall now consider covering morphisms

of topological groupoids.

We start by recording the following definition which has been given in Brown [1]. This definition will also serve to establish the notation we use throughout the section.

## 3.6.1. Definition.

Let  $\widetilde{G}$  and G be topological groupoids over  $\widetilde{X}$  and Xrespectively. A <u>covering morphism</u>  $(P,\overline{P}): \widetilde{G} \longrightarrow G$  is a morphism of topological groupoids such that  $P: \operatorname{St}_{\widetilde{G}} \widetilde{X} \longrightarrow \operatorname{St}_{\widetilde{G}} \widetilde{P} \widetilde{X}$  is a homeomorphism for each  $\widetilde{X}$  in  $\widetilde{X}$ .

Given a covering morphism  $(P,\overline{P})$ , we can form the fibred product  $G \times_X \widetilde{X} = \{(\alpha,\widetilde{x}) \in G \times \widetilde{X} ; \pi(\alpha) = \overline{P}(\widetilde{x})\}$  topologised as a subspace of  $G \times \widetilde{X}$ . We have a natural map  $S_p : G \times_X \widetilde{X} \longrightarrow \widetilde{G}$  defined by  $S_p(\alpha,\widetilde{x}) = \widetilde{\alpha}_{\widetilde{X}}$ , where  $P(\widetilde{\alpha}_{\widetilde{X}}) = \alpha$  and  $\pi(\widetilde{\alpha}_{\widetilde{X}}) = \widetilde{x} \cdot S_p$  is bijective and in fact,  $S_p^{-1}(\widetilde{\alpha}) = (P(\widetilde{\alpha}), \pi(\widetilde{\alpha}))$ . Clearly  $S_p^{-1}$  is continuous.

Seemingly stronger conditions on P than appear in 3.6.1 have been given by R. Brown and J.P.L. Hardy in Hardy [1], and we record their definition as :-

## 3.6.2. Definition.

A topological covering morphism  $P: \widetilde{G} \longrightarrow G$  is a covering morphism of topological groupoids for which  $S_p$  is continuous.

Their definition is designed to obtain an equivalence between the categories TCov(G) and  $TO_{p}(G)$ , see Hardy [1]. However, we shall see that 3.6.1 is no less interesting in that in studying covering morphisms as in 3.6.1, one is, essentially, studying morphisms **O**f principal bundles. In fact, in several important cases these definitions coincide, as we show in the following theorem. Note that b) of the following theorem has an analogue for Borel groupoids, see Chapter 4. <u>3.6.3</u>. Theorem.

Let  $P: \widetilde{G} \longrightarrow G$  be a covering morphism of topological groupoids. Then :

a) If  $\tilde{G} = X \times G$  is the topological groupoid of 2.2.5 and P

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is the natural covering morphism  $(x,g) \mapsto g$ , then P is a topological covering morphism.

- b) If  $\widetilde{G}$  is compact Hausdorff and G is Hausdorff, then P is a topological covering morphism.
- c) If  $\widetilde{G}$  and G are locally trivial and the composition functions in  $\widetilde{G}$  and G are open maps, then P is a topological covering morphism.

Proof.

For a) see Hardy [1].

b) Under these hypotheses G×<sub>X</sub> X is Hausdorff, and so S<sub>P</sub> is a homeomorphism by a well-known result of elementary point set topology.
c) We can suppose G and G are transitive, that X ε X and x = P(X) ε X. Since G and G are locally trivial, the commutativity of the diagram

where the homeomorphism  $P_{O} = P |_{St_{\widetilde{G}}} \widetilde{x}_{\circ}$ , and the openness of the maps  $\pi'$  imply that  $\overline{P}$  is an open map.

Let  $\{\widetilde{U}_i, \widetilde{x}_o, \widetilde{\lambda}_i\}$  and  $\{U_j, x_o, \lambda_j\}$  be local trivialisations for  $\widetilde{G}$  and G, and choose (reorder if necessary) indices i, j such that  $\overline{P}(\widetilde{U}_i) \subseteq U_i$  and  $\overline{P}(\widetilde{U}_j) \subseteq U_j$ . Thus,  $P: \widetilde{G}(\widetilde{U}_i, \widetilde{U}_j) \longrightarrow G(U_i, U_j)$ . Now consider the diagram

 $= (\bar{P}(\tilde{x}), \bar{P}(\tilde{y}), \lambda_{j}(\bar{P}(\tilde{y}))P_{o}(\tilde{\lambda}_{j}(\tilde{y})^{-1})P_{o}(\tilde{\alpha})P_{o}(\tilde{\lambda}_{i}(\tilde{x})^{-1})^{-1}\lambda_{i}(\bar{P}\tilde{x}), \tilde{\tilde{x}}).$ Since  $\bar{P}$  is open,  $P_{o}$  is a homeomorphism and the compositions are open, it follows that  $\Theta$  is open. Whence  $\ell_{p}$  is open and is, therefore, a homeomorphism. This completes the proof.

Observe that in proving Lemma 3.4.5, we did not make essential use of the invertibility of  $\Gamma$  and we can generalise this result to obtain : <u>3.6.4</u>. <u>Proposition</u>.

Suppose S and S' are locally trivial principal bundles with structural sheaves. Then a covering morphism  $(\Gamma, \overline{\Gamma})$ :  $G(S) \longrightarrow G(S')$ of locally trivial topological groupoids restricts to a bundle map St  $G(S)^{X} \longrightarrow St G(S')^{\overline{\Gamma}}x$ , for each element x of the base space of S.

Suppose (S, P, B) and (S', P', B') are (topological) principal bundles with structural sheaves  $(\Sigma, \sigma, Y)$  and  $(\Sigma', \sigma', Y')$ , and suppose  $(f, \overline{f}) : S \longrightarrow S'$  is a bundle map with respect to  $(\kappa, \overline{\kappa}) : \overline{\Sigma} \longrightarrow \overline{\Sigma}'$ . If  $b \in B_y$ ,  $f : P^{-1}(b) \longrightarrow P'^{-1}(\overline{f}(b))$  is a homeomorphism into, and  $\kappa(s', \overline{f}(b))^{-1}f \kappa(s, b)(h) = \omega(s, s')\kappa_y(h)$ for all  $h \in \overline{\Sigma}_y$ , where  $\omega(s, s') \in H'$  for all choices of s and s' and  $\kappa(s, b)$  denotes the map of §3. This fact follows, essentially, from 3.4.4. Thus, we now see that  $\kappa_y$  is an isomorphism into of topological groups, that is,  $\kappa_y$  is a homeomorphism into.

We next make precise the notion of equivalence of bundle maps which we alluded to at the very end of Chapter 1.

3.6.5. Definition.

Let (S, P, B) and (S', P', B') be as above, let (R, q, C)and (R', q', C') be principal bundles with sheaves  $(\mathfrak{F}, \mathfrak{C}, \mathbb{Z})$  and  $(\mathfrak{F}, \mathfrak{C}', \mathbb{Z}')$  and let  $(g, \overline{g})$  be a bundle map  $\mathbb{R} \longrightarrow \mathbb{R}'$  with respect to  $(\mathfrak{L}, \overline{\mathfrak{L}}) : (\mathfrak{F}, \mathfrak{C}, \mathbb{Z}) \longrightarrow (\mathfrak{F}', \mathfrak{C}', \mathbb{Z}')$ . Then an <u>equivalence</u>  $(f, \overline{f}) \sim (g, \overline{g})$  of <u>bundle maps</u> consists of bundle isomorphisms  $(h, \overline{h}) : S \longrightarrow \mathbb{R}$  and  $(h', \overline{h'}) : S' \longrightarrow \mathbb{R}'$ , with respect to isomorphisms  $(m, \overline{m}) : \Sigma \longrightarrow \mathfrak{F}$  and  $(n, \overline{n}) : \Sigma' \longrightarrow \mathfrak{F}'$  of sheaves, respectively, such that  $(g,\overline{g}) \circ (h,\overline{h}) = (h',\overline{h'}) \circ (f,\overline{f})$ . Where the composites involved are the composites in the category of principal bundles with structural sheaves, and so the left hand side of the equality has  $(\ell,\overline{\ell}) \circ (m,\overline{m})$  for its homomorphism of sheaves, whilst that on the right hand side has  $(n,\overline{n}) \circ (\kappa,\overline{\kappa})$ .

We now have :

# <u>3.6.6</u>. <u>Theorem</u>.

Let  $(f,\overline{f})$ :  $(S, P, B) \longrightarrow (S', P', B')$  be a map of locally trivial principal bundles with respect to  $(\kappa, \overline{\kappa})$ :  $(\Sigma, \sigma, Y) \longrightarrow (\Sigma', \sigma', Y')$ . Then  $(f,\overline{f})$  is equivalent to a bundle map  $(g,\overline{g})$ :  $(R, q, C) \longrightarrow (R', q', C')$ , of locally trivial principal bundles, with respect to

 $(2,\overline{2}):(\beta, e, Z) \longrightarrow (\beta, e', Z')$ , where :

i) (R, q, C) and (R',q',C') are regular in the sense of 1.5.8. ii) If  $\mathscr{B}_{\mathbf{y}} = q^{-1}(c)$  and  $\mathscr{B}_{\overline{\mathbf{z}},\mathbf{y}} = q'^{-1}(\overline{\mathbf{g}}(c))$ , then  $\mathbf{g} : q^{-1}(c) \longrightarrow q'^{-1}(\overline{\mathbf{g}}(c))$  is a group homomorphism which coincides

with  $l_y$ . This condition holding for all  $y \in Z$ . Proof.

This can be proved by considering 1.5.6, 1.5.7 and 1.5.8 as far as the algebraic part is concerned, and by working locally, as we have already done many times, to prove continuity of the required functions.

By analogy with our previous terminology, we call  $(g,\overline{g})$  a <u>regular representative</u> of  $(f,\overline{f})$ .

We can now prove the main result of this section :

3.6.7. Theorem.

Suppose  $\widetilde{G}$  and G are transitive locally trivial topological groupoids over  $\widetilde{X}$  and X respectively, and suppose we have a pair of maps

 $P_{g}: St_{\tilde{g}}\tilde{x}_{s} \longrightarrow St_{\tilde{g}}x_{s}$  and  $\overline{P}: \tilde{X} \longrightarrow X$ 

such that :

a)  $\tilde{P}_{o} = P_{o} | \tilde{G} \{ \tilde{x}_{o} \} \longrightarrow G \{ x_{o} \}$  is an isomorphism into of topological groups.

b)  $(P_o, \overline{P}) : (St_{\tilde{G}}\tilde{x}_o, \pi', \tilde{x}) \longrightarrow (St_{G}x_o, \pi', x)$  is a map of locally trivial principal bundles with respect to  $\tilde{P}_o$ .

Then P can be extended uniquely to a covering morphism  $P: \widetilde{G} \longrightarrow G$  of topological groupoids.

Proof.

By the results of §6, Chapter 1, P<sub>o</sub> can be extended uniquely to a covering morphism P:  $\tilde{G} \longrightarrow G$  of abstract groupoids. The final part of the proof of 1.6.2 shows that  $St_{\tilde{G}}P$  and P<sub>o</sub> are equivalent in the sense of 3.6.5, thus  $St_{\tilde{G}}P$  is a homeomorphism for each object  $\tilde{x}$ of  $\tilde{G}$ .

It remains to prove that P is continuous, for  $ob P = \overline{P}$  is certainly continuous by hypothesis. However, the continuity of P is provided, essentially, by the diagram and argument of 3.6.3 c) except that we are dealing with continuity rather than openness, and we are not taking fibre products.

This result can immediately be generalised to the extent that we can drop the requirement of transitivity. By use of 3.6.6, 3.6.7and the argument of 1.6.3 we now see that a map  $P: S \longrightarrow S'$  of locally trivial principal bundles, with structural sheaf, induces a unique covering morphism  $G(P): G(S) \longrightarrow G(S')$  of locally trivial topological groupoids.

To summarise, our results show that the study of locally trivial topological groupoids and covering morphisms, is equivalent to the study of locally trivial principal bundles, with structural sheaf, and bundle maps.

# Chapter 4. INVARIANT MEASURES FOR GROUPOIDS

<u>80</u>. Introduction.

It is well known that if G is a locally compact Hausdorff topological group, then G admits an essentially unique Baire measure (left Haar measure) which is preserved under left translation by elements of G. A similar statement holds for right translations, although a left invariant Haar measure need not be simultaneously right invariant, and conversely. This fact is of central importance in the analysis of the representations of G, and also in many branches of mathematics in which G spaces occur.

The purpose of this chapter is to establish analogous results for locally compact Hausdorff topological groupoids. However, we present an account of the more measure-theoretic aspects of the problem in the first three sections of the chapter. The later sections are concerned with an application of these results to construct, for groupoids, versions of the convolution algebras  $C_c(G)$  and L'(G) associated with a group G: if G is a locally compact, Hausdorff, locally trivial, topological groupoid, then  $C_c(G)$  is a convolution algebra but L'(G) need not be. Nevertheless, L'(G) has sufficiently many algebra-like properties to be of interest and we investigate these in Section 5.

§1. Borel Groupoids.

In this section we record some facts about Borel spaces and Borel groupoids which we will need later on. Our terminology in respect of Borel spaces is that of Mackey[3]but, as this terminology is not as universal as it might be, we record some of the elementary facts below for the convenience of the reader. Again, Mackey [3] contains all the statements up to and including 4.1.7.

4.1.1. Definition.

Let S be any set. By a Borel structure on S we mean a family  $\beta$  of subsets of S such that :

i) S and  $\phi$  (the empty set) both belong to  $\beta$  .

ii)  $\bigcup_{n=1}^{\infty} E_n$  and  $\bigcap_{n=1}^{\infty} E_n$  both belong to  $\beta$  whenever

 $E_1, E_2, \dots, E_n, \dots$  are elements of  $\beta$ .

iii) If  $E \in \beta$ , then  $S \setminus E \in \beta$ .

By a <u>Borel space</u> we mean a pair  $(S,\beta)$  where  $\beta$  is a Borel structure on the set S; we call the elements of  $\beta$  the <u>Borel subsets</u> of S.

It is to be observed that a Borel structure on S is precisely what is more often termed a  $\sigma$ -algebra of subsets of S.

As usual, we often suppress the " $\beta$  " and refer simply to the Borel space S .

Given any family  $\mathcal{F}$  of subsets of S, there is a unique smallest Borel structure on S which contains  $\mathcal{F}$  and called the Borel structure generated by (or associated with) the family  $\mathcal{F}$ . In particular, if S is a topological space and  $\mathcal{F}$  the family of all open (or closed) sets, then the Borel structure obtained is called the Borel structure generated by the topology of S.

## <u>4.1.2</u> Definition

Let  $(S_1, \beta_1)$  and  $(S_2, \beta_2)$  be two Borel spaces. Then a function  $f: S_1 \longrightarrow S_2$  is called a <u>Borel function</u> if  $f^{-1}(B_2) \in \beta_1$ for all elements  $B_2$  of  $\beta_2$ . If f is invertible and both f and  $f^{-1}$  are Borel functions, we call f a <u>Borel isomorphism</u> and in this case S\_ and S\_ are said to be <u>isomorphic</u>.

If  $S_1$  and  $S_2$  are topological spaces and  $\beta_1$  and  $\beta_2$  are the Borel structures generated by the respective topologies, then any continuous function  $f: S_1 \longrightarrow S_2$  is a Borel function. Further, any homeomorphism is a Borel isomorphism. The converse of these statements is not true of course.

4.1.3 Constructions on Borel spaces.

Let  $(S,\beta)$  be a Borel space and E a subset of S, then  $\beta|_E = \{E \cap B; B \in \beta\}$  is a Borel structure on E and we call

# $(E,\beta|_E)$ a Borel subspace of S .

If  $\{S_{\alpha}\}_{\alpha \in A}$  is any family of Borel spaces, we can form the disjoint union or <u>sum</u>  $S = \bigsqcup_{\alpha \in A} S_{\alpha}$  of the  $S_{\alpha}$  and give S a natural Borel structure as follows. A subset B of S will be called a Borel subset of S if, and only if,  $B = \bigsqcup_{\alpha \in A} B_{\alpha}$ , where  $B_{\alpha}$  is a Borel set in  $S_{\alpha}$  for all  $\alpha \in A$ . Also, we can form the (Cartesian) <u>product</u>  $S = \prod_{\alpha \in A} S_{\alpha}$ and give S the Borel structure generated by the elementary rectangles in the usual way. Observe that the Borel structure generated by the topology of the Cartesian product of a family of topological spaces, is identical with the Cartesian product of the Borel structures generated by the topologies in the factors.

Finally, if  $f: S_1 \longrightarrow S_2$  where  $S_1$  is a Borel space and  $S_2$  is a set, we can make  $S_2$  into a Borel space by declaring  $B \subset S_2$  to be a Borel set if, and only if,  $f^{-1}(B)$  is a Borel set in  $S_1$ . We call this structure the <u>quotient</u> of  $S_1$  by f. Clearly f is now a Borel function.

## <u>4.1.4</u>. <u>Definition</u>

A Borel space S will be called <u>standard</u> if S is Borel isomorphic to the Borel space associated with a Borel subset of a complete separable metric space.

It turns out that a countable product or sum of standard Borel spaces is itself standard. Also, any finite set S with the discrete Borel structure is a standard space.

## 4.1.5. Definition

A Borel space (S,  $\beta$ ) is called <u>countably generated</u> if :

- i) S is separated in that given two points p and q of S, with  $p \neq q$ , there exists a Borel set E of S such that  $p \in E$  and  $q \notin E$ .
- ii) There is a sequence  $E_1$ ,  $E_2$ , ...,  $E_n$ , .... of Borel subsets of S which generates  $\beta$ .

Any standard Borel space is countably generated, although the converse is false. We now quote Theorem 3.2 of Mackey [3] which we will need to use on several occasions.

## <u>4.1.6</u>. <u>Theorem</u>

Let  $f: S_1 \longrightarrow S_2$  be a 1 - 1 Borel function with  $S_1$  a standard space and  $S_2$  a countably generated space. Then the range of f is a Borel subset of  $S_2$  and f is a Borel isomorphism of its domain with its range. In particular, the range of f is X standard.

Finally, we quote :

## <u>4.1.7</u>. <u>Corollary</u>

A subset of a standard Borel space is a standard subspace if, and only if, it is defined by a Borel subset.

Having recorded all the facts we need about Borel spaces, we now turn to Borel groupoids.

4.1.8. Definition

By a <u>Borel groupoid</u> G over X we mean a groupoid in which both G and X are Borel spaces and all the functions  $\pi$ ,  $\pi'$ , u, composition and inverse are Borel functions.

This definition has appeared, essentially, in Ramsey [1] and in Mackey [1] and [2].

Again, we remark that it is to be understood that  $\mathfrak{D} \subset G \times G$ has the subspace Borel structure inherited from the product Borel structure on  $G \times G$ . And, unless otherwise stated, all subsets of G such as  $G\{x\}$ ,  $cost_{G}x$  etc. have the subspace Borel structure.

If G is a Borel groupoid over X, then it is immediate that  $u: X \longrightarrow I(G)$  is a Borel isomorphism, inverse : G \longrightarrow G is a Borel isomorphism and that  $G\{x\}$  is a Borel group for each object x of G. It is also immediate that any topological groupoid gives rise to a Borel groupoid by taking the Borel structure generated by the topology.

Many of the results recorded for topological groupoids in

Chapter 2 have valid interpretations in terms of Borel groupoids, also the proofs of such interpretations generally parallel those of the topological result, and will therefore be omitted. For example, we have :

- 4.1.9. Proposition.
  - Let G be a Borel groupoid over X. Then :
  - a) If  $U \subset X$  is a Borel set then G(U) is a Borel subset of G. If, further, X is standard
  - b) G(x,y), G{x}, St<sub>G</sub>x and cost<sub>G</sub>x are Borel sets in G for all objects x and y of G.
  - c) U G{x} is a Borel set in G.
     x ε X
     Finally, if G is a standard space, then
  - d) 🕀 is a Borel set in G × G . 👹

Proposition 2.1.5 also has a Borel analogue obtained merely by replacing the words "topological" and "homeomorphic" by "Borel" and "Borel isomorphic" respectively. Also, all the examples of §2 of Chapter 2 can be turned into Borel groupoids in the obvious way. In particular, we draw attention to the Borel version of 2.2.5. Finally, we remark that a product  $TT = G_a$  or a sum  $\Box = G_a$ , of a family of  $a \in A$   $a \in A$ Borel groupoids  $\{G_a\}_{a \in A}$ , can be turned into a Borel groupoid by using the product Borel structure and the sum Borel structure respectively. §2. Invariant Measures.

Suppose G is a topological group. It is a fact of some considerable importance that an element S of G determines a homeomorphism of G onto itself simply by left multiplication. Moreover, if G is locally compact and Hausdorff, then it is the famous theorem of Haar-Weil that G admits a Baire measure (see S4)  $\mu$  with the properties

i)  $\mu(s.E) = \mu(E)$  for any element  $s \in G$  and Baire set E of G. ii)  $\mu$  is not identically zero.

Condition i) is paraphrased by saying  $\mu$  is left invariant.

Furthermore, it is the (also famous) result of von-Neumann that if  $\mu'$ is any other Baire measure on G with properties i) and ii), then there exists a constant c > 0 such that  $\mu' = c \mu$ .

An element s c G also determines a homeomorphism of G onto itself by right multiplication, and the above theorems of Haar-Weil and von-Neumann hold for right invarian measures. However, the map  $x \mapsto x^{-1}$ is a homeomorphism of G onto itself which interchanges right and left. There is, therefore, a 1-1 correspondence between right Haar measures and left Haar measures. This is not to say that a left invariant measure is automatically right invariant, and conversely. This need not be true. Indeed, there is a relationship between  $\mu(E)$  and  $\mu(E.s)$  , for a left Haar measure  $\mu$ , determined by the equation  $\mu(E \cdot s^{-1}) = \Delta(s)\mu(E)$  for all Baire sets E of G and elements s  $\varepsilon$  G, see Berberian [1]. Chapter 9. In this equation  $\Delta(s)$  is a positive real number and the function  $\Delta: G \longrightarrow R$  is called the modular function of G. It is a continuous homomorphism of G into the multiplicative group of non zero real numbers. If  $\Delta \equiv 1$  , then G is called unimodular and this condition is equivalent to the condition that any left invariant Haar measure be also right invariant. If G is compact, discrete or Abelian, then it is unimodular.

One more well known definition we shall need is the following. Two measures  $\mu$  and  $\mu'$  on X are said to be <u>equivalent</u> if, and only if, they have the same null sets. This relation is an equivalence relation and the equivalence class,  $[\mu]$ , of  $\mu$  is called the <u>measure</u> <u>class</u> of  $\mu$ , see Mackey [3]. In terms of measure classes, von-Neumann's result implies that "the measure class of Haar measure on G is unique".

Guided by the foregoing discussion we now turn our attention to groupoids, first to Borel groupoids and then to topological groupoids in  $\mathfrak{g}_{+}$ . Thus, let G denote a Borel groupoid over X and let  $s \in G(x,z)$ . We define  $L_s : \operatorname{cost}_G x \longrightarrow \operatorname{cost}_G z$  by  $L_s(\alpha) = s\alpha$  and call  $L_s$  the

<u>left translation by the element</u> s. Similarly, we define the <u>right</u> <u>translation</u>,  $R_s : St_G x \longrightarrow St_G z$  by  $R_s(\alpha) = \alpha s^{-1}$ . Both  $L_s$  and  $R_s$ are Borel isomorphisms and, in fact,  $(L_s)^{-1} = L_{s^{-1}}$  and  $(R_s)^{-1} = R_{s^{-1}}$ . The purpose in working with costars to define  $L_s$  is that these give the maximum amount of composability on the left, whilst when defining  $R_s$ stars give the maximum amount of composability on the right. Clearly  $L_s$  and  $R_s$  are natural generalisations of left and right multiplication, respectively, in a group G.

With these definitions we propose:

## <u>4.2.1</u>. Definition.

Let G be a Borel groupoid over X. A <u>system of left invariant</u> (Borel) measures for G consists of a non trivial Borel measure m on G, a non trivial Borel measure  $\mu$  on X and non trivial Borel measures  $\mu_{\chi}$ on cost<sub>c</sub>x, for each  $x \in X$ , satisfying:

i) The function X → R<sup>≥0</sup> defined by x → µ<sub>x</sub>(E ∩ cost<sub>G</sub>x)
 is µ -measurable for each Borel set E of G, where R<sup>≥0</sup>
 denotes the non negative extended real line.

ii)  $m(E) = \int_{X} \mu_{x}(E_{\cap} \operatorname{cost}_{G} x) d\mu$ iii)  $\mu_{z}(L_{s}(E_{x})) = \mu_{z}(sE_{x}) = \mu_{x}(E_{x})$  for each  $s \in G(x,z)$  and Borel set  $E_{x}$  in  $\operatorname{cost}_{G} x$ .

We refer to Mackey [3] or Bartle [1] for the definition of Borel measure etc. In fact, Bartle [1] serves as a general reference for the measure theory of §2 and §3 of this chapter.

Observe that the conditions i) and ii) of the above definition are analogous to those defining a canonical system of measures as in Rohlin [1]; the integral in ii) being understood in the sense of Lebesgue. The reason for their inclusion in the definition will become apparent later on when defining  $L^{1}(G)$  for a topological groupoid.

One defines a <u>right invariant system</u> in a similar way with the necessary modifications. One is that each measure  $\mu_x$  is defined on  $St_G^{\times}$ , and the other is that iii) is replaced by iii)  $\mu_z(R_gE_x) =$ 

 $\mu_z(E_x s^{-1}) = \mu_x(E_x)$  for each  $s \in G(x,z)$  and Borel set  $E_x$  in  $st_G x$ . Since the inverse map is a Borel isomorphism  $G \xrightarrow{inv} G$  and the diagram



is commutative, any left invariant system of measures induces a right invariant system, and conversely. That is, there is a 1-1 correspondence between left invariant systems and right invariant systems. For this reason we shall study only left invariant systems except in one or two instances where right invariant ones are easier to handle. Thus, unless otherwise stated, the term "invariant system of measures" means left invariant system. Such a system will be denoted by  $\{m, \mu, \mu_x; x \in X\}$ .

Considering Definition 4.2.1, if condition i) is satisfied for some measure  $\mu$  on X, then we can define m by ii) and so ii) is effectively redundent. m is of course determined by  $\mu$  and the  $\mu_{\rm X}$ , x  $\varepsilon$  X, by ii). Moreover,  $\mu$  can always be chosen to satisfy i) as we shall see later in this section. However, the choice of  $\mu$  will depend to some extent on circumstances, at least, as far as the applications we shall consider are concerned. Conditions i) and ii) will play a crucial role in constructing L'(G) later, and iii) is needed to obtain associativity of the convolution product.

Note that if G is a group, then 4.2.1 collapses to the usual definition of a (left) invariant Borel measure on G.

In terms of integrals, condition iii) of Definition 4.2.1 becomes: let  $f: \operatorname{cost}_{G} x \longrightarrow R$  be any  $\mu_{x}$  integrable function and, for  $s \in G(x,z)$ , define  $f^{s}: \operatorname{cost}_{G} z \longrightarrow R$  by  $f^{s}(\beta) = f(s^{-1}\beta)$ , then  $f^{s}$  is  $\mu_{z}$  integrable and  $\int_{\operatorname{cost}_{c} x} f d\mu_{x} = \int_{\operatorname{cost}_{c} z} f^{s} d\mu_{z}$ .

The two problems which immediately present themselves are: firstly, consider the existence of such invariant systems. Secondly, to

what extent are such systems unique? We shall attempt to settle these questions and we remark that the first can be resolved for the most interesting groupoids. That is, such a system exists for these groupoids. However, the uniqueness is less satisfactory in that such systems are not usually unique! We shall phrase the results in terms of measure classes, and it turns out that the non uniqueness can be measured in terms of the measure classes of Borel measures on X , at least, for the groupoids of main interest. Naturally, we obtain successively sharper results by imposing successively more conditions on G . We shall consider both problems simultaneously, our first observation being:-

## 4.2.2. Proposition.

Suppose G is any Borel groupoid over X and that  $\{m, \mu, \mu_x; x \in X\}$  and  $\{m', \mu', \mu'_x; x \in X\}$  are two systems of invariant measures for G. If  $\mu$  is equivalent to  $\mu'$  and  $\mu_x$  is equivalent to  $\mu'_x$  for each x in X, then m is equivalent to m'. <u>Proof</u>.

