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## Crossed modules and their higher dimensional analogues

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# CROSSED MODULES AND THEIR HIGHER DIMENSIONAL ANALOGUES 

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Thesis submitted to the University of. Wales in support of the application for the degree of Philosophiae Doctor.

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## IH:IM ARA'ITION

The work of this thesis has been carried out by the candidate and contains the results of his own investigations. The work has not. alkeady been accepted in substance for any degree, and is not being concurrently submitted in candidature for any degree. All sources of information have been acknowledged in the text.

I Claire. Hebddi hi, ni fyddai'r thesis hwn wedi cael ei gwbihau.

## ACKNOWLEDGEMENTS

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For a fairly general algebraic category $C$ (possible interpretations of $C$ include the categories of groups, rings (associative, commutative), algebras (associative, commutative, Lie or Jordan)) we give various alternative descriptions of an $n$-fold category internal to $C$. One of these descriptions we call a "crossed n-cube in $C$ ". Crossed l-cubes are better known as "crossed modules" (this latter term being due to Whitehead [Wl]). Crossed 2-cubes in the category of groups are originally due to Loday [L].

We give a combinatorial description of crossed n-cubes for $n=1,2,3$ and $C$ equal to the category of groups, Lie algebras, commutative algebras and ( $n=1,2$ ) associative algebras.

The study of certain universal crossed 2-cubes leads us to notions of non-abelian tensor, exterior and antisymmetric products of groups and of Lie and commutative algebras. The tensor and exterior products of groups are originally due to Brown and Loday [B-L]. We also look at the crossed 3-cube analogue of the tensor product of groups.

We study the relevance of crossed modules and crossed 2-cubes to the homology of groups and Lie algebras. In particular we prove
 with im $\partial=N$, then $H_{2}(N) \cong \operatorname{ker} a \cap[M, M]$.

THEOREM If M,N are normal subgroups of a group $G$ such that $G=M N$, then there is an exact sequence

$$
\begin{aligned}
& \pi_{3}(M \wedge N) \rightarrow H_{2}(G) \rightarrow H_{2}(G / M) \oplus H_{2}(G / N) \rightarrow M \cap N /[M, N] \rightarrow \\
& \rightarrow H_{1}(G) \rightarrow H_{1}(G / M) \oplus H_{1}(G / N) \rightarrow 1
\end{aligned}
$$

where $\pi_{3}(M \wedge N)$ is the kernel of a map $M \Lambda N \rightarrow M$ from the exterior product of $M$ and $N$.

THHOREM If, in the preceding theorem, $G=N$, then we can extend the exact sequence by two terms:

$$
H_{3}(G) \rightarrow H_{3}(G / M) \rightarrow \pi_{3}(M \wedge N) .
$$

The second two theorems are originally due to Brown and Loday [B-L] who obtained them as a corollary to their van Kampen type theorem for squares of maps. The proofs in this thesis are purely algebraic.

We give the analogue of the second theorem in which $H_{2}(G)$ is replaced by the group $H_{2}(G)$, this group $H_{2}(G)$ being the one introduced by Dennis [D] as a kind of "second homology group suitable for algebraic K-theory".

We give Lie algebra versions of the first two theorems.

For many common algebraic categories there exists a useful theory of crossed modules. In this thesis we introduce several equivalent notions of a higher dimensional crossed module, and we develop certain aspects of the resulting higher dimensional theory. This work is motivated by a recent result of R. Brown and J.-T. Loday, and also by the theory of crossed modules themselves, as we shall now explain. In parts, this introduction relies heavily on [B].

Recall that a group homomorphism $a: M \rightarrow P$ is said to be a crossed P-module (in groups) if there is an action of $P$ on $M,(p, m) \rightarrow P_{m}$, which satisfies $\partial\left(p_{m}\right)=p(\partial m) p^{-1}, \quad \partial m_{m}{ }^{\prime}=$ $m m^{\prime} m^{-1}$ for $m, m^{\prime} \in M, p \in P$. Standard examples of crossed modules are:
(i) the inclusion $N \rightarrow P$ of a normal subgroup $N$ of the group $P$, with the action of $P$ on $N$ given by conjugation (throughout this thesis we shall keep to the convention that, if $x, y$ are elements of some group then, the congugate of $x$ by $y$ is the elementt $y x y^{-1}$ );
(ii) the zero morphism $0: M \rightarrow P$ in which $M$ is a $P$-module in the usual sense;
(iii) the morphism $x: M \rightarrow$ Aut $M$ from $M$ to the group of automorphisms of $M$ in which $x m$ is the inner automorphism determined by $m \in M$, together with the standard action of Aut $M$ on $M$;
(iv) the boundary map $a: \pi_{2}\left(X, Y, X_{0}\right) \rightarrow \pi_{1}\left(Y, X_{0}\right)$ from the
second relative homotopy group to the fundamental group, with the standard action of $\pi_{1}\left(Y, X_{0}\right)$ on $\pi_{2}\left(X, Y, X_{0}\right)$.

As this last example suggests, crossed modules can be used to model certain homotopy types. In particular there is a functor $B:(c r o s s e d$ modules) $\rightarrow$ (CW-complexes) such that if $\partial: M \rightarrow P$ is a crossed $P$-module then $B(M \rightarrow P)$ has fundamental group coker $a$ and second homotopy group ker $d$ [B-Hl,L]. Further, any pointed, connected CW-complex X with $\pi_{i}(X)=0$ for $i>2$ is of the homotopy type of some $B(M \rightarrow P)[M L-W]$.

Also, a van Kampen type theorem for $\pi_{2}\left(X, Y, x_{0}\right)$ considered as a crossed $\pi_{2}\left(Y, X_{O}\right)$-module has been found [B-H2].

From the standpoint of homotopy theory, crossed modules should perhaps be viewed as 2 -dimensional groups. It is reasonable to ask then, what are the higher-dimensional groups (or crossed modules)? J.H.C. Whitehead gave a partial answer to this by introducing what he called "homotopy systems", but what are now called crossed complexes. These gadgets consist of a sequence of groups

$$
\rightarrow C_{n} \rightarrow \partial_{n} \ldots C_{3} \rightarrow a_{3} \quad C_{2} \rightarrow \partial_{2} \quad c_{1}
$$

in which:
(i) $C_{n}$ is abelian for $n \geqslant 3$;
(ii) $\partial_{n-1} \partial_{n}=0$;
(iii) $C_{1}$ acts on $C_{n}, n \geqslant 2$, and $a_{2} C_{2}$ acts trivially on $C_{n}$, $n \geqslant 3 ;$
(iv) $\partial_{2}$ is a crossed module, and each $\partial_{n}$ is an equivariant map.

The standard example of a crossed complex is obtained from a pointed filtered space $X \supset \ldots X_{n} \supset \ldots X_{2} \supset X_{1} \supset$ $\left\{x_{0}\right\}$ by setting $C_{1}=\pi_{1}\left(X_{1}, x_{0}\right), C_{n}=\pi_{n}\left(X_{n}, X_{n-1}, x_{0}\right)$ and taking each $\partial_{n}$ to be the boundary operator. Crossed complexes give certain partial generalisations to the homotopy theoretic results mentioned above involving crossed modules. However, the abelian nature of crossed complexes is a bar to the obvious full generalisations.

Note that crossed complexes arise in the cohomology of groups [ML] since, if $P$ is a group and $M$ is a $P$-module, then $H^{n+1}(P ; M)$ can be obtained as equivalence classes of $n-d i m e n s i o n a l$ crossed complexes in which ker $a_{n}=M$, coker $\partial_{2}=P$, ker $\partial_{i} / i m \partial_{i+1}=0$ for $2 \leqslant i \leqslant n-1$. It seems reasonable to expect that other notions of higher dimensional crossed modules might also be of relevance to (co-)homology.

A more recent and important reformalation of the fact that $a: \pi_{2}\left(X, Y, X_{0}\right) \rightarrow \pi_{1}\left(Y, X_{O}\right)$ has a crossed module structure is that, if $F \rightarrow E \rightarrow B$ is a fibration, then the induced map $\pi_{1} F \rightarrow \pi_{1} E$ is a crossed module. This is one of the reasons for the use of crossed modules in algebraic k-theory [L,GW-L]. Recall that if $\Lambda$ is a ring (with unit) then $\mathrm{F} \Lambda$ is defined as the homotopy fibre of the inclusion BGL $\wedge \rightarrow$ $(B G L \Lambda)^{+}$. Now $\pi_{1} F$ is the Steinberg group $S t \Lambda$, and $\pi_{1} B G L \Lambda=$ GLA ; thus we have a crossed module St $\Lambda \rightarrow$ GLA. The study of bi-relative Steinberg groups has led to the definition of a type of 2 -dimensional crossed module, which is called a "crossed square" [GW-L]. With a few formal modifications
this definition states that a crossed square consists of a commutative diagram of groups

$$
\begin{array}{rll}
L & \rightarrow \lambda & N \\
\lambda^{\prime} \downarrow & & \iota^{\prime} \delta^{\prime} \\
M & \rightarrow B^{\circ} & P
\end{array}
$$

together with actions of $P$ on $L, M$ and $N$ (hence $M$ acts on $L$ and N via $\delta$, and N acts on L and M via $\mathrm{B}^{\prime}$ ), and a function $h: M \times N \rightarrow$ L such that:
(i) each of the maps $\lambda, \lambda^{\prime}, \delta, \delta^{\prime}$ and the composite $\delta^{\prime} \lambda^{\prime}$ are crossed modules;
(ii) the maps $\lambda, \lambda^{\prime}$ preserve the actions of $P$;
(iii) $h\left(m^{\prime}, n\right)=m_{h}\left(m^{\prime}, n\right) h(m, n)$,

$$
h\left(m, n n^{\prime}\right)=h(m, n) n_{h\left(m, n^{\prime}\right) ;}
$$

(iv) $P_{h}(m, n)=h\left(P_{m}, p_{n}\right)$;
(v) $\quad \lambda h(m, n)=m_{n} n^{-1}, \quad \lambda \cdot h(m, n)=m n_{m}^{-1}$;
(vi) $h(m, \lambda l)=m_{1} I^{-1}, h\left(\lambda^{\prime} l, n\right)=1 n_{1}^{-1} ;$
for all $l \in L, m, m^{\prime} \in M, n, n^{\prime} \in N, p \in P$.
The standard examples of crossed modules (see above) can be extended to examples of crossed squares:
(i) if $M, N$ are normal subgroups of the group $p$, then the diagram of inclusions

$$
\begin{array}{ccc}
\mathrm{MNN} & \rightarrow & \mathrm{~N} \\
\downarrow & & \downarrow \\
\mathrm{M} & \rightarrow & \mathrm{P}
\end{array}
$$

together with the actions of $P$ on $M, N$ and MnN given by
conjugation, and the function $h: M \times N \rightarrow M \cap N,(m, n) \rightarrow[m, n]$, is a crossed square (throuought this thesis we shall keep to the convention that, if $x, y$ are elements of some group then, the comutator $[x, y]$ is the element $x y x^{-1} y^{-1}$ );
(ii) if $M, N$ are ordinary $P$-modules and $A$ is an arbitrary abelian group on which $P$ is assumed to act trivially, then the diagram