Suppose m(E) = 0. Then, by ii) of 4.2.1, we have  $\int \mu_x(E \cap \operatorname{cost}_G x) d\mu = 0$ . Hence,  $\mu_x(E \cap \operatorname{cost}_G x) = 0$  for all  $x \in X \setminus A$  where A is some  $\mu$ -null subset of X. By the hypothesis,  $\mu'(A) = 0$  and also  $\mu'_x(E \cap \operatorname{cost}_G x) = 0$  for all  $x \in X \setminus A$ . Consequently,  $m'(E) = \int \mu'_x(E \cap \operatorname{cost}_G x) d\mu' = 0$ . Similarly, if m'(E) = 0 we conclude that m(E) = 0 and we have proved the proposition.

We shall say that two systems  $\{m, \mu, \mu_x; x \in X\}$  and  $\{m', \mu', \mu'_x; x \in X\}$  of invariant measures for G are <u>equivalent</u> if m is equivalent to m',  $\mu$  is equivalent to  $\mu'$  and  $\mu_x$  is equivalent to  $\mu'_x$ , for each  $x \in X$ . Proposition 4.2.2 now asserts that  $\{m, \mu, \mu_x; x \in X\}$  is equivalent to  $\{m', \mu', \mu'_x; x \in X\}$  if, and only if,  $\mu$  is equivalent to  $\mu'$  and  $\mu_x$  is equivalent to  $\mu'_x$  for each  $x \in X$ . Thus, the problem of classifying all such systems up to equivalence amounts to classifying all systems  $\{\mu_x; x \in X\}$  and measures  $\mu$ 

on X subject to i) and iii) of 4.2.1.

If G is any Borel groupoid over X and  $x_o \in X$ , then there is a natural Borel action of  $G\{x_o\}$  on the left of  $\operatorname{Cost}_G x_o$  defined by composition. That is,  $G\{x_o\} \times \operatorname{cost}_G x_o \longrightarrow \operatorname{cost}_G x_o$  is defined by  $\alpha \cdot \beta = \alpha\beta$ . This action is effective in the sense that  $\alpha \cdot \beta = \alpha' \cdot \beta$ implies  $\alpha = \alpha'$  (see Chapter 1), and the orbit  $G\{x_o\} \cdot \beta$  is the set  $G(\pi(\beta), x_o)$ . Thus, the orbit set  $\operatorname{cost}_G x_o / G\{x_o\}$  is precisely the set X and the natural surjection  $\operatorname{cost}_G x_o \longrightarrow \operatorname{Cost}_G x_o / G\{x_o\}$  coincides with the initial map  $\pi : \operatorname{Cost}_G x_o \longrightarrow X$ .

Apart from the purely measure-theoretic aspects i) and ii) of Definition 4.2.1, the existence of an invariant system of measures can be reduced to the problem of finding a measure  $\mu_x$  on  $\operatorname{cost}_{G^X}$  which is preserved by the natural action of  $G\{x\}$  on  $\operatorname{Cost}_{G^X}$ . This is the content of :

<u>4.2.3</u> Proposition.

÷.

Let G be a Borel groupoid over X and let Y be a section of the transitive components of G. Then a system of Borel measures  $\{\mu_x; x \in X\}$  satisfying condition iii) of an invariant system exists for G if, and only if, for each  $y \in Y$ , there is a Borel measure  $\mu_y$ on  $\operatorname{Cost}_{G} y$  preserved by the natural action of  $G\{y\}$ . Proof.

It clearly suffices to suppose G is transitive and  $Y = \{x_n\}$ , also the necessity of the condition is clear.

Suppose, conversely, that there is such a measure  $\mu_{X_0}$  on  $\operatorname{cost}_{G_{X_0}}$  and let  $T \subset G$  be a wide tree subgroupoid with  $\mathcal{T}_{X} \in \mathcal{T}(X, X_0)$ . Define Borel measure  $\mu_{X}$  on  $\operatorname{cost}_{G_{X}}$ , for each  $x \in X$ , by  $\mu_{X}(E_{X}) = \mu_{X_0}(L_{\mathcal{T}_{X}}(E_{X}))$ , for each Borel set  $E_{X}$  of  $\operatorname{cost}_{G_{X}}$ . If  $s \in G(x,z)$  and  $E_{X}$  is a Borel set in  $\operatorname{cost}_{G_{X}}$ , then  $\mu_{Z}(sE_{X}) = \mu_{X_0}(L_{\mathcal{T}_{Z}}(sE_{X})) = \mu_{X_0}(\mathcal{T}_{Z}(sE_{X}))$ . But  $s = \mathcal{T}_{Z}^{-1}s_{O_{X}}\mathcal{T}_{X}$  for some unique  $s_{O} \in G\{x_{O}\}$ . Consequently, we have  $\mu_{Z}(sE_{X}) = \mu_{X_{O}}(s_{O}\mathcal{T}_{X} E_{X}) = \mu_{X_{O}}(\mathcal{T}_{X} E_{X})$ on using the  $G\{x_{O}\}$  invariance of  $\mu_{X_{O}}$ . Thus,  $\mu_{Z}(sE_{X}) = \mu_{X}(E_{X})$  and
so  $\{\mu_x; x \in X\}$  satisfies iii) of 4.2.1.

We can prove rather more than is stated in the conclusion, for if T' is another wide tree subgroupoid of G with  $\tau'_x \in T'(x,x_o)$ , then  $\tau'_x = a' \tau_x$  for some unique  $a' \in G\{x_o\}$ , thus we have  $\mu_x(\tau'_x E_x) = \mu_{x_o}(a' \tau_x E_x) = \mu_{x_o}(\tau_x E_x)$ . This means that the system  $\{\mu_x; x \in X\}$  is independent of the choice of T.

With the notation of the previous proposition we have

#### 4.2.4. Corollary.

G admits an invariant system  $\{m,\mu,\mu_x; x \in X\}$  of measures if, and only if, there is a  $G\{y\}$  invariant measure  $\mu_y$  on  $cost_Gy$ , for each  $y \in Y$ .

#### Proof.

Again the necessity is clear. So suppose conversely we have  $G\{y\}$  invariant measures  $\mu_y$  on  $\operatorname{cost}_G y$ , for each  $y \in Y$ . Using 4.2.3, we obtain a system  $\{\mu_X; x \in X\}$  of Borel measures satisfying iii) of 4.2.1. Now let  $\mu$  be an indiscrete measure on X with one point support (or even countable support). Thus,  $\mu(E) = 1$  if  $p \in E$ and  $\mu(E) = 0$  if  $P \notin E$  for each Borel set E of X, where P is a distinguished point of X. Then for any Borel subset E of G, the function  $x \longmapsto \mu_x (E \cap \operatorname{cost}_G x)$  is  $\mu$ -measurable. Thus, we can define m on G by  $m(E) = \int \mu_x(E \cap \operatorname{cost}_G x) d\mu$  to obtain the required conclusion.

Needless to say, indiscrete measures  $\mu$  on X are not very interesting. Nevertheless, since an indiscrete measure  $\mu$  with one point support is not equivalent to one with two point support on a standard space X, for instance, it is now apparent that systems of invariant measures on G are not unique even up to equivalence.

Our next task is to construct an invariant measure on  $cost_{G}x$ from one on  $G\{x\}$  and, ultimately, to obtain the general form of such measures, with suitable restrictions on G. It will suffice to suppose

G is transitive in what follows.

4.2.5.

# The General Construction of an Invariant System.

Let G be a transitive Borel groupoid over X and let  $T \subset G$ be a wide tree subgroupoid with  $T_x \in T(x, x_o)$  as usual. For each  $x \in X$  define  $\phi_x : G(x, x_o) \longrightarrow G[x_o]$  by  $\phi_x(a) = a \tau_x^{-1}$ . Then  $\phi_x$ is a Borel isomorphism which commutes with the natural left actions of  $G\{x_{o}\}$  on  $G(x,x_{o})$  and  $G\{x_{o}\}$ . Thus, the left invariant Borel measures on  $G\{x_{o}\}$  are in 1-1 correspondence with the left invariant Borel measures on G(x,x). More precisely, if  $\gamma$  is a left invariant Borel measure on  $G\{x_o\}$ , then  $v_x$ , defined by  $v_x(E) = v(\phi_x(E))$ , is easily seen to be a left invariant Borel measure on  $G(x,x_{a})$  . Conversely, given a left invariant Borel measure  $\mathcal{V}_{\mathbf{x}}$  on  $G(\mathbf{x},\mathbf{x}_{\mathbf{y}})$ , then  $\mathcal{V}$ , defined by  $\mathcal{V}(\mathbf{E}) = \mathcal{V}_{\mathbf{x}}(\phi_{\mathbf{x}}^{-1}(\mathbf{E}))$ , is a left invariant Borel measure on  $G\{\mathbf{x}_o\}$ . We denote these measures by  $\mathcal{V}_{x} = \phi_{x}^{-1}(\nu)$  and  $\mathcal{V} = \phi_{x}(\nu_{x})$  respectively. Next suppose that  $\omega_x$  is a  $G\{x_o\}$  invariant measure for each  $x \in X$ , and choose a Borel measure  $\mu$  on X such that the function X  $\longrightarrow$  R defined by  $x \mapsto \omega_x(E \cap G(x,x))$  is  $\mu$ -measurable for each Borel subset E of  $cost_{g}x$  . Such a measure  $\mu$  always exists, for an indiscrete measure with countable support will do, and what is more it will do even if G is not transitive; which means that we do not need the hypothesis of transitivity in our next result (4.2.6). Now define Borel measure  $\mu_o$  on  $\operatorname{cost}_{G^X}$  by

$$\mu_{o}(E) = \int_{X} \omega_{x}(E \cap G(x, x_{o})) d\mu.$$

If  $a \in G\{x_o\}$ , then  $\mu_o(a.E) = \int \omega_x (a.E \cap G(x,x_o)) d\mu$   $= \int \omega_x (a(E \cap G(x,x_o))) d\mu = \int \omega_x (E \cap G(x,x_o)) d\mu = \mu_o(E) .$ Thus  $\mu_o$  is  $G\{x_o\}$  invariant. We now prove: <u>4.2.6</u>. <u>Theorem</u>.

Let G be any Borel groupoid over X. Then G admits an invariant system of measures.

Proof.

Υ.

As already remarked, we can suppose G is transitive. So, with the above notation, define  $\gamma$  on  $G\{x_o\}$  by  $\mathcal{V}(E)$  = the number of elements of E if E is finite.

= +  $\infty$  otherwise.

Then  $\gamma$  is a left invariant Borel measure on  $G\{x_o\}$ . Let  $\omega_x = \phi_x^{-1}(\gamma)$ be defined on  $G(x,x_o)$  and let  $\mu_o$  be defined on  $\operatorname{cost}_{G} x_o$  by  $\mu_o(E) = \int \omega_x (E \cap G(x,x_o)) d\mu$  for a suitable choice of  $\mu$ . Then  $\mu_o$ is  $G\{x_o\}$  invariant and so we have the result by Corollary 4.2.4.

Given  $G\{x_o\}$  invariant measures  $\omega_x$  on  $G(x,x_o)$ , for each  $x \in X$ , we have seen that we can choose a suitable measure  $\mu$  on X and define a  $G\{x_o\}$  invariant measure on  $\cot_x c_x$  by  $\mu_o(E) = \int \omega_x (E \cap G(x,x_o)) d\mu$ . Note that this construction does depend on the choice of T in general, unlike that of 4.2.3, see 4.4.5. On the other hand, given any Borel measure  $\mu$  on X and  $G\{x_o\}$  invariant measures  $\omega_x$  on  $G(x,x_o)$  for each  $x \in X$ , we can always choose a function  $c: X \longrightarrow R$  such that, for each Borel set E of  $\cot_x c_o$ , the function  $x \longmapsto c(x) \omega_x (E \cap G(x,x_o))$  is  $\mu$ -measurable. We then define  $\mu_o$  on  $\cot_x c_o$  by  $\mu_o(E) = \int c(x) \omega_x (E \cap G(x,x_o)) d\mu$ . Then  $\mu_o$  is a  $G\{x_o\}$  invariant Borel measure. We will show that this is the general form of such measures, with suitable restrictions on G and our invariant systems.

The measure  $\gamma$  defined in proving Theorem 4.2.6 (termed a <u>discrete measure</u>) is not, of course,  $\sigma$ -finite. If we impose the very natural condition of  $\sigma$ -finiteness on each measure in an invariant system, it is not clear that Theorem 4.2.6 then holds. Indeed, with the degree of generality which holds in the hypothesis of Theorem 4.2.6 the task of classifying all invariant systems seems a hopeless one. We do manage, however, to obtain below a complete description of  $G\{x\}$  invariant measures on  $cost_{Gx}$ , with suitable restrictions, and again when discussing  $\star c$  is non-negative of course.

Haar systems in §4.

4.2.7.

Suppose G is a transitive standard Borel groupoid over a standard Borel space X and consider a system  $\{m, \mu, \mu_x; x \in X\}$  of invariant measures for G which is finite in the sense that each measure in the system is finite. We need this assumption to apply theorems of Rohlin [1] . Since X is a standard space and  $\pi : \operatorname{cost}_{G} x_{\circ} \longrightarrow X$  is a Borel function, the decomposition of  $cost_{G}x_{o}$  into the sets  $G(x, x_{o})$ is a measurable decomposition in the sense of Rohlin [1] . Thus, by the results of Rohlin [1], there are finite Borel measures  $\omega_x$  on G(x, x) for all  $x \in X \setminus N$  such that  $\mu_{x_o}(E) = \int \omega_x(E \cap G(x, x_o)) d\mu$  for all Borel sets E of  $cost_{G}x_{o}$ , where  $\tilde{\mu}$  denotes the quotient of  $\mu_{x}$ by  $\pi$  and  $\widetilde{\mu}(N) = 0$ . The measures  $\omega_x$  are, moreover, unique  $\widetilde{\mu}$  mod 0. For the sake of notation we shall write  $E_x$  for  $E \cap G(x, x_o)$ . Now  $\mu_{o} = \mu_{x_{o}}$  is  $G\{x_{o}\}$  invariant and so, for  $\beta \in G\{x_{o}\}$ , we have  $\mu_{o}(\beta \cdot E) = \mu_{o}(E) \cdot Thus, \quad \int \omega_{x}(\beta \cdot E_{x}) d\tilde{\mu} = \int \omega_{x}(E_{x}) d\tilde{\mu} \cdot E_{x}$  $\mathtt{But}$  $\int \omega_{\mathbf{x}}(\boldsymbol{\beta} \cdot \mathbf{E}_{\mathbf{x}}) d\widetilde{\boldsymbol{\mu}} = \int \omega_{\mathbf{x}}^{\boldsymbol{\beta}} (\mathbf{E}_{\mathbf{x}}) d\widetilde{\boldsymbol{\mu}} , \quad \text{where} \quad \omega_{\mathbf{x}}^{\boldsymbol{\beta}} (\mathbf{E}_{\mathbf{x}}) = \omega_{\mathbf{x}}(\boldsymbol{\beta} \cdot \mathbf{E}_{\mathbf{x}}) , \text{ this}$ holding for all Borel sets  $E_x$  of G(x, x). Thus, the relation  $\int \omega_{\mathbf{x}}(\mathbf{E}_{\mathbf{x}}) d\tilde{\boldsymbol{\mu}} = \int \omega_{\mathbf{x}}^{\beta} (\mathbf{E}_{\mathbf{x}}) d\tilde{\boldsymbol{\mu}}, \text{ for all Borel sets } \mathbf{E} \text{ of } \operatorname{cost}_{\mathbf{G}^{\mathbf{x}}_{\mathcal{O}}},$ implies  $\omega_x^{\beta}(E_x) = \omega_x(\beta \cdot E_x) = \omega_x(E_x)$  for all  $x \in X \setminus N'$  by uniqueness of the measures  $\omega_{\mathbf{x}}$  , where N' is a  $\widetilde{\mu}$  -null set containing N . other words the measures  $\omega_x$  are  $G\{x_o\}$  invariant for all  $x \in X \setminus N'$ . By considering a tree subgroupoid of G(N') and a single point  $y \in X \setminus N'$ , we can carry  $\omega_y$  to  $G\{x_o\}$  invariant measures  $\omega_x$  on  $G(x, x_o)$  for each  $x \in N'$ , and the relation  $\mu_{c}(E) = \int_{V} \omega_{x}(E_{x}) d\mu$  still holds. In fact, we use the technique of 4.2.5 to do this. This means that we can assume  $N' = \phi$  for our purposes. Let  $\omega_0$  denote the measure thus obtained on  $G\{x_{o}\}$  - it is left invariant. If T is the wide tree subgroupoid of 4.2.5 and  $\varphi_x$  is defined by  $\varphi_x(\alpha) = \alpha \tau_x^{-1}$ , then by

Theorem B of Halmos [1], §60, there exists a constant C(x) such that  $\phi_x(\omega_x) = C(x)\omega_o$ . So if  $v_x = \phi_x^{-1}(\omega_o)$ , we have  $\omega_x = C(x)v_x$ . We have now proved:

#### 4.2.8. Theorem.

Suppose G is a transitive standard Borel groupoid over a standard space X and  $\mu_o$  is a finite  $G\{x_o\}$  invariant Borel measure on  $\operatorname{Cost}_{G} x_o$ . Then there exists a wide tree subgroupoid T of G, a finite function c : X  $\longrightarrow$  R, a/Borel measure  $\mu$  on X and a/left invariant Borel measure  $\gamma$  on  $G\{x_o\}$  such that:

i) The function  $x \mapsto c(x) \omega_x(E \cap G(x, x_o))$  is  $\mu$  -measurable for each Borel set E of  $Cost_G x_o$ .

ii)  $\mu_{o}(E) = \int_{X} c(x) \omega_{x}(E \cap G(x, x_{o})) d\mu$ , where  $\omega_{x}$  denotes the measure  $\varphi_{x}^{-1}(v)$  determined by  $T \cdot \mathbf{m}$ 

In the circumstances we have just considered, any two  $G\{x_o\}$ -invariant finite measures  $\omega_x$  and  $\nu_x$  on  $G(x, x_o)$  are related by  $\omega_x = c(x) \gamma_x$  for some constant c(x). Using this fact and the argument of 4.2.2 we have immediately:

#### 4.2.9. Corollary.

Assume the hypothesis of 4.2.8, then the measure class of  $\mu_o$ depends only on the measure class of  $\mu$  and the set  $\{x \in X ; c(x) = 0\}$ .

#### 4.2.10

Our results, thus far, enable us to construct in principle all finite invariant systems for a standard Borel groupoid G over a standard space X. We summarise the procedure as follows for a transitive groupoid, the general case being an obvious modification of this procedure.

i) We first choose a finite left invariant Borel measure  $\gamma$  on some vertex group  $G\{x_o\}$  (if there is no such measure on  $G\{x_o\}$ , then there are no finite invariant systems on G by 4.2.8) and a finite Borel measure  $\mu_1$  on X. Next choose a wide tree subgroupoid T of G to obtain measures  $\omega_x = \phi_x^{-1}(\gamma)$  on  $G(x, x_o)$  as in 4.2.8, and, finally, choose a function  $c : X \longrightarrow R$  satisfying

A) The function  $x \mapsto c(x) \omega_x(E \cap G(x, x_o))$  is  $\mu_i$  -measurable for each Borel set E of  $Cost_G x_o$ .

Now define  $\mu_o$  on  $\operatorname{Cost}_{\mathbf{G}} \mathbf{x}_o$  by  $\mu_o(\mathbf{E}) = \int c(\mathbf{x}) \omega_{\mathbf{x}} (\mathbf{E} \cap \mathbf{G}(\mathbf{x}, \mathbf{x}_o)) d\mu_i$ , to obtain the most general form of  $G\{\mathbf{x}_o\}$  invariant measures on  $\operatorname{Cost}_{\mathbf{G}} \mathbf{x}_o$ . ii) Next we construct the system  $\{\mu_{\mathbf{x}}; \mathbf{x} \in \mathbf{X}\}$  as we did in 4.2.3. We can use T again to do this, but the definition of  $\mu_{\mathbf{x}}, \mathbf{x} \in \mathbf{X}$ , does not depend on T. To complete the construction, we now choose a finite Borel measure  $\mu_2$  on X satisfying

B) The function  $x \mapsto \mu_x(E \cap Cost_G x)$  is  $\mu_z$ -measurable for each Borel set E of G.

We now have the system  $\{m, \mu_2, \mu_x; x \in X\}$  where

$$m(E) = \int_{X} \mu_{x}(E \cap \operatorname{Cost}_{G} x) d \mu_{z} \cdot \square$$

This procedure depends ostensibly on many choices, but we are interested in measure classes rather than measures. By Theorem B of Halmos [1], §60, the construction does not depend on the choice of  $\gamma$ as far as measure classes are concerned. For the same reason, the measure class of  $\mu_o$  does not depend on the choice of the tree T. Indeed, in many cases the value of  $\mu_o$  does not depend on T, see §4; in particular, see 4.4.5. The measure class of  $\mu_o$  does depend, however, on the choice of the function c, and we will now investigate the effect of changing c. First we note that the measure class of  $\mu_o$ is actually independent of the choice of c up to the set of non-zero values of c. That is, if we change c, to c<sub>2</sub> where c<sub>2</sub>(x)  $\neq 0$  if, and only if, c<sub>1</sub>(x)  $\neq 0$  and c<sub>2</sub> satisfies A), then the measure class of  $\mu_o$  is unchanged. We show next that a greater change than this amounts to changing the measure  $\mu_i$  and leaving c<sub>1</sub> unaltered. To see this, we proceed as follows. Firstly, note that in the procedure 4.2.10, we can

always start by taking the measure  $\gamma$  to be a normalised measure on  $G{x_{o}}$ . It then follows that  $\omega_{x}$  is a probability measure on  $G(x, x_{o})$ for each  $x \in X$ . Now take  $E = Cost_{g}x_{o}$ , then  $\mu_{o}(cost_{g}x_{o}) = \int c_{1}(x)d\mu_{1}$ and so we conclude that  $c_1$  is  $\mu_1$ -measurable and  $\int_x c_1 d\mu_1$  exists and is non-zero if  $\mu_o$  is non trivial. Now define  $c_1^{-1}$ : X ---- R by  $c_{1}^{-1}(x) = \frac{1}{c_{1}(x)}$  if  $c_{1}(x) \neq 0$  and  $c_{1}^{-1}(x) = 0$  if  $c_{1}(x) = 0$ . Then  $c_1^{-1}$  is  $\mu_1$ -measurable and the function  $x \mapsto c_1^{-1}(x) c_1(x) \omega_x (E_1 G(x,x_1))$  $\mu_i$ -measurable for each Borel set E . This means that we can replace c is by a characteristic function  $\chi_{\gamma}$ , where  $Y = \{x \in X ; c, (x) \neq 0\}$  and  $\chi_{\gamma}$ is  $\mu_1$ -measurable, and this does not change the null sets of  $\mu_o$  . That is,  $\mu_o$  is equivalent to the measure  $\mu_o'$  where  $\mu'_{o}(E) = \int \chi_{y} \omega_{x}(E \cap G(x, x_{o})) d\mu_{y}, \text{ where } \mu_{y}(Y) \neq 0. \text{ If we now}$ change  $c_1$  to  $c_2$ , then by this last observation we need only consider  $c_2 = \chi_z$  where  $\mu_1(z) \neq 0$ . Consequently, if we now define  $n_z$  on  $\operatorname{Cost}_{G} x_{o}$  by  $n_{o}(E) = \int_{U} \chi_{z} \omega_{x}(E \cap G(x, x_{o})) d\mu_{i}$ , then  $n_{c}(E) = \int_{C} x_{y} \omega_{x}(E \cap G(x, x_{c})) d\mu_{z}, \text{ for Borel sets } E, \text{ where}$  $\mu_2 = (\mu_1(z)/\mu_1(y)) \cdot \mu_1$ , and no is equivalent to  $\int c_1(x) \omega_x (E \cap G(x, x_o)) d\mu_z$ 

Finally, condition B) is a condition on the choice of  $\mu_2$ and so these remarks together with 4.2.10 lead us ultimately to: 4.2.11. Theorem.

Let G be a standard Borel groupoid over a standard space X. Then the number of inequivalent finite systems of invariant measures is not greater than the number of pairs  $(. [\mu_1], [\mu_2])$ , where  $\mu_1$ , and  $\mu_2$  are finite Borel measures on X.

We shall now discuss two interesting examples of Borel groupoids for which an invariant system of measures can be constructed in a very natural way. To do this we need the following definition. 4.2.12. Definition.

Suppose G is a transitive Borel groupoid over X. We say G is <u>Borel globally trivial</u> if there is a distinguished point  $x_o \in X$ and a Borel function  $\lambda : X \longrightarrow G$  such that  $\lambda(x) \in G(x, x_o)$  for each  $x \in X$ .

Conditions under which this is satisfied include those given in the following result.

4.2.13. Proposition.

Let G be a transitive Borel groupoid over X for which G and X are standard spaces. The following conditions on G are equivalent:

a) G is Borel globally trivial.

b) There is a point  $x_o \in X$  and a Borel set E in G meeting each of the sets  $G(x, x_o)$  in precisely one point.

c) There is a wide tree subgroupoid of G which is a Borel set in G. Proof. To show a) implies b).

Suppose  $\lambda : X \longrightarrow G$  is a Borel global trivialisation with  $\lambda(x) \in G(x, x_o)$ , for each  $x \in X$ . By 4.1.6,  $\lambda$  is a Borel isomorphism of X onto  $E = \lambda(X)$ . E is a Borel set and E meets each set  $G(x, x_o)$  in precisely one point.

To show b) implies c).

Suppose E satisfies b), then E is a standard subspace of G. Define  $\lambda : X \longrightarrow G$  by  $\lambda(x) \in E \cap G(x, x_{\circ})$ . Clearly  $\lambda^{-1} = \pi|_E$ and so  $\lambda^{-1}$  is a Borel function. Thus, by 4.1.6 again,  $\lambda$  is a Borel function since E is standard. Let T be the wide tree subgroupoid of G defined by  $T(x,y) = \{\lambda(y)^{-1} \lambda(x)\}$  and let  $\Gamma : X \times X \longrightarrow G$  be defined by  $\Gamma(y,x) = \lambda(y)^{-1} \lambda(x)$ . Then  $\Gamma$  is an injective Borel function whose range is T and so T is a Borel subset of G.

Finally, we show c) implies a).

Suppose T is a wide tree subgroupoid of G which is a Borel set of G and let  $\mathcal{T}_{XY} \in T(x,y)$ . Define  $w : T \longrightarrow X \times X$  by  $w(\mathcal{T}_{XY}) = (x,y)$ , then w is a Borel isomorphism. Choose any point

Our first example is :

#### <u>4.2.14</u>. Example.

Let G be a standard globally trivial Borel groupoid over a standard space X, and let  $\lambda : X \longrightarrow G$  be a Borel global trivialisation of G with  $\lambda(x) \in G(x, x_0)$ . Then the function  $\Gamma : G \longrightarrow X \times X \times G\{x_0\}$  defined by  $\Gamma(\alpha) = (\pi(\alpha), \pi'(\alpha), \lambda(\pi'(\alpha)) \alpha \lambda(\pi(\alpha))^{-1})$  is a Borel isomorphism whose restriction  $\Gamma_x : \operatorname{Cost}_{G} x \longrightarrow X \times G\{x_0\}$ , defined by  $\Gamma_x(\alpha) = (\pi(\alpha), \lambda(x) \alpha \lambda(\pi(\alpha))^{-1})$ , is also a Borel isomorphism.

Let  $\mu_1$  and  $\mu_2$  be  $\sigma$ -finite Borel measures on X and suppose  $\mathcal V$  is a left invariant  $\sigma$  -finite Borel measure on  $G\{x_{\sigma}\}$ . The natural action  $G\{x_{\delta}\} \times (X \times G\{x_{\delta}\}) \longrightarrow X \times G\{x_{\delta}\}$ , defined by  $(a, (x, \beta)) \mapsto (x, \alpha\beta)$ , preserves the product  $\mu_2 \times \nu$ on  $X \times G\{x_o\}$  and, since  $\prod_x$  is equivariant,  $\mu_x = \prod_x^{-1} (\mu_z \times v)$  is preserved by the natural action of  $G\{x \}$  on  $Cost_Gx$ . An argument exactly like that used to prove 4.2.3 shows that the system  $\{\mu_x; x \in X\}$  satisfies the condition iii) of 4.2.1. Finally, define m on G by  $m = \Gamma'(\mu_1 \times \mu_2 \times \nu)$  then by Fubini's theorem (see Bartle [1]) conditions i) and ii) of 4.2.1 hold. Thus,  $\{m, \mu, \mu_x; x \in X\}$  is a system of invariant measures for G . The main point to note here is that the global triviality of G allows us to apply Fubini's theorem and to then deduce the  $\mu_i$  -measurability of  $x \mapsto \mu_x$  (E  $\cap \operatorname{Cost}_G x$ ), for any Borel set E of G . This statement holding for any  $\sigma$  -finite measure  $\mu_1$  .