$$
\begin{array}{lll}
\mathrm{A} & \rightarrow & \mathrm{~N} \\
\downarrow & \downarrow \\
\mathrm{M} & \rightarrow \mathrm{P}
\end{array}
$$

in which each map is a zero map, together with the zero map $0: M \times N \rightarrow A, i s$ a crossed square;
(iii) the diagram

where $\chi \mathrm{m}$ is the inner automorphism determined by $m \in M$ and where $l$ is the inclusion of the inner automorphism subgroup, together with the standard actions and the function $h: \operatorname{Inn} M \times \operatorname{Inn} M \rightarrow M,\left(x m, x m^{\prime}\right) \rightarrow\left[m, m^{\prime}\right]$, is a crossed square;
(iv) [B-L] if $U, V$ are subspaces of $X$ with a point $x_{0}$ in common, then the diagram of boundary maps

$$
\begin{array}{ccc}
\pi_{3}\left(X ; U, V, x_{0}\right) & \rightarrow & \pi_{2}\left(V, U \cap V, x_{0}\right) \\
\downarrow & \downarrow \\
\pi_{2}\left(U, U \cap V, x_{0}\right) & \rightarrow & \pi_{1}\left(U \cap V, x_{0}\right)
\end{array}
$$

in which $\pi_{3}\left(X ; U, V, X_{0}\right)$ is the triad homotopy group, together with the standard actions and the triad Whitehead product $h: \pi_{2}\left(U, U \cap V, X_{0}\right) \times \pi_{2}\left(V, U \cap V, x_{0}\right) \rightarrow \pi_{3}\left(X ; U, V, X_{0}\right)$, is a crossed square.

It is worth noting that the crossed complexes of length 3 are the crossed squares of the form


In this thesis we shall be very much concerned with crossed squares and their higher dimensional counterparts.

Let $d: M \rightarrow P$ be a crossed module. Since $P$ acts on $M$ we may form the semi-direct product MXP. Let $s, b: M \underline{P} \rightarrow P$ be given respectively by $(m, p) \rightarrow p,(m, p) \rightarrow(a m) p$. The group MXP acquires a category structure, with $s, b$ the source and target maps, and with category composition given by $(m, p) \circ\left(m^{\prime},(\partial m) p\right)=\left(m m^{\prime}, p\right)$. The crossed module axioms are equivalent to this category structure making MxP a category internal to the category of groups (a result noted by several people and published in $[B-S]$ ).

This suggests how to define a crossed module internal to other algebraic categories: consider an internal category
object $C$ with source and target maps $s, b: C \rightarrow P$; the associated "crossed module" is the restriction of $b$ to ker $s \rightarrow P$. This process is analysed in [L-R,P1] and in Chapter I of this thesis. In his work on deformation theory, Gerstenhaber [G] developes a cohomology based on crossed modules. Also, Lue [Lul,2] (developing the work of Gerstenhaber and work of Frohlich [F]) uses "crossed modules" in varieties of algebras. The commutative algebra version of crossed modules has been used in essence rather than in name in [L-S], and has recently been shown to be closely related to Kozul complexes [P2].

In view of the widespread use of crossed modules in other algebraic categories, it is reasonable to expect that notions of higher dimensional crossed modules might also find use in these other categories.

The equivalence between crossed modules and categories internal to the category of groups suggests, as a possible notion of an $n$-dimensional crossed module, an n-fold category internal to the category of groups. Indeed, such $n$-fold categories have been introduced by Loday [L] as a model of truncated homotopy types. Loday gives them (or more precisely, a slightly reformulated version of them) the name "n-cat-group"; however, we shall follow the more recent [B-L] and use the more accurate term catn-group.

Given a catn-group $G$ one can form its iterated nerve, an
 called the classifying space of of $G$. Conversely, any pointed, connected CW-complex $X$ with $\pi_{i} X=0$ for $i>n+1$ is
itself of the homotopy type of some BG [L].
Recently a van Kampen type theorem has been found [B-L] for the "fundamental catn-group of an $n$-cube of spaces". Here an $n$-cube of spaces is just a functor, from the $n$-fold product of the category associated with the ordered set $0<1$, to the category of pointed topological spaces. Clearly cat $^{n}$-groups are a reasonable generalisation of crossed modules.

In order to apply the $n$-dimensional van Kampen type theorem, one needs to compute colimits of catn-groups. For such computations a more combinatorial version of catn-groups is required. For $n=1$ crossed modules prove to be sufficiently combinatorial. It turns out that cat ${ }^{2}$-groups are equivalent to crossed squares [L], and that crossed squares are just the version needed for applications of the 2-dimensional theorem. For higher dimensions a notion of a "crossed n-cube" is clearly needed.

A striking fact about the algebraic theory of crossed modules is that many results on crossed modules in groups, for instance the crossed complex description of cohomology (see above), carry over to other algebraic categories. (In fact, the crossed complex description of cohomology was first given for varieties of algebras [Lul], and then rediscovered for the case of groups.) It is likely that (topologically motivated) results on "crossed n-cubes in groups" will also carry over to other algebraic categories, provided that the various algebraic versions of "crossed n-cubes" exist.

In Chapter I of this thesis we give several equivalent notions of a higher dimensional crossed module. Because of the many different algebraic categories in which these notions are likely to be of interest, we adapt P.J. Higgin's definition [H] of a category of groups with multiple operators, to obtain a fairly general algebraic category $C$ which we call a "category of $\Omega$-groups". We work in C throughout the chapter. Possible interpretations of $C$ include the categories of groups, rings (associative or commutative), and algebras (associative, commutative, Lie or Jordan). The notions of higher dimensional crossed modules which we introduce, and prove equivalences between, are:
(i) n-fold categories internal to $C$;
(ii) catn_objects in $C$;
(iii) crossed $n$-cubes in $C$;
(iv) n-simplicial objects in $C$ whose normal complexes are of length 1 ;
(v) $n$-fold crossed modules in $C$.

Crossed $n$-cubes will be of most interest to us. For $n=1$ they are just crossed modules; for $n=2$ (and $C$ the category of groups) they are crossed squares (see above). The observation that simplicial groups whose normal complex is of length 1 are equivalent to categories internal to the category of groups, is well known and has led Conduché [C] to the definition of a "crossed module of length 2"; such a 'crossed module' being equivalent to a simplicial group whose normal complex is of length 2. It turns out that there is a functor from crossed squares to crossed modules
of length 2.
In Chapter $\mathfrak{I}$ we give detailed descriptions of some low dimensional crossed n-cubes for $C$ equal to the category of groups, Lie algebras, commutative algebras, and associative algebras.

In Chapter III we look at certain colimits of crossed 2-cubes, and obtain non-abelian generalisations of some standard constructions: let $M, N$ be groups which act on each other (and on themselves by conjugation); following [B-L] we obtain a non abelian tensor product $M \otimes N$, which is the group generated by elements $m \otimes n$ for $m \in M, n \in N$, subject to the relations

$$
\begin{aligned}
& m m^{\prime} \otimes n=\left(m_{m} \otimes \otimes m_{n}\right)(m \otimes n), \\
& m \otimes n n^{\prime}=(m \otimes n)\left(n_{m} \otimes n_{n^{\prime}}\right) .
\end{aligned}
$$

We obtain a non-abelian exterior product $M \wedge N$ (again originally due to $[B-L]$ ), and a non-abelian anti-symmetric product $M \perp N$ (a special case of which has been used in [D]). The Lie and commutative algebra versions of these constructions are also given. We consider a certain colimit of crossed 3-cubes (in groups) which leads us to the definition of a "cubical tensor product". In addition the chapter contains various exact sequences involving the non abelian constructions.

The relevance of crossed modules to cohomology has been mentioned above. Surprisingly, little work has been done on the dual situation of crossed modules in homology. In Chapter IV we show that if $N$ is a group and $a: M \rightarrow P$ is a projective crossed module with im $\alpha=N$, then $H_{2}(N) \cong$ ker $\partial \mathrm{n}[\mathrm{M}, \mathrm{M}]$ (this is joint work with T.Porter [E-P]).

This formula should perhaps be seen as a crossed version of Hopf's formula for $H_{2}(N)$. We give a weaker version of the formula for the case of Lie algebras. It is worth noting that our methods give a new and simpler proof of the key lemma 2.l of [R]. We go on to investigate the link between crossed squares and homology. Let $R \rightarrow F \rightarrow G$ be a free presentation of a group G. We obtain, by algebraic means, two isomorphisms $H_{2}(G) \cong \operatorname{ker}(G \wedge G \rightarrow G), H_{3}(G) \cong \operatorname{ker}(F \wedge R$ $\rightarrow$ F). We combine these new descriptions of $\mathrm{H}_{2}(\mathrm{G}), \mathrm{H}_{3}(\mathrm{G})$ with certain of the exact sequences of Chapter III to obtain:

THEOREM If $\mathrm{M}, \mathrm{N}$ are normal subgroups of a group G such that $G=\mathbb{M N}$, then there is an exact sequence
$\pi_{3}(M \wedge N) \rightarrow H_{2}(G) \rightarrow H_{2}(G / M) \oplus H_{2}(G / N) \rightarrow M n N /[M, N] \rightarrow$
$\rightarrow H_{l}(G) \rightarrow H_{l}(G / M)+H_{l}(G / N) \rightarrow l$
where $\pi_{3}(M \wedge N)$ is the kernel of a canonical map $M \wedge N \rightarrow M$.

THEOREM $[f$, in the preceding theorem, $G=N$, then we can extend the exact sequence by two terms:
$\mathrm{H}_{3}(\mathrm{G}) \rightarrow \mathrm{H}_{3}(\mathrm{G} / \mathrm{M}) \rightarrow \pi_{3}(\mathrm{M} \Lambda \mathrm{N})$.

These two theorems are originally due to Brown and Loday [B-L] who obtained them as a corollary to their 3-dimensional van Kampen type theorem. Our proofs are purely algebraic, and consequently we are able to give the Lie algebra version of the first theorem. We also give an analogue of the first theorem in which $H_{2}(G)$ is replaced by the group $\mathrm{H}_{2}(G)$ : the group $\mathrm{H}_{2}(G)$ being the group
introduced by Dennis [D] as a kind of "second homology group suitable for algebraic k-theory".

Chapter $V$ is a collection of miscellaneous comments.

VARIOUS ALTERNATIVE DESCRIPTIONS OF INTERNAL
n-FOLD CATEGORIES

## O. INTRODUCTION

We begin this chapter by defining a "category of n-groups" C. Interpretations of $C$ include the categories of groups, rings (associative, commutative), and algebras (associative, commutative, Lie and Jordan). Thus the theory of $\Omega$-groups provides a convenient setting in which to work. In $\$ 2,3,5,6$ we introduce, and prove equivalences between:
(i) $n$-fold categories internal to $C$;
(ii) catn-objects in $C$;
(iii) crossed n-cubes in C;
(iv) n-simplicial objects in $C$ whose normal complexes are of length 1.
(v) n-fold crossed modules in C.