Suppose, conversely, that  $\{m', \mu', \mu'_x; x \in X\}$  is any finite invariant system of measures for G. Theorem 4.2.8 (applied to  $X \times G\{x_o\}$  rather than  $Cost_G x$ ) shows that the measures  $\mu'_x$  need not be products  $\mu_2 \times \nu$  as constructed above. However, this theorem does show that  $\mu'_x$  is equivalent to such a measure. And it is now apparent that in this case, 4.2.11 becomes. "Suppose G admits one finite invariant system of measures, then the number of inequivalent finite invariant systems is equal to the number of pairs  $([\mu_1], [\mu_2])$ , where  $\mu_1$  and  $\mu_2$  are finite Borel measures on X".

# 4.2.15. Example.

The next example we shall consider is the groupoid  $\tilde{G}$  of 2.2.5.

Let G be a Borel group and S a right Borel G-space in which the evaluation map  $S \times G \longrightarrow S$  is a Borel function, then, as already remarked,  $\widetilde{G} = S \times G$  is a Borel groupoid over S, see Mackey [1] and [2]. We shall consider right invariant systems for  $\widetilde{G}$  rather than left invariant ones, simply because  $St_{\widetilde{G}}s = s \times G$ , for any  $s \in S$ , and is easier to deal with than  $Cost_{\widetilde{C}}s$ .

Suppose we have a  $\sigma$ -finite Borel measure  $\mu$  on S and  $\sigma$ -finite left invariant Borel measure  $\nu$  on G (note that we do not ask that  $\mu$  be G-invariant). Let m be the product  $\mu \times \nu$  on  $\tilde{G}$ . Now, for each  $s \in S$ , we have a Borel isomorphism  $\Theta_s : G \longrightarrow St_{\tilde{G}}s$  defined by  $\Theta_s(g) = (s,g)$  which carries  $\nu$  to a Borel measure  $\mu_s$  on  $St_{\tilde{G}}s$ . We claim that the system  $\{m, \mu, \mu_s; s \in S\}$  is a right invariant system for  $\tilde{G}$ . Indeed, conditions i) and ii) follow immediately by Fubini's theorem again. If  $\alpha = (s,g) \in \tilde{G}(x,z)$ , so that x = s and  $z = s \cdot g$ , then  $R_\alpha : St_{\tilde{G}}x \longrightarrow St_{\tilde{G}}z$  is defined by  $R_\alpha((x,h)) = (z,g^{-1}h)$ . If  $E_x$  is a Borel set in  $St_{\tilde{G}}x$  them Examples theoremise  $\mu_z(z,g^{-1}E) = \nu(g^{-1}E) = \nu(E) = \mu_x(E_x)$ , using left invariance of  $\nu$ . Thus, we have established condition iii) of 4.2.1 for  $\{m, \mu, \mu_s; s \in S\}$  and with it our claim.

Conversely, suppose  $\{m, \mu, \mu_s; s \in S\}$  is an invariant system of measures for  $\tilde{G}$ . Then, for each  $s \in S$ ,  $\Theta_s^{-1}$  carries  $\mu_s$  to a

measure  $\vartheta_s$  on G. If x, y \in S and g \in G is such that  $x \cdot g = y$ , the invariance property of the  $\mu_s$  implies that

$$\gamma_{\mathbf{x}}(g \cdot E) = \gamma_{\mathbf{y}}(E)$$

for any Borel set E of G. In particular, if  $g \in \widetilde{G} \{x\}$  we have  $\mathcal{V}_{\mathbf{x}}(g \cdot E) = \mathcal{V}_{\mathbf{x}}(E)$  for any Borel set E of G. It does not follow from these relations that the measures  $\nu_{\mathrm{x}}$  are left invariant, unless G acts trivially nor need  $\mathcal{V}_{\mathbf{x}}$  and  $\mathcal{V}_{\mathbf{y}}$  coincide. For consider the following example. Let  $S = G = \{e, a\}$  be the cyclic group of order 2 with the discrete topology and Borel structure and let G act on S by right multiplication. Since any groupoid with the discrete topology is a topological groupoid and, hence, a Borel groupoid, it follows that  $\tilde{G} = \{e, a\} \times \{e, a\}$  is a globally trivial, standard Borel groupoid over {e, a}. Define  $\mu$  on S by  $\mu(\{e\}) = 1$  and  $\mu(\{a\}) = 1$ , define  $\mu_e$  on Stree by  $\mu_e((e,e)) = 0$ ,  $\mu_e((e,a)) = 1$  and define  $\mu_a$  on St<sub>G</sub> a by  $\mu_a((a, a)) = 0$ ,  $\mu_a((a, e)) = 1$ . Finally, define m on  $\widetilde{G}$  by  $m(E) = \int \mu_s(E \cap st_{\widetilde{G}}s) d\mu$  to obtain the system  $\{m, \mu, \mu_s; s \in S\}$ . It is easily seen to be an invariant system. However, the image  $\gamma_e$  of  $\mu_e$  on G (under  $\theta_e$ ) is clearly not an invariant measure on G, neither is  $\gamma_a$  and, moreover,  $\gamma_e$  +  $\gamma_a$ .

Since there are precisely three measure classes of (non trivial) Borel measures on  $S = \{e, a\}$  and  $\tilde{G}$  is globally trivial, Theorem 4.2.11 yields "there are exactly nine inequivalent systems of invariant measures on  $\tilde{G}$  ", see 4.2.14.

We remark that Example 4.2.15 shows that  $\pi$  and  $\pi'$  need not be  $(m, \mu)$  measure preserving.

# 83. Covering Morphisms and Invariant Measures.

Suppose  $P: \widetilde{G} \longrightarrow G$  is a covering morphism of Borel groupoids (see below for the definitions) and G admits a system of invariant measures. One is led naturally to ask if P induces or lifts this system to an invariant system on  $\widetilde{G}$ ? In this section, we answer this question affirmatively with mild restriction on  $\widetilde{G}$  and G, see Theorem 4.3.4. <u>4.3.1.</u> <u>Definition</u>.

Suppose  $\widetilde{G}$  and G are Borel groupoids with object sets  $\widetilde{X}$  and X respectively. A morphism  $P: \widetilde{G} \longrightarrow G$  is called a <u>Borel morphism</u> if both P and the induced map ob  $P: \widetilde{X} \longrightarrow X$  are Borel functions.

If  $P: \widetilde{G} \longrightarrow G$  is a morphism of groupoids, we shall denote by  $\overline{P}$  the induced map ob P on objects.

Now suppose P is a covering morphism of abstract groupoids, P:  $\widetilde{G} \longrightarrow G$ , and form the fibred product  $G \times_{X} \widetilde{X} = \{(\alpha, \widetilde{x}) \in G \times \widetilde{X}; \pi'(\alpha) = \widetilde{P}(\widetilde{x})\}$ . Since P is star bijective, it is also costar bijective in the sense that the induced map  $\cot_{\widetilde{G}} P: \cot_{\widetilde{G}} \widetilde{x} \longrightarrow \cot_{\widetilde{G}} \widetilde{P}(\widetilde{x})$  is a bijection for each  $\widetilde{x}$  in  $\widetilde{X}$ . Thus, there is a natural map  $s_{P}: G \times_{X} \widetilde{X} \longrightarrow \widetilde{G}$ , see Section 6 Chapter 3, where  $S_{P}$  is defined by  $S_{P}(\alpha, \widetilde{x}) = \widetilde{\alpha}_{\widetilde{X}}$ , where  $\widetilde{\alpha}_{\widetilde{X}}$  is the unique element of  $\widetilde{G}$  which covers  $\alpha$ and ends at  $\widetilde{X}$ . That is to say,  $P(\widetilde{\alpha}_{\widetilde{X}}) = \alpha$  and  $\pi'(\widetilde{\alpha}_{\widetilde{X}}) = \widetilde{x}$ . We make the following:

#### 4.3.2. Definition.

A <u>Borel covering morphism</u>  $P: \widetilde{G} \longrightarrow G$  is a Borel morphism which is also an abstract covering morphism satisfying:

- a)  $\operatorname{St}_{\widetilde{G}} P : \operatorname{St}_{\widetilde{G}} \widetilde{x} \longrightarrow \operatorname{St}_{\widetilde{G}} \overline{P}(\widetilde{x})$  is a Borel isomorphism for each  $\widetilde{x}$  in  $\widetilde{X}$ .
- b) The natural map  $S_p : G \times_X \widetilde{X} \longrightarrow \widetilde{G}$  is a Borel function, where  $G \times_X \widetilde{X}$  has the subspace Borel structure of  $G \times \widetilde{X}$ .

Observe that  ${\rm cost\,}_{\widetilde{G}}\,P\,$  is a Borel isomorphism if, and only if, St $_{\widetilde{G}}\,P\,$  is one.

It might appear, at first sight, that the conditions defining a Borel covering morphism are very restrictive ; this is not the case for we have

4.3.3. Proposition.

Suppose  $\widetilde{G}$ , G,  $\widetilde{X}$  and X are all standard Borel spaces and let P:  $\widetilde{G} \longrightarrow G$  be a Borel morphism which is also a covering morphism of

abstract groupoids. Then P is a Borel covering morphism.

Proof.

By hypothesis,  $\operatorname{St}_{\widetilde{G}} P : \operatorname{St}_{\widetilde{G}} \widetilde{X} \longrightarrow \operatorname{St}_{\widetilde{G}} \widetilde{P} \widetilde{X}$  is a Borel function and is also bijective for each  $\widetilde{X} \in \widetilde{X}$ . Since stars are Borel subsets, they are standard subspaces (see 4.1.7 and 4.1.9) and so  $\operatorname{St}_{\widetilde{G}} P$  is a Borel isomorphism by 4.1.6. Thus P satisfies a) of 4.3.2.

It is easily seen that  $S_p$  is a bijective function and that  $S_p^{-1}$  is defined by  $S_p^{-1}(\tilde{a}) = (P(\tilde{a}), \pi'(\tilde{a}))$ . Since the composite of  $S_p^{-1}$  with each of the projections on  $G \times \tilde{X}$  is a Borel function, it follows that  $S_p^{-1}$  is a Borel function and hence, by 4.1.6 again, a Borel isomorphism. Consequently,  $S_p$  is a Borel function and the proof is complete.

Note that the proof of 4.3.3 also shows that  $G \times_X \widetilde{X}$  is a standard subspace of  $G \times \widetilde{X}$  and is, therefore, a Borel subset of  $G \times \widetilde{X}$ . Next we prove :

4.3.4. Theorem.

Let  $\tilde{G}$  and G be standard Borel groupoids with  $\tilde{X}$  and Xstandard Borel spaces and let  $P: \tilde{G} \longrightarrow G$  be a Borel covering morphism. Suppose  $\{m, \mu, \mu_X; x \in X\}$  is an invariant system of measures on G and  $\tilde{\mu}$  is a Borel measure on  $\tilde{X}$  which is such that  $\overline{P}^{-1}(A)$  is  $\tilde{\mu}$ -measurable for each  $\mu$ -measurable set A of X. Then P induces an invariant system  $\{\tilde{m}, \tilde{\mu}, \mu_{\tilde{X}}; \tilde{X} \in \tilde{X}\}$  on  $\tilde{G}$ . Proof.

Suppose  $\tilde{x} \in \tilde{X}$ , then  $S_{p}^{-1}(\cos t_{\tilde{G}} \tilde{x}) = \cos t_{\tilde{G}} \tilde{P} \tilde{x} \times {\tilde{x}}$ , and if  $P_{i}$  denotes projection, we have  $P_{i} S_{p}^{-1} = \cot t_{\tilde{G}} P$ . Define  $\mu_{\tilde{X}}$  on  $\cot t_{\tilde{G}} \tilde{x}$  by the relation  $\mu_{\tilde{X}} (E) = \mu_{\tilde{P}\tilde{X}} (\cot t_{\tilde{G}} P(E))$  and define  $\mathcal{V}_{\tilde{X}}$ on  $\cot t_{\tilde{G}} \tilde{P} \tilde{x} \times {\tilde{x}}$  by  $\mathcal{V}_{\tilde{X}} (E) = \mu_{\tilde{P}\tilde{X}} (P_{i}(E))$ . Then  $S_{p}^{-1}$ preserves  $\mu_{\tilde{X}}$  and  $\mathcal{V}_{\tilde{X}}$ .

We wish to show that the function  $\tilde{\phi} : \widetilde{X} \longrightarrow \mathbb{R}^{> \circ}$  defined by  $\tilde{\phi}(\widetilde{x}) = \mu_{\widetilde{X}}(\widetilde{E} \cap \operatorname{cost}_{\widetilde{G}} \widetilde{x})$  is  $\tilde{\mu}$ -measurable for each Borel set  $\widetilde{E}$ in  $\widetilde{G}$ . To do this, it suffices to consider a basic Borel set of the form  $(G \times \widetilde{X}) \cap (E \times \widetilde{Y}) = E \times \widetilde{Y}$ 

in  $G \times_X \tilde{X}$ , where  $E \subset G$  and  $\tilde{Y} \subset \tilde{X}$  are Borel sets, and to show  $\tilde{\mu}$ -measurability of  $\phi : \tilde{X} \longrightarrow \mathbb{R}^{\geqslant \circ}$  defined by  $\phi(\tilde{x}) = h_{\tilde{X}} ((E \times_X \tilde{Y}) \cap \operatorname{cost}_{G} \overline{P} \tilde{x} \times \{\tilde{x}\}) \cdot \operatorname{Since}$   $(E \times_X \tilde{Y}) \cap (\operatorname{cost}_{G} \overline{P} \tilde{x} \times \{\tilde{x}\}) = (E \cap \operatorname{cost}_{G} \overline{P} \tilde{x}) \times \{\tilde{x}\}, \phi(\tilde{x})$   $= \mu_{\overline{P} \tilde{X}} (E \cap \operatorname{cost}_{G} \overline{P} \tilde{x}) \text{ and so } \phi \text{ is } \tilde{\mu} \text{ -measurable by hypothesis and}$ the fact that  $\{m, \mu, \mu_X; x \in X\}$  is an invariant system of measures for G.

Next, define Borel measure  $\gamma$  on  $G \times_{X} \tilde{X}$  by  $\gamma(E) = \int_{X} \gamma_{\tilde{X}} (E \cap (\cot_{G} \overline{P} \tilde{X} \times \{\tilde{X}\})) d \tilde{\mu}(\tilde{X})$  for each Borel set E of  $\tilde{X}$   $G \times_{X} \tilde{X}$  and, finally, let  $\tilde{m}$  be the image of  $\gamma$  under  $S_{p}$ . We claim that  $\{\tilde{m}, \tilde{\mu}, \mu_{\tilde{X}}; \tilde{x} \in \tilde{X}\}$  is an invariant system for  $\tilde{G}$  and we have already dealt with conditions i) and ii) of 4.2.1. To deal with the third condition, let  $\tilde{x} \in \tilde{G}(\tilde{x}, \tilde{z})$  and suppose  $E_{\tilde{X}}$  is a Borel set in  $\cot_{\tilde{G}} \tilde{x}$ , then  $\mu_{\tilde{Z}}(\tilde{s} E_{\tilde{X}}) = \mu_{\tilde{P}\tilde{Z}}(P(\tilde{s} E_{\tilde{X}})) = \mu_{\tilde{P}\tilde{Z}}(P(\tilde{s})P(E_{\tilde{X}}))$   $= \mu_{\tilde{P}\tilde{X}}(P(E_{\tilde{X}}))$  by the invariance of the  $\mu_{X}$ . But, by definition, we have  $\mu_{\tilde{P}\tilde{X}}(P(E_{\tilde{X}})) = \mu_{\tilde{X}}(E_{\tilde{X}})$  and so  $\mu_{\tilde{Z}}(\tilde{s} E_{\tilde{X}}) = \mu_{\tilde{X}}(E_{\tilde{X}})$ . This establishes the required property iii) and completes the proof of the theorem.

Let  $\tilde{G}$  be the groupoid of 4.2.15 and let  $P: \tilde{G} \longrightarrow G$  be defined by P((s,g)) = g. Then P is a Borel covering morphism of Borel groupoids. Suppose S and G are standard spaces,  $\mu$  is any  $\sigma$ -finite measure on S and  $\gamma$  is a left invariant measure on G. Then an application of Theorem 4.3.4 gives exactly the system on  $\tilde{G}$  as we obtained in 4.2.15. (except that we worked with right invariant systems there).

St. Haar Measures for Groupoids.

This section is to some extent a continuation of §2 except that we shall now consider topological groupoids; in fact, we shall consider locally compact Hausdorff topological groupoids. All the results of §2 hold for such groupoids but, as we shall see, we can sharpen some results. In particular, when we consider locally trivial unimodular groupoids there is a canonical invariant measure whose form is very satisfactory. We first make some remarks concerning our terminology.

If Y is any locally compact Hausdorff space, we shall observe the terminology of Berberian [1] in respect of measure-theoretic concepts relating to Y. That is to say, we consider the class & of all compact  $G_{\zeta}$ 's of Y and the  $\sigma$ -ring S generated on Y by  $\mathcal{L}$ . The elements of S are called the Baire sets of Y . We use the terms "Baire function" and "Baire measure" with the meaning given to them by Berberian [1]. Thus, in particular, a <u>Baire measure</u>  $\mu$  is a measure on S which is such that  $\mu(c) < \infty$  for all  $C \in \mathcal{L}$ . We shall not have occasion to consider the elements of the  $\sigma$  -ring generated by the class of all compact sets of Y - the so called Borel sets of Y . For one thing, this latter concept does not usually coincide with the usage of "Borel set" we have already introduced in §1 and §2. More important, however, is the technical fact that the  $\sigma$  -ring of Baire sets in the product topological space  $X \times Y$  , of two locally compact Hausdorff spaces, is precisely the cartesian product of the  $\sigma$ -ring of Baire sets in X with that of Y. This fact need not hold for the elements of the  $\sigma$  -ring generated by the compact sets. Since we will need to use Fubini's theorem on several occasions, we consider only Baire sets.

Thus, by a "Haar measure" on a locally compact group G we mean a Baire measure on G with left (or right) invariance, rather than a Borel measure. This amounts to considering the Baire contraction of a Haar measure and loses nothing.

Two facts which we will specifically need are :

i) The cartesian product of two Baire measures is a Baire measure.
ii) Every Baire measure is σ -finite; every Baire set is

 $\sigma$ -bounded, in fact every Baire set is contained in the union of a sequence of compact G 's.

One particular Baire measure we will need to use on several occasions (in order to make non-vacuous certain general statements) is the indiscrete measure  $\mu$  with finite support, as defined in 4.2.4.

Let G be any locally compact Hausdorff topological groupoid over X. Since a closed (or open) subspace of a locally compact space is itself locally compact, it follows by a) of Proposition 2.1.3 that  $G\{x\}$  is a locally compact Hausdorff topological group, and also  $\cot_{C}x$ is a locally compact space for each  $x \in X$ . It follows also, by a) of Proposition 2.1.4, that X is a locally compact Hausdorff space. If, further, G is locally trivial, then it follows from Proposition 2.4.3 that the transitive components of G (and their object sets) are locally compact. For any topological groupoid G and  $s \in G(x, z)$ , both  $R_g$ and  $L_g$  are homeomorphisms. Thus, if G is locally compact Hausdorff, then  $L_g$  and  $R_g$  are Baire measurability preserving. The appropriate version of a system of invariant measures for a locally compact Hausdorff topological groupoid is

#### <u>4.4.1.</u> Definition.

Let G be a locally compact Hausdorff topological groupoid over X. By a <u>left Haar system</u> of measures for G we mean a Baire measure m on G, a Baire measure  $\mu$  on X and a Baire measure  $\mu_x$  on  $\operatorname{cost}_{G^X}$  for each  $x \in X$  satisfying:

i) The function  $X \longrightarrow R^{\geq 0}$  defined by  $x \longmapsto \mu_x(E \cap \operatorname{cost}_{G} x)$ is  $\mu$ -measurable for each Baire set E of G.

ii) 
$$m(E) = \int \mu_x (E \cap \operatorname{cost}_G x) d\mu$$
.  
×

iii)  $\mu_z(L_sE_x) = \mu_x(E_x)$  for any  $s \in G(x, z)$  and any Baire set  $E_x$ of  $cost_G x$ .

#### 4.4.2.

Note that we include the condition that m be a Baire measure in our definition, it does not follow from the other conditions as the following example shows.

\* m ≠0, μ ≢0, μx ≠0.

<u>Example</u>. Let the additive group of the real line, R, act trivially on the right of the space [0,1] = X, and let  $\tilde{G} = [0,1] \times R$ . Let  $\mu$  be ordinary Lebesgue measure on [0,1], let  $\lambda$  be ordinary Lebesgue measure on R and, for each  $x \in [0,1]$ , let  $\mu_x$  be defined on  $\cot_{\tilde{G}} x = x \times R$  by  $\mu_x = f(x)\lambda$  where  $f:[0,1] \longrightarrow R$  is defined by  $f(x) = \frac{1}{x}$  if  $x \neq 0$ , f(0) = 1. Then, for any Baire set E of  $\tilde{G}$ , the function  $x \longmapsto \mu_x(E \cap \cot_{\tilde{G}} x)$  is  $\mu$ -measurable and we can define m on  $\tilde{G}$  by  $m(E) = \int \mu_x(E \cap \cot_{\tilde{G}} x)d\mu$ . Now  $\mu$  is a [0,1]

Baire measure and so is  $\mu_x$  for each  $x \in X$ . But, if  $E = [0, 1] \times [0, 1]$ , then E is a compact  $G_g$  and  $m(E) = \int_{C_{0,1}} f(x) d\mu = \int_{O} \frac{1}{x} dx \neq \infty$ .

Thus, m is not a Baire measure.

Observe, however, that if  $\mu$  is an indiscrete measure with finite support, then m is necessarily a Baire measure.

One defines "right invariant Haar system" in the obvious way and, again, such systems are in 1-1 correspondence with left invariant ones. For this reason, we usually consider just left invariant Haar systems and, in future, the term Haar system means left Haar system. Indeed, all the results of §2 hold for a Haar system, but we shall now sharpen some of them and consider the effect of the imposition of a topology and, in particular, the imposition of local triviality.

One definition we need is :

4.4.3. Definition.

A locally compact Hausdorff topological groupoid will be called <u>unimodular</u> if each vertex group is unimodular.

Of course, if G is transitive, then G is unimodular if, and only if, any one vertex group is unimodular.

4.4.4. General Construction of a Haar System.

Let G be a transitive locally compact Hausdorff topological groupoid over X, let T be a wide tree subgroupoid and let  $\mathcal{T}_{x} \in \mathcal{T}(x, x)$ 

for each  $x \in X$ . Let  $\phi_x : G(x, x_o) \longrightarrow G\{x_o\}$  be the homeomorphism defined by  $\phi_x(\alpha) = \alpha \tau_x^{-1}$  as in 4.2.5. If  $\gamma$  is a left invariant Haar measure on  $G\{x_o\}$ , then  $\gamma_x$ , defined by  $\gamma_x(E) = \gamma (\phi_x(E))$ , is a left invariant Baire measure on  $G(x, x_o)$ . Suppose now that T' is another wide tree subgroupoid of G with  $\tau'_x \in T'(x, x_o)$ , then, with the obvious meaning, we have

$$\nu'_{x}(E) = \nu (\phi'_{x}(E)) = \nu (E \tau'_{x}^{-1})$$

$$= \nu (E \tau_{x}^{-1} \cdot \tau_{x} \cdot \tau'_{x}^{-1}) = \nu ((E \tau_{x}^{-1}) \tau_{x} \tau'_{x}^{-1})$$

$$= \Delta (\tau'_{x} \tau_{x}^{-1}) \nu_{x} (E) .$$

$$\nu'_{x}(E) = \Delta (\tau'_{x} \tau_{x}^{-1}) \nu_{x} (E) . .... * ,$$

That is

where  $\Delta$  is the modular function of  $G\{x_0\}$ , see Berberian [1] page 260. Thus, we immediately obtain

#### 4.4.5. Proposition.

The extensions of  $\mathcal{V}$  to  $G(x, x_o)$ , as above, are in 1-1 correspondence with the distinct values of the modular function  $\Delta$  on  $G\{x_o\}$ . If G is unimodular, then the extension is unique.

These facts have been observed by Westerman [2]. He introduces the notion of a "continuous system of measures" which consists of a family of measures each defined on a set G(x, y), and subject to a smoothness condition. His system is different from ours in several respects. Firstly, his invariance condition, not being defined on costars or stars, does not reflect the maximum amount of invariance. This means that his system, when dealing with  $\tilde{G} = X \times G$ , amounts to considering invariant measures on the stability subgroups of the action, rather than those on G, and seems unlikely to give useful information about  $\tilde{G}$ . Secondly, Westman does not include " $\mu$ " or "m" in his definition and, hence, cannot attempt to give L'(G) a convolution product structure as we do in the next section. If we mimic his smoothness condition we would include the following in the definition of a Haar system : for each real valued continuous function f on G, with compact support, the function 111.

 $X \longrightarrow R$  defined by  $x \longmapsto \int f d\mu_x$  is continuous and has compact  $\operatorname{cost}_{G} x$ 

support. This condition is not needed for any of the constructions we carry out and, in any case, is satisfied for most of the examples we consider. In particular, it is satisfied if G is locally trivial or if G is the groupoid  $\tilde{G}$  determined by a G-space ; at least, in the case when the invariant system of measures is the one constructed in 4.2.15. Finally, if one includes this smoothness condition, it is not clear that a Haar system always exists.

By considering the extension of a Haar measure on a vertex group to each set  $G(x, x_o)$  and an indiscrete measure  $\mu$  with countable support, Theorem 4.2.6. yields :

4.4.6. Theorem.

Let G be any locally compact Hausdorff topological groupoid . over X . Then G admits a Haar system of measures.

#### 4.4.7.

If G is transitive, T is a wide tree subgroupoid of G,  $\gamma$ is a left Haar measure on  $G\{x_o\}$ ,  $\omega_x$  is the image of  $\gamma$  under  $\phi_x$ and  $c: X \longrightarrow R$  is any function, then the measure  $\mu_o$  defined on  $\cot_G x_o$  by  $\mu_o(E) = \int_X c(x) \omega_x (E \cap G(x, x_o)) d\mu$  is  $G\{x_o\}$ invariant, where  $\mu$  is chosen to make the integrand measurable. By 4.4.2,  $\mu_o$  need not be a Baire measure however. On the other hand, given  $\mu$  on X we can choose  $c: X \longrightarrow R$  to make  $x \longmapsto \mu_x (E \cap G(x, x_o)) \mu$ -measurable and, again, obtain  $\mu_o$  as above. It is our present aim to show that, with suitable restrictions, every  $G\{x_o\}$  invariant Baire measure has this form. <u>4.4.8</u>.

With the hypothesis of 4.4.4, suppose now that  $\mathcal{V}_{\mathbf{x}}$  is a left invariant Baire measure on  $G(\mathbf{x}, \mathbf{x}_{o})$ . Then  $\mathcal{V}$ , defined on  $G\{\mathbf{x}_{o}\}$  by  $\mathcal{V} = \phi_{\mathbf{x}}(\mathcal{V}_{\mathbf{x}})$ , is a left Haar measure. Thus, if  $\mathcal{V}$  is a fixed left Haar measure on  $G\{x_o\}$ ,  $\omega_x$  is any left invariant Baire measure on  $G(x, x_o)$  and  $\nu_x = \varphi_x(\omega_x)$ , then by the von-Neumann theorem there exists a constant  $c(x, \tau_x)$  such that  $\nu_x = c(x, \tau_x) \gamma$ . Then, by \* of 4.4.4 we have

$$c(x, \tau'_{x}) = \Delta(\tau'_{x}\tau_{x}^{-1})c(x, \tau_{x}) \dots **$$

### 4.4.9.

The next step is to apply disintegration theorems to a  $G\{x_o\}$ invariant measure on  $\operatorname{cost}_G x_o$ . In the case of a Borel groupoid, we needed standardness of the groupoid, essentially, and finiteness of the measures to apply theorems of Rohlin. In this case we shall appeal to theorems of Bourbaki. To do this, we do not need finiteness of the measures but we do need conditions on G . It will be convenient to assume rather more than we need; we shall assume that G is  $\sigma$ -bounded, metrizable and complete in addition to being locally compact.<sup>\*</sup> With these restrictions, the concepts of Baire set and Borel set, as in §1, coincide; indeed G is separable and is a standard Borel space.

Thus, with the aforementioned conditions on G, suppose  $\mu_o$ is a  $G\{x_o\}$  invariant Baire measure on  $\operatorname{cost}_G x_o$ . Let  $\mu$  be the quotient  $\pi(\mu_o)$  of  $\mu_o$  on X, which must nowbe a Baire measure, and apply Théorème 1 (or 2) of §3, Bourbaki [1] to obtain a family  $\omega_x$  of measures on  $G(x, x_o)$  such that  $\mu_o(E) = \int_X \omega_x(E \cap G(x, x_o)) d\mu$ , for each Baire set E of  $\operatorname{cost}_G x_o$ . (we can assume that  $\omega_x$  exists for each  $x \in X$  by the argument of 4.2.7). Since  $\mu_o$  is  $G\{x_o\}$ invariant, 4.2.7 shows that each  $\omega_x$  is  $G\{x_o\}$  invariant. We need: 4.4.10. Lemma.

Suppose G is a locally compact Hausdorff topological group, and let  $\mu$  be a left invariant measure defined on the Baire sets of G with the property that  $\mu(u) < \infty$  for some non empty open Baire subset U of G. Then  $\mu(C) < \infty$  for each compact G C in G. In particular,  $\mu$  is a Baire measure. Suppose also that  $\pi$ :  $\operatorname{cost}_{G^{X_0}} \longrightarrow X$  is a proper map, see pp 139. Proof.

The set  $\{g, U; g \in G\}$  of translates of U covers G. Thus, if C is a compact  $G_{\xi}$ , there are finitely many elements  $g_1, g_2, \dots, g_n$  of G such that  $C \subset \bigcup_{i=1}^n g_i U$ . Thus, by left invariance of  $\mu$  we have  $\mu(C) \leq \sum_{i=1}^n \mu(g_i U) \leq n\mu(U)$ .

Using this lemma, we show next that  $\mu$  almost all of the  $\omega_{\rm r}$ of 4.4.9 are Baire measures. To do this, suppose U is any open Baire set in G such that  $\mu_o(\mathtt{U}) < \infty$  . The formula  $\mu_{o}(U) = \int \omega_{x}(U \cap G(x, x_{o})) d\mu \text{ means that } \omega_{x}(U \cap G(x, x_{o})) < \infty$ for all  $x \in \pi(U)$  except possibly on a  $\mu$ -null subset N of  $\pi(U)$  . Since U  $\cap$  G(x, x) is an open non-empty Baire set for all  $x \in \pi(U)$ , the Lemma 4.4.10 yields that  $\omega_x$  is a Baire measure for all  $x \in \pi(U) \setminus N$ . (4.4.10 is actually applied to the image  $v_x = \phi_x(\omega_y)$ of course). To obtain the conclusion, we note that the conditions we have placed on G mean that there is a sequence  $E_1, E_2, E_3, \dots, E_n, \dots$ of compact G 's which covers cost G . Thus, by the Baire Sandwich Theorem (Berberian [1], page 176) there exists open Baire sets  $V_n$  and compact  $G_{\varsigma}$ 's  $C_n$  such that  $E_n \subset V_n \subset C_n$  for each n. Since  $\mu_{\varsigma}$ is a Baire measure,  $\mu_o(C_n)$  is finite for all n. This means that there are countably many open Baire sets  $V_n$  which cover  $cost_G x_s$  and each of which has finite  $\mu_o$  measure. If  $N_n \subset \pi(V_n)$  denotes the null set of the previous argument, then  $\bigcup_{n=1}^{\infty} \mathbb{N}_{n}$  is  $\mu$ -null and, for all  $x \in X \setminus \bigcup_{n=1}^{\omega} N_n$ ,  $\omega_x$  is a Baire measure. Thus, we may assume all the  $\omega_{\perp}$  are Baire measures and this observation means that we have proved the

x anologue for Haar systems of 4.2.8 for standard Borel groupoids:-4.4.11. Theorem.

Let G be a transitive locally compact topological groupoid over X for which G is  $\sigma$ -bounded, metrizable and complete, and suppose also that TT : cost x  $\longrightarrow$  X is a proper map and, finally, suppose  $\mu_o$  is a  $G\{x_o\}$  invariant Baire measure on  $\operatorname{cost}_{G} \times_o$ . Then there exists a wide tree subgroupoid T of G, a function  $c : X \longrightarrow R$ a measure  $\mu$  on X and a left Haar measure  $\gamma$  on  $G\{x_o\}$  such that

i) The function  $x \mapsto c(x) \omega_x (E \cap G(x, x_o))$  is  $\mu$ -measurable for each Baire set E of  $cost_G x_o$ .

ii)  $\mu_o(E) = \int_X c(x) \omega_x (E \cap G(x, x_o)) d\mu$ , where  $\omega_x = \phi_x^{-1}(v)$ .

Note that  $\mu$  must be a Baire measure.

Again because of \*\* of 4.4.8, we have

#### 4.4.12. Corollary.

Assume the hypotheses of the previous theorem. Then the measure class of  $\mu_o$  depends only on the measure class of  $\mu$  and the set  $\{x \in X ; c(x) = 0\}$ .

The procedure of 4.2.10 can now be applied to construct in principle all Haar systems for groupoids satisfying the conditions of 4.4.9.

Suppose now that G is a compact metrizable topological groupoid over X. Then G is necessarily complete, separable and  $\sigma$ -bounded. Further, the remarks made before Theorem 4.2.11, concerning the measure classes of  $\mu_o$ , apply equally well here and we obtain the following theorem.

#### 4.4.13. Theorem.

Suppose G is a compact metrizable topological groupoid over X. Then the number of inequivalent Haar systems for G is not greater than the number of pairs  $(\lceil \mu_1 \rceil, \lceil \mu_2 \rceil)$ , where  $\mu_1$  and  $\mu_2$  are Baire measures on X.

Locally compact Hausdorff versions of the two examples of 4.2.14 and 4.2.15 can now be considered, and Haar systems constructed in exactly the same way.

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# <u>4.4.14.</u> A construction for Locally Trivial Groupoids.

We next consider locally trivial topological groupoids and present a construction of a  $G\{x_{s}\}$  -invariant measure on  $cost_{G}x_{s}$ which utilises the local triviality. Thus, let G denote a transitive, locally trivial, locally compact, Hausdorff, topological groupoid over X, and let  $\{U_i, \lambda_i, x_o\}$  be a local trivialisation of G. For  $x \in U_i$ , write  $c_i(x) = c(x, \lambda_i(x))$  as in 4.4.8. Here  $c(x, \tau_x) = 1$  and  $c(x, \tau'_{x}) = c(x, \lambda_{i}(x)) = \Delta(\lambda_{i}(x)\tau_{x}^{-1}) = c_{i}(x) . \text{ Thus, if } x \in U_{i} \cap U_{i},$ \*\* of 4.4.8 shows that  $c_j(x) = \Delta (\lambda_j(x) \lambda_i(x)^{-1}) c_i(x)$ . Since  $\lambda_i$ ,  $\lambda_{j}$  and  $\Delta$  are continuous, it follows that  $c_{j}$  is measurable on  $U_i \cap U_i$  if, and only if,  $c_i$  is . Let  $\mu$  be a Baire measure on Х for which  $c_i$  is  $\mu$  -measurable on  $U_i$ . Let f be a continuous function with compact support on  $\pi^{-1}(U_i) \cap \operatorname{cost}_{G^{\times}}$ . Since the map  $U_i \times G\{x_o\} \longrightarrow \pi^{-1}(U_i) \cap \operatorname{cost}_G x_o$ , defined by  $(x, a) \longmapsto a \lambda_i(x)$ , is a homeomorphism, the function  $x \mapsto \int f(a \lambda_i(x)) d\nu(a)$  is continuous, where  $\mathcal Y$  denotes the fixed Haar measure of 4.4.8. Thus  $\int d\mu(x) \frac{1}{c_{i}(x)} \int f(\alpha \lambda_{i}(x) d\nu(\alpha) \text{ exists and defines a Baire measure}$ 

 $\mu_{o}^{i} \text{ on } \pi^{-1}(U_{i}) \cap \operatorname{cost}_{G} x_{o} \text{ which is } G\{x_{o}\} \text{ invariant by 4.4.7. Now,}$ if  $\omega_{i}$  is the left invariant Baire measure defined on  $G(x, x_{o})$  by  $\omega_{i}(E_{x}) = \gamma(E_{x}\lambda_{i}(x)^{-1})$ , then it is easy to see that  $\int f(a \lambda_{i}(x))d\gamma(a) = \int f(\beta)d\omega_{i}(\beta) \cdot \text{ If } x \in U_{i} \cap U_{j}, \text{ we can}$  $G\{x_{o}\} \qquad G(x, x_{o})$ 

also define left invariant Baire measure  $\omega_j$  on  $G(x,x_o)$  in the same way and by 4.4.4 we have  $\omega_j(E_x) = \Delta(\lambda_j(x)\lambda_i(x)^{-1})\omega_i(E_x)$  for any

Baire set  $E_x$  of  $G(x,x_o)$ . Whence we have :  $\int f(\alpha \lambda_j(x)) d \gamma(\alpha) = \Delta (\lambda_j(x) \lambda_i(x)^{-1}) \int f(\alpha \lambda_i(x)) d \gamma(\alpha) .$   $G\{x_o\}$   $G\{x_o\}$ 

Hence, using the relation  $c_j(x) = \Delta(\lambda_j(x)\lambda_i(x)^{-1})c_i(x)$ , we have

$$\frac{1}{c_{j}(x)} \int_{G\{x_{o}\}} f(\alpha \lambda_{j}(x)) d\nu(\alpha) = \frac{\Delta(\lambda_{j}(x)\lambda_{i}(x)^{-1})}{\Delta(\lambda_{j}(x)\lambda_{i}(x)^{-1})c_{i}(x)} \int_{G\{x_{o}\}} f(\alpha \lambda_{i}(x)) d\nu(\alpha)$$
That is,  $\frac{1}{c_{j}(x)} \int_{G\{x_{o}\}} f(\alpha \lambda_{j}(x)) d\nu(\alpha) = \frac{1}{c_{i}(x)} \int_{G\{x_{o}\}} f(\alpha \lambda_{i}(x)) d\nu(\alpha)$ 

which implies that  $\mu_o^i = \mu_o^j$  on  $U_i \cap U_j$ . Under these conditions, see Chapter 3, §2 , Prop.1 of Bourbaki [2], there exists a measure  $\mu_o$  on cost  $_{\rm G}$  whose restriction to  $\pi^{-1}(U_{\rm i})$  is  $\mu_o^{\rm i}$ . It is clearly  $G{x_i}$  invariant. In fact, if E is a compact  $G_{s}$  of  $\operatorname{cost}_{G} x_{o}$  with  $\operatorname{E} \subset \pi^{-1}(U_{\kappa})$ , then  $\mu_{o}(\operatorname{E})$  is given by

$$\mu_{o}(\mathbf{E}) = \int \frac{1}{c_{\kappa}(\mathbf{x})} \omega_{\kappa}(\mathbf{E} \cap \mathbf{G}(\mathbf{x},\mathbf{x})d_{\mu}(\mathbf{x}))$$

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Now let  $V \subset U_{\kappa}$  and  $H \subset G\{x_{o}\}$  be compact  $G_{s}$ 's, so that V × H is a compact  $G_{S}$ . Let  $\phi_{\kappa}$  :  $U_{\kappa} \times G\{x_{s}\} \longrightarrow \operatorname{cost}_{G} x_{s} \cap \pi^{-1}(U_{\kappa})$ be the usual homeomorphism defined by  $\phi_{\kappa}(x,\alpha) = \alpha \lambda_{\kappa}(x)$ , and let E be the Baire set of  $\operatorname{cost}_{G} \times \operatorname{defined}$  by  $E = \mathscr{P}_{\kappa} (V \times H)$ . Then  $E_x = E \cap G(x,x_o) = \phi_{\kappa,x}(H) = H \lambda_{\kappa}(x)$ . But  $\omega_{\kappa}(E_x) = \omega_{\kappa}(E_x)$  $\nu (\mathbf{E}_{\mathbf{x}} \lambda_{\mathbf{k}}(\mathbf{x})^{-1}) = \nu (\mathbf{H}) .$ Thus,  $\mu_{o}(E) = \int \frac{1}{c_{\kappa}(x)} \mathcal{V}(H) d\mu = \mathcal{V}(H) \int \frac{1}{c_{\kappa}(x)} d\mu$  $= \mu(\mathbf{V}) \, \gamma(\mathbf{H}) \, \mathbf{A} \, (c_{\mathbf{v}}^{-1}) \, ,$ 

where  $A(c_{\kappa}^{-1})$  denotes the average value of  $\frac{1}{C_{\kappa}}$  on V. This shows that the construction here somewhat resembles a product construction, but a suitable correcting factor is needed. This is supplied by  $A(c_{\kappa}^{-1})$  over compact  $G_{\varsigma}$ 's V. In the next construction (4.4.15) we shall consider Unimodular groupoids ; in which circumstances  $C_{\kappa} \equiv 1$  and so no correcting factor will be needed. Of course, if G is globally trivial,  $c_{\kappa} \equiv 1$  on X and no correcting factor is needed in this case either. This fact is used in proving Theorem 4.5.9.

Next, we define  $\mu_x$  by  $\mu_x = L_{\lambda_i}(x)^{-1}(\mu_o)$ . As we showed

in 4.2.3,  $\mu_x$  does not depend on the index i because of the  $G\{x_n\}$ invariance of  $\mu_o$ . That is, if  $x \in U_i \cap U_j$ , then  $L_{\lambda_{i}(x)}^{-1}(\mu_{o}) = L_{\lambda_{i}(x)}^{-1}(\mu_{o})$ . Finally, let  $\mu_{i}$  be another Baire measure on X and define m by  $m(E) = \int \mu_x (E \cap \operatorname{cost}_G x) d\mu_i$ . The  $\mu_i$  -measurability of the integrand here is a consequence of Fubini's Theorem and we give details in the next construction. In fact, m is locally.a product, see 4.4.15, and we have  $m(\mathcal{L}_i(W \times E)) = \mu_i(W).\mu_o(E)$ for Baire sets W and E where  $\psi_i$  is defined as in 4.4.15.

follows that  $\{m, \mu, \mu_x; x \in X\}$  is a Haar system for G, the invariance being a consequence of 4.2.3. If E is the Baire set defined above and W is a compact  $G_{s}$  in X, then  $m(\mathcal{L}_{i}(W \times E)) = \mu_{i}(W)\mu_{s}(E)$ 

=  $\mu_{i}(W) \mu(V) \nu(H) A(C_{-1}^{-1})$ .

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#### Haar Systems which are locally a product. 4.4.15.

To close this section, we shall now apply a construction of A. Goetz [1] to construct a Haar system which is locally a product. In order to carry out the construction, we need to assume that G is unimodular and the need for this will become apparent as we proceed. It is because of Theorem 3.2.5 essentially.

Suppose  $P_F: S_F \longrightarrow B$  is a locally trivial fibre bundle with group H in which the fibre F and base B are locally compact, so that  $S_F$  is locally compact. Suppose  $\mu$  is a Baire measure on B and  $\nu$ is a Baire measure on F , and denote by  $\mu \times \nu$  the product measure on B × F.

# <u>4.4.16</u>. <u>Definition</u>. (Goetz [1])

A Baire measure m on S<sub>F</sub> is called the product measure of  $\mu$ and  $\gamma$  in the fibre bundle  $S_F$  if for every choice of atlas  $\{U_i, \phi_i\}$ and for each Baire set  $Z \subset U_i \times F$  the equality  $m(\phi_i(Z)) = (\mu \times \nu)(Z)$ holds.

We shall paraphrase this by saying that "m is locally a product of  $\mu$  and  $\gamma$  ."

We need:

# <u>4.4.17.</u> Theorem.

Let G be a transitive locally trivial topological groupoid over X and let  $x \in X$ . Then G can be regarded as a locally trivial fibre bundle with fibre  $\operatorname{cost}_{G} x_{o}$ , projection  $\pi'$  and group  $G\{x_{o}\}$ acting on the left of  $\operatorname{cost}_{G} x_{o}$  in the natural way determined by the composition in G.

#### Proof.

We shall give the essential details of this in proving the next theorem.

We shall call a groupoid G over X <u>countably disconnected</u> if it has at most countably many transitive components.

We now prove:

4.4.18. Theorem.

#### Proof.

It suffices to consider the transitive case, for Corollary 2 to 2.4.3 shows that G is the topological and measure theoretic sum of its transitive components.

Sufficiency.

Let  $\{U_i, \lambda_i, x_o\}$  be a local trivialisation for G, then  $\phi_i : U_i \times G\{x_o\} \longrightarrow \pi^{-1}(U_i) \wedge \operatorname{cost}_G x_o$  defined by  $\phi_i(x, a) = a \lambda_i(x)$ is a chart over  $U_i$  for  $\operatorname{cost}_G x_o$ , see Theorem 3.2.4. Here the transition function  $h_{ji}(x) = \phi_{j,x}^{-1} \phi_{i,x}$  corresponds to right multiplication by the element  $\lambda_i(x) \lambda_j(x)^{-1}$  on the fibre  $G\{x_o\}$ .

Let  $\mu_i$  be a Baire measure on X and let  $\mathcal{V}$  be a Haar measure on  $G\{x_o\}$ . By the hypothesis of unimodularity,  $\mathcal{V}$  is both left and right invariant. Now form the product  $\mu_i \times \mathcal{V}$  on  $X \times G\{x_o\}$ . Let  $x \in U_i$  and define a Baire measure  $\omega_x$  on  $G(x,x_o)$  (which is the fibre of  $\operatorname{cost}_G x_o$  over x) by  $\omega_x(E_x) = \mathcal{V}(\phi_{i,x}^{-1}(E_x))$ . By Theorem 1 of Goetz [1], the definition of  $\omega_x$  is independent of the choice of coordinate neighbourhood containing x. It is also independent of the choice of atlas  $\{U_i, \phi_i\}$  and, hence,  $\omega_x$  is well defined. The proof of these statements uses the right invariance of  $\mathcal{V}$ . The next step is to define Baire measure  $\mu_o$  on  $\operatorname{cost}_G x_o$  by  $\mu_o(E) = \int_X \omega_x(E \cap G(x,x_o))d\mu_i$ .

Goetz shows that  $\mu_o$  is locally a product. We shall now show that it is  $G\{x_o\}$  invariant. To do this, let  $\alpha \in G\{x_o\}$  and let E be a Baire set. Since  $(\alpha E)_x = \alpha E_x$  we have  $\mu_o(\alpha E) = \int_X \omega_x (\alpha E_x) d\mu_c$ . But

$$\begin{split} & \omega_{\mathbf{x}}(\mathbf{a}\mathbf{E}_{\mathbf{x}}) = \mathcal{V}(\mathbf{a}\mathbf{E}_{\mathbf{x}}\lambda_{\mathbf{i}}(\mathbf{x})^{-1}) \quad \text{for any i such that } \mathbf{x} \in \mathbf{U}_{\mathbf{i}} \cdot \mathbf{Thus} \\ & \omega_{\mathbf{x}}(\mathbf{a}\mathbf{E}_{\mathbf{x}}) = \mathcal{V}(\mathbf{E}_{\mathbf{x}}\lambda_{\mathbf{i}}(\mathbf{x})^{-1}) = \omega_{\mathbf{x}}(\mathbf{E}_{\mathbf{x}}) \quad \text{by the left invariance of } \mathcal{V} \cdot \mathbf{Hence} \quad \mu_{\mathbf{o}}(\mathbf{a}\mathbf{E}) = \mu_{\mathbf{o}}(\mathbf{E}) \cdot \mathbf{I} \end{split}$$

Next, define  $\mathscr{L}_{i}: U_{i} \times \operatorname{cost}_{G} \times \longrightarrow \pi'^{-1}(U_{i})$  by  $\mathscr{L}_{i}(x, \alpha) = \lambda_{i}(x)^{-1} \alpha$ . One easily shows that  $\mathscr{L}_{i}$  is a chart for G over  $U_{i}$  and, in fact, the transition function  $g_{ji}(x) = \mathscr{L}_{j,x} \mathscr{L}_{i,x}$ corresponds to the operation of the element  $\lambda_{j}(x) \lambda_{i}(x)^{-1}$  of  $G\{x_{o}\}$ . Using the functions  $\mathscr{L}_{i,x}: \operatorname{cost}_{G} \times \longrightarrow \operatorname{cost}_{G} \times$ , we obtain well defined measures  $\mu_{x} = \mathscr{L}_{i,x}(\mu_{o})$  on  $\operatorname{cost}_{G} \times$  for each  $x \in X$ . Clearly, each  $\mu_{x}$  is locally a product. Now choose another Baire measure  $\mu_{2}$  on X and define Baire measure m on G by

$$\mu(E) = \int \mu_{x}(E \cap \operatorname{cost}_{G} x) d\mu_{z} \cdot$$

By Goetz [1], m is locally a product.

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To complete the proof of the sufficiency, we need to show that the system  $\{\mu_x; x \in X\}$  is invariant in the sense of iii) of the definition. However, the definition of  $\mathcal{V}_{i,x}$  shows that the system  $\{\mu_x; x \in X\}$  coincides with that obtained in proving the sufficiency in 4.2.3. Thus, the invariance follows and so does the sufficiency of the theorem. That is,  $\{m, \mu_2, \mu_x; x \in X\}$  is a Haar system in which m and  $\mu_x$ ,  $x \in X$ , is locally a product. Necessity.

Suppose  $\{m, \mu_{2}, \mu_{x}; x \in X\}$  is a Haar system in which m and each  $\mu_{x}$  is locally a product. If  $x \in X$ , then  $\mu_{x}$  is locally a product of a measure  $\mu$  on X and a measure  $\nu$  on  $G\{x_{o}\}$ . By Theorem 1 of Goetz [1] and the bundle structure of  $\operatorname{cost}_{G}x_{o}$ ,  $\nu$  is a right invariant Haar measure. But  $\mu_{x_{o}}$  is invariant under the action of  $G\{x_{o}\}$  from which it easily follows that  $\nu$  is also left invariant. Thus,  $G\{x_{o}\}$  is unimodular and so, therefore, is G.

The proof of the theorem is now complete.

Notice that no restriction was placed on the choice of  $\mu_1$  and  $\mu_2$  in proving Theorem 4.4.18 other than the condition of being Baire measures. The measurability of the two integrands occuring in the definition of  $\mu_0$  and m, respectively, is a consequence of a local form of Fubini's Theorem, see Goetz [1]. Since any compact Hausdorff topological groupoid is unimodular and countably disconnected, we have:

### 4.4.19. Corollary.

Let G be a locally trivial, compact, metrizable, topological groupoid over X. Then the inequivalent Haar systems on G are in 1-1 correspondence with the pairs  $(\Box_{\mu_1}, \Box_{\mu_2})$ , where  $\mu_1$  and  $\mu_2$  are Baire measures on X.

To conclude our discussion of invariant measures for groupoids, we remark that the construction in 4.4.18 is independent of the choice of the coordinate systems, and can be regarded as a canonical representative of each of the equivalence classes of Haar systems.

The construction of 4.4.14 shows that for locally trivial groupoids m can always be chosen to be locally a product, 4.4.18 shows that  $\mu_x$  can be/chosen if, and only if, G is unimodular.

### §5. Convolution Algebras for Groupoids.

Suppose G is a locally compact Hausdorff topological group and let m denote Haar measure on G. Associated with G are the two function spaces  $C_{c}(G) = \{f : G \longrightarrow \mathcal{L} ; f \text{ is continuous and} \}$ has compact support  $\}$ , and L'(G) defined by  $L'(G) = \{f : G \longrightarrow \mathcal{L} ; f \text{ is an m integrable Baire function } \}$ . Here  $\mathcal{L}$  denotes the complex field and, as usual, two elements of L'(G) will be identified if they only differ on an m-null set.

It is an important fact in much of analysis that both  $C_{c}(G)$ and L'(G) have natural multiplications or convolutions which turn them into complex associative algebras. In particular, L'(G) is a Banach algebra with the usual L, norm, and has been extensively analysed of late in an effort to gain insight into the structure of G.

The precise definition of convolution is as follows. If f and g are both elements of  $C_c(G)$  or of L'(G) we define their convolution f \* g by f \* g(a) =  $\int f(\beta)g(\beta^{-1}a)dm(\beta)$ . In the case G

of L'(G), this formula only defines f \* g = m almost everywhere, see Berberian [1]. It is the purpose of this section to attempt a generalisation of these facts for groupoids. We then give applications of the results we obtain in Sections 6 and 7.

Throughout this section G will denote a locally compact Hausdorff topological groupoid over X and  $\{m, \mu, \mu_x; x \in X\}$  will denote a Haar system for G. We define two function spaces associated with G. Firstly, we define

 $C_{c}(G) = \{f : G \longrightarrow \mathcal{C} ; f \text{ is continuous and has compact} support \}$  and

secondly  $L'(G) = \{f : G \longrightarrow \mathcal{L} ; f \text{ is an } m \text{ integrable Baire function} \}$ . We shall always identify two m integrable Baire functions f and g if the set of elements of G on which they differ is m-null. Both of these spaces are, of course, complex vector spaces with the usual pointwise operations of addition and scalar multiplication.

For two functions f and g in  $C_c(G)$  we define their <u>convolution</u> f \* g : G  $\longrightarrow \mathcal{L}$  by the formula

 $\mathbf{f} * \mathbf{g}(\alpha) = \int_{\operatorname{cost}_{G}} \mathbf{f}(\beta) \mathbf{g}(\beta^{-1}\alpha) d\mu_{\pi'(\alpha)}(\beta)$ .

We observe here that the integrand is a continuous function with compact support, and so the integral above exists and f \* g is a well defined function on G. We will also define a convolution product on  $L^{1}(G)$ but will need several preliminary lemmas. As with  $L^{1}(G)$ , for G a group, the convolution f \* g of two functions f and g in  $L^{1}(G)$ is only defined m-almost everywhere by the above formula; this causes no real difficulty however.

As we have already noted, a concept of Invariant Measures has been given by Westman [2], which differs essentially from ours. Westman also defines a convolution for  $C_c(G)$  and this too differs essentially from ours. Indeed, Westman's definition does not permit a discussion of L'(G).

We shall denote by || || the uniform norm on  $C_c(G)$  so that  $||f|| = \sup_{x \in G} |f(x)|$  where  $f \in C_c(G) \cdot C_c(G)$  now becomes an incomplete normed vector space with norm || || . We also define the L norm on  $L^1(G)$  by  $||f||_1 = \int_{G} |f| dm$  as usual.  $L^1(G)$  now

becomes a Banach space with this norm. If G is a group, then L'(G)is a Banach algebra, that is, the inequality  $||f * g||_{1} \leq ||f||_{1} || g||_{1}$ holds on L'(G). We shall see that this inequality can fail when G is a groupoid.

Our present task is to define f \* g for functions f and gin L'(G). To this end we shall prove a series of lemmas starting with: <u>4.5.1</u>. Lemma.

Let  $f: G \longrightarrow C$  be an m-integrable Baire function on G, then: a)  $f |_{cost_G x}$  is a Baire function on  $cost_G x$  for all  $x \in X$ .

for 
$$\mu$$
 almost all x. Also,  $\int f dm = m(E) = \int \int f d\mu_x d\mu$ .  
G  $\times \operatorname{cost}_{G}^{x}$ 

Thus, b) and c) are verified for the case of a characteristic function, and the usual arguments extend b) and c) to the case of a simple function.

For the general case, we first observe that f is integrable if, and only if, |f| is integrable and so we can suppose f is real valued and non-negative. Define  $\phi : X \longrightarrow R^{>0}$  by  $\phi(x) = \int_{cost_{G}x} f d\mu_{x}$ ,

thus  $\phi$  can take extended real values. Our task is to show that  $\phi$ is  $\mu$ -integrable. To do this, we shall suppose otherwise and derive a contradiction. Thus, let A denote the set of  $x \in X$  for which  $\phi(x)$ is infinite valued, and suppose  $\mu(A) > 0$ ; we show that this forces  $\int f dm$  to be infinite.

Let  $f_n = \sum_{j=1}^{K_n} a_j^n \chi_n$  be a sequence of simple Baire functions

such that

i)  $0 \leq f_n \leq f_{n+1}$  for all n.

ii) f<sub>n</sub> † f pointwise.