In $\$ 4$ we give a result on colimits of crossed $n$-cubes in $C$.

1. CATEGORIES OF $\Omega$-GROUPS

Our definition of a "category of $\Omega$-groups" is adapted from [H].

A pointed set $X$ is said to admit a set $\Omega$ of finitary operations if to each $\omega \in \Omega$ is attached a non-negative integer $n=n(\omega)$ called its weight and, for this $n$, there is a pointed map of sets $X^{n} \rightarrow X$ from the $n$-fold product of X to X .

A pointed set $X$ which admits a set $\Omega$ of finitary
operations is called an $\Omega$-group if the following five axioms hold:
(i) the set $\Omega$ contains no operations whose weights are greater than 2; there is precisely one operation (written 0 ) of weight 0 , and precisely two operations (written + , *) of weight 2; there is a prefered operation (written -) of weight 1 ;
(ii) the operations $0,-$, + satisfy the axioms of $a$ (non abelian) group;
(iii) for all $x, y, z \in X$, and unitary operations $\omega$, we have

$$
\omega(x * y)=\omega x * y=x * \omega y
$$

(iv) and, provided $\omega$ is not the prefered unitary operator -,

$$
\omega(x+y)=\omega x+\omega y ;
$$

(v) and $(y+z) * x=(y * x)+(z * x)$,

$$
x *(y+z)=(x * y)+(x * z)
$$

A morphism of $\Omega$-groups is a set map which preserves the operations. Any category whose objects are $\Omega$-groups for some fixed $\Omega$, and whose morphisms are precisely the morphisms of $\Omega$-groups, will be called a category of $\Omega$-groups.

EXAMPLE(1.1.1) Let $\Omega=\{0,-,+, *\}$. Then the category of groups is a category of $\Omega$-groups in which the operation * is trivial (i.e. has constant value 0 )
is trivial. The category of $r$ ings is a category of ת-groups in which the operation * is non trivial.

EXAMPLE(I.1.2) Let $A$ be a commutative $r$ ing (with unit) and let $\Omega=\left\{0,-,^{+}, *\right\} \cup\{a \in A\}$. Then the categories of associative, commutative, Lie and Jordan algebras over A
are categories of $\Omega$-groups. in which $a \in A$ is scalar multiplication.

EXAMPLE(1.1.3) A category of interest (in the sense of Orzech [O]) which has only two binary operations is a category of $\Omega$-groups. We could equally well work with a notion of $\Omega$-groups which allows more than two binary operations, but have no examples to motivate this generalisation.

EXAMPLE(1.1.4) We note that an $\Omega$-group has precisely one operation of weight 0 . Thus, for instance, the category of associative rings with unit is not a category of $\Omega$-groups.

For the remainder of this chapter we fix a category $C$ of $\Omega$-groups.
2. CATn_OBJECTS IN C

Recall that a category internal to consists of: a pair of objects $G, P$ in $C$; and four morphisms s:G $\rightarrow \mathrm{P}, \mathrm{b}: \mathrm{G}$ $\rightarrow P, i: P \rightarrow G, 0: G X P G \rightarrow G(h e r e ~ G X P G=\{(x, Y) \in G X P G: b x=$ sy\}) such that;
(i) $\quad$ si $=$ bi $=$ identity;
(ii) (isx) $0 x=x, x \circ(i b x)=x$;
(iii) $s(x \circ y)=s x, \quad b(x \circ y)=b y$;
(iv) $x \circ(y \circ z)=(x \circ y) \circ z$;
(whenever these last two equations are defined).
A map of categories internal to $C$ is a pair of structure preserving morphisms $\phi: G \rightarrow G ', \psi: P \rightarrow P^{\prime}$.

Note that, since the category composition o is a morphism in $C$, for $(u, v),(x, y) \in G X_{P G}$ we have
(l) $(u \circ v)+(x \circ y)=(u+x) \circ(v+Y)$,
(2) (u $\circ \mathrm{v}) *(x \circ y)=(u * x) \circ(v * y)$.

Also we can write the category composition in terms of the group structure on $G$, since
(3) $u \circ v=(i b u-i b u+u) 0(v+0)$
$=((i b u) \circ v)+((-i b u+u) \circ 0) \quad(b y l)$
$=v-i b u+u$.

Suppose now we are given an arbitrary triple of morphisms s:G $\rightarrow P, b: G \rightarrow P, i: P \rightarrow G$ in $C$ which satisfy $s i=$ bi $=$ identity. For $(u, v) \in \operatorname{GxpG}$ we can define $u \circ v=$ v - ibu + u. It is readily seen that this partial operation, together with the three morphisms, constitute a category internal to $C$ if and only if equations (1) and (2) hold. But we have:
equation (1)

```
# (v - ibu + u) + (y - ibx + x)
= v+y-ibx - ibu + u + x
m (-ibu + u) + (y - isy)
= (y - isy) + (-ibu + u).
```

Let ker $s$, ker $b$ be the kernels of $s, b$, and denote by [ker $b$, ker $s$ ] the subobject of $G$ generated by the commator elements $p+q-p-q$ with $p \in$ ker $b, q \in$ ker s. Then equation (1) states precisely that [ker $b, \operatorname{ker} s]=0$. Under the assumption that [ker b, ker $s$ ] $=0$, we also have:
equation (2)
$\equiv(v-i b u+u) *(y-i b x+x)$
$=(v * x)-(i b u * i b x)+(u * x)$
$\pm((v-i s v) *(x-i b x))+((u-i b u) *(y-i s y))$
$=0$.
Denote by <ker b,ker $s\rangle *$ the subobject of $G$ generated by the elements $p * q, q * p$ with $p \in \operatorname{ker} b, q \in$ ker $s$. Then equation (2) states precisely that $\langle\operatorname{ker} b, \operatorname{ker} s\rangle *=0$.

We are thus led to
DEFINITION(1.2.1) A cat ${ }^{1}$-object $G$ in $C$ consists of an object $G$ in $C$ and a subobject $P$ of $G$; and two morphisms s,b:G $\rightarrow \mathrm{P}$ such that;
(i) $s l_{\mathrm{P}}=\mathrm{b} l_{\mathrm{P}}=$ identity;
(ii) and [ker b, ker s] $=0$, 〈ker b;ker $s\rangle *=0$.

A map of catl-objects is a morphism $\phi: G \rightarrow G^{\prime}$ such that $\phi s=$ $s^{\prime} \phi$ and $\phi b=b^{\prime} \phi$.

We have immediately
PROPOSITION(1.2.2) There is an equivalence of categories, (cat ${ }^{1}$-objects in $C$ ) (categories internal to $C$ ).

We now aim to generalise this proposition.

DEFINITION(1.2.2) A catn-object $G$ in $C$ consists of a family of catl-objects in $C, s_{i}, b_{i}: G \rightarrow P_{i}, l \leqslant i \leqslant n$, such that
(i) $s_{i} s_{j}=s_{j} s_{i}, b_{i} b_{j}=b_{j} b_{i}$, and $s_{i} b_{j}=b_{j} s_{i}(i \neq j)$. (Here $s_{i} s_{j}$ is the composition of the map $s_{j}$ with the map obtained by restricting $s_{i}$ to $P_{j}$.)

A map of catn-objects is a morphism $\phi: G \rightarrow G^{\prime}$ such that $\phi s_{i}$ $=s_{i}{ }^{\prime} \phi$ and $\phi b_{i}=b_{i}{ }^{\prime} \phi$.

The notion of a catn-group is due to Loday [L], although he used the term "n-cat-group". The term "catn-group" is used in [B-L].

The definition of a category internal to $C$ can be restated in purely categorical language. More precisely, axioms (ii), (iii), (iv) can be replaced by
(ii)' so $=s \pi_{1}$, bo $=b \pi_{2}$ (where $\pi_{i}: G \times p G \rightarrow G$ is the $i$ th projection);
(iii)' $O(1 \times 0)=O(O \times 1): G \times P G \times P G \rightarrow G ;$
(iv)' $0(1 \times i b)=0(i s \times 1)=1 G$

Thus for an arbitrary category $C$ ' we can form the category $C T^{1}\left(C^{\prime}\right)$ of categories internal to $C^{\prime}$. Inductively we define the category $C T^{n}\left(C^{\prime}\right)$ of $n$-fold categories internal to $C^{\prime}$ to be the category $\operatorname{CT}^{n}\left(C^{\prime}\right)=\operatorname{CT}^{1}\left(C^{n-1}\left(C^{\prime}\right)\right)$.

PROPOSITION(1.2.3) There is an equivalence of categories, (catn-objects in $C) \approx(n$-fold categories internal to $C$ ). PROOF We have already proved the proposition for $n=1$. Assume it is true for a particular value $n$. Then there is an equivalence, $C T^{n+1}(C) \propto C T^{l}\left(c a t^{n}\right.$-objects in $C$ ). So we need to prove an equivalence between $C T^{l}$ (cat ${ }^{n}$-objects in $C$ ) and (cat ${ }^{n+1}$-objects in $C$ ).

Suppose given a category internal to the category of catn-objects in $C$. Thus we have four maps $s, b: \underline{G} \rightarrow \underline{P}, i: \underline{P} \rightarrow$ $\underline{G}, 0: \underline{G} x_{\underline{G}} \rightarrow \underline{G}$ of catn-objects. Suppose that the cat ${ }^{n_{-}}$ object $\underline{G}$ consists of maps $s_{i}, b_{i}: G \rightarrow P_{i}, l \leqslant i \leqslant n$. We
obtain a cat ${ }^{n+1}$-object from $\underline{G}$ by setting $P_{i+1}=P$ the underlying group of $\underline{P}$, and setting $s_{i+1}=s, b_{i+1}=b$.

Conversely suppose given a cat ${ }^{n+1}$-object which consists of the maps $s_{i}, b_{i}: G \rightarrow P_{i}, l \leqslant i \leqslant n+1$. Let $G^{\prime}$ be the cat ${ }^{\text {n-object consisting of the maps } s_{i}, b_{i}: G \rightarrow P_{i}, l \leqslant i \leqslant}$ $n$. Let $\underline{p}^{\prime}$ be the cat ${ }^{n}$-object consisting of the restricted maps $\left.s_{i}\right|_{P_{n+1}},\left.b_{i}\right|_{P_{n+1}}: P_{n+1} \rightarrow P_{i} \cap P_{n+1}, l \leqslant i \leqslant n$. Then the maps $s_{n+1}, b_{n+1}: \underline{G}^{\prime} \rightarrow \underline{Q}^{\prime}$, together with the inclusion $\underline{P}^{\prime} \rightarrow \underline{G}^{\prime}$ and the map $\underline{G}^{\prime} x_{\underline{P}}{ }^{\prime} \underline{G}^{\prime} \rightarrow \underline{G}^{\prime}$ given by $(u, v) \rightarrow v-b_{n+1 u}+u$, constitute a category internal to the category of cat ${ }^{\text {n-objects }}$ in C .