Such a sequence exists, see Berberian [1], §16. Then  $\int f_{n-1} dm \leq \int f_n dm = \sum_{j=1}^n a_j^n m(E_j^n) \leq \int f dm .$ Thus, for all n,  $\sum_{j=1}^{K_n} a_j^n m(E_j^n) \leq \int f dm$  and so  $\int f_n dm$  is a bounded increasing sequence. Define  $\phi^n : X \longrightarrow R \ge 0$  by  $\phi^{n}(\mathbf{x}) = \int_{\operatorname{cost}_{n}\mathbf{x}} f_{n} d\mu_{\mathbf{x}} \cdot$ Then  $\phi^n(\mathbf{x}) = \int_{\text{cost } \mathbf{x}} \sum_{j=1}^{K_n} a_j^n \chi_{\mathbf{E}_j^n} d\mu_{\mathbf{x}} = \sum_{j=1}^{K_n} a_j^n \mu_{\mathbf{x}}(\mathbf{E}_j^n \wedge \text{cost}_{\mathbf{G}}^{\mathbf{x}})$ . Since  $f_n$  is monotone we have  $\phi^n(x) \leq \phi^{n+1}(x) \leq \phi(x)$  for all n, and  $\phi^n$  is  $\mu$  -measurable by i) of the definition of a Haar system.  $\int_{X} \phi^{n}(\mathbf{x}) d\mu = \int_{j=1}^{K} a_{j}^{n} \mu_{\mathbf{x}}(\mathbf{E}_{j}^{n} \cap \operatorname{cost}_{\mathbf{g}} \mathbf{x}) d\mu =$ Also  $= \sum_{j=1}^{n} a_{j}^{n} \int_{X} \mu_{x}(E_{j}^{n} \cap \operatorname{cost}_{G} x) d\mu = \sum_{j=1}^{K_{n}} a_{j}^{n} m(E_{j}^{n}) \text{ using } b) \text{ and } c) \text{ as}$ verified for simple functions. Thus, the  $\phi^n$  form an increasing sequence of integrable functions. Next we show that  $\lim_{n\to\infty} \phi^n(x) = \phi(x)$ . If  $x \in X \setminus A$ ,  $\phi(x)$  is a real number and  $\phi^n(x) \leqslant \phi(x)$  for all n, so by the Beppo Levi Theorem, Bartle [1],  $\lim \phi^n(x) = \lim \int_{cost_{\mathcal{C}}x} f_n d\mu_x = \int_{cost_{\mathcal{C}}x} f d\mu_x$ 

 $= \oint (x) \cdot \text{If } x \in A, \text{ then } \oint^{n}(x) \longrightarrow \infty \quad \text{as } n \longrightarrow \infty \quad \text{for if}$  $\phi^{n}(x) \text{ is bounded, then the Monotone Convergence Theorem of Berberian [1],}$ page 94, shows that  $\int_{cost_{c}x} f d\mu_{x} \text{ is finite which is a contradiction.}$ 

Thus  $\phi$  is a  $\mu$ -measurable function on X and our supposition that  $\mu(A) > 0$  means that  $\int_{X} \phi(x) d\mu$  is infinite. But this means that  $\int_{X} \phi^{n}(x) d\mu$  cannot be bounded which means, in turn, that  $\sum_{j=1}^{Kn} a_{j}^{n} m(E_{j}^{n})$  is not bounded which contradicts the boundedness of the integrals of the  $f_{n}$ . This proves b) and c) follows easily from this.

For any two functions f and g defined on G and taking values in  $\mathcal{C}$ , we define f  $\nabla$  g mapping  $\operatorname{cost}_{G} x \times \operatorname{cost}_{G} x$  into  $\mathcal{C}$ by f  $\nabla$  g ( $\beta$ ,  $\alpha$ ) = f( $\beta$ )g ( $\beta^{-1}\alpha$ ), for each  $x \in X$ . We have: <u>4.5.2</u>. Lemma.

Suppose f and g are Baire functions on G. Then f  $\nabla$  g is a Baire function on  $cost_G x \times cost_G x$  for each  $x \in X$ . Proof.

 $f \lor g$  is really the composite  $\operatorname{cost}_{G} x \times \operatorname{cost}_{G} x \longrightarrow \operatorname{cost}_{G} x \times G \xrightarrow{fg} \mathcal{L}$  where the first map is  $(\beta, \alpha) \longmapsto (\beta, \beta^{-1} \alpha)$ , and is continuous, and  $fg(\beta, \alpha) = f(\beta)g(\alpha)$ . Since it is a standard fact that fg is a Baire function, see Berberian [1], §84, we have the result.

Let f and g be complex functions on G so that, for each x, we have  $f \nabla g : \operatorname{cost}_{G} x \times \operatorname{cost}_{G} x \longrightarrow \mathcal{L}$ . Now fix a in the second factor to obtain  $(f \nabla g)^{\alpha} : \operatorname{cost}_{G} x \longrightarrow \mathcal{L}$  defined by  $(f \nabla g)^{\alpha}(\beta) = f \nabla g (\beta, \alpha)$ , and fix  $\beta$  in the first factor to obtain  $(f \nabla g)_{\beta}^{\alpha} : \operatorname{cost}_{G} x \longrightarrow \mathcal{L}$  defined by  $(f \nabla g)_{\beta}(\alpha) = f \nabla g (\beta, \alpha)$ . These functions are, of course, the <u>sections</u> of  $f \nabla g$ . For a function  $h: \mathbb{C} \longrightarrow \mathcal{L}$  and an element  $\alpha \in \operatorname{cost}_{G} x$ , we define  $h^{\alpha} : \operatorname{cost}_{G} x \longrightarrow \mathcal{L}$  by  $h^{\alpha}(\beta) = h(\beta^{-1}\alpha)$  and for an element  $\beta \in \operatorname{cost}_{G} x$  we define  $h_{\beta} : \operatorname{cost}_{G} x \longrightarrow \mathcal{L}$  by  $h_{\beta}(\alpha) = h(\beta^{-1}\alpha)$ . These two functions will be called the <u>translates</u> of h. We remark that our notation for sections and translates will never cause confusion simply because one cannot take sections of a function of one variable; on the other hand we will never be concerned with translates of functions of two variables. In any case, it will be clear from the context whether we are discussing translates or sections.

We now give explicit formulas for the sections of f  $\nabla$  g in:

4.5.3. Lemma.

Let f and g be complex valued functions on G then : a)  $(f \nabla g)^{\alpha} = fg^{\alpha}$ , that is,  $(f \nabla g)^{\alpha}(\beta) = f(\beta)g^{\alpha}(\beta)$ for all  $\beta \in \operatorname{cost}_{c} x$ .

b)  $(f \nabla g)_{\beta} = f(\beta)g_{\beta}$ , that is,  $(f \nabla g)_{\beta}(\alpha) = f(\beta)g_{\beta}(\alpha)$ for all  $\alpha \in \operatorname{cost}_{G} x$ .

Proof.

The proof is straightforward and will be omitted.

This lemma immediately gives :-

<u>4.5.4</u>. Lemma.

If f and g are Baire functions on G, then every section  $(f \nabla g)^{\alpha}$  and every section  $(f \nabla g)_{\beta}$  is a Baire function.

If f and g are m-integrable Baire functions on G, then 4.5.1 asserts that there is a set  $Y \subset X$  such that  $\mu(Y) = 0$  and  $f \Big|_{cost_{G}X}$  and  $g \Big|_{cost_{G}X}$  are both  $\mu_{X}$  integrable Baire functions for

all  $x \in X \setminus Y$ . If, further, H is the full subgroupoid of G over Y, then by ii) of the definition of Haar system of measures we have m(H) = 0. With this notation we obtain :

4.5.5. Lemma.

Let f and g be m integrable Baire functions on G, then: i) For each  $\beta \in G \setminus H$  (f  $\nabla g$ )<sub> $\beta$ </sub> is a  $\mu_x$  integrable Baire function on cost<sub>G</sub>x, where  $x = \pi'(\beta)$ .

ii) For each  $x \in X \setminus Y$  the iterated integral  $\iint (f \nabla g) d\mu_x d\mu_x$ exists.

#### Proof.

i) For any  $\beta \in G \setminus H$   $(f \lor g)_{\beta}(\alpha) = f(\beta)g_{\beta}(\alpha)$  for all  $\alpha \in \operatorname{cost}_{G} x$ , by Lemma 4.5.3, and so i) follows from the fact that  $\alpha \longmapsto \beta^{-1} \alpha$ is a homeomorphism which preserves the system  $\{\mu_{x}; x \in X\}$  in the sense of iii) of the definition of a Haar system. ii) This follows from i), Fubini's theorem and the invariance of the system  $\{\mu_x; x \in X\}$ .

The final result we need to enable us to define a convolution on  $L^{1}(G)$  is :

4.5.6. Theorem.

Suppose f and g are m-integrable Baire functions on G, then, for all  $x \in X \setminus Y$ ,  $(f \lor g)^{\alpha}$  is a  $\mu_x$  integrable Baire function on  $\operatorname{cost}_{G} x$  for  $\mu_x$  almost all  $\alpha \in \operatorname{cost}_{G} x$ . Indeed, there is a Baire set  $A_x$  in  $\operatorname{cost}_{G} x$ , for all  $x \in X \setminus Y$ , and a  $\mu_x$  integrable Baire function h such that :

i)  $\mu_{x}(A_{x}) = 0$ .

ii) For all  $a \in \operatorname{cost}_{G} \times \setminus A_{x}$ ,  $(f \vee g)^{\alpha}$  is a  $\mu_{x}$  integrable Baire function and  $h(\alpha) = \int_{\operatorname{cost}_{G} x} (f \vee g)^{\alpha} d \mu_{x}$ .

Proof.

Let  $\mu_X \times \mu_X$  be the product Baire measure on  $\operatorname{cost}_G x \times \operatorname{cost}_G x$  for each  $x \in X \setminus Y$ . By Lemma 4.5.1 and the definition of Y,  $f \Big|_{\operatorname{cost}_G x}$  and  $g \Big|_{\operatorname{cost}_G x}$  are  $\mu_X$  integrable Baire functions on  $\operatorname{cost}_G x$ , and by 4.5.2  $f \nabla g$  is a Baire function on  $\operatorname{cost}_G x \times \operatorname{cost}_G x$ . By 4.5.5, the iterated integral  $\iint (f \nabla g)(\beta, \alpha) d\mu_X d\mu_X$  exists and so by Lemma 4.5.4 we can apply the converse of Fubini's theorem (see 41.2 of Berberian [1]) and we see that  $f \nabla g$  is  $\mu_X \times \mu_X$  integrable. Thus, by Fubini's theorem,

$$\int (\mathbf{f} \nabla \mathbf{g}) d(\mu_{\mathbf{x}} \times \mu_{\mathbf{x}}) = \iint (\mathbf{f} \nabla \mathbf{g})(\beta, \alpha) d\mu_{\mathbf{x}} d\mu_{\mathbf{x}}$$
$$= \iint \left\{ \int (\mathbf{f} \nabla \mathbf{g})^{\alpha}(\beta) d\mu_{\mathbf{x}}(\beta) \right\} d\mu_{\mathbf{x}}(\alpha) ,$$

and so by the definition of "iterated integral" the desired  $A_x$  and h exist.

Observe finally that the value of  $\int h d\mu_x$  is the iterated integral  $\iint f \nabla g d\mu_x d\mu_x \cdot \blacksquare$ 

We are now in a position to define the convolution of two m-integrable Baire functions on G. Let f and g be two such functions on G, taking complex values, and let  $Y \subset X$  and  $A_{\chi} \subset \operatorname{cost}_{G} x$ have their usual meanings. Then for  $\alpha \in \operatorname{cost}_{G} x \setminus A_{\chi}$  we define the <u>convolution</u> f \* g of f and g by :-

$$\mathbf{f} * \mathbf{g}(\alpha) = \int (\mathbf{f} \nabla \mathbf{g})^{\alpha} d\mu_{\mathbf{x}} = \int \mathbf{f}(\beta) \mathbf{g}(\beta^{-1} \alpha) d\mu_{\mathbf{x}} \cdot \mathbf{g}(\beta^{-1} \alpha) d\mu_{\mathbf{x}}$$

Our next task is to show that f \* g, defined as above, is defined m almost everywhere on G . Recall that H is the full subgroupoid of G over Y .

#### 4.5.7. Proposition.

Let  $A = \bigcup_{x \in X \setminus Y} A_x$ , where  $A_x \subset \operatorname{cost}_G x$  has its usual meaning,

then  $H \cup A$  is contained in an m-null set of G.

<u>Proof</u>. Let  $f = \varkappa_A$  be the characteristic function of A and suppose  $\phi$  is a simple Baire function on G which is such that  $0 \le \phi \le f$ . We can write  $\phi = \sum_{\kappa=1}^{h} a_{\kappa} \varkappa_{B_{\kappa}}^{\kappa}$ , where the sets  $B_{\kappa}$  are pairwise disjoint

Baire sets with  $\bigcup_{\kappa=1}^{n} B \leq A$  and  $0 < a_{\kappa} \leq 1$ , for  $\kappa = 1, 2, ..., n$ .

Now, by 4.5.1 we have

$$\int \phi dm = \int \int \phi d\mu_x d\mu = \int \int \int \sum_{\substack{\text{cost}_{G^x} \\ \text{K} = 1}} \int \int \phi d\mu_x d\mu = \int \int \sum_{\substack{\text{cost}_{G^x} \\ \text{K} = 1}} \int \int \sigma d\mu_x d\mu$$
$$= \int \int \int \sigma d\mu_x d\mu d\mu$$
$$= \int \int \int \sigma d\mu_x d\mu d\mu d\mu$$

But  $\mathcal{B}_{\kappa} \cap \operatorname{cost}_{\mathcal{G}} x \subseteq A_{x}$  and so  $\mu_{x}(\mathcal{B}_{\kappa} \cap \operatorname{cost}_{\mathcal{G}} x) = 0$  for all x, whence  $\int_{\mathcal{G}} \phi \, \mathrm{dm} = 0$ . Thus  $\sup \left\{ \int_{\mathcal{G}} \phi \, \mathrm{dm} \right\} = 0$  where the supremum is taken over all simple Baire functions  $\phi$  on  $\mathcal{G}$  such that  $0 \leq \phi \leq f$ .
Proposition 4.5.7 implies that the formula  $f * g(a) = \int (f \nabla g)^a d\mu \cos t_c x$ 

defines f \* g m-almost everywhere and we can then define f \* g to be zero on a null set of G to obtain  $f * g : G \longrightarrow C$ .

For any topological space X we can form the space C(X) of continuous complex valued functions on X, with a similar notion existing for real instead of complex valued functions. If X is compact, we again have the uniform norm on C(X), denoted by || || and defined by  $|| f || = \sup_{X \in X} || f(x) ||$ . Of course, if X is compact, then  $C(X) = C_c(X)$ . We shall need the following lemma (Lemma 1, page 45, Nachbin [1]). 4.5.8. Lemma.

Let E be a compact space, F a topological space and  $f: E \times F \longrightarrow R$  a function. Suppose that for each  $y \in F$  the function  $E \longrightarrow R$  defined by  $x \longmapsto f(x,y)$  is continuous. Let  $f': F \longrightarrow C(E)$  be defined by f'(y)(x) = f(x,y). Then f' is continuous if, and only if,  $f: E \times F \longrightarrow R$  is continuous.

It is easy to modify this lemma to allow complex valued functions instead of real valued ones.

We next prove:

#### 4.5.9. Theorem.

Let G be a locally compact, locally trivial, Hausdorff topological groupoid over X which is countably disconnected. Then  $C_c(G)$  can be given the structure of a complex convolution algebra with \* as its multiplication.

Proof.

With the stated hypotheses, we can assume G transitive and

use 4.4.14 to obtain the Haar system  $\{m, \mu_1, \mu_x; x \in X\}$  constructed there, and we can take  $\mu = \mu_1$  in that construction for simplicity. The proof of the theorem will be divided into three well defined steps. The steps dealing with the vector space aspects are well known and will be omitted, there are three non trivial ones concerning the convolution product.

i) We show f \* g has compact support. So suppose  $f \neq 0$  has support contained in A and that  $g \neq 0$  has support contained in B, with A and B compact. Since G is Hausdorff,  $\mathfrak{D}$  is closed in  $G \times G (2.1.4)$  and so  $A \times B \cap \mathfrak{D}$  is compact. Let A  $\cdot B$  be the image comp.( $A \times B \cap \mathfrak{D}$ ) of  $A \times B \cap \mathfrak{D}$  under the composition function. A  $\cdot B$  is compact. We show that support of  $f * g \subseteq A \cdot B$ . To do this, we show that  $a \in G \setminus A \cdot B$  implies f \* g(a) = 0, which is equivalent to showing that  $f * g(a) \neq 0$  implies  $a \in A \cdot B$ . So suppose  $f * g(a) \neq 0$ , then we have  $\int_{cost} f(\beta)g(\beta^{-1}a)d\mu_{\pi'}(a) \neq 0$ 

which means that  $f(\beta)g(\beta^{-1}\alpha) \neq 0$  on some set  $E \subset G \cap \operatorname{cost}_{G}\pi'(\alpha)$ of positive  $\mu_{\pi'(\alpha)}$ -measure. So for  $\beta \in E$ ,  $\beta \in$  support of  $f \subseteq A$ and  $\beta^{-1}\alpha \in$  support of  $g \subseteq B$ . So  $\alpha \in \beta B$  for  $\beta \in A$ , whence  $\alpha \in A \cdot B$  and we conclude that support of  $f * g \subseteq A \cdot B$ . ii) Next we prove that f \* g is continuous.

Since continuity is a local property, we can now suppose G is globally trivial. Then the construction as given in 4.4.14 reduces to forming  $\mu \times \mu \times \nu$  on  $X \times X \times G\{x_o\}$ , as in example 4.2.14, and we are now asking for the continuity of the function

$$h = f * g : X \times X \times G\{x_{\circ}\} \longrightarrow C \text{ defined by}$$

$$h(x,y,a) = \int f(z,y,\beta)g(x,z,\beta^{-1}a) d(\mu \times \nu)$$

$$X \times G\{x_{\circ}\}$$

$$= \int \int f(z,y,\beta)g(x,z,\beta^{-1}a) d\nu(\beta)d\mu(z)$$

$$\times G\{x_{\circ}\}$$

Since h is of compact support, we can suppose  $X \times X \times G\{x_o\}$  is

compact that is, that both X and  $G\{x_o\}$  are compact and that both  $\mu$ and  $\gamma$  are finite. Take E = X and  $F = X \times G\{x_o\}$  in Lemma 4.5.8 and consider the function, for a choice  $(y,a) \in F$ ,  $X \longrightarrow \mathcal{L}$  defined by  $x \longmapsto \int f(z,y,\beta)g(x,z,\beta^{-1}a)d(\mu \times \gamma)$ . Given  $\varepsilon > 0$ ,  $X \times G\{x_o\}$ 

the continuity of g ensures the existence of a neighbourhood  $U_{\varepsilon}$  of x such that  $|g(x,z,\beta^{-1}\alpha) - g(x',z,\beta^{-1}\alpha)| < \varepsilon$  whenever  $x' \in U_{\varepsilon}$ . Basic properties of the integral then yield that

$$x \longmapsto \int f(z,y,\beta)g(x,z,\beta^{-1}\alpha)d(\mu \times \nu) \text{ is a continuous function.}$$
$$X \times G\{x_o\}$$

The next step is to show continuity of the function  $h' : X \times G\{x_o\} \longrightarrow C(X)$ defined by h'(y,a)(x) = h(x,y,a). Again let  $\varepsilon > 0$  be arbitrary, then by the continuity of the function  $f(z,y,\beta)g(x,z,\beta^{-1}a)$ , for x, y, afixed, we can find a neighbourhood V about (y,a) such that whenever  $(y',a') \in V$ 

$$\left|f(z,y,\beta)g(x,z,\beta^{-1}\alpha)-f(z,y',\beta)g(x,z,\beta^{-1}\alpha')\right| < \frac{\varepsilon}{\mu(X)\gamma(G\{x_{0}\})}$$

for all  $x \in X$ . This implies that

 $\int \int f(z,y,\beta)g(x,z,\beta^{-1}\alpha) - f(z,y',\beta)g(x,z,\beta^{-1}\alpha') dy d\mu |$ zeX  $\beta \in G\{x_{j}\}$ 

$$< \frac{\varepsilon}{\mu(X)\nu(G\{x_{o}\})} \int \int I \, d\nu d\mu = \varepsilon$$

$$\times G\{x_{o}\}$$

this being true for all x . Thus, we must have

 $\sup_{\mathbf{x} \in \mathbf{X}} \left| h'(\mathbf{y}, \alpha)(\mathbf{x}) - h'(\mathbf{y}', \alpha')(\mathbf{x}) \right| \leq \varepsilon \text{ and so we have}$ 

$$\| h (y,a) - h (y',a') \| \leq \varepsilon$$

and so h' is continuous. Hence, by Lemma 4.5.8, h is continuous in the two variables x and (y,a). Another application of this lemma then shows that h is continuous in x, y and a, and this observation completes the proof of the continuity of h = f \* g.

iii) To complete the proof that  $C_c(G)$  is a complex algebra, we show that associativity holds.

Suppose f, g, h  $\varepsilon$  C<sub>c</sub>(G), it is required to prove that we have :

$$(f * g) * h (a) = f * (g * h)(a)$$

for all  $\alpha \in G$ ; the triple products being defined by i) and ii). By definition of \* we have :-

$$(f * g) * h (\alpha) = \int_{\operatorname{cost}_{G} \pi'(\alpha)} (f * g)(\beta) h (\beta^{-1}\alpha) d \mu_{\pi'(\alpha)}(\beta)$$

$$= \int_{\operatorname{cost}_{G} \pi'(\alpha)} \left\{ \int_{\operatorname{cost}_{G} \pi'(\beta)} f(\forall)g(\sqrt[x]{\beta}) d \mu_{\pi'(\beta)}(\forall) \right\} h(\beta^{-1}\alpha) d \mu_{\pi'(\alpha)}(\beta) \cdot$$

$$\operatorname{Since} \beta \varepsilon \operatorname{cost}_{G} \pi'(\beta) = \pi'(\alpha) \text{ so that}$$

$$(f * g) * h(\alpha) = \int_{\operatorname{cost}_{G} \pi'(\alpha)} \int_{\operatorname{cost}_{G} \pi'(\alpha)} f(\forall)g(\sqrt[x]{\beta}) h (\beta^{-1}\alpha) d \mu_{\pi'(\alpha)}(\forall) d \mu_{\pi'(\alpha)}(\beta)$$

Noting that the integrand here is a continuous complex function on  $G \times G$ with compact support, and that  $\mu_{\pi'(\alpha)} \times \mu_{\pi'(\alpha)}$  is a Baire measure on  $\cot_{G}\pi'(\alpha) \times \cot_{G}\pi'(\alpha)$ , we can interchange the order of integration by Fubini's theorem so that

$$(f*g)*h(\alpha) = \int_{cost_{G}} f(\gamma) \int_{cost_{G}} g(\gamma^{-1}\beta)h(\beta^{-1}\alpha)d\mu_{\pi'}(\alpha)^{d}\mu_{\pi'}(\alpha) ---(1)$$

Next, we have by definition of \*

$$f * (g * h) (a) = \int_{cost_{G}} f(\chi) g * h(\chi^{-1} \alpha) d \mu_{\pi'(\alpha)}(\chi)$$

$$= \int_{\operatorname{cost}_{G}} f(\vartheta) \int_{\operatorname{cost}_{G}} g(\vartheta) h(\overline{\varsigma}^{1} \overline{\vartheta}^{1} \alpha) d\mu_{\pi'}(\overline{\vartheta}^{1} \alpha)^{(\vartheta)} d\mu_{\pi'(\alpha)}(\vartheta) .$$

Since  $\pi'(z^{-1}\alpha) = \pi(-z)$ , we have

$$f^*(g^*h)(a) = \int_{\operatorname{cost}_{G}} f(\gamma) \int_{\operatorname{cost}_{G}} g(\varsigma)h(\varsigma^{-1} \sigma^{-1} a)d\mu_{\pi}(\gamma)^{d}\mu_{\pi}'(a) --- (2)$$

Now consider 
$$\int_{cost_{G}} g(\chi^{-1}\beta)h(\beta^{-1}\alpha) d\mu_{\pi'(\alpha)}(\beta) \text{ in (1)}$$

and make the substitution  $\chi^{-1}\beta = \delta$ , so that  $\beta = \Im \delta$  and  $\beta^{-1} = \delta^{-1}\chi^{-1}$  and we note that  $\delta \in \operatorname{cost}_{G}\pi(\Im)$  and  $L_{\chi} : \operatorname{cost}_{G}\pi(\chi) \longrightarrow \operatorname{cost}_{G}\pi'(\alpha)$  preserves the appropriate measures.

Thus, 
$$\int_{cost_{G}} g(\varsigma^{-1}\beta) h(\beta^{-1}\alpha) d\mu_{\pi'(\alpha)}(\beta) \text{ becomes}$$

$$\int_{cost_{G}} g(\varsigma) h(\varsigma^{-1}\gamma^{-1}\alpha) d\mu_{\pi'(\alpha)}(\gamma \beta)$$

$$cost_{G}\pi'(\alpha) = \int_{cost_{G}} g(\varsigma) h(\varsigma^{-1}\gamma^{-1}\alpha) d\mu_{\pi(\gamma)}(\varsigma) \text{ by the}$$

$$\int_{cost_{G}} g(\varsigma) h(\varsigma^{-1}\gamma^{-1}\alpha) d\mu_{\pi(\gamma)}(\varsigma) + \int_{cost_{G}} g(\varsigma) h(\varsigma) + \int_{cost_{G}} g(\varsigma) +$$

invariance properties of a Haar system. T Consequently we have (1) = (2), whence (f\*g)\*h = f\*(g\*h) and so the proof is complete.

#### 4.5.10.

and

We now turn our attention to  $L^{1}(G)$  and its convolution product. Again we shall suppose G is locally trivial and we will also suppose that G is itself a Baire set. Since any Baire set in G is contained in a countable union of sets of the form  $(\pi \times \pi')^{-1}(U_{i} \times U_{j})$ , Goetz [1], this last assumption means that G is countably disconnected. Thus, we will suppose G is globally trivial in the following details. We will suppose G is equipped with the Haar system we used in proving 4.5.9. Thus, up to an m-null set of G, we have

$$f * g(x,y,\alpha) = \int \int f(z,y,\beta)g(x,z,\beta^{-1}\alpha)dy(\beta)d\mu(z)$$

$$\times G\{x_{\alpha}\}$$

The operation  $(f,g) \mapsto f \nabla g$  is easily seen to be bilinear. That is, we have

> $(f_{1} + f_{2}) \nabla g = f_{1} \nabla g + f_{2} \nabla g$ f  $\nabla (g_{1} + g_{2}) = f \nabla g_{1} + f \nabla g_{2}.$ (c f  $\nabla g$ ) = f  $\nabla c.g = c(f \nabla g)$  for any scalar

 $c \in C$ . Precisely the same statement holds for the operation (f,g)  $\mapsto$  f \* g due to the linearity of the integral.

Let  $E = E_1 \times E_2 \times H_1$  and  $F = F_1 \times F_2 \times H_2$  be Baire sets in  $X \times X \times G\{x_o\}$  and suppose  $f = \mathcal{X}_E$  and  $g = \mathcal{X}_F$  are m-integrable. Then a straightforward calculation shows that

 $f * g(x,y,\alpha) = \mu(E_1 \cap F_2) \gamma(H_1 \cap \alpha H_2^{-1}) \chi_{F_1 \times E_2 \times H_1 H_2}^{\chi}$ where  $H_1 H_2 = \{h_1 h_2; h_1 \in H_1, h_2 \in H_2\}$ . Thus, f \* g is an + See Theorems 5 and 6. 666. Berberian [1]. m-integrable Baire function on G. So, by the bilinearily properties above, if f and g are simple m-integrable Baire functions on G, then f \* g is an m-integrable Baire function on G. Moreover, if

$$f = \underset{i \times e_{z} \times H_{1}}{x} \leq f' = \underset{i \times e_{z} \times H_{1}}{x} \text{ and}$$

$$g = \underset{i \times e_{z} \times H_{z}}{x} \leq g' = \underset{i \times e_{z} \times H_{z}}{x} \text{ so that}$$

$$E_{i} \leq E_{i}', E_{2} \leq E_{z}' \text{ etc. the calculation above shows that } f^{*}g \leq f'^{*}g'.$$
Now suppose f and g are any two m-integrable Baire functions  
on G, and that  $f_{n} \uparrow f$  and  $g_{n} \uparrow g$  are sequences of simple m  
integrable Baire functions converging pointwise to f and g respectively.  
Then  $f_{n} \ast g_{n}$  is an increasing sequence of m-integrable Baire functions and  

$$\lim (f_{n} \ast g_{n}) (\alpha) = \lim \int_{cost_{G}} f_{n}(\beta)g_{n}(\beta^{-1}\alpha)d \mu_{\pi'}(\alpha)$$

$$= \int_{\operatorname{cost}_{G}} f(\beta) g(\beta^{-1} \alpha) d\mu_{\pi'(\alpha)} = f * g(\alpha) \text{ by the}$$

Beppo Levi Theorem, Bartle [1] . Consequently, f \* g is a Baire function on G . However, f \* g need not be m-integrable as the following example shows :

Example.