This correspondence between categories internal to the category of cat ${ }^{n}$-objects in $c$, and cat ${ }^{n+1}$-objects in $c$, gives rise to a pair of quasi-inverse functors. $\nabla$
3. CROSSED n-CUBES IN C

Our aim now is a definition of a "crossed module in $\mathbf{C "}$ (we shall also use the term "crossed l-cube in C"), and a proof that such a gadget is equivalent to a catl-object in C.

Suppose that we are given a split, short exact sequence $M \stackrel{i}{=} \mathrm{G} \underset{\mathrm{j}}{\frac{1}{5}} \mathrm{P}$ in C. Thus $\mathrm{M}, \mathrm{G}, \mathrm{P}$ are $\Omega$-groups; the maps $i_{0}, i$ are injective (and so we consider $M, P$ to be subobjects of G) ; and $M=$ ker $s$, and $s i=$ identity. Let $\alpha^{+}, \alpha^{*}, \alpha^{* 0}: \operatorname{PXM} \rightarrow$ $M$ be, respectively, the functions

$$
\begin{aligned}
(p, m) \rightarrow\left(p^{+}\right)_{m} & =p+m-p, \\
(p, m) \rightarrow\left(p^{*}\right)_{m} & =p * m, \\
(p, m) \rightarrow\left(p^{*}\right)_{m} & =m * p .
\end{aligned}
$$

A triple of functions obtained in this manner will be called a C-action of $P$ on $M$.

Suppose now we have an arbitrary triple of functions PXM $\rightarrow$ P, denoted by $\alpha^{+}, \alpha^{*}, \alpha^{* 0}$ (we are not assuming that these functions are a c-action). The underlying set of $M \times P$ can be made to admit the set $\Omega$ of finitary operations (recall that $c$ is a category of $\Omega$-groups) by defining:

$$
\begin{aligned}
& (m, p)+\left(m^{\prime}, p p^{\prime}=\left(m+(p+)^{\prime}, p+p^{\prime}\right),\right. \\
& (m, p) *\left(m^{\prime}, p^{\prime}\right)=\left(\left(m * m^{\prime}\right)+\left(p^{*}\right)_{m^{\prime}}+\left(p^{\prime * 0}\right)_{m, p *} p^{\prime}\right), \\
& -(m, p)=\left(-(-p+)_{m,-p)}\right. \\
& \omega(m, p)=(\omega m, \omega p) \text { for each unitary operation } \omega \text { except }-.
\end{aligned}
$$ The resulting $\Omega$-group is the semi-direct product of $M$ with $P$ and will be denoted MxP.

PROPOSITION(1.3.1) Let M, P be objects in C and let $\alpha^{+}, \alpha^{*}, \alpha^{* 0}: P \times M \rightarrow M$ be three functions. These functions are a C-action if and only if the semi-direct product MxP is an object in $\mathbf{C}$.

PROOF If the functions are a c-action then they are derived from a split, short exact sequence $M \rightarrow G \leftrightarrows P$, and $G$ is isomorphic to MxP. Conversely, if MxP is an object in $C$, then the functions are a c-action since they are derived
 $(0, p), s(m, p)=p$. $\nabla$

This proposition is essentially due to Orzech [O]. As an application we give

EXAMPLE(1.3.2) Let $C$ be the category of multiplicative
groups with identity $e$, let $M, P$ be groups, and let $\alpha^{+}: P \times M \rightarrow$ $(r, m) \rightarrow P_{m}$ M be a function (we assume that both $\alpha^{*}, \alpha^{* 0}: P \times M \rightarrow M$ are the zero map). In the semi-direct product Mxp multiplication is given by $(m, p)\left(m^{\prime}, p^{\prime}\right)=\left(m m^{\prime}, p p^{\prime}\right)$, the identity is $(e, e)$, and $(m, p)^{-1}=\left(p^{-1} m^{-1}, p^{-1}\right)$. It is routine to check that MXN is a group if and only if
(i) $e_{m}=m$,
(ii) $p\left(p^{\prime} m\right)=\left(p p^{\prime}\right)_{m}$,
(ii) $\left.\mathrm{P}_{(\mathrm{mm}} \mathrm{m}^{\prime}\right)=\left(\mathrm{Pm}_{\mathrm{m}}\right)\left(\mathrm{Pm}^{\prime}\right)$,
for all $m, m^{\prime} \in M, p \in P$. Thus in this case a C-action coincides with the usual notion of a group action.

We can now state the crucial
DEFINITION(1.3.3) A crossed module in $\mathbf{C}$ (or a crossed 1-cube in C) consists of: a pair of objects $M, P$ in $C$; $a$ morphism $a: M \rightarrow P$; and a C-action of $P$ on $M$, such that;
(1) $a((p+) m)=p+\partial m-p$,
$a\left(\left(p^{*}\right) m\right)=p * \partial m$,
$a\left(\left(p^{* 0}\right) m\right)=\partial m * p ;$
(ii) $(\partial m+)_{m^{\prime}}=m+m^{\prime}-m$, $\left(\partial m^{*}\right)_{m^{\prime}}=m * m^{\prime}$, $\left(\partial m^{* 0}\right) m^{\prime}=m^{\prime} * m ;$
for $m, m^{\prime} \in M, p \in P$.
A map of crossed modules is a pair of morphisms $\psi: M \rightarrow M^{\prime}$, $\phi: P \rightarrow P^{\prime}$ such that $\partial^{\prime} \psi=\phi \partial$ and $\left.\psi(p \omega)_{m}\right)=(\phi p \omega)_{\psi m}$ for $\omega=+, *, *^{\circ}$.

When $C$ is the category of groups this definition reduces to the classical definition of a crossed module [Wl]. The
definition also reduces to give the algebraic cases of a crossed module defined in [L-R]. Our general notion of a crossed module is essentially the same as the one given in [P3].

PROPOSITION(1.3.4) There is an equivalence of categories, (crossed modules in $C$ ) $\approx$ (cat¹-objects in $C$ ).

PROOF Suppose given a catl-object $s, b: G \rightarrow P$. Conjugation and multiplication in $G$ give rise to a $C$-action of $P$ on ker $s$. The restriction of $b$ to ker $s \rightarrow P$ clearly satisfies axiom (1.3.3.i). We must check axiom (1.3.3.ii).

Suppose now we are given a crossed module $a: M \rightarrow P$. We can construct two maps $s^{\prime}, b^{\prime}: M x P \rightarrow P$ by defining $s^{\prime}(m, p)=$ $p, b^{\prime}(m, p)=(\partial m) p$. Axiom (1.3.3.i) ensures that $b^{\prime}$ is $a$ morphism in C. The maps $s^{\prime}, b^{\prime}$ clearly satisfy axiom (1.2.1.i). We must check axiom (1.2.1.ii).

We shall show that axiom (1.3.3.ii) is equivalent to axiom (1.2.1.ii). Let $x \in$ ker $s^{\prime}, y \in k e r b^{\prime} ;$ then $x=$ $(m, 0), Y=(-n, \partial n)$ for some $m, n \in M$. We have:
axiom (1.2.1.ii)

```
x+y=y+x
Ex*y=0
y*x=0
```

```
(m-n,an)=(-n+(an+)m,an)
(-(m*n)+(an*0)m,0)=(0,0)
(-(n*m)+(\partialn*)m,0)=(0,0)
```

$$
\begin{aligned}
& n+m-n=(\partial n+)_{m} \\
& m * n=\left(\partial n^{\circ}\right)_{m} \\
& n * m=\left(\partial n^{*}\right)_{m}
\end{aligned}
$$

$=$ axiom (1.3.3.ii).

We have thus given a correspondence between crossed modules in $C$ and catl-objects in $C$. This correspondence gives rise to an equivalence of categories. $\quad \nabla$

We can combine propositions (1.2.2) and (1.3.4) to obtain an equivalence between categories internal to $C$ and crossed modules in C. For $C$ equal to the category of groups this equivalence has been known for some time; it first seems to have appeared in print in [B-S]. The equivalence is given in $[L-R]$ for $C$ equal to various common algebraic categories. Porter [P3] gives the equivalence in the general setting of a category of groups with multiple operations.

We now aim for a definition of a "crossed n-cube in C" and a proof that such a gadget is equivalent to a catn-object in $C$. We begin by generalising the notion of a C-action.

DEFINITION(1.3.5) An n-fold split short exact sequence in C (abbreviated to n.s.s.e.s) is an object $G$ in $C$ together with $n$ subobjects $P_{i}$ and endomorphisms $s_{i}: G \rightarrow P_{i}, l \leqslant i \leqslant$ $n$, such that $s_{i} \mid P_{i}=$ identity and $s_{i} s_{j}=s_{j} s_{i}$

So a O.s.s.e.s. is just an object in $C$; a l.s.s.e.s. is the standard notion of a split, short exact sequence.

In order to handle n.s.s.e.s.'s we introduce some notation. Let $|n|$ denote the set $\{1,2, \ldots, n\}$, and let $r|n|$ be the poset consisting of the subsets of $|n|$. For each $\gamma$ $\epsilon \Gamma|n|$, let $\gamma^{\prime}=|n| \backslash \gamma$. Let the largest number in $\gamma$ be denoted max $\gamma$. Let $10, \iota_{n}: \Gamma|n-1| \rightarrow \Gamma|n|$ be the poset maps given respectively by $\gamma \rightarrow \gamma, \gamma \rightarrow \gamma U\{n\}$.

Suppose we are given an n.s.s.e.s. as above. For each $\gamma$ є $\Gamma|n|$ construct the multiple intersection


Thus if $n=1$, we have $Y\{1\}=$ ker $s_{1}, Y \emptyset=P_{1}$. If $n=2$, we have $Y\{1,2\}=\operatorname{ker} s_{1} \cap \operatorname{ker} s_{2}, Y\{1\}=\operatorname{ker} s_{1} \cap P_{2}$, $Y\{2\}=\operatorname{ker} s_{2} \cap P_{1}, Y \emptyset=P_{1} \cap P_{2}$.

PROPOSITION(1.3.6) If $\gamma_{1} \subset \gamma_{2} \in \Gamma|n|$, then there is a C-action of $\mathrm{Y}_{1}$ on $\mathrm{Y}_{2}$.

PROOF For each $x \in Y \gamma_{1}, y \in Y \gamma_{2}$ define $\left(X^{+}\right)_{y}=x+y-x$, $\left(x^{*}\right) y=x * y,\left(x^{* 0}\right) y=y * x$. It is readily seen that $\left(x^{+}\right) y,\left(x^{*}\right) y,\left(x^{*}\right)_{y} \in Y_{2} . \quad \nabla$

We also have
PROPOSITION(1.3.7) Let $\gamma_{1}, \nu_{2} \in$ 「Inl be such that $\gamma_{1} \not \ddagger \gamma_{2}$, $\gamma_{2} \ddagger \gamma_{1}$. Then there are three functions $h^{+}, h^{*}, h^{* 0}: Y \gamma_{1} \times$ $Y \gamma_{2} \rightarrow Y\left(y_{1} U \gamma_{2}\right)$ given, respectively, by $(x, y) \rightarrow x+y-x-$ $y,(x, y) \rightarrow x * y,(x, y) \rightarrow y^{*} x . \quad \nabla$

Note that there are also functions $h^{+1}, h^{*}, h^{* 0}$; $\mathrm{Y} \gamma_{2} \times \mathrm{Y} \gamma_{1} \rightarrow \mathrm{Y}\left(\gamma_{2} \mathrm{U} \gamma_{1}\right)$, and that $h^{+}(x, y)=-h^{+}(y, x)$, $h^{*}(x, y)=h^{* 0}(y, x), h^{* 0}(x, y)=h^{*}(y, x)$.