4

Suppose G has compact vertex groups and  $\mathcal{V}(G\{x_o\}) = 1$ . For f, g, h  $\in L^1(X)$ , define f  $\times$  g : X  $\times X \times G\{x_o\} \longrightarrow \mathbb{C}$  by  $(f \times g)(x,y,a) = f(x)g(y)$ , then f  $\times$  g and h  $\times$  g  $\in L^1(G)$ . For such functions we have  $(f \times g) * (h \times f)(x,y,a)$ 

$$= \int (f \times g) (z,y,\beta) (h \times f) (x,z,\beta^{-1}a) d \nu(\beta) d \mu(z)$$
  

$$X \times G\{x_o\}$$
  

$$= \int f(z)g(y)h(x)f(z) d \mu(z)$$
  

$$= \int (h \times g) (x,y,a) f^{2}(z) d \mu(z)$$
  

$$= (h \times g) (x,y,a) \int f^{2}(z) d \mu(z).$$

This means that 
$$\int (\mathbf{f} \times \mathbf{g}) * (\mathbf{h} \times \mathbf{f})(\mathbf{x}, \mathbf{y}, \mathbf{a}) d(\mu \times \mu \times \mathcal{V})$$
$$\mathbf{X} \times \mathbf{X} \times G\{\mathbf{x}_{o}\}$$
$$= \left\{ \int \mathbf{f}^{2}(\mathbf{z}) d\mu(\mathbf{z}) \right\} \int (\mathbf{h} \times \mathbf{g})(\mathbf{x}, \mathbf{y}, \mathbf{a}) d(\mu \times \mu \times \mathcal{V}) \cdot \mathbf{X} \times \mathbf{X} \times G\{\mathbf{x}_{o}\}$$

But, if  $\mu$  is not atomic with countable support, we can always find  $f \ge 0$  such that  $f \in L'(X)$  and  $f^2 \notin L'(X)$ , that is,  $\int_X f^2 d\mu = \infty$ .

Thus,  $(f \times g) * (h \times f)$  is then not integrable.

This means that the operation \* is not closed in general and so we have \* :  $L'(G) \times L'(G) \longrightarrow M(G)$ , where M(G) is the set of complex valued Baire functions on G. We summarise the properties of (L'(G), \*) as follows :

i) L'(G) is a complex vector space.

ii) \*: L'(G) × L'(G) → M(G) is a/binary operation which satisfies :a) if f, g, h ∈ L'(G) and (f\*g) \* h and f \* (g\*h) are defined,
then (f\*g) \* h = f \* (g\*h)

b) f\*(g+h) = f\*g+f\*h and (f+g)\*h = f\*h+g\*h, for all f, g, h ∈ L<sup>i</sup>(G).

c) (cf\*g) = f\*cg = c(f\*g) for  $c \in \mathbb{C}$ .

These properties hold up to an m-null set of G, of course, and so (L'(G), \*) has algebra-like properties and only fails to be an algebra in that \* is not closed and, therefore, associativity does not always hold.

The previous example shows the following.

Suppose, in addition to the prevailing hypotheses, that G has compact vertex groups and L'(G) is an algebra, then  $\mu$  is atomic with countable support. Conversely, if  $\mu$  is atomic with finite support, then it is easily seen that L'(G) is an algebra. If  $\mu$  is atomic with countably infinite support, then it may or may not happen that L'(G) is an algebra. This depends on the values of  $\mu$ .

Observe, also, that the identity function identifies L'(G)T this notation means that  $\star$  is defined on a subset of  $L'(G) \times L'(G)$ . with a subspace of M(G). One might call (L'(G), \*) a <u>hyperalgebra</u> by analogy with recent work of Dunkl [1], who considers hypergroups. These are (locally compact) spaces H together with a map  $\lambda : H \times H \longrightarrow M_p(H)$  satisfying certain group-like axioms, where  $M_p(H)$  denotes the space of regular probability measures on H. Here, also, the map  $x \longmapsto S_x$  - the measure with mass 1 at x - identifies

To conclude this section, we give an example to show that the inequality  $\|f * g\|_{1} \leq \|f\|_{1} \|g\|_{1} can fail, even if <math>f * g \in L'(G)$ . Example

Let  $E = E_1 \times E_2 \times H_1$  and  $F = F_1 \times F_2 \times H_2$  be Baire sets in G and suppose  $f = \chi_E$  and  $g = \chi_F$  are m-integrable, then

$$f * g (x,y,\alpha) = \mu (E_1 \cap F_2) \vee (H_1 \cap \alpha H_2) \overset{\sim}{F_1} \times E_2 \times H_1 H_2$$
  
pose G{x, } is compact and  $\mathcal{V}(G\{x, \}) = 1$  and take

Now suppose  $G[x_0]$  is compact and  $\mathcal{Y}(G[x_0]) = 1$  and take  $H_1 = H_2 = G[x_0]$ , then  $|| f^* g ||_1 = \mu(E_1 \land F_2) \mu(F_1) \mu(E_2)$ . If we take  $E_1 = F_2$  with  $0 < \mu(E_1) = \mu(F_2) < 1$  and take  $F_1 = E_2$  such that  $\mu(F_1) = \mu(E_2) = 1$ , then  $|| f^* g ||_1 = \mu(E_1)$ . On the other hand,  $|| f ||_1 || g ||_1 = \mu(E_1)^2$  and so  $|| f ||_1 || g ||_1 < || f^* g ||_1$ . In fact, it is clear that there need not exist a positive real number r such that  $|| f^* g ||_1 \leq r || f ||_1 || g ||_1$  holds for all f,  $g \in L^1(G)$ , even when  $f^* g \in L^1(G)$ . Similarly, if we take the norm completion of  $C_c(G)$  with respect to the uniform norm, then  $C_c(G)$  is a Banach space but it too fails to be a Banach algebra in general.

### 86. An Application to Differential Geometry.

H with a subspace of  $M_{p}(H)$  .

Let X be a connected differentiable n-manifold and let TX be the tangent bundle of X. We can regard TX as a locally trivial fibre bundle over X with fibre  $\mathbb{R}^n$  and group GL(n) - the general linear group of  $\mathbb{R}^n$  - acting faithfully on  $\mathbb{R}^n$ . Now form G(TX)with its topology as in Chapter 3; the elements of G(TX) are linear isomorphisms between tangent planes of TX. Then  $G_X(TX)$  is locally

compact, Hausdorff and locally trivial. It is in fact unimodular since GL(n) is unimodular, see Nachbin [1] . Thus, we can take a Haar system  $\{m, \mu, \mu_x; x \in X \}$  on G(TX) which is locally a product, see 4.4.18. Let G = G(TX) and apply Theorem 4.5.9 with  $\{m, \mu, \mu_x; x \in X\}$  to conclude that  $C_{c}(G(TX))$  is a complex convolution algebra. In this way we can associate with X a complex convolution algebra  $C_{c}(G(TX))$ . We show next that  $C_{c}(G(TX))$  is an invariant for the appropriate diffeomorphisms on X

Suppose X and X' are connected differentiable n-manifolds and let f: X --- X' be a diffeomorphism of X onto X'. Then, see Lang [1], f induces an isomorphism Tf : TX --- TX of tangent bundles. It is a consequence of Theorem 3.4.6 and the form of a fibre bundle isomorphism that Tf induces an isomorphism  $\phi$ :  $G(TX) \longrightarrow G(TX')$ of locally trivial topological groupoids. Suppose now that  $\{m, \mu, \mu_x; x \in X\}$  and  $\{m', \mu', \mu'_x; x \in X'\}$  are Haar systems for G(TX) and G(TX') each of which is locally a product. We shall say that f preserves these two systems if : f: X - X preserves  $\mu$  and  $\mu'$ i)  $\phi$  : G(TX)  $\longrightarrow$  G(TX') preserves m and m'. ii) iii)  $\phi_x : \operatorname{cost}_{\mathcal{G}}(\mathrm{T} X)^x \longrightarrow \operatorname{cost}_{\mathcal{G}}(\mathrm{T} X')^{\mathrm{f}(x)}$  preserves  $\mu_x$  and  $\mu'_{f(x)}$  for all  $x \in X$ .

Note that i) and iii) imply ii).

If f preserves the systems  $\{m, \mu, \mu_x; x \in X\}$  and  $\{m',\mu',\mu'_x; x \in X'\}$ , then the function  $\Theta$  : C<sub>c</sub>(G(TX))  $\longrightarrow$  C<sub>c</sub>(G(TX'))

defined by  $\Theta(g)(\alpha') = g(\phi^{-1}(\alpha'))$  is easily seen to be an isomorphism of convolution algebras. Thus  $C_c(G(TX))$  is an invariant for Haar system preserving diffeomorphisms. Remarks.

**i**)

It would be interesting to interpret geometric features of Х into algebraic facts about  $C_c(G(TX))$ , and conversely. One is also led to ask "to what extent does  $C_c(G(TX))$  characterise X ?"

This work also raises the interesting question of reducing the structure group of TX to the smallest possible unimodular group. This is a variant of the problem of "G-structures", see Chern [1].

iii) It would be of some interest to try and replace continuous functions with compact support by differentiable functions with compact support.

# 87. A Remark Concerning G-spaces.

In this final section, we discuss briefly the groupoid  $\widetilde{G}$ , associated with a transformation group, and the convolution algebra  $C(\widetilde{G})$ .

Let G be a compact Hausdorff topological group acting continuously on the right of a compact Hausdorff space X, and form the compact Hausdorff topological groupoid  $\tilde{G}$ , see 2.2.5. Let  $\gamma$  be the unique probability Haar measure on G, let  $\mu$  be a Baire measure on X and construct the Haar system  $\{m, \mu, \mu_X; x \in X\}$  for  $\tilde{G}$  as in 4.2.15. This system is a right invariant Haar system and  $\mu_X$  is defined on  $\operatorname{St}_{\tilde{G}} x = x \times G$ . Let  $C(\tilde{G}) = \{f: \tilde{G} \longrightarrow \mathcal{L} ; f \text{ is continuous}\}$ . Since the Haar system we are considering is right invariant, the convolution product on  $C(\tilde{G})$  is defined as follows. For  $\Theta$ ,  $\phi \in C(\tilde{G})$ , we define  $\Theta * \phi(\alpha) = \int \Theta(\alpha\beta^{-1}) \phi(\beta) d\mu_{\pi(\alpha)}(\beta)$  so that  $\operatorname{St}_{\tilde{G}} \pi(\alpha)$ 

 $\Theta * \phi : \widetilde{G} \longrightarrow \mathcal{L} \quad \text{In fact, if } a = (x,h) \text{ and } \beta = (x,g) \in \operatorname{St}_{\widetilde{G}} \pi(a),$ then  $\Theta * \phi ((x,h)) = \int \Theta (x \cdot g, g^{-1}h) \phi (x,g) d_{\mathcal{H}_{X}}(g)$ St  $\widetilde{G} x$ 

$$= \int \Theta(\mathbf{x} \cdot \mathbf{g}, \mathbf{g}^{-1}\mathbf{h}) \phi(\mathbf{x}, \mathbf{g}) d \gamma(\mathbf{g}) \cdot \mathbf{G}$$

Let  $f = \Theta * \phi$ . Then, for h fixed, the map  $X \longrightarrow C$  defined by  $x \longmapsto \int \Theta(x \cdot g, g^{-1}h) \phi(x, g) dv(g)$  is continuous. That is,  $x \mapsto f(x,h)$  is continuous. Let  $f': G \longrightarrow C(X)$  be defined by f'(g)(x) = f(x,g). An argument like that used to prove step ii) in the proof of Theorem 4.5.9 shows that f' is continuous. Whence, by 4.5.8,  $f = \Theta * \phi$  is continuous. Thus, \* is a closed operation. Again associativity of \* follows by the argument used in proving 4.5.9, it does not, of course, need  $\tilde{G}$  to be locally trivial and depends only on the invariance property of a Haar system. In consequence of these facts, we have proved that  $C(\tilde{G})$  is a complex convolution algebra.

# Remark concerning 4.4.9.

In 4.4.9, we have assumed that  $\pi$ :  $\operatorname{cost}_{G^X} \longrightarrow X$  is a proper map. Thus,  $\pi(K)$  is compact for each compact set K of X, see Eourbaki, General Topology, Chap. 1, § 10.3. This means that  $\pi(\mu_o)$  is a Baire measure. This condition is slightly stronger than that needed to apply the disintegration theorem (Théorème 1 ) of Bourbaki [1]; the condition in Bourbaki's theorem is that  $\pi$  be  $\mu_o$ - propre, see Bourbaki [1]. Also, the condition that G be a Baire set, that is, G is  $\sigma$ - bounded is more than is needed (this is needed in 4.4.11), but we do need that G be metrizable, separable and complete.

We can in fact remove the condition that TT be proper and apply Théorème 2, § 3, № 3, Bourbaki [1]. The arguments of 4.4.9, 4.4.11 and 4.4.12 are otherwise unchanged except that we cannot conclude that  $\mu$  (a pseudo-image of  $\mu_o$  by TT) is a Baire measure. Notice that  $\mu$  exists since we are assuming G to be  $\sigma$ -bounded, see Proposition 1, § 3, № 2, Bourbaki [1].

Chapter 5.

#### REPRESENTATIONS OF GROUPOIDS.

SO. Introduction.

In this chapter, we lay the foundations of a theory of representations of topological groupoids. We shall formulate our later definitions and theorems with locally compact Hausdorff topological groupoids in mind, and the representations we shall consider are fibre spaces with Hilbert space fibres. Of course, many of our definitions and statements will generalise beyond these restrictions even to the extent of purely abstract groupoids.

Our results will enable us to obtain an analogue, for compact locally trivial Lie groupoids, of the classical Peter-Weyl theory for compact Lie groups.

The approach we adopt is to consider the concept of a groupoid G acting on a fibre space S, but an alternative is to consider a representation as a homomorphism from G into the groupoid G(S) of admissible maps, as indeed Westman does in Westman [1]. Unfortunately, the lack of a natural topology for G(S), in the case of a non locally trivial fibre bundle S, forces Westman to assume local triviality throughout and renders this approach somewhat restrictive. Our approach has the advantage that none of the definitions we make need the condition of local triviality and many of our theorems will be proved without this requirement.

Finally, we comment that we shall restrict attention to transitive groupoids throughout though, again, it is evident that this restriction is not an essential one for our theory.

S1. Groupoid Actions.

We begin this chapter with a brief discussion of the notion of a groupoid acting on a fibre space, this concept is a special case of Ehresmann's more general notion of a category acting on a fibre space, see Ehresmann [2]. Throughout, G denotes a transitive groupoid over X. Let  $P: S \longrightarrow X$  be a surjective function and form the fibred product

$$\times X S = \left\{ (\alpha, s) \in G \times S ; p(s) = \pi(\alpha) \right\}$$

we make :

G

5.1.1. <u>Definition</u>. (Ehresmann) G is said to <u>act on the left of S via P</u> if there is a function  $\cdot : G \times_X S \longrightarrow S$ , called the <u>evaluation map</u>, such that i) The diagram



is commutative, where P denotes the projection; thus, P(a.s) =  $\pi'(a)$ .

ii) For all  $s \in S$  and  $\beta, \alpha \in G$  we have  $\beta \cdot (\alpha \cdot s) = \beta \alpha \cdot s$  and  $I \cdot s = s$ , for any identity I, whenever these are defined.

In the case G a topological groupoid and P : S  $\longrightarrow X$  a continuous function (so that S is a <u>fibre space</u> over X), we give  $G \times X$  S the subspace topology of G × S and we make : <u>5.1.2</u>. <u>Definition</u>.

G is said to act on the left of S in a <u>strongly continuous</u> manner via P if G acts on the left of S via P and we have :a) for each fixed s ∈ S the function St<sub>G</sub>P(s) → S, defined by a → a.s, is continuous.

b) for each fixed  $a \in G$  the function  $\phi_a : P^{-1}(\pi(a)) \longrightarrow P^{-1}(\pi'(a))$ , defined by  $\phi_a(s) = a \cdot s$ , is continuous.

We say G acts <u>continuously</u> on the left of S if the evaluation map is continuous (Ehresmann).

We point out that the condition G acts in a continuous fashion is actually stronger than the condition G acts in a strongly continuous fashion. We amplify this comment and explain our terminology in 5.4.3.

#### 5.1.3. Remarks.

i) If  $\alpha \in G(x,y)$  and G acts in a strongly continuous manner, then it immediately follows that  $\phi_{\alpha} : P^{-1}(x) \longrightarrow P^{-1}(y)$  is a homeomorphism with inverse  $(\phi_{\alpha})^{-1} = \phi_{\alpha-1}$ . Since we are supposing G is transitive, it now follows that all the fibres of S are homeomorphic, thus, S is a fibre space with fibre.

ii) The definition of groupoid action collapses to that of group action in the case G has one object. In any case, there is an induced action of each vertex group  $G\{x\}$  on the fibre  $P^{-1}(x)$ . This induced action is continuous, respectively strongly continuous, if the action of G is continuous, respectively strongly continuous.

iii) One can define <u>right actions</u> of a groupoid G on S in the obvious way and, as with group actions, there is a 1-1 correspondence between left and right actions. For this reason, we shall only discuss left actions.

We have the following groupoid analogues of effective and transitive group actions.

# 5.1.4. Definition.

Let G be a groupoid acting on S via P. We say the action is <u>effective</u> if the following holds:-

for all  $\alpha, \beta \in G$  and  $s \in S$  whenever we have  $\alpha \cdot s = \beta \cdot s$ , we must have  $\alpha = \beta$ . We say the action is <u>transitive</u> if given any pair  $s, s, \epsilon S$ , there exists  $\alpha \in G$  such that  $\alpha \cdot s = s_2$ .

Observe that if G is any groupoid over X, there is a natural action of G on X via the identity map  $P: X \longrightarrow X$  defined by  $a \cdot x = \pi'(a)$ . This action is transitive if, and only if, G is a transitive groupoid.

#### 5.1.5.

Suppose G is any transitive groupoid over X acting on the left of S via P. Let  $x_o \in X$  and let  $T \subset G$  be a wide tree subgroupoid of G. We show next that the action of G on S can be recovered from the induced action of T on S and the induced action of  $G\{x_o\}$  on  $P^{-1}(x_o)$ . This result parallels 1.2.6. Let  $a \in G$  and let  $s \in P^{-1}(x)$ , where  $x = \pi(a)$ . If  $\tau_x$ 

denotes the unique element of  $T(x_o, x)$ , then  $\alpha = \tau_y \alpha_o \tau_x^{-1}$  for some unique element  $\alpha_o$  of  $G\{x_o\}$ , where  $y = \pi'(\alpha)$ . Consequently,  $\alpha \cdot s = (\tau_y \alpha_o \tau_x^{-1}) \cdot s = \tau_y \alpha_o \cdot (\tau_x^{-1} \cdot s) = \tau_y \cdot (\alpha \cdot (\tau_x^{-1} \cdot s))$  using ii) of 5.1.1. Thus, knowing the effect of  $\tau_x^{-1}$ ,  $\alpha_o$  and  $\tau_y$  we know the effect of  $\alpha$  and so the action of G is determined by those of Tand  $G\{x_o\}$ .

It follows immediately that we have

# 5.1.6. Proposition.

Suppose G is a transitive groupoid over X acting on S via P. Then G acts effectively (transitively) if, and only if, the induced action of some one vertex group is effective (transitive).

Moreover, with the notation of 5.1.5 and writing

 $\begin{aligned} \mathcal{T}_{xy} &= \mathcal{T}_{y} \mathcal{T}_{x}^{-1} \in \mathbf{T}(x, y) \text{, we have the diagram} \\ & G\{x\} \times \bar{P}(x) \xrightarrow{} \bar{P}(x) \\ & \theta \times \phi \downarrow & \downarrow \phi \\ & G\{y\} \times \bar{P}(y) \xrightarrow{} \bar{P}(y) \\ & \text{where } \theta(\alpha) = \mathcal{T}_{xy} \alpha \mathcal{T}_{yx} \text{ and } \phi(s) = \mathcal{T}_{xy} \cdot s \end{aligned}$ 

 $(a, s) \in G\{x\} \times P^{-1}(x)$ , then  $\phi(a \cdot s) = \tau_{xy} \cdot (a \cdot s)$ . On the other hand,  $\theta(a) \cdot \phi(s) = \tau_{xy} \cdot \tau_{yx} \cdot (\tau_{xy} \cdot s) = \tau_{xy} \cdot (a \cdot s) \cdot \tau_{xy} \cdot (a \cdot s)$ .

If

Thus, the above diagram is commutative and represents an equivalence of group actions. It follows from this that we have :

### 5.1.7. Proposition.

Let G be a transitive groupoid acting on the left of S via P. Then the induced action of each vertex group is effective (transitive) if, and only if, the induced action of any one vertex group is effective (transitive). It will be convenient to record two definitions.

Firstly	:	•	•	
			· · ·	• • .
5.1.8.	Definition.	· ·		

Let (S, P, X) and (S', P', X) be fibre spaces. A map  $\theta: S \longrightarrow S'$  is an <u>isomorphism</u> if  $\theta$  is a homeomorphism and  $P'\theta = P$ . And secondly:

### 5.1.9. Definition.

An action of the groupoid G on the left of S is <u>equivalent</u> (or <u>isomorphic</u>) to an action of G on the left of S' if there exists a fibre space isomorphism  $\Theta: S \longrightarrow S'$  and a groupoid isomorphism  $\Gamma: G \longrightarrow G$  such that the diagram

 $\begin{array}{c} G \times_{X} S \xrightarrow{\bullet} S \\ \hline \\ \Gamma \times \theta \\ G \times_{X} S' \xrightarrow{\bullet} S' \end{array}$ 

is commutative.

# 5.1.10. A General Construction.

We have seen that an action of a groupoid G on a space S induces an action of each vertex group on the appropriate fibre. It is our intention to show, conversely, that, ignoring topological considerations for the present, an action of  $G\{x_n\}$  extends to an action of G and that this extension is unique up to equivalence.

Suppose G is a transitive groupoid over X and  $x \in X$ . Suppose, also, that we are given an action  $G\{x_o\} \times F \xrightarrow{\cdot} F$  of  $G\{x_o\}$ on the left of a space F; we have a natural action  $St_Gx_o \times G\{x_o\} \xrightarrow{\cdot} St_Gx_o$ , as usual. Now form  $(St_Gx_o \times F) \times G\{x_o\} \xrightarrow{\cdot} St_Gx_o \times F$  defined by  $(\beta, f) \cdot \alpha = (\beta \alpha, \alpha^{-1}, f)$ , let  $S = (St_Gx_o \times F) / (\beta f) = \pi'(\beta)$ ;  $(\beta, f) \cdot G\{x_o\}$  of S. Define P:  $S \longrightarrow X$  by  $P([\beta, f]) = \pi'(\beta)$ ; then P:  $S \longrightarrow X$  is a fibre space. We shall define an action of G on S via P. Here  $G \times_X S = \{(\alpha, [\beta, f]); \pi'(\beta) = \pi(\alpha)\}$ ; we

define  $\cdot : G \times_{\chi} S \longrightarrow S$  by  $\alpha \cdot [\beta, f] = [\alpha\beta, f]$ . Suppose  $[\beta, f] = [\beta', f']$ , then there exists  $\gamma \in G\{x_n\}$  such that  $(\beta', f') = (\beta, f) \cdot \forall = (\beta \forall, \forall \cdot f)$ . Hence,  $\beta' = \beta \forall$  and  $f' = \forall \cdot f$ . Consequently,  $\alpha\beta' = \alpha\beta\gamma$  and so  $(\alpha\beta', f') = (\alpha\beta, f) \cdot \gamma$ , whence  $\left[ \alpha\beta, f \right] = \left[ \alpha\beta', f' \right]$ . This means that  $\alpha \cdot \left[ \beta, f \right] = \alpha \cdot \left[ \beta', f' \right]$  and so "." is well defined. We show next that "." is an action. Consider  $P(a \cdot [\beta, f])$ , since  $a \cdot [\beta, f] = [\alpha\beta, f]$ , we have  $P(a \cdot [\beta, f]) = \pi'(\alpha\beta)$ =  $\pi'(\alpha)$  and so i) of 5.1.1 holds. For ii), suppose we form  $\alpha \cdot [\beta, f]$ (to do this, we need  $\pi'(\beta) = \pi(\alpha)$ ) and then form  $\delta \cdot (\alpha \cdot [\beta, f])$  (to do this, we need  $\pi'(\alpha\beta) = \pi(\beta)$ . Now  $\beta \cdot (\alpha \cdot [\beta, f]) = \beta \cdot [\alpha\beta, f]$ =  $[\delta \alpha \beta, f]$ . On the other hand,  $\delta \alpha$  is defined and  $\pi(\delta \alpha) = \pi(\alpha)$ =  $\pi'(\beta)$  and so  $\delta a \cdot [\beta, f]$  is defined and equals  $[\delta a\beta, f]$ . Thus, S.  $(a \cdot [\beta, f]) = Sa \cdot [\beta, f]$  whenever defined. It is easily seen that  $I \cdot [\beta, f] = [\beta, f]$  if  $I = I_{\pi'(\beta)}$  and so ii) of 5.1.1 holds and we do indeed have an action of G on S. This extends the given action of  $G{x_3}$  to one of G(see 5.4.12).

We now demonstrate the uniqueness part of our claim. Suppose G is a transitive groupoid over X and G acts on the fibre space  $q: E \longrightarrow X$ . Let  $x_o \in X$ , take  $F = q^{-1}(x_o)$ , so that  $G\{x_o\}$  acts on F in the induced manner, and form  $S = (St_gx_o \times F)/G\{x_o\}$  as above. Now define the extended action of G on S via P: S  $\longrightarrow X$ . We will show that the actions of G on E and S are equivalent. To do this, we define  $\Theta: S \longrightarrow E$  by  $\Theta([\beta, f]) = \beta \cdot f$ . Suppose  $[\beta, f] = [\beta', f']$ , then there exists  $\forall \in G\{x_o\}$  such that  $(\beta', f') = (\beta, f) \cdot \forall$ . Hence,  $\beta' \cdot f' = \beta \forall \cdot \forall^{-1}f = \beta \cdot f$  and so  $\Theta([\beta, f]) = \Theta([\beta', f'])$  and  $\Theta$  is well defined. Moreover,  $q(\beta \cdot f) = \pi'(\beta)$ , since G acts on E via q, and so  $\beta \cdot f \in q^{-1}(\pi'(\beta))$ , that is,  $\Theta: P^{-1}(x) \longrightarrow q^{-1}(x)$  or, in other words,  $q\Theta = P$ . Suppose  $\Theta([\beta, f]) = \Theta([\beta', f'])$ , then  $\beta \cdot f = \beta' \cdot f'$  and so  $\beta'^{-1}\beta \cdot f = f'$ . Hence,  $(\beta, f) \cdot (\beta^{-1}\beta') = (\beta', \beta'^{-1}\beta \cdot f) = (\beta', f')$  which implies that  $[\beta, f] = [\beta', f']$ , and so  $\Theta$  is injective. Let  $e \in q^{-1}(x)$  be any element of E. Pick  $\beta \in G(x_{\sigma}, x)$  and define  $f = \beta^{-1} \cdot e \in q^{-1}(x_{\sigma}) = F$ . Then  $[\beta, f] \in S$  and  $\Theta([\beta, f]) = \beta \cdot f = \beta \cdot \beta^{-1} \cdot e = e$ . Consequently,  $\Theta$  is bijective and is (algebraically) an isomorphism of fibre spaces.

Finally, we show  $\Theta$  is an equivalence of actions. Take  $\Gamma$  to be the identity in Definition 5.1.9 and consider

 $\begin{array}{cccc} G \times_X S & \stackrel{\bullet}{\longrightarrow} S \\ I \times \theta & & & \downarrow \theta \\ G \times_X E & \stackrel{\bullet}{\longrightarrow} E \end{array}$ 

If  $s = [\beta, f]$ , then  $\Theta(\alpha \cdot s) = \Theta(\alpha \cdot [\beta, f]) = \alpha\beta \cdot f$ . On the other hand,  $I(\alpha) \cdot \Theta(s) = \alpha \cdot (\beta \cdot f) = \alpha\beta \cdot f$ . Thus, the diagram commutes and this establishes our claim.

Observe that by setting  $F = G\{x_o\}$  (with left multiplication), we obtain a natural action of G on  $st_G x_o$ , via  $\pi'$ , which is defined by composition.

# 82. Invariant Sets.

In this section, we investigate the nature of subspaces of a fibre space which are invariant under a groupoid action. This investigation is in preparation for the study of the irreducible representations of G. 5.2.1. <u>Definition</u>.

Suppose G acts on the space S via  $P: S \longrightarrow X$ . We call a subset  $S' \subset S$  <u>G-invariant</u> or <u>G-stable</u> if  $a \cdot s' \in S'$  for all  $s' \in S'$ and  $a \in G$  for which  $a \cdot s'$  is defined.

#### 5.2.2.

Since we are supposing G to be transitive, it is immediate that P(S') = X for any G-invariant subset S' of S. It is also immediate that  $S' \cap P^{-1}(x)$  is  $G\{x\}$ -invariant for each  $x \in X$ , this remark being true whether or not G is transitive.