In the light of propositions (1.3.6),(1.3.7), and this last observation, we make the following DEFINITION(1.3.8) An n-action $\underline{Y}, n \geqslant 0$, consists of:
(a) an object $Y \gamma$ in $C$ for each $\gamma \in \Gamma|n|$;
(b) three functions $\alpha^{+}, \alpha^{*}, \alpha^{* 0}: \mathrm{Y} \gamma_{1} \times \mathrm{Y} \gamma_{2} \rightarrow \mathrm{Y} \gamma_{2}$ for each $\gamma_{1}$ c $\gamma_{2} \in \Gamma|n|$;
(c) three functions $h^{+}, h^{*}, h^{* 0}: Y \gamma_{3} \times Y \gamma_{4} \rightarrow Y\left(\gamma_{3} \gamma_{4}\right)$ for each $\gamma_{3}, \gamma_{4} \in\left[|n|\right.$ such that $\gamma_{3} \notin \gamma_{4}, \gamma_{4} \notin \gamma_{3}$ and $\max \left(\gamma_{3} \backslash\left(\gamma_{3} n_{4}\right)\right)$ < $\max \left(\gamma_{4} \backslash\left(\gamma_{3} n^{n}\right)\right)$;

An n-action which is derived from an n.s.s.e.s. will be called a c-n-action.

For $n=0$, a c-0-action is just an object in $C$. For $n=$ 1 , a C-1-action is just a c-action. For $n=2$, a 2-action consists of four objects $Y\{1,2\}, Y\{1\}, Y\{2\}, Y \phi$ together with three functions $\alpha^{+}, \alpha^{*}, \alpha^{* 0}: \mathrm{Y}_{1} \times \mathrm{Y} \gamma_{2} \rightarrow \mathrm{Y} \gamma_{2}$ for each $\gamma_{1}$ c $\gamma_{2}$, and precisely three functions $h^{+}, h^{*}, h^{* 0}: Y\{1\} \times Y\{2\} \rightarrow$ $\mathrm{Y}\{1,2\}$.

Given a 1-action $\alpha^{+}, \alpha^{*}, \alpha^{* 0}: P \times M \rightarrow M$ we can construct a 0 -action by forming the semi-direct product MxP. The l-action is easily retrieved from the 0-action. More generally, given an n-action $\underline{Y}$ we can construct an ( $n-1$ )-action RY without loosing information. The details are as follows.

Suppose given an n-action $\mathbb{Y}$. Recall the poset maps $10, \iota_{n}: \Gamma|n-1| \rightarrow \Gamma|n|$. For each $\gamma \in \Gamma|n-1|$ define $R Y y=Y \iota_{n} \gamma \underline{x} Y \iota_{0 \gamma}$.

For each $\gamma_{1} \subset \gamma_{2} \in \Gamma|n-1|$ we can construct three maps $\alpha^{+}, \alpha^{*}, \alpha^{\star 0}: R Y \gamma_{1} \times R Y \gamma_{2} \rightarrow R Y \gamma_{2}$ as follows: set
$L=Y \ln ^{\gamma}{ }^{\gamma} 2$,
$M=Y$ YOV $_{2}$,
$N=Y \iota_{n}{ }^{\gamma} l^{\prime}$,
$\mathrm{P}=\mathrm{YLON}$ : $;$
thus $R Y \gamma_{1}=N X P, R Y \gamma_{2}=L X M$; it is readily seen that there are three functions $h^{+}, h^{*}, h^{* 0}: M \times N \rightarrow L$ and that, for each ( $n, p, 1, m$ ) $\in R Y \gamma_{1} \times R Y \gamma_{2}$, we can define

$$
\begin{aligned}
& \alpha^{+}(n, p, 1, m)=\left(\left(n^{+}\right)\left(\left(p^{+}\right) 1\right)-n^{+}\left(\left(p^{+}\right)_{m, n}\right),\left(p^{+}\right)_{m}\right) \\
& \left.\alpha^{*}(n, p, 1, m)=\left(n^{*}\right)_{1}+\left(p^{*}\right)_{1}+h^{* 0}(m, n),\left(p^{*}\right)_{m}\right) \\
& \left.\alpha^{* 0}(n, p, 1, m)=\left(n^{* 0}\right) 1+\left(p^{* 0}\right) 1+h^{*}(m, n),\left(p^{* 0}\right)_{m}\right)
\end{aligned}
$$

For each $\gamma_{3}, \gamma_{4} \in \Gamma|n-1|$ such that $\gamma_{3} \notin \gamma_{4}, \gamma_{4} \notin \gamma_{3}$, $\max \left(\gamma_{3} \backslash\left(\gamma_{3} \cap_{4}\right)\right)<\max \left(\gamma_{4} \backslash\left(\gamma_{3} \gamma_{\gamma_{4}}\right)\right)$, we can construct three functions $h^{+}, h^{*}, h^{* 0}: R Y \gamma_{3} \times R Y \gamma_{4} \rightarrow R Y\left(y_{3} \mathcal{V H}_{4}\right)$ as follows: set $K=Y \ln _{n}\left(\gamma_{3} U \gamma_{4}\right)$,
$L=Y \ln _{n} \boldsymbol{V}_{3}$,
$M=Y \ln _{n} \boldsymbol{V}_{4}$,
$N=Y \operatorname{IO}\left(\gamma_{3} \mathrm{Ur}_{4}\right)$,
$P=Y$ Yor $_{3}$,
$Q=$ Ylor4;
thus $\mathrm{RY}_{3}=\operatorname{LXP}, \mathrm{RY}_{4}=\mathrm{MXQ}, \mathrm{RY}\left(\boldsymbol{\gamma}_{3} \mathrm{Ur}_{4}\right)=\mathrm{KXN}$; it is readily checked that there are six triples of functions
$h^{+}, h^{*}, h^{* 0}: Q \times L \rightarrow K$,
$h^{+}, h^{*}, h^{* 0}: P \times M \rightarrow K$,
$h^{+}, h^{*}, h^{* 0}: L \times M \rightarrow K$,

$$
\begin{aligned}
& h^{+}, h^{*}, h^{* 0}: N \times L \rightarrow K, \\
& h^{+}, h^{*}, h^{* 0}: N \times M \rightarrow K, \\
& h^{+}, h^{*}, h^{* 0}: P \times Q \rightarrow N,
\end{aligned}
$$

and that for each ( $1, p, m, q$ ) $\in \mathrm{RY}_{3} \times \mathrm{RY}_{4}$ we can define

$$
\begin{aligned}
& h^{+}(1, p, m, q)= \\
& \left((1+) h^{+}(p, m)+h^{+}(1, m)-(m+) h^{+}\left(h^{+}(p, q), 1\right)\right. \\
& -h^{+}\left(h^{+}(p, q), m\right)-\left(h^{+}(p, q)+\right)\left(\left(m+h^{+}(q, 1)\right), h^{+}(p, q)\right), \\
& h^{*}(1, p, m, q)= \\
& \left(h^{*}(1, m)+h^{*}(p, m)+h^{* 0}(q, 1), h^{*}(p, q)\right), \\
& h^{* 0}(1, p, m, q)= \\
& \left(h^{* 0}(1, m)+h^{* 0}(p, m)+h^{*}(q, 1), h^{* 0}(p, q)\right) .
\end{aligned}
$$

We have thus completed the construction of RY.

PROPOSITION(1.3.9) An n-action $\underline{Y}, n \geqslant 1$, is a $C-n$-action if and only if RY is a C -( $\mathrm{n}-1$ )-action.

PROOF Suppose $\underline{Y}$ is a $C$-n-action. Then $\underline{Y}$ is derived from some n.s.s.e.s. which consists of maps $s_{i}: G \rightarrow P_{i}, 1 \leqslant i \leqslant$ $n$, say. It is routine to check that $R Y$ is derived from the $(n-1) . s . s . e . s . c o n s i s t i n g$ of the maps $s_{i}: G \rightarrow P_{i}, l \leqslant i \leqslant$ n-1.

Conversely, suppose RY is a $C-(n-1)$-action which is derived from a ( $n-1$ ).s.s.e.s. consisting of the maps $s_{i}: G \rightarrow$ $P_{i}, 1 \leqslant i \leqslant n-1$. For each $y \in[|n-1|$ the object $R y \gamma$ is a semi-direct product; these semi-direct products give rise to a semi-direct product structure on $G$, say $G=G_{n} \times G_{0}$. By
setting $s_{n}: G \rightarrow G_{0}$ equal to the canonical projection we obtain an n.s.s.e.s. from which $\underline{Y}$ is derived. $\nabla$

In Chapter II we use this proposition to obtain explicit descriptions of C -n-actions for specific choices of C and low values of $n$.

We are now in a position to make the main definition of this section.

DEFINITION(1.3.10). A crossed $n$-cube in C consists of a contravariant functor $\underline{Y}: \Gamma|n| \rightarrow C$ and an n-action structure on the set $\{Y y: \gamma \in \Gamma|n|\}$ (where $Y y$ denotes the image of $\gamma$ under $\underline{Y}$ ), such that:
(i) the n -action is a c -n-action;
(ii) for each $\gamma_{1} \subset \gamma_{2} \in \Gamma|n|$ the c-action of $\mathrm{Y} \gamma_{1}$ on $\mathrm{Y} \gamma_{2}$ is via the map $\mathrm{Y} \gamma_{1} \rightarrow Y \varnothing$ (in fact we shall assume that for any $\gamma^{\prime} \gamma^{\prime} \in \Gamma|n|$, there is a c-action of $Y \gamma$ on $\mathrm{Y}^{\prime}{ }^{\prime}$ via the map $\mathrm{Y} \boldsymbol{\gamma} \rightarrow \mathrm{Y} \varnothing$ );
(iii) for each $\gamma_{1} \subset \gamma_{2} \in \Gamma|n|$, the map $Y \gamma_{2} \rightarrow Y \gamma_{1}$ is a crossed module, and it preserves the actions of $\mathrm{Y} \varnothing$; (iv) let $\gamma_{3}, \gamma_{4}, \gamma_{5}, \gamma_{6} \in \Gamma|n|$ be such that $\nu_{3} \supset \gamma_{5}, \gamma_{4} \supset \gamma_{6}$, $\gamma_{3} \notin \gamma_{4}, \gamma_{4} \notin \gamma_{3}, \gamma_{5} \notin \gamma_{6}, \gamma_{6} \notin \gamma_{5}$ and $\max \left(\gamma_{3} \backslash \gamma_{3} \cap \gamma_{4}\right)$ < $\max \left(\gamma_{4} \backslash \gamma_{3} \cap_{\gamma_{4}}\right)$; thus we have maps $\delta_{3}: \mathrm{Y}_{3} \rightarrow \mathrm{Y}_{5}, \mathrm{O}_{4}: \mathrm{Y}_{4} \rightarrow$ $Y_{\gamma_{6}}, \delta: Y\left(\gamma_{3} \mathcal{U r}_{4}\right) \rightarrow Y\left(\gamma_{5} u_{6}\right)$ (we are allowing the possibility that certain of these maps may be identity maps) and functions $h^{+}, h^{*}, h^{* 0}: Y \gamma_{3} \times Y^{\prime} \gamma_{4} \rightarrow Y\left(\gamma_{3} u_{4}\right)$; then $\left.\begin{array}{l}\delta h^{+}(x, y)=h^{+}\left(\delta_{3} x, \delta_{4 y}\right) \\ \delta h^{*}(x, y)=h^{*}\left(\delta_{3} x, \delta_{4 y}\right) \\ \delta h^{* 0}(x, y)=h^{*}\left(\delta_{3} x, \delta 4 y\right)\end{array}\right\}$
$\left.\begin{array}{l}8 h^{+}(x, y)=-h^{+}\left(\delta_{4} y, \delta_{3} x\right) \\ \delta h^{*}(x, y)=h^{* \circ}\left(\delta_{4 y,} \delta_{3} x\right) \\ \delta h^{* \circ}(x, y)=h^{*}\left(\delta_{4} y, \delta_{3} x\right)\end{array}\right\}$ if $\max \left(\gamma_{6} \backslash \gamma_{5} \cap \gamma_{6}\right)<\max \left(\gamma_{5} \backslash \gamma_{5} \cap \gamma_{6}\right)$;
(v) for each $\gamma_{3}, \gamma_{4} \in \Gamma|n|$ such that $\gamma_{3} \notin \gamma_{4}, \gamma_{4} \notin \gamma_{3}$ and $\max \left(\gamma_{3} \backslash \gamma_{3} \cap \gamma_{4}\right)<\max \left(\gamma_{4} \backslash \gamma_{3} \cap \gamma_{4}\right)$, the functions
$h^{+}, h^{*}, h^{* 0}: Y \gamma_{3} \times Y \gamma_{4} \rightarrow Y\left(\gamma_{3} \cup \gamma_{4}\right)$ and the morphisms $\delta: Y\left(\gamma_{3} \cup \gamma_{4}\right)$
$\rightarrow Y \gamma_{3}, \delta^{\prime}: Y\left(\gamma_{3} U_{4}\right) \rightarrow Y \gamma_{4}$ satisfy
$8 h^{+}(x, y)=x-\left(y^{+}\right) x$,
$O h^{*}(x, y)=\left(y^{* 0}\right) x$,
$8 h^{* 0}(x, y)=\left(y^{*}\right) x$,
$\delta^{\prime} h^{+}(x, y)=(x+) y-y$,
$0^{\prime} h^{*}(x, y)=\left(x^{*}\right) y$,
$0^{\prime} h^{* 0}(x, y)=\left(x^{* 0}\right) y ;$
(vi) and also

$$
\begin{aligned}
& h^{+}(\delta z, y)=z-\left(y^{+}\right) z \\
& h^{*}(\delta z, y)=\left(y^{* 0}\right) z \\
& h^{* 0}(\delta z, y)=\left(y^{*}\right)_{z} \\
& h^{+}\left(x, \delta^{\prime} z\right)=\left(x+z_{z}-z,\right. \\
& h^{*}\left(x, \delta^{\prime} z\right)=\left(x^{*}\right)_{z} \\
& h^{* 0}\left(x, \delta^{\prime} z\right)=\left(x^{* 0}\right)_{z}
\end{aligned}
$$

for all $x \in \mathrm{Y}_{3}, \mathrm{Y} \in \mathrm{Y} \gamma_{4}, \mathrm{z} \in \mathrm{Y}\left(\boldsymbol{\gamma}_{3} \mathrm{Ur}_{4}\right)$.
A map $\Phi: \underline{Y} \rightarrow \underline{Y}^{\prime}$ of crossed $n$-cubes is a family of structure preserving morphisms $\left\{\phi_{\gamma}: Y y \rightarrow Y^{\prime} \gamma\right\}$.

We shall frequently use the term crossed module instead of crossed 1-cube, and crossed square instead of crossed 2-cube.

For many practical purposes axiom (1.3.10.i) is not combinatorial enough. The fact that $\underline{Y}$ is a $C-n$-action
needs to be expressed in terms of rules governing the various " $h$ " functions (cf. the description of a crossed square of groups, i.e. a crossed 2-cube in the category of groups, which is given in the introduction). In chapter II we will give such a combinatorial description of crossed n-cubes for various particular choices of $C$ and low values of $n$.

PROPOSITION(1.3.11) There is an equivalence of categories, (crossed $n$-cubes in $C$ ) $\propto$ (cat ${ }^{n}$-objects in $C$ ). PROOF The proposition is trivial for $n=0$, so let $n \geqslant 1$. Suppose given a catn-object consisting of the maps $s_{i}, b_{i}: G \rightarrow P_{i}, l \leqslant i \leqslant n$. The maps $s_{i}, l \leqslant i \leqslant n$, constitute an n.s.s.e.s. and thus give rise to a c-n-action Y. Given $\gamma_{1} \subset \gamma_{2} \in \Gamma|n|$ satisfying $\gamma_{2} \backslash \gamma_{1}=\{k\}$ for some $k$, axiom (1.2.1.i) and the commutlivity conditions of a catn-object ensure that the morphism $b_{k}: G \rightarrow P_{k}$ restricts to a map $Y \gamma_{2} \rightarrow Y \gamma_{1}$, and that the resulting diagram is commutative. Thus $\underline{Y}$ is a contravariant functor from $\Gamma|n|$ to $C$. It is readily verified that $\underline{Y}$ is a crossed n-cube.

Conversely, suppose given a crossed n-cube $Y$.
Considering $\underline{Y}$ as a $C$-n-action, we can construct a $C-(n-1)$-action RY (see above). This construction can be extended to give a contravariant functor RY: $\Gamma|n-1| \rightarrow C$ since, for each $\gamma_{1} \subset \gamma_{2} \in \Gamma|n-1|$, the maps $\delta_{n}: Y \iota_{n} \gamma_{2} \rightarrow$ $Y \ln _{n} \mathcal{I}^{\prime}, 80: Y \operatorname{lo\gamma }_{2} \rightarrow Y \operatorname{lor}_{1}$ in the image of $Y$ give rise to a morphism RYy $y_{2} \rightarrow R Y y_{1},(x, y) \rightarrow\left(\delta_{n} x, \delta_{0 Y}\right)$.

CLAIM The functor RY is a crossed ( $n-1$ )-cube.
The verification of this claim is routine and we omit
it. Note that by iterating $m$ times, $1 \leqslant m \leqslant n$, the construction $\underline{R}$ we obtain a crossed ( $n-m$ )-cube $\underline{R}^{m}{ }_{Y}$ say.

For $1 \leqslant i \leqslant n$ let $\theta_{i} \prime:\{1, \ldots, n\} \rightarrow\{1, \ldots, n\}$ be the set map which interchanges $i$ with $n$ and leaves the other elements of $\{1, \ldots, n\}$ unchanged. This map $\theta_{i}$ induces an endofunctor $\theta_{i}: \Gamma|n| \rightarrow \Gamma|n|$. The composite functor $\underline{y} \theta_{i}$ is clearly a crossed n-cube. We can thus construct the crossed l-cube $R^{n-1} Y_{i}$, or the equivalent catl-object $s_{i}, b_{i}: R^{n-1} Y \theta_{i}\{1\} \underline{x} R^{n-1} Y \theta_{i} \phi \rightarrow R^{n-1} Y \theta_{i} \phi$. For convenience let $G_{i}=R^{n-1} Y_{1}\{1\} \underline{X} R^{n-1} Y \theta_{i} \phi$, and let $P_{i}=R^{n-1} Y \theta_{i} \phi$. We shall show how the n such catlobjects combine to form a cat ${ }^{n}$-object.

For each $\gamma \in$ 「|nl there is a canonical inclusion $\mathrm{Y} \boldsymbol{\gamma} \rightarrow$ $G_{i}$. There is also an isomorphism $\psi_{i}: G_{n} \rightarrow G_{i}$ whose restriction to $\mathrm{Y} \gamma \rightarrow \mathrm{Y} \gamma$ is the identity. Let $\mathrm{P}_{\mathrm{i}}{ }^{\prime}=\psi_{i}{ }^{-1} \mathrm{P}_{\mathrm{i}}$ and let $s_{i}{ }^{\prime}, b_{i} \prime: G_{n} \rightarrow P_{i}$ ' be the composite maps $s_{i}{ }^{\prime}=$ $\psi_{i}{ }^{-1} s_{i_{i}}, b_{i}{ }^{\prime}=\psi_{i}-l_{b_{i}} \psi_{i}$. For each $x \in Y \gamma$ we have $s_{i}{ }^{\prime} s_{j}{ }^{\prime} x$
 Hence the maps $s_{i}{ }^{\prime}, b_{i} \prime: G \rightarrow P_{i}, 1 \leqslant i \leqslant n$, constitute $a$ cat ${ }^{n}$-object.

We have thus given a correspondence between catn-objects and crossed n-cubes. This correspondence gives rise to an equivalence of categories, completing the proof. $\nabla$

As an illustration of the equivalence between crossed n-cubes and cat ${ }^{n}$-objects we give the following EXAMPLE(1.3.12) Suppose given a crossed square in groups (as described in the introduction):

with $h: M \times N \rightarrow[$. Form the semi-direct products $L \underline{X} M, N X P$ and define a group action of $N X P$ on $L X M$ by $(n, p)(1, m)=$ $\left({ }^{n}\left(P_{1}\right) h\left(P_{m, n}\right)^{-1}, P_{m}\right)$. The maps $s_{2}, b_{2}:\left(L_{X} \underline{M}\right) \underline{x}(N \underline{x} P) \rightarrow N \underline{P}$ given respectively by $(1, m, n, p) \rightarrow(n, p),(l, m, n, p) \rightarrow$ ( $\lambda 1 \mathrm{~m}_{\mathrm{n}},(\delta \mathrm{m}) \mathrm{p}$ ) constitute a catl${ }^{l}$ group.

Now the above diagram of groups together with the function $h^{\prime}: N \times M \rightarrow L, h^{\prime}(n, m)=h(m, n)^{-1}$, is also a crossed square. Thus we can similarly form a cat ${ }^{l}$ group $s_{1}, b_{1}:(L \underline{X} N) \underline{x}(M \times P) \rightarrow M \underline{P}$.