Suppose G acts on the left of S via P: S  $\longrightarrow$  X and S is G-invariant. Let  $x_o \in X$ , let  $T \subset G$  be a wide tree subgroupoid of G and let  $T_{xy}$  denote the unique element of T(x,y). If  $S'_x$  denotes  $S' \cap P^{-1}(x_{o})$ , then  $S'_{x_{o}}$  is  $G[x_{o}]$  -invariant and we can form the set  $S'' = \bigcup_{x \in X} T_{x_{o}x} \cdot S'_{x_{o}}$ . We claim that S' = S'' and indeed it is obvious that  $S'' \in S'$  since S' is G invariant. Now suppose  $s \in S' \cap P^{-1}(x)$ , then  $T_{xx_{o}} \cdot s \in S'_{x_{o}}$  and so  $T_{x_{o}x} \cdot (T_{xx_{o}} \cdot s) = s \in S''$ . Consequently, S' = S'' and we see that the G-invariant subsets S' of Sare determined by the  $G[x_{o}]$  invariant subsets of  $P^{-1}(x_{o})$ , for any  $x_{o} \in X \cdot \blacksquare$ 5.2.3.

Now suppose G is a topological groupoid and  $P: S \longrightarrow X$ is a fibre space with G acting on S via P. Suppose, also, that  $S' \subset S$  is a G-invariant subset of S. Since we have  $(G \times_X S) \cap (G \times S') = G \times_X S'$ , the induced action  $\cdot : G \times_X S' \longrightarrow S'$ of G on S' is the restriction of the action of G on S. Thus, the action of G on S' is continuous, respectively strongly continuous, if that of G on S is continuous, respectively strongly continuous.

This analysis exposes the set theoretic nature of a G-invariant set  $S' \subset S$  in terms of the subsets which are invariant under the induced action of any one vertex group.

# 83. Actions of locally trivial Groupoids.

We shall now turn to the consideration of actions of locally trivial topological groupoids on a fibre space S.

5.3.1.

Let G be a transitive locally trivial topological groupoid over X and let  $\{U_i, \lambda_i, x_o; i \in I\}$  be a local trivialisation for G based at  $x_o$ . Suppose G acts in a strongly continuous manner on the left of the fibre space P: S  $\longrightarrow$  X and let  $S_o = P^{-1}(x_o)$ . Next define

$$\phi_{j}: U_{j} \times S_{o} \longrightarrow P^{-1}(U_{j})$$

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$$\beta_{j}(x,s) = \lambda_{j}(x)^{-1} \cdot s$$

$$\phi_{j,x}: S_{o} \longrightarrow P^{-1}(x)$$

defined by  $\phi_{j,x}(s) = \phi_j(x,s)$  is a homeomorphism, and, if  $x \in U_i \cap U_j$ , the homeomorphism

$$g_{ji}(x) = \phi_{j,x}^{-1} \cdot \phi_{i,x} : S_{o} \longrightarrow S_{o}$$

coincides with the action of the element  $\lambda_j(x) \lambda_i(x)^{-1}$  of  $G\{x_o\}$ . The continuity of the functions  $\lambda_i$  and of the operations in G imply that the assignment

$$g_{ji}: U_i \cap U_j \longrightarrow G\{x_s\}$$
,

of x to  $g_{ji}(x) = \lambda_j(x) \lambda_i(x)^{-1}$ , is continuous. In fact, the system  $\{g_{ji}\}$  is exactly the system of transition functions of 3.1.2.

Now suppose  $S' \subset S$  is a  $G\{x_i\}$  -invariant subspace of S so that

$$g_{ji}(x) \cdot s'_{o} = s'_{o} \cdot \phi_{j,x} \cdot \phi_{i,x} (s'_{o}) = s_{o}^{-1}$$

That is,

and so we have the relation

$$\phi_{j,x}(s'_{o}) = \phi_{j,x}(s'_{o})$$

for all i, j such that  $x \in U_i \cap U_j$ 

If we now form the set

$$S' = \bigcup_{\substack{x \in X \\ i \in I}} \phi_{i,x}(S') = \bigcup_{\substack{x \in X \\ i \in I}} \lambda_i(x)^{-1} S'$$

---- (\* )

then S' is a G-invariant subset of S. To see this, let  $\alpha \in G(x,y)$ and  $s \in S' \cap P^{-1}(x)$  and choose i,  $j \in I$  such that  $x \in U_i$  and  $y \in U_j$ , then  $\alpha = \lambda_j(y)^{-1}\alpha_o \lambda_i(x)$  for some  $\alpha_o \in G\{x_o\}$ and also  $s = \lambda_i(x)^{-1}$ .  $s_o$  for some  $s_o \in S'_o$ . So  $\alpha \cdot s = \lambda_j(y)^{-1}\alpha_o \lambda_i(x) \cdot \lambda_i(x)^{-1}$ .  $s_o$  $= \lambda_j(y)^{-1} \cdot (\alpha_o \cdot s_o)$ 

which is an element of S since a .s  $\epsilon$  S . The argument of §2 shows,

we have proved :-

5.3.2. Proposition.

With the notation of 5.3.1, a subset  $S \subset S$  is G-invariant if, and only if,

 $S = \bigcup \phi_{i,x} (S' \cap P^{-1}(x_{o})) \cdot \square$ ieI

We record the following comments :-

5.3.3. Remarks.

i) If G acts continuously on S, then the functions  $\phi_{i}$  are homeomorphisms and so in this case S is a locally trivial fibre bundle with fibre S<sub>o</sub>, group G{x<sub>o</sub>} and transition functions { $g_{ji}$ }. Note that the transition functions are entirely determined by G.

ii) The significance of the relation (\*) in 5.3.1 and Proposition 5.3.2 is that we obtain S' as follows. To construct S'  $\cap P^{-1}(U_j)$ , we keep the index j fixed and form  $\bigcup_{x \in U_j} \varphi_{j,x} (S' \cap P^{-1}(x_j))$ , and

similarly, to construct  $S' \cap P^{-1}(U_i)$  we keep the index i fixed and form  $\bigcup_{x \in U_i} \phi_{i,x} (S' \cap P^{-1}(x_i))$ . The point of (\*) is that over

 $U_i \cap U_j$  these constructions are compatible, that is

 $\bigcup_{x \in U_{i} \cap U_{j}} \phi_{j,x} (S' \cap P^{-1}(x_{o})) = \bigcup_{x \in U_{i} \cap U_{j}} \phi_{i,x} (S' \cap P^{-1}(x_{o})) .$ From this fact and Remark i) it follows that if G acts continuously
on S, then S' is a locally trivial sub-bundle of S.

iii) Let (S, P, X) be any locally trivial fibre bundle with fibre F and group H. Then there is a natural continuous action of G(S)on S. In fact we define  $G(S) \times X S \xrightarrow{} S$ by  $\overline{S} \cdot s = \overline{S}(s)$ .

That this defines an action of G(S) on S via P is obvious, and we need only prove the continuity of this function.

Let  $\{U_i, \phi_i\}$  be an atlas for S, then it suffices to work

Let  $\eta_{ij}$  denote the usual homeomorphism

$$\begin{aligned} & \underset{ij}{\overset{ij}{\mapsto}} : \mathcal{G}(S)(U_{i},U_{j}) \xrightarrow{} U_{i} \times U_{j} \times H, \\ & \mathcal{G}(S)(U_{1},U_{j}) \times_{X} \overline{P}'(U_{i}) \xrightarrow{} \overline{P}'(U_{j}) \\ & \overbrace{u_{1}}^{i} \times \varphi_{i} & \uparrow \varphi_{j} \\ & (u_{i} \times U_{j} \times H) \times_{X} (U_{i} \times F) \xrightarrow{} U_{j} \times F \end{aligned}$$

represents an equivalence of actions where the "bottom" map is (.) defined by

and is clearly continuous since the two functions

$$((x,y,h), (x,f)) \longmapsto (x,y,h) \longmapsto y$$
$$((x,y,h), (x,f)) \longmapsto (h,f) \longmapsto h \cdot f$$

are both continuous. The commutativity of the diagram above then gives us the required continuity of the action.

This proves

# 5.3.4. Proposition.

and

then the diagram

Let S'C S be a subset of S, let Y = P(S') and let  $Z = P^{-1}(Y)$ . Then S' is a locally trivial sub bundle of S if, and only if, S' is G(Z) invariant.

In the light of these results we make the following observation. 5.3.5. <u>Comment</u>.

It is interesting to observe that we can view fibre bundles as fibre spaces  $P: S \longrightarrow X$  together with a continuous action of a topological groupoid G on S via P. If G is locally trivial, it then follows that S is locally trivial. Proposition 5.3.4 shows that the sub bundles of S are precisely the G-invariant subsets of S. This viewpoint could be taken in an attempt to cast the whole theory of fibre bundles in terms of fibre spaces and groupoid actions. 5.3.6.

Suppose G is a compact Hausdorff locally trivial topological groupoid over X and  $\{m, \mu, \mu_X; x \in X\}$  is a Haar system of measures which is locally a product. Suppose G acts continuously on the fibre space P: S  $\longrightarrow$  X, then S is a locally trivial fibre bundle by 5.3.3. Suppose, also, that each fibre  $P^{-1}(x)$  has a metric  $d_x$  continuously assigned to it in the usual way from a metric d on the fibre, say  $P^{-1}(x_o)$ , see 5.4.7 iii). As a typical application of the use of the system  $\{m, \mu, \mu_X; x \in X\}$ , we show that each fibre can be equipped with a metric with respect to which the maps  $\phi_{\alpha}$ ,  $\alpha \in G$ , are isometric. Define  $d_x^*$  on  $P^{-1}(x)$  by

$$d_{\mathbf{x}}^{*}(s_{1},s_{2}) = \int_{\operatorname{cost}_{G}^{\mathbf{x}}} d_{\pi(\alpha)}(\alpha^{-1} s_{1},\alpha^{-1} s_{2}) d\mu_{\mathbf{x}}(\alpha)$$

By the hypothesis, the integrand is continuous and  $\cot_{\mathbf{G}} x$  is compact. Thus, the integral exists and  $d_{\mathbf{x}}^*$  is well defined. If  $d_{\mathbf{x}}^*(s_1,s_2) = 0$ , then the integrand is zero almost everywhere and so  $d_{\mathbf{\pi}(\mathbf{a})}(\mathbf{a}^{-1}s_1,\mathbf{a}^{-1}s_2) = 0$ for some  $\mathbf{a} \in \mathbf{G}$ . Hence,  $\mathbf{a}^{-1}s_1 = \mathbf{a}^{-1}s_2$  and, thus,  $\mathbf{s}_1 = \mathbf{s}_2$ . The other metric axioms are easily verified and so  $d_{\mathbf{x}}^*$  is a metric on  $\mathbf{P}^{-1}(\mathbf{x})$ . Moreover, by choice of  $\{\mathbf{m}, \mathbf{\mu}, \mathbf{\mu}_{\mathbf{x}}; \mathbf{x} \in \mathbf{X}\}$  as locally a product  $d_{\mathbf{x}}^*$ is continuously assigned from a metric  $\mathbf{d}^*$  on the fibre  $\mathbf{P}^{-1}(\mathbf{x}_0)$ . Since  $\mathbf{d}^*(\mathbf{s}_1, \mathbf{s}_2) = \int_{\mathbf{cost}_{\mathbf{G}}^*} \mathbf{d}_{\mathbf{\pi}(\mathbf{a})} (\mathbf{a}^{-1}s_1, \mathbf{a}^{-1}s_2) \mathbf{d}_{\mathbf{\mu}_{\mathbf{x}}}(\mathbf{a})$  and  $\mathbf{d}_{\mathbf{x}}$  is equi- $\cot_{\mathbf{G}}^*$ .

valent to d, it follows that d\* and d are equivalent metrics. Therefore, by the local triviality of S, changing  $d_x$  to  $d_x^*$  does not affect the continuity of P: S  $\longrightarrow X$  or the continuity of the action of G on S. Finally, we show that the  $\varphi_a$  are isometric with respect to the metrics  $d_x^*$ . Suppose  $\beta \in G(x,z)$ , then  $d_z^*(\beta \cdot s_1, \beta \cdot s_2) = \int d_{\pi(X)}(x^{-1}(\beta \cdot s_1), x^{-1}(\beta \cdot s_2)) d\mu_z$ 

=  $\int_{cost_G x} d_{\pi(\alpha)}(\alpha^{-1}s_1, \alpha^{-1}s_2) d\mu_x$  by the invariance property of a Haar system.

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Thus, 
$$d_z^*(\beta \cdot s_1, \beta \cdot s_2) = d_x^*(s_1, s_2) \cdot \blacksquare$$

# 84. Representations of Groupoids.

Throughout the remainder of this chapter, unless otherwise stated, G will denote a transitive locally compact Hausdorff topological groupoid over X . Further conditions will be imposed on G as required.

We are now prepared to consider representations of groupoids G, and we begin with the following definition which is modelled on that of a family of vector spaces, as given in Atiyah [1].

### 5.4.1. Definition.

A continuous function  $P : S \longrightarrow X$  is called a <u>family of</u> <u>Hilbert spaces</u> over X if

a) For each  $x \in X$ ,  $P^{-1}(x)$  is a complex separable Hilbert space equipped with a norm  $\|\cdot\|_{x}$  and an inner product <,  $>_{x}$ , related in the usual way, and which are compatible with the topology induced by S on  $P^{-1}(x)$ .

b) The operations of addition and scalar multiplication are compatible with the topology on S in the sense that  $+: S \times_X S \longrightarrow S$ and (.):  $\mathcal{C} \times S \longrightarrow S$ , defined respectively by  $+(s_1, s_2) = s_1 + s_2$ and (.)( $\kappa, s$ ) =  $\kappa \cdot s$ , are continuous. Here,  $S \times_X S$  is defined by  $S \times_X S = \{(s_1, s_2) \in S \times S; P(s_1) = P(s_2)\}$ .

We shall frequently refer to the family of Hilbert spaces  $\ensuremath{\mathsf{S}}$  over X .

Recall that a map  $\phi$ : H,  $\longrightarrow$  H<sub>2</sub> is a unitary operator if  $\phi$ is an inner product preserving, linear isomorphism of Hilbert spaces H, and H<sub>2</sub>.

Our basic definition is :-

## 5.4.2. Definition.

A (strongly) continuous linear representation of G is a family P: S  $\longrightarrow$  X of Hilbert spaces over X, together with a (strongly) continuous action of G on the left of S via P such that each of the maps  $\phi_{\alpha}$ : P<sup>-1</sup>( $\pi(\alpha)$ )  $\longrightarrow$  P<sup>-1</sup>( $\pi'(\alpha)$ ) is linear. If each of the maps  $\phi_a$  is a unitary operator, we call S a (strongly) continuous unitary representation of G.

5.4.3. Comment.

Suppose H is a Hilbert space and L(H,H) is the algebra of all bounded linear maps H  $\longrightarrow$  H with its norm or uniform topology. Let Aut(H) denote the subset of all invertible elements of L(H,H) with the subspace topology. Let f be a fixed element of H and define  $T_f: L(H,H) \longrightarrow H$  by  $T_f(A) = A(f)$ , then the topology on L(H,H) generated by the family  $\{T_f\}_{f \in H}$  is called the strong operator topology on L(H,H). Again, Aut(H) can be given the strong operator

subspace topology. We remark that the strong operator topology is weaker or coarser than the uniform topology.

If G is a locally compact topological group and H a complex Hilbert space, it is usual to define a representation of G on H to be a homomorphism L: G  $\longrightarrow$  Aut(H) which is strongly continuous, that is, continuous in the strong operator topology on Aut(H), see Loomis [1] or Hewitt and Ross [1] . This amounts to considering an action  $G \times H \longrightarrow H$ with the property : for each fixed  $h \in H$ , the map  $G \longrightarrow H$  defined by  $g \longmapsto g \cdot h$  is continuous, and each operation of G is a unitary operator, or at least, a linear invertible operator. These comments provide the motivation for our definition and the terminology we use. However, we are interested in continuous representations as well as strongly continuous ones.

Finally, observe that a (strongly) continuous (linear) unitary representation of G induces, in the usual way, a (strongly) continuous (linear) unitary representation of each vertex group.

5.4.4. Definition.

A (strongly) continuous representation (S, P, X) of G is called <u>reducible</u> if there exists a proper G-invariant subspace  $S' \subset S$ such that (S',P,X) is a (strongly) continuous representation of G in

the induced structure. We call such a representation (S', P, X) a <u>subrepresentation</u> of G. If no such proper subrepresentations exist, we call (S, P, X) <u>irreducible</u>.

All our definitions, thus far, collapse to the usual ones for a group in the case of a groupoid G over one object.

We next prove what may be considered as the first part of a Peter-Weyl theory for groupoids.

5.4.5. Theorem.

Let G be a transitive locally compact Hausdorff topological groupoid over X and suppose G has compact vertex groups. Then any irreducible strongly continuous unitary representation of G is finite dimensional.

#### Proof.

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Let (S, P, X) be any strongly continuous unitary representation of G, let  $x_o \in X$  and let  $S_o = P^{-1}(x_o)$ . By hypothesis,  $G\{x_o\}$  is compact and we have an induced strongly continuous unitary representation of  $G\{x_o\}$  on  $S_o$ . Thus, by Theorem 22.13 of Hewitt and Ross [1], there is a finite dimensional subspace  $S'_o \subseteq S_o$  which is a strongly continuous unitary representation of  $G\{x_o\}$ .

Let T be a wide tree subgroupoid of G with  $\mathcal{T}_{xy}$  denoting the unique element of T(x,y) and form the set  $S' = \bigcup_{x \in X} \mathcal{T}_{x,x} \cdot S'$ .

By 5.2.2, S' is G-invariant and, since the  $\tau_{x,x}$  act as unitary operators, we see that S'  $\cap P^{-1}(x)$  is a finite dimensional subspace of  $P^{-1}(x)$  of the same dimension as S', for each  $x \in X$ .

The relation  $S' \times_X S' = (S \times_X S) \cap (S' \times S')$  shows that the addition function  $+ : S' \times_X S' \longrightarrow S'$  is continuous, being the obvious restriction, and similarly for scalar multiplication. Thus,  $P : S' \longrightarrow X$  is a family of (finite dimensional) Hilbert spaces over X. Given an element  $s' \in S'$ , we certainly have that the function  $\theta : St_CP(s') \longrightarrow S'$  defined by  $\theta(a) = a \cdot s'$  is continuous, since

this is true for an element s  $\varepsilon$  S .

Finally, it is clear that each of the operators  $\phi_{\alpha} : P^{-1}(\pi(\alpha)) \cap S' \longrightarrow P^{-1}(\pi'(\alpha)) \cap S'$  (restricted to  $P^{-1}(\pi(\alpha)) \cap S'$  in the usual way) is unitary, and so S' is a strongly continuous finite dimensional unitary representation of G. Consequently, any irreducible representation of G must be finite dimensional and the proof is complete.

# 5.4.6. Corollary.

If G satisfies the hypothesis of Theorem 5.4.5 and has Abelian vertex groups, then the irreducible representations of G are one dimensional. <u>Proof</u>.

The irreducible representations of a compact Abelian group are one dimensional, see Hewitt and Ross [1] .

5.4.7. Remarks.

i) If G is locally trivial and S is a locally trivial fibre bundle in the structure of 5.3.1, then the relation (\*) of 5.3.1 shows that the set S' constructed in the proof of 5.4.5 is given by  $S' = \bigcup_{\substack{x \in X \\ i \in I}} \phi_{i,x}(S'), \text{ see 5.3.2, and so it follows,}$ 

in this case, that S' is a locally trivial finite dimensional representation of G, that is, S' is a vector bundle.

- ii) We have shown that a representation S of G is irreducible if, and only if, S  $\cap P^{-1}(x)$  is an irreducible representation of  $G\{x\}$  for each  $x \in X$ .
- iii) Recall that a fibre bundle (S, P, X), with Hilbert space fibre F, is called a bundle with norm and inner product if the coordinate functions are isometric with respect to the inner product and norm on each fibre and the inner product and norm on F.

We next prove :

### 5.4.8. Theorem.

Let G be a transitive locally trivial topological groupoid over X and let  $x_o \in X$ . Then any strongly continuous unitary representation (S, P, X) of G can be given the structure of a locally trivial fibre bundle with group  $G\{x_o\}$ , fibre  $S_o = P^{-1}(x_o)$ , on which  $G\{x_o\}$ acts in the induced manner, and transition functions  $\{g_{ji}\}$  as in 5.3.1. Moreover, S can be equipped with norm and inner product. Proof.

Let  $\{U_i, \lambda_i, x_o\}_{i \in I}$  be a local trivialisation for G based at  $x_o$ . Following 5.3.1, we define  $\phi_j : U_j \times S_o \longrightarrow P^{-1}(U_j)$  by  $\phi_j(x,s) = \lambda_j(x)^{-1} \cdot s$ , and the  $\phi_j$  are bijective. In fact,  $\phi_j^{-1}(s) = (P(s), \lambda_j(P(s)) \cdot s)$ . If the action of G is continuous, then the  $\phi_j$  are homeomorphisms and the proof is complete. If nct, we retopologise S by taking the  $\phi_j$  as homeomorphisms, then it is immediate that the addition and scalar multiplication functions are still continuous since S is now locally a product.

Now if  $x \in U_j$ , then  $\phi_{j,x} : S_o \longrightarrow P^{-1}(x)$  is defined by  $\phi_{j,x}(s) = \lambda_j(x)^{-1} s$  and if we write  $a = \lambda_j(x)^{-1}$ , then  $\phi_{j,x} = \phi_a$ . Thus, if  $\langle , \rangle_o$  and  $\parallel \parallel_o$  are the given inner product and norm on  $S_o$ , we have  $\langle s_o, s' \rangle_o = \langle \phi_a(s_o), \phi_a(s'_o) \rangle_x$  for all  $s_o, s' \in S_o$  since the  $\phi_a$  are unitary. Hence, we have  $\langle s_o, s'_o \rangle_o = \langle \phi_{j,x}(s_o), \phi_{j,x}(s'_o) \rangle_x$  and similarly for the norms. That is, S has norm and inner product induced from those of the fibre  $S_o$ .

Finally, we show that after retopologising S in this fashion, we still have a strongly continuous representation of G. Let  $s \in S$ be fixed with P(s) = x, we have to show that  $\Theta$  :  $St_Gx \longrightarrow S$  is continuous in the new topology of S, where  $\Theta(a) = a \cdot s$ . Let  $\Theta_j = \Theta_{\pi'}^{-1}(U_j) : \pi'^{-1}(U_j) \cap St_Gx \longrightarrow P^{-1}(U_j)$ , then

 $\phi_j^{-1} \Theta_j : \pi'^{-1}(U_j) \cap \operatorname{St}_G x \longrightarrow U_j \times S_o$  is defined by

 $a \longmapsto (P(a \cdot s), \lambda_{j}(P(a \cdot s)) \cdot (a \cdot s)) \cdot Thus, \text{ if } P_{i} \text{ and } P_{2}$ denote the projections, then  $P_{i}\phi_{j}^{-1}\Theta_{j}(a) = P(a \cdot s) = \pi'(a)$  and so  $P_{i}\phi_{j}^{-1}\Theta_{j}$  is continuous. Also,  $P_{2}\phi_{j}^{-1}\Theta_{j}(a) = (\lambda_{j}(\pi'(a))a) \cdot s$ which is the composite  $\pi'^{-1}(U_{j}) \cap St_{g}x \xrightarrow{(\pi', I)} U_{j} \times \pi'^{-1}(U_{j}) \xrightarrow{(\lambda_{j}, I)} G \times \pi'^{-1}(U_{j}) \xrightarrow{Comp} G \xrightarrow{\bullet} S_{a}$ 

$$a \longmapsto (\pi'(a),a) \longmapsto (\lambda_{j}(\pi'(a)),a) \longmapsto \lambda_{j}(\pi'(a))a \longmapsto \lambda_{j}(\pi'(a))a \ldots$$
$$\lambda_{j}(\pi'(a))a \cdot s$$

and each map in this series is continuous, the last map  $(\cdot)$  being continuous by hypothesis and the fact that the topology of  $S_0$  is unchanged. Thus,  $\Theta$  is continuous and the proof of the theorem is complete.

### 5.4.9. Remarks.

i) We have a slight inconsistency in respect of our terminology in Theorem 5.4.8 in that the group  $G\{x_o\}$  acts on the fibre  $S_o$  in a strongly continuous fashion, rather than a continuous fashion. Our terminology "fibre bundle" has always previously assumed a continuous action, but it will be convenient in future to use this slightly more general terminology. This will cause no confusion and it will be clear from the context whether the action of the structure group is continuous or strongly continuous.

ii) If G is compact as well as being transitive and locally trivial, then using the argument of 5.3.6 with an inner product rather than a metric, from any strongly continuous linear representation of G we can obtain a strongly continuous unitary representation. In doing this, we can first apply Theorem 5.4.8 and take a locally trivial linear representation. The argument is now the same as used in 5.3.6 noting that although we assumed a continuous action there, a strongly continuous one is enough, for, in defining  $\langle s_1, s_2 \rangle_X^*$  by  $\langle s_1, s_2 \rangle_X^* = \int_{x} \langle \alpha^{-1} s_1, \alpha^{-1} s_2 \rangle_{\pi(\alpha)} \cdot d\mu_X(\alpha)$ 

s, and s, are fixed and so the integrand is continuous and <,> \* x is a well defined inner product.

iii) Observe that Theorem 5.4.8 provides a necessary condition that a fibre space (S, P, X) admit a strongly continuous representation of G. We show later that these conditions are also sufficient ones. Note also that in changing the topology of S as we did in proving Theorem 5.4.8, the representation S does not become a continuous one, of course, since  $G_{x}$  acts in the same way, that is, strongly continuously.

We need :

5.4.10. Definition.

Let  $P: S \longrightarrow X$  and  $P': S' \longrightarrow X$  be two families of Hilbert spaces over X. An <u>isomorphism</u>  $h: (S, P, X) \longrightarrow (S', P', X)$ is a homeomorphism  $h: S \longrightarrow S'$  such that :

a) P'h = P

b) For each  $x \in X$ ,  $h_x = h_{P^{-1}(x)} : P^{-1}(x) \longrightarrow P'^{-1}(x)$  is a unitary operator.

### 5.4.11. Definition.

Two representations (S, P, X) and (S', P', X) of G are <u>equivalent</u> if there is an isomorphism  $h: (S, P, X) \longrightarrow (S', P', X)$ such that  $h \phi_a \bar{\chi}' = \phi'_a$  for all  $a \in G$ .

These definitions are consistent with 5.1.8 and 5.1.9 and collapse to the usual ones if G is a group. Note, also, that if (S, P, X) is a representation of G and  $L : (S, P, X) \longrightarrow (S', P', X)$ is an isomorphism, then L induces an action of G on S' and (S, P, X) and (S', P', X) are then equivalent representations of G.

We now turn to the task of proving one of our main results which is that of proving a topological version of the construction given in 5.1.10. We emphasise that conclusion a) does not need the hypothesis of local triviality. And we observe, also, that the condition of local compactness is not used in an essential way.