Let $G_{2}=(L \underline{X} M) \underline{X}(N \underline{X} P), G_{1}=(L \underline{X} N) \underline{X}(M \underline{X} P), P_{2}=N \underline{X} P, P_{1}^{\prime}=$ MxP (we consider $P_{1}, P_{2}$ ' as subgroups of $G_{2}$ ). There is an isomorphism $W_{1}: G_{2} \rightarrow G_{1},(1, m, n, p) \rightarrow(\ln (m, n), n, m, p)$. Let $s_{1} \prime b_{1}: G_{2} \rightarrow P_{1}$ ' be the composite maps $s_{1} \prime=\psi_{1}-l_{s_{1}} W_{1}, b_{1}{ }^{\prime}$ $=\psi_{1}{ }^{-l_{b_{1}} \psi_{1}}$. The four maps $s_{1}, b_{1},^{\prime}, s_{2}, b_{2}$ constitute a cat ${ }^{2}$-group.
4. ON n-PUSHOUTS OF CROSSED n-CUBES

The computation of colimits of crossed $n$-cubes (in the category of groups) is of relevance to the higher dimensional van Kampen theorem of [B-L]. In this section we look at one particular type of colimit which, following Brown and Loday, we call an "n-pushout".

The following definitions will appear in [B-L2] (see also [Wa]). Let $C^{\prime}$ be an arbitrary category. An n-cube in
$C^{\prime}$ is a contravariant functor $Y^{\prime}$ from 「Inl to C'. An n-corner in $C^{\prime}$ is a contravariant functor $\underline{y}$ from $\Gamma \ln \| \emptyset$ to $C^{\prime}$. (Here $\Gamma|n| \backslash \emptyset$ denotes the poset $\Gamma|n|$ with the empty set removed.) Suppose colim $\underline{Y}$ of such a corner exists. Then $\underline{Y}$ and the natural transformation $\underline{Y} \rightarrow \operatorname{colim} \underline{Y}$ define an $n$-cube $\underline{Y}^{\prime}$ in $C^{\prime}$. The object colim $\underline{Y}$ is called the pushout of the corner $\underline{Y}$, and $\underline{Y}^{\prime}$ is called an n-pushout in $C^{\prime}$.

It will be convenient to have some functors from crossed $n$-cubes to crossed $m$-cubes $m \neq n$.

For $1 \leqslant i \leqslant n$ let $|n|_{i}=\{1,2, \ldots, n\} \backslash\{i\}$ and let $\left[|n|_{i}\right.$ be the poset consisting of the subsets of $|n| i$. The map $|n-1| \rightarrow|n|_{i}$ which takes $j$ to $j$ for $1 \leqslant j \leqslant i-1$, and $j$ to $j+1$ for $i \leqslant j \leqslant n-1$, induces an isomorphism of posets $\pi: \Gamma|n-1| \rightarrow \Gamma|n|_{i} . \quad$ There $i s$ a canonical inclusion $\zeta: \Gamma|n|_{i} \rightarrow$「|n|. Given a crossed n-cube $\underline{Y}: \Gamma|n| \rightarrow C$, we can restrict to obtain a crossed $(n-1)$-cube $a^{i} \underline{Y}=\underline{Y} \zeta: \Gamma|n-1| \rightarrow C$.

Given a crossed ( $n-1$ )-cube $\underline{Y}$ in $C$, let $\tau^{i} \underline{Y}$ be the unique crossed $n$-cube such that $\partial^{i} \tau^{i} \underline{Y}=\underline{Y}$ and $Y \gamma=0$ whenever i $\epsilon$ $y \in \Gamma|n|$. Let $\xi^{i} \underline{y}$ be the unique crossed $n$-cube with $a^{i}{ }^{i} \underline{Y}$ $=\underline{Y}$ and $Y y=Y(\gamma \backslash\{i\})$ whenever $i \in \gamma \in \Gamma|n|$ and where: the $\operatorname{map} Y \gamma \rightarrow Y(\gamma \backslash\{i\})$ is the identity; and whenever $i \in \gamma \mathcal{I}^{\mathrm{C}}$ $\gamma_{2} \in\left\lceil|n|\right.$ the map $Y \gamma_{2} \rightarrow Y \gamma_{1}$ is the same as the map $Y\left(\gamma_{2} \backslash\{i\}\right) \rightarrow Y\left(\gamma_{1} \backslash\{i\}\right)$.

The constructions $a^{i}, T_{i}^{i} \xi^{i}$ are functorial.

Let $\underline{Y}$ be an $n$-corner in the category of crossed n-cubes. So for each $\beta \in \Gamma|n| \backslash$ we have a crossed n-cube
$\underline{Y}$. For $\gamma \in \Gamma|n|$ we shall denote by $Y \beta \gamma$ the image of $\gamma$ under $\underline{Y} \beta$.

Let us suppose that $Y \beta y$ is trivial whenever the intersection $\beta \cap \gamma$ is non empty. For all $\beta_{1}, \beta_{2} \in \Gamma|n|$ such that the intersections $\beta_{1} \cap \gamma, \beta_{2} \cap \gamma$ are both empty, let us suppose that $Y \beta_{1} \gamma=Y \beta_{2} \gamma$ and that the maps from $Y \beta_{1} \gamma$ are the same as those from $\mathrm{Y}_{2} \gamma$.

Let us also suppose that all maps of crossed n-cubes in the corner $\underline{Y}$ are the canonical inclusions. Then

PROPOSITION(1.4.1) Let $\underline{Y}: \Gamma|n| \rightarrow C$ be a crossed n-cube. The following conditions on $\underline{Y}$ are equivalent:
(i) $\underline{Y}=$ colim $\underline{Y}$ is the pushout of the corner $\underline{Y}$;
(ii) for $1 \leqslant i \leqslant n$ we have
(a) $a\{i\} \underline{Y}=a\{i\} \underline{Y}\{i\}$ and,
(b) given any other crossed n-cube $\underline{Y}^{\prime}$ satisfying a\{i\}Y' = $a\{i\} Y\{i\}$, the identity maps $Y \gamma \rightarrow Y^{\prime} \gamma$, for $|n| \neq \gamma \in \Gamma|n|$, extend uniquely to a map of crossed $n$-cubes $\underline{Y} \rightarrow Y^{\prime}$. PROOF FOR $1 \leqslant i \leqslant n$ the functor $a^{i}$ has as left and right adjoints the functors $\tau^{i}, \xi^{i}$. Hence $a^{i}$ preserves colimits, and it follows that (i) implies (ii.a). The rest of the proof is trivial. $\nabla$

For $C$ the category of groups and $n=2$, this proposition is given in [B-L]. Our proof is just a generalisation of the one given there.

To illustrate the proposition, take $n=2$. Then the following two statements are equivalent:
(i) the diagram of crossed squares in C (i.e. crossed

2-cubes in C)

| 0 | $\rightarrow 0$ |  | 0 | $\rightarrow$ | N |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\downarrow$ | $\downarrow$ | $\rightarrow{ }^{\text {l }}$ | $\downarrow$ |  | ${ }^{1} 81$ |
| 0 | $\rightarrow \mathrm{P}$ |  | 0 | $\rightarrow$ | $P$ |
|  | $\iota^{\prime}{ }^{\prime}$ |  |  | 1 |  |
| 0 | $\rightarrow 0$ |  | L | $\rightarrow$ | $N^{\prime}$ |
| $\downarrow$ | $\downarrow$ | $\rightarrow$ | $\downarrow$ |  | ${ }^{1} 8$ |
| M | $\rightarrow{ }^{8} \mathrm{P}$ |  | $M^{\prime}$ | - | $P^{\prime}$ |

in which $6,6^{\prime}$ are the canonical inclusions, is a pushout; (ii) $M^{\prime}=M, N^{\prime}=N, P^{\prime}=P$ and, given any crossed square

$$
\begin{aligned}
& L^{\prime} \rightarrow \mathrm{N} \\
& \downarrow \\
& { }^{\prime} \delta^{\prime} \\
& \mathrm{M} \rightarrow \mathrm{P}
\end{aligned}
$$

there is a unique morphism $\alpha: L \rightarrow L^{\prime}$ such that the quadruple $\left(\alpha, l_{M}, l_{N}, l_{P}\right)$ is a map of crossed squares.
5. n-SIMPLICIAL OBJECTS IN C

We shall now show that catn-objects in $C$ are equivalent to certain types of $n$-fold simplicial objects in $C$, thus highlighting the fact that simplicial techniques are applicable to the theory of higher dimensional crossed modules.

Let $C^{\prime}$ be any category. Recall that a simplicial object $K_{\#}$ in $C^{\prime}$ consists of a family $K_{n}, n \geqslant 0$, of objects in $C^{\prime}$ and morphisms $d_{i}: K_{n} \rightarrow K_{n-1}, v_{i}: K_{n} \rightarrow K_{n+1}, 0 \leqslant i \leqslant n$, satisfying:
(i) $\quad d_{i} d_{j}=d_{j-1} d_{i} \quad 0 \leqslant i<j \leqslant n+1$,
(ii) $\quad v_{i} v_{j}=v_{j+1} v_{i} \quad 0 \leqslant i \leqslant j \leqslant n-l_{\text {, }}$
(iii) $d_{i} v_{j}=v_{j-1} d_{i} \quad i<j$,
$\mathrm{d}_{\mathrm{i}} \mathrm{v}_{\mathrm{j}}=\mathrm{v}_{\mathrm{j}} \mathrm{d}_{\mathrm{i}-1} \quad \mathrm{i}>j+\mathrm{l}_{\mathrm{r}}$
$d_{i} v_{j}=$ identity $i=j$ or $j+1$.
A map $\psi K_{\#} \rightarrow K_{\#}$ 'of simplicial objects is a family of structure preserving morphisms $\psi_{n}: K_{n} \rightarrow K_{n}$. Suppose that the category $C^{\prime}$ has kernels. Then recall that the normal complex of a simplicial object $K_{\#}$ is obtained by taking for each $n$ the subobject ${ }_{i} n_{1}$ ker $d_{i}$ of $K_{n}$ i the restriction of $d_{0}$ to this subobject is the differential of the complex. The complex is said to be of length $r$ if it is non-trivial in dimension $r$ and trivial in each dimension greater than $r$. We shall denote by SMPlC' the category of simplicial objects in $C^{\prime}$ whose normal complexes are of length 1. Inductively we define $\operatorname{SMP}^{n^{\prime}}{ }^{\prime}=\operatorname{SMP}^{1}\left(\operatorname{SMP}^{n-1} \mathbf{C l}^{\prime}\right)$. An object in $\operatorname{SMP}^{\prime}{ }^{\prime}$ ' will be called an n-simplicial object in $C^{\prime}$ whose normal complexes are of length 1.

PROPOSITION(1.5.1) There is an equivalence of categories, (n-simplicial objects in $C$ whose normal complexes are of length 1) $\propto$ (catn-objects in $C$ ).