The theorem we are concerned with is :-

5.4.12. Theorem. (Extension Theorem).

Let G be a transitive locally compact Hausdorff topological

groupoid over X and let  $x_{\sigma} \in X$ . Suppose we are given an action of  $G\{x_{\sigma}\}$  on the left of a Hilbert space F such that  $G\{x_{\sigma}\}$  acts as a group of unitary operators. Then

a) If  $G\{x_o\}$  acts continuously on F, there is a continuous unitary representation P: E  $\longrightarrow X$  of G whose induced representation of  $G\{x_o\}$  on  $P^{-1}(x_o)$  is equivalent to that of  $G\{x_o\}$  on F. b) If G is locally trivial and the action of  $G\{x_o\}$  is strongly continuous, then there is a locally trivial strongly continuous unitary representation P: E  $\longrightarrow X$  of G whose induced representation of  $G\{x_o\}$ on  $P^{-1}(x_o)$  is equivalent to that of  $G\{x_o\}$  on F. Moreover, P: E  $\longrightarrow X$  is a bundle with norm and inner product induced from those of F.

c) If  $G\{x_o\}$  acts continuously in b), then the representation P: E  $\longrightarrow X$  is unique up to equivalence (the equivalence  $\Theta$  of 1.5.10). Proof.

a) We construct the fibre space  $P: E \longrightarrow X$  with the natural action of G on E, as in 5.1.10. In this case, the action of  $G\{x_o\}$ on F is continuous and, in any case, the action of  $G\{x_o\}$  on the right of  $St_Gx_o$ , defined by composition, is continuous. We give E the quotient topology of  $St_Gx_o \times F$  by r . In this topology, P is continuous. For consider the diagram :

This commutes, where  $P_i = Projection$ , and  $\pi' P_i$  is continuous. Hence, P is continuous by the universal property of quotients. Thus,  $P : E \longrightarrow X$ is a fibre space with Hilbert space fibre. We put a Hilbert space structure on each fibre as follows. Now  $P^{-1}(x) = \{[\beta, f]; \pi'(\beta) = x\}$ . So, if  $[\beta, f]$  and  $[\beta', f'] \in P^{-1}(x)$ , then  $\beta$  and  $\beta'$  belong to G(x, x) and so  $\beta' = \beta a$  for some unique  $a \in G\{x, z\}$ . Hence  $(\beta', f') = (\beta, a \cdot f') \cdot a$  which implies that  $[\beta', f'] = [\beta, \alpha \cdot f']$ . We now define an addition + on  $P^{-1}(x)$  by the rule :-

$$\beta, \mathbf{f} + [\beta', \mathbf{f}'] = [\beta, \mathbf{f}] + [\beta, \alpha \cdot \mathbf{f}']$$
$$= [\beta, \mathbf{f} + \alpha \cdot \mathbf{f}'].$$

We need to show that "+" is a well defined operation. Suppose  $\begin{bmatrix} \beta, f \end{bmatrix} = \begin{bmatrix} \beta_{i}, f_{i} \end{bmatrix} \text{ and } \begin{bmatrix} \beta', f' \end{bmatrix} = \begin{bmatrix} \beta_{i}', f_{i}' \end{bmatrix} \text{ Then there exists } \alpha_{i} \text{ and } \alpha_{i}' \in G\{x_{o}\} \text{ such that } (\beta, f) = (\beta_{i}, f_{i}) \cdot \alpha_{i} \text{ and } (\beta', f') = (\beta_{i}', f_{i}') \cdot \alpha_{i}' \text{ and } \alpha_{i}' \in G\{x_{o}\} \text{ such that } (\beta, f) = (\beta_{i}, f_{i}') \cdot \alpha_{i} \text{ and } (\beta', f') = (\beta_{i}', f_{i}') \cdot \alpha_{i}' \text{ and } \alpha_{i}' \in G\{x_{o}\} \text{ such that } (\beta, f) = (\beta_{i}, f_{i}') \cdot \alpha_{i} \text{ and } (\beta', f') = (\beta_{i}', f_{i}') \cdot \alpha_{i}' \text{ and } \alpha_{i}' \text{ and } (\beta_{i}', f_{i}') = (\beta_{i}', f_{i}') + (\beta_{i}', f_{i}') = (\beta_{i}', f_{i}') \cdot \alpha_{i}' \text{ and } \alpha_{i}' \text{ and } \alpha_{i}' \text{ and } (\beta_{i}', f_{i}') = [\beta_{i}, f_{i}' + \beta_{i}' \cdot f_{i}'] \text{ and } \alpha_{i}' \text{ and } \alpha_{i}' \text{ and } (\beta_{i}', f_{i}') = \alpha_{i}' \cdot \alpha_{i}' + \alpha_{i}' \cdot \alpha_{i}' \cdot \alpha_{i}' \text{ by linearity} \text{ of the action of } G\{x_{o}\} \text{ But } \alpha_{i}^{-1} \cdot f_{i}' = f \text{ consider } \beta_{i}' + \beta_{i}' \cdot \beta_{i}' + \alpha_{i}' \cdot \beta_{i}' + \beta_{i}' \cdot \beta_{i}' + \alpha_{i}' \cdot \beta_{i}$ 

$$[\beta, f] = [\beta, \kappa f]$$
, for  $\kappa \in \mathbb{C}$ .

If  $[\beta, f] = [\beta', f']$ , then  $(\beta, f) = (\beta', f') \cdot \alpha$  for some  $\alpha \in G\{x_o\}$ . By definition,  $\kappa[\beta', f'] = [\beta', \kappa f']$ . But  $(\beta', \kappa f') \cdot \alpha = (\beta' \alpha, \alpha^{-1} \cdot \kappa f')$  $= (\beta' \alpha, \kappa \alpha^{-1} \cdot f')$ 

since the elements of  $G\{x_{\alpha}\}$  act as unitary operators. Thus,  $(\beta', \kappa f') \cdot \alpha = (\beta, \kappa f)$  which implies that  $[\beta', \kappa f'] = [\beta, \kappa f]$ . Whence, the scalar multiplication is well defined on  $P^{-1}(x)$ . It is routine to show that these operations, of addition and scalar multiplication, endow  $P^{-1}(x)$  with a complex vector space structure.

As usual, we can form the space  $E \times_{P} E$  where  $E \times_{P} E = \left\{ ([\beta, f], [\beta', f']) \in E \times E ; P([\beta, f]) = P([\beta', f']) \right\}$  $= \left\{ ([\beta, f], [\beta', f']) ; \pi'(\beta) = \pi'(\beta') \right\}$ , and then we obtain a

well defined map

 $+: E \times_p E \longrightarrow E$ .

Similarly, we obtain a well defined map

 $\mathcal{L} \times E \longrightarrow E$ , defined by  $(\kappa, \lceil \beta, f \rceil) \longmapsto \kappa \lceil \beta, f \rceil$ , and we need to investigate the continuity of these functions. We do this shortly; first, however, we define norms and inner products on  $P^{-1}(x)$ .

Let <,> and || || denote the inner product and norm, respectively, on F. If  $[\beta, f]$  and  $[\beta', f'] \in P^{-1}(x)$ , then  $\beta' = \beta \alpha$ , for some  $\alpha$ , and  $[\beta', f'] = [\beta, \alpha \cdot f']$ . Define <,><sub>x</sub> on  $P^{-1}(x)$  by < $[\beta, f], [\beta', f'] >_x = <[\beta, f], [\beta, \alpha \cdot f'] >_x$ =  $< f, \alpha \cdot f' > .$ 

We show  $\langle , \rangle_x$  is well defined. With the same choices as we made to verify well-definedness of addition, we have

$$\langle [\beta_{i}, f_{i}] \rangle_{x} = \langle f_{i}, \forall_{i} \cdot f_{i} \rangle$$

$$and \langle [\beta, f], [\beta', f'] \rangle_{x} = \langle f, \forall_{i} \cdot f' \rangle$$

$$now \ \alpha_{i}^{-1} \cdot f_{i} = f \ and \ \alpha_{i}^{-1} \cdot \forall_{i}' \cdot f_{i}' = \forall_{i} \cdot f' \cdot Thus,$$

$$\langle f, \forall_{i} \cdot f' \rangle = \langle \alpha_{i}^{-1} \cdot f_{i}, \alpha_{i}^{-1} \cdot \forall_{i}' \cdot f_{i}' \rangle$$

$$= \langle f_{i}, \forall_{i}' \cdot f_{i}' \rangle$$

since  $G\{x_o\}$  acts as a group of unitary operators. This means that  $\langle , \rangle_x$  is well defined. Also, we define  $\| \|_x$  on  $P^{-1}(x)$  by  $\| [\beta, f] \|_x = \| f \|$ .

This is well defined. It is easily seen that  $\langle , \rangle_{x}$  is an inner product on  $P^{-1}(x)$  and  $\|\|_{x}$  is a norm on  $P^{-1}(x)$ . Also, the relation  $\langle [\beta, f], [\beta, f] \rangle_{x} = \|[\beta, f]\|_{x}^{2}$ 

is clear.

Observe that if we choose  $\beta \in G(x, x)$ , we have a unitary operator  $\kappa(\beta, x) : F \longrightarrow P^{-1}(x)$  defined by  $\kappa(\beta, x)(f) = [\beta, f]$ , see §3, Chapter 3. Thus, E induces the correct topology on  $P^{-1}(x)$ , for each x in X. Observe, also, that if we choose  $\beta = I_{x_o} \in G\{x_o\}$ , then  $\kappa(I_{x_o}, x_o) : F \longrightarrow P^{-1}(x_o)$  determines an equivalence between the (given) action of  $G\{x_o\}$  on F and that induced on  $P^{-1}(x_o)$ . Let  $\alpha \in G(x, y)$ , then  $\mathscr{A}_{\alpha} : P^{-1}(x) \longrightarrow P^{-1}(y)$  is defined by  $\mathscr{A}_{\alpha}([\beta, f]) = [\alpha\beta, f]$  and it is obvious that  $\mathscr{A}_{\alpha}$  is unitary with respect to  $\langle , \rangle_{x}$  and  $\langle , \rangle_{y}$ . Thus, G acts on E as a groupoid of unitary operators.

The next step is to show that addition and scalar multiplication are continuous, so that  $P: E \longrightarrow X$  is a family of Hilbert spaces. To do this, we first observe that r is an open map; this is proved in exactly the way 1.4, Chapter 4, Husemoller [1], is proved, we do not even need continuity of the action for this, only that  $G\{x_n\}$  act as a group of homeomorphisms. Consequently,  $r \times r$  is a quotient map since it, too, is an open map. Define the space

 $(\operatorname{st}_{G} x_{o} \times F) \times_{\times} (\operatorname{st}_{G} x_{o} \times F) = E' \text{ by}$  $E' = \left\{ ((\beta_{i}, f_{i}), (\beta_{2}, f_{2})); \pi'(\beta_{i}) = \pi'(\beta_{2}) \right\}.$ 

Since

commutes and X is Hausdorff, E' is closed in  $(St_{G}x_{o} \times F) \times (St_{G}x_{o} \times F)$ . Moreover,  $\mathbf{r} \times \mathbf{r}$  : E'  $\longrightarrow$  E  $\times_{p}$  E and is surjective and also E' is  $\mathbf{r} \times \mathbf{r}$  - saturated, that is,  $(\mathbf{r} \times \mathbf{r})^{-1}(\mathbf{E} \times_{p} \mathbf{E}) = \mathbf{E}'$ . Thus, by results of Brown [1], page 98,  $\mathbf{r} \times_{X}\mathbf{r} = \mathbf{r} \times \mathbf{r} \mid_{E'}$  is an identification map. Now consider +:  $(St_{G}x_{o} \times F) \times (St_{G}x_{o} \times F) \longrightarrow St_{G}x_{o} \times F$ defined by  $((\beta_{i}, f_{i}), (\beta_{2}, f_{2})) \longmapsto (\beta_{1}, f_{1} + \beta_{1}^{-1}\beta_{2} \cdot f_{2})$ . This function is continuous, since  $G\{x_{o}\}$  acts continuously, and so is its restriction to E'. Finally,

$$E' \xrightarrow{+} st_{G_{a} \times F}$$

$$r \times r \downarrow \qquad \qquad \downarrow r$$

$$E \times_{a} E \xrightarrow{+} E$$

commutes and, hence,  $+ : E \times_{P} E \longrightarrow E$  is continuous by the universal property of quotients. Next we deal with the continuity of scalar product  $\mathcal{L} \times E \longrightarrow E$ . Since the scalar product on F is continuous, the map  $\mathcal{C} \times (\operatorname{St}_{G} \times F) \longrightarrow \operatorname{St}_{G} \times F$  defined by  $(\kappa, (\beta, f))$  $\longmapsto (\beta, \kappa f)$  is continuous. Thus, the continuity of scalar multiplication follows from the commutativity of the diagram



where  $I : \mathbb{C} \longrightarrow \mathbb{C}$  is the identity map. Thus,  $P : E \longrightarrow X$  is a family of Hilbert spaces over X.

It remains to prove that the action of G on E is continuous. Let  $G \times_{X} (St_{G}x_{o} \times F) = \{ (\alpha, (\beta, f)); \pi'(\beta) = \pi(\alpha) \}$  and define  $\ni : G \times_{X} (St_{G}x_{o} \times F) \longrightarrow St_{G}x_{o} \times F$  by  $\ni (\alpha, (\beta, f)) = (\alpha\beta, f)$ . Since  $G \times_{X} (St_{G}x_{o} \times F) = (G \times_{X} St_{G}x_{o}) \times F$ ,  $\ni$  is the restriction of composition  $\times$  Identity and is, therefore, continuous. Clearly  $I \times r : G \times_{X} (St_{G}x_{o} \times F) \longrightarrow G \times_{X} E$  and  $(I \times r)^{-1} (G \times_{X} E) = G \times_{X} (St_{G}x_{o} \times F)$ . Since X is Hausdorff,  $G \times_{X} (St_{G}x_{o} \times F)$  is closed in  $G \times (St_{G}x_{o} \times F)$  and  $I \times r$ , being open, is a quotient map. Therefore  $I \times_{X} r = I \times r \int_{G \times_{X} (St_{G}x_{o} \times F)}^{G \times_{X} (St_{G}x_{o} \times F)}$ 

is a quotient. Finally,

$$\begin{array}{c} G_{X}(St_{G^{X_{o}}} \times F) \xrightarrow{\partial} St_{G^{X_{o}}} \times F \\ \downarrow \\ I_{X}r \\ G_{X}E \xrightarrow{} E \\ \end{array}$$

is commutative and so "." is continuous. This completes the proof of a). b) Again we consider the space  $P: E \longrightarrow X$  as in a), we give each fibre  $P^{-1}(x)$  the norm  $\|\|_{x}$  and inner product  $\langle , \rangle_{x}$  as above, and we define addition and scalar multiplication as in a). Let  $\{U_i, \lambda_i, x_o\}$  be a local trivialisation for G and let  $E_x = P^{-1}(x_o)$ . For each index j we define

$$\phi_{j}: U_{j} \times E_{x_{j}} \longrightarrow P(U_{j})$$
  
$$\phi_{j}(x, [\beta, f]) = \lambda_{j}(x)^{-1} \cdot [\beta, f]$$

Thus,  $\varphi_j(x, [\beta, f]) = [\lambda_j(x)^{-1}\beta, f]$ . We now topologise the space E by taking the maps  $\varphi_j$  as homeomorphisms. Thus, E is locally a product and P: E  $\longrightarrow X$  is continuous in the new topology. Since E is now a vector bundle, the operations of addition and scalar multiplication are continuous; this is proved by working locally. Thus, P: E  $\longrightarrow X$ is a family of Hilbert spaces over X. Notice also that  $\varphi_{j,x} = \varphi_{\lambda_j}(x)^{-1}$ and is, therefore, a unitary operator. Thus, P: E  $\longrightarrow X$  is a bundle with norm and inner product.

Finally, G acts on E in a strongly continuous manner and the proof of this is exactly the same as that used in proving Theorem 5.4.8. This completes the proof of b).

c) This follows by 5.1.10, 5.3.3 i) and the uniqueness theorem, Theorem 2.7, of Chapter 5, Husemoller [1].

The proof of the theorem is now complete.

We have the following corollary to 5.4.12.

5.4.13. Corollary.

by

Suppose G is a transitive locally compact Hausdorff topological groupoid over X and let  $x_o \in X$ . Suppose we are given a strongly continuous action of  $G\{x_o\}$ , as a group of unitary operators on the left of a finite dimensional Hilbert space F. Then there is a continuous unitary representation (S, P, X) of G whose induced representation of  $G\{x_o\}$  on  $P^{-1}(x_o)$  is equivalent to the given one of  $G\{x_o\}$ on F.

Proof.

Since F is finite dimensional, it is locally compact. Thus,
by Ellis [1], the action of  $G\{x_o\}$  on F is necessarily continuous. The conclusion now follows by a) of Theorem 5.4.12.

If G is not locally trivial, then the extension of the action of a vertex group to an action of G need not be unique. This is shown by:-

5.4.14. <u>Example</u>.

Let G be the cyclic group  $\{+1, -1\}$  with the discrete topology and let  $X = \{x_1, x_2\}$  with the indiscrete topology. Let G act on the left of X according to the rules:

> +1 •  $x_1 = x_1$  +1 •  $x_2 = x_2$ -1 •  $x_1 = x_2$  -1 •  $x_2 = x_1$

This determines a continuous action and the resulting groupoid (see 2.2.5) is a transitive compact topological groupoid over X. Let  $S_1 = X \times C$ and let  $P_1 : S_1 \longrightarrow X$  be the projection, since  $S_1$  is a trivial bundle,  $P_1 : S_1 \longrightarrow X$  is a family of Hilbert spaces over X. We define an action of  $\tilde{G}$  on  $S_1$  as follows: if  $\alpha \in \tilde{G}\{x_1\}$  or  $\tilde{G}\{x_2\}$ , then  $\alpha$  must act trivially; if  $\alpha \in \tilde{G}(x_1, x_2)$ , then  $\alpha \cdot (x_1, z) = (x_2, z)$ and if  $\alpha \in \tilde{G}(x_2, x_1)$ , then  $\alpha \cdot (x_2, z) = (x_1, z)$ . This action is clearly unitary and must be strongly continuous since each star is discrete.

Now let  $S_2 = x_1 \times C$   $\sqcup$   $x_2 \times C$  and define  $P_2 : S_2 \longrightarrow X$ by  $P_2((x_1,z)) = x_1$  and  $P_2((x_2,z)) = x_2$ . Then it is easily seen that  $P_2 : S_2 \longrightarrow X$  is a family of Hilbert spaces over X. Define an action of  $\tilde{G}$  on  $S_2$  in the same way as we defined that on  $S_1$ . Again we obtain a strongly continuous unitary representation of  $\tilde{G}$ .  $S_1$  and  $S_2$ are not homeomorphic and, hence, are inequivalent representations of  $\tilde{G}$ ; however, they induce the same representation of each vertex group.

Theorems 5.4.8 and 5.4.12 together yield

## 5.4.15. Theorem.

Let G be a locally compact, transitive, locally trivial Hausdorff topological groupoid over X and let  $x_{o} \in X$ . Then a fibre space P: S  $\longrightarrow$  X, with Hilbert space fibre F, admits a strongly

continuous unitary representation of G if, and only if, P : S  $\longrightarrow X$ can be given the structure of a Hilbert fibre bundle, with norm and inner product, group  $G\{x_o\}$  acting in a strongly continuous manner on F, as a group of unitary operators, and transition functions  $\{g_{ji}\}$  as usual.

# 5.4.16. Definition.

We shall call a representation (S, P, X) of G <u>faithful</u> if, for all  $\alpha,\beta \in G$ , the relation  $\phi_{\alpha} = \phi_{\beta}$  implies  $\alpha = \beta$ .

If (S, P, X) is a representation of a groupoid G, then the set  $G(S) = \{ \phi_{\alpha} : \alpha \in G \}$  is a groupoid of unitary operators between Hilbert spaces, as in 1.2.2, and the function  $\Theta : G \longrightarrow G(S)$  defined by  $\alpha \longmapsto \phi_{\alpha}$  is a homomorphism of groupoids. To ask that (S, P, X)be a faithful representation of G is precisely the same thing as to ask for  $\Theta$  to be injective. Notice that if (S, P, X) is the representation constructed in 5.1.10, then  $\phi_{\alpha}$  is an admissible map as defined in §3, Chapter 3.

We shall call a continuous unitary representation of G on a finite dimensional fibre space S a "representation of G as a groupoid of matrices". We then have the following Peter-Weyl Theorem for groupoids. 5.4.17. Theorem.

Let G be a transitive locally compact Hausdorff topological groupoid with compact Lie vertex groups. Then G admits a faithful representation as a groupoid of matrices.

#### Proof.

By the classical Peter-Weyl theory for compact Lie groups, any vertex group  $G\{x_o\}$  admits a faithful representation as a group of matrices, see Chevalley [1]. Thus, there is a continuous action  $G\{x_o\} \times F \longrightarrow F$  of  $G\{x_o\}$ , on a finite dimensional Hilbert space F, as a group of unitary operators and this action is faithful. Applying Theorem 5.4.12 a), we obtain a representation of G as a groupoid of matrices. The technique of proving 5.1.6 shows that this representation is faithful.

## \$5. Operations on Representations.

Let G be a transitive, locally compact, Hausdorff topological groupoid over X. Let U(G) denote the set of equivalence classes of strongly continuous unitary representations of G, equivalence being in the sense of Definition 5.4.11, and let  $\overline{U}(G)$  denote the subset of U(G) consisting of the continuous unitary representations of G. Let [S, P, X] denote the equivalence class generated by the representation (S, P, X) and let  $S |_{G\{x_o\}}$  denote the representation of  $G\{x_o\}$  on  $P^{-1}(x_o), x_o \in X$ , obtained by restiction from S. Then we have a map  $\Gamma: U(G) \longrightarrow U(G\{x_o\})$  defined by  $\Gamma([S, P, X]) = [S|_{G\{x_o\}}]$ .

If G is locally trivial, or we consider only finite dimenensional representations in U(G), then Theorem 5.4.12 asserts that  $\Gamma$  is surjective. In the case G locally trivial,  $\Gamma$  restricted to  $\overline{U}(G)$  is injective. But  $\Gamma$  is not generally injective as the Example 5.4.14 shows. 5.5.1. Sums of Representations.

If we have a sequence  $H_1$ ,  $H_2$ , ...,  $H_n$ , ... of Hilbert spaces and  $H_n$  has inner product  $<, >_n$ , then it is possible, as is well known, to form their direct sum  $H = \bigoplus_{n=1}^{\infty} H_n$  as follows : H is the subspace of  $\prod_{n=1}^{\infty} H_n$  consisting of those elements  $h = (h_1, h_2, \dots, h_n, \dots)$ such that  $\sum_{n=1}^{\infty} < h_n$ ,  $h_n >_n < \infty$ . We then define an inner product <, > on H by  $< h, h' > = \sum_{n=1}^{\infty} < h_n, h'_n >_n$ .

Suppose now that  $(S_1, P_1, X)$  and  $(S_2, P_2, X)$  are two families of Hilbert spaces over X, then we can form a new family  $(S_1 \oplus S_2, q, X)$ of Hilbert spaces over X called their <u>direct sum</u>. To construct  $S_1 \oplus S_2$ , we proceed as follows. Let  $S_1 \oplus S_2 = \{(s_1, s_2) \in S_1 \times S_2; P_1(s_1) = P_2(s_2)\}$ and define  $q: S_1 \oplus S_2 \longrightarrow X$  by  $q((s_1, s_2)) = P_1(s_1) = P_2(s_2)$ . We give  $S_1 \oplus S_2$  the subspace topology and then q is continuous. Since  $q^{-1}(x) = P_1^{-1}(x) \oplus P_2^{-1}(x)$ , we give  $q^{-1}(x)$  the inner product defined in

 $(S \oplus S_2, q, X)$  is a family of Hilbert spaces.

Now suppose  $(S_1, P_1, X)$  and  $(S_2, P_2, X)$  are strongly continuous unitary representations of G, then  $(S_1 \oplus S_2, q, X)$  can be made into a strongly continuous unitary representation as follows. If  $\alpha \in G(x,y)$ and  $(s_1, s_2) \in q^{-1}(x)$ , we define

$$(s_1, s_2) = (a \cdot s_1, a \cdot s_2) -----*$$

For a fixed element  $(s_1, s_2) \in S_1 \oplus S_2$  we have that the functions  $\Theta_1 : \operatorname{St}_{G} \operatorname{P}_1(s_1) \longrightarrow S_1$  and  $\Theta_2 : \operatorname{St}_{G} \operatorname{P}_2(s_2) \longrightarrow S_2$  are both continuous, where  $\Theta_1(\alpha) = \alpha \cdot s_1$  and  $\Theta_2(\alpha) = \alpha \cdot s_2 \cdot \operatorname{Thus}$ , the function  $\Theta : \operatorname{St}_{G} \operatorname{q}(s_1, s_2) \longrightarrow S_1 \oplus S_2$  defined by  $\Theta(\alpha) = \alpha \cdot (s_1, s_2)$  is continuous.

Finally, it is easy to check that \* defines an action of G on  $S_1 \oplus S_2$  via q and that each of the maps  $\phi_{\alpha} : q^{-1}(\pi(\alpha)) \longrightarrow q^{-1}(\pi'(\alpha))$  is unitary. We call  $S_1 \oplus S_2$  the <u>direct sum representation</u> of G.

This definition generalises in the obvious way to a sequence  $S_1, S_2, \ldots, S_n, \ldots$  of families of Hilbert spaces, and if each  $S_n$  is a representation of G, we obtain the direct sum representation  $\bigoplus_{n=1}^{\infty} S_n$ .

If all the spaces  $S_n$  are locally trivial, this construction coincides precisely with the Whitney sum of Vector bundles. We now prove: 5.5.2. <u>Theorem</u>.

Let G be a transitive, locally trivial, locally compact, Hausdorff topological groupoid with compact vertex groups. Then any continuous unitary representation (S, P, X) of G is the direct sum of the irreducible subrepresentations of G (up to natural equivalence). Proof.

First note that each irreducible subrepresentation of G is finite dimensional by Theorem 5.4.5.

Let (S, P, X) be any continuous unitary representation of G, let  $x \in X$  and let  $S = P^{-1}(x)$ . Now  $G\{x_o\}$  is compact and S is a continuous unitary representation of  $G\{x_o\}$ , thus,  $S_o$  is, up to natural equivalence, the direct sum  $F = F_o^{\dagger} \oplus F_o^2 \oplus \cdots \oplus F_o^n \oplus \cdots \oplus F_o^n \oplus \cdots$  where the action of  $G\{x_o\}$  on F is as defined by \* of 5.5.1 and  $F_o^{\dagger}, F_o^2, \cdots, F_o^n$ ,  $\cdots$  are the irreducible subrepresentations of  $S_o$ , see Mackey [4], [5].

Let E denote the representation of Theorem 5.4.12 with fibre F and let  $E^n$  denote that with fibre  $F_o^n$ , then  $E = \bigoplus_{n=1}^{\infty} E^n \cdot By \cdot c$ ) of Theorem 5.4.12, we have an equivalence  $\Theta^{-1}: S \longrightarrow E = \bigoplus_{n=1}^{\infty} E^n$  of representations. Now let  $S^n = \Theta(E^n)$  for each n, then  $S^n$  is an irreducible subrepresentation of S by 5.4.7 ii). Since  $S = \bigoplus_{n=1}^{\infty} S^n$ , n=1

the proof is complete.

Collecting together the results of 5.4.5 and 5.5.2 we have: 5.5.3. <u>Theorem</u>. (Peter-Weyl Theorem for groupoids).

Let G be a transitive, locally compact, Hausdorff topological groupoid with compact vertex groups and let x be any object of G. Then: a) Every strongly continuous irreducible unitary representation of G is finite dimensional.

b) If G is locally trivial, then every continuous unitary representation of G is the direct sum of (finite dimensional) irreducible subrepresentations.

### 5.5.4. Theorem.

Let G be a locally compact locally trivial Hausdorff topological groupoid over X. Let (S, P, X) be a continuous unitary representation of G and suppose  $S = \bigoplus_{n=1}^{\infty} E_n$  and  $S = \bigoplus_{n=1}^{\infty} F_n$  are two decompositions of S into direct sums of irreducible subrepresentations of S. Then there exists a permutation  $\pi$  of indices such that  $E_n$ is equivalent to  $F_{\pi(n)}$  for each n. Proof.

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This is true for the representations of any vertex group, see Mackey [5], and so the technique of the previous result, Theorem 5.5.2, establishes the result.

Thus, to within equivalence, the two decompositions contain the same irreducible representations and each one that occurs, occurs in both the same number of times.

# 5.5.5. Concluding Comments.

We shall close with some remarks concerning one of the directions in which this work may proceed further.

Suppose H, and H<sub>2</sub> are Hilbert spaces with inner products <,> and  $<,>_2$  respectively, then we can form the tensor product H<sub>1</sub>  $\otimes$  H<sub>2</sub>, see Maurin [1] page 82, with inner product <,> defined by  $< h_1 \otimes h_2, h_1' \otimes h_2' > = < h_1, h_1' > < h_2, h_2' >_2$ .

If G is a locally compact group and  $H_1$  and  $H_2$  are strongly continuous unitary representation of G, we can make G act on  $H_1 \otimes H_2$  by defining  $g \cdot (h_1 \otimes h_2) = g \cdot h_1 \otimes g \cdot h_2$  and this defines a unitary action. In fact,  $H_1 \otimes H_2$  becomes a strongly continuous unitary representation of G.

Now let G be a transitive locally compact topological groupoid over X and let  $(S_1, P_1, X)$  and  $(S_2, P_2, X)$  be strongly continuous unitary representations of G . We can form the tensor product  $(S_1 \otimes S_2, q, X)$  of  $S_1$  and  $S_2$  and, essentially, this construction amounts to considering  $(St_{G}x_{x} \times (P_{1}^{-1}(x_{x}) \otimes P_{2}^{-1}(x_{x}))) / G[x_{x}]$  as in Under suitable conditions, for example G locally trivial and 5.1.10. S, and S continuous unitary representations,  $(S \otimes S, q, X)$  will be a strongly continuous unitary representation of G and, in such circumstances, U(G) becomes a semi-ring with operations  $\oplus$ and Ø Now let R(G) denote the ring completion of U(G), see Husemoller [1], we call R(G) the ring of unitary representations of G (or just the representation ring of G). We can now extend  $\Gamma$  to a ring homomorphism  $\Gamma$ : R(G)  $\longrightarrow$  R(G{x<sub>o</sub>}), the study of which seems worthwhile.

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