PROOF We shall first consider the case $n=1$. Suppose given a simplicial object $K_{\#}$ in $C$ whose normal complex is of length 1. By restricting to dimensions 0,1 we obtain an
inclusion $K_{0} \rightarrow K_{1}$ and two maps $d_{0}, d_{1}: K_{1} \rightarrow K_{0}$. Axiom (1.2.1.i) is clearly satisfied. In order to verify axiom (l.2.l.ii) let $x \in \operatorname{ker} \mathrm{~d}_{\mathrm{l}}, \mathrm{y} \in \mathrm{ker} \mathrm{d}_{0}$. Then

$$
\begin{aligned}
& x+y-x-y=d_{0}\left(v_{0} x+v_{O Y}-v_{1 Y}-v_{0 x}+v_{1 Y}-v_{O Y}\right), \\
& x * y=d_{0}\left(v_{0} x *\left(v_{O Y}-v_{1 Y}\right)\right), \\
& y * x=d_{0}\left(\left(v_{O Y}-v_{1 Y}\right) * v_{O X}\right) .
\end{aligned}
$$

It is now a simple matter to check that the elements $x+y-x-y, x * y, y * x$ all lie in the $\int_{\text {linge of thection }}^{\text {in }}$ ker $\mathrm{d}_{1} \cap \mathrm{ker} \mathrm{d}_{2}$ and are hence trivial. Thus $\mathrm{d}_{\mathrm{O}}, \mathrm{d}_{1}: \mathrm{K}_{1} \rightarrow \mathrm{~K}_{0}$ is a catl-object.

Conversely suppose given a catl-object $s, b: G \rightarrow P$. By taking the nerve of the associated category we obtain a simplicial object $\operatorname{Ner}(G, P) \#$. That is $\operatorname{Ner}(G, P) O=P$, $\operatorname{Ner}(G, P) \downarrow=G, \operatorname{Ner}(G, P)_{n}=\left\{\left(g_{1}, \ldots, g_{n}\right) \in G^{n}: b g_{i}=\right.$ $\left.3 g_{i+1}\right\}$, and the maps are:

```
di}(\mp@subsup{g}{1}{\prime},\ldots,gn)
\begin{tabular}{ll}
\(b g_{1}\), & \(i=0, n=1\) \\
s \(g_{1}\), & \(i=1, n=1\) \\
\(\left(g_{2}, \ldots, g_{n}\right)\), & \(i=1, n\rangle 1\) \\
\(\left(g_{1}, \ldots, g_{i-1}, g_{i+1}-b g_{i}+g_{i}, g_{i+2}, \ldots, g_{n}\right)\), & \(1<i<n\) \\
\(\left(g_{1}, \ldots g_{n-1}\right)\), & \(i=n, n\rangle 1\)
\end{tabular}
vi
    (sgl,g1,...,gn), i=0
    (g1,\ldots..gi,bgi,gi+1,\ldots,gn), i}\geqslantn
```

(Recall that $g_{i+1}-\operatorname{bg}_{i}+g_{i}$ is the category composition $\left.g_{i} \circ g_{i+1}.\right)$ The normal complex of $\operatorname{Ner}(G, P) \#$ is of length 1. We now consider the case $n \geqslant 1$. Suppose given an n-simplicial object in $C$ whose normal complexes are of length 1. By restricting to dimensions 0,1 we have, for
"each of the $n$ directions", an associated catl-object. The n such catl-objects clearly satisfy the commutativity conditions of definition (1.2.2) and thus constitute a cat ${ }^{n}$-object.

Conversely suppose given a cat ${ }^{\text {n-object consisting of the }}$ maps $s_{i}, b_{i}: G \rightarrow P_{i}, l \leqslant i \leqslant n$. By taking the nerve of the category associated to the catlobject $s_{1}, b_{1}: G \rightarrow P_{1}$ we obtain a simplicial object $\operatorname{Ner}\left(G, P_{1}\right) \#$ whose normal complex is of length 1 . Now the catlobject $\mathrm{s}_{2}, \mathrm{~b}_{2},: \mathrm{G} \rightarrow \mathrm{P}_{2}$ induces a category structure on $\operatorname{Ner}\left(G, P_{1}\right) \#$. By taking the nerve of this induced category structure we obtain a 2 -simplicial object whose normal complexes are of length 1 . Iterating the process we obtain an n-simplicial object whose normal complexes are of length 1.

This correspondence between cat ${ }^{n}$-objects in C and n-simplicial objects in $C$ whose normal complexes are of length 1 , gives rise to an equivalence of categories. $\nabla$

Proposition (1.5.1), for $C$ the category of groups and $n$ $=1$, is given in [L].
6. n-FOLD CROSSED MODULES

A crossed n-cube is, in some sense, equivalent to an "n-fold crossed module", i.e. a "crossed module in the category of ( $n-1$ )-fold crossed modules". A precise definition of an " $n$-fold crossed module" is desirable since it will enable inductive arguments to be applied to crossed n-cubes.

In this section we give a definition of an "n-fold
crossed module in $C^{\prime \prime}$, and we outline a proof that such an entity is equivalent to an $n$-fold category internal to $C$.

Let $C^{\prime}$ be an arbitrary category with kernels and a null object 0 .

DEFINITION(1.6.1) A 1-fold crossed module in $C$ ' consists of:
(a) an object $E$ in $C^{\prime}$ with two subobjects $M, P$;
(b) four morphisms $a: M \rightarrow P, s: E \rightarrow P, \nu: E \rightarrow P \times P, \eta: M \times M \rightarrow E$ such that;
(i) $\quad M=\operatorname{ker} s$ and the restriction of $s$ to $P$ is the identity;
(ii) the diagram

| M | $\rightarrow$ | E | $\leftarrow$ | P |
| :---: | :---: | :---: | :---: | :---: |
| $a^{1}$ |  | ${ }^{\dagger} \nu$ |  | I |
|  | $(1 \mathrm{p}, 0)$ |  | (1p,1p) |  |
| P | $\rightarrow$ | PXP | $\cdots$ | P |

is commutative and;
(iii) the diagram

|  | $(1 \mathrm{M}, 0)$ |  | ( $1_{M}, 1_{M}$ ) |  |
| :---: | :---: | :---: | :---: | :---: |
| M | $\rightarrow$ | M $\times$ M | - | M |
| \\| |  | ${ }^{1} \eta$ |  | ${ }^{\prime}$ |
| M | $\rightarrow$ | E | - | P |

is commutative.
A map of l-fold crossed modules is a structure preserving morphism $\psi: E \rightarrow E^{\prime}$.

Essentially this definition is given in [Pl] for $C$ ' the category of groups, and in [AG] for $C '$ a category of interest.

PROPOSITION(1.6.2) There is an equivalence of categories, (crossed modules in $C$ ) $\propto$ (1-fold crossed modules in $C$ ). PROOF Suppose given a crossed module $d: M \rightarrow P$ in C. Then set $E=M \underline{X} P$, let $s: M \underline{P} P \rightarrow P$ be the map $s(m, p)=p$, let $v: M \underline{P}$ $\rightarrow P \times P$ be the $\operatorname{map} \nu(m, p)=(a m p, p)$, and let $\eta: M \times M \rightarrow M \times P$ be the map $\nu\left(m, m^{\prime}\right)=\left(m m^{\prime-1}, \partial m^{\prime}\right)$.

Conversely suppose given a l-fold crossed module. Then there is a map $a: M \rightarrow P$, and the split short exact sequence $M \rightarrow E \pm P$ gives rise to a C-action of $P$ on $M$.
[t is readily seen that axioms (1.6.1,ii,iii) are equivalent, respectively, to axioms (1.3.3,i,ii). So we have a correspondence between crossed modules in C and l-fold crossed modules in $C$; this gives rise to the required equivalence of categories. $\nabla$

DEFINITION(1.6.3) An n-fold crossed module in $C$ ', $n>1$, is a l-fold crossed module in the category of ( $n-1$ )-fold crossed modules in C.

In order to prove the $n$-dimensional version of proposition (1.6.2) let us digress for a moment, and recall some results on algebraic theories. A general reference for this digression is [S]

The category $C$ of $\Omega$-groups is an algebraic category and
so, for any category B with a terminal object, we can form the category $C(B)$ of $\Omega$-groups over $B$. For example, when $C$ is the category of groups and $B$ is the category of pointed topological spaces, then $C(B)$ is the category of topological groups.
Suppose $B$ has kernels and a nall olject.
LLet us denote by $\mathrm{x}^{\mathrm{n}}(\mathrm{C}(\mathrm{B})), \mathrm{CT}^{\mathrm{n}}(\mathrm{C}(\mathrm{B}))$ respectively the category of $n$-fold crossed modules in $C(B)$ and the category of $n$-fold categories internal to $\mathrm{C}(\mathrm{B})$. The objects of both of these categories are many sorted, partial algebraic theories over B.

Let $A(B)$ be an arbitrary category of many sorted, partial algebraic theories over $B$. Let $\hat{B}$ be the category whose objects are the functors from $\mathrm{B}^{\circ p p}$ to the category of sets, and whose morphisms are the natural transformations of such functors. The Yoneda embedding induces an embedding $L: A(B) \rightarrow A(\hat{B})$.

There is a canonical equivalence $K: A(\hat{B}) \propto(A(s e t s))^{B O p p}$, where (A(sets)) $\mathrm{B}^{\circ p p}$ is the category of functors from Bopp to the category $A(s e t s)$, and natural transformations of such functors.

We can now prove

```
PROPOSITION(1.6.4) There is an equivalence of categories,
(n-fold categories in C(B)) & (n-fold crossed modules in
C(B)).
PROOF First let us consider the case n = l. There is a
diagram of functors
```

$$
\begin{aligned}
& \operatorname{CT}^{\mathbf{l}}(\mathrm{C}(\mathrm{~B})) \rightarrow^{\iota} \operatorname{CT}^{\mathbf{l}}(\mathrm{C}(\hat{\mathrm{~B}})) \rightarrow^{\mathrm{k}}\left(\mathrm{CT}^{\mathbf{l}}(\mathrm{C})\right)^{\mathrm{BOPp}} \\
& { }^{1} \lambda \\
& \mathrm{X}^{\mathbf{l}}(\mathrm{C}(\mathrm{~B})) \rightarrow \rightarrow^{\prime} \mathrm{X}^{\mathbf{l}}(\mathrm{C}(\hat{B})) \quad \rightarrow^{\mathrm{K}^{\prime}}\left(\mathrm{X}^{\mathbf{l}}(\mathrm{C})\right)^{\text {Bopp }}
\end{aligned}
$$

in which $\left\llcorner, L^{\prime}\right.$ are the Yoneda embeddings, $k, k^{\prime}$ are the canonical equivalences, and $\lambda$ is the equivalence induced by the equivalence $\mathrm{CT}^{\mathrm{l}}(\mathrm{C}) \propto \mathrm{x}^{\mathbf{l}}(\mathrm{C})$. The image of the composite functor $\lambda k \iota$ is equivalent to the image of the composite functor $\kappa^{\prime}{ }^{\prime}$ '. This proves the proposition for $\mathrm{n}=1$.

Let us assume that the proposition has been proved in dimension $n-1$. Then we have a sequence of equivalences

$$
\begin{aligned}
\operatorname{CT}^{n}(C(B)) & =\operatorname{CT}^{1}\left(\operatorname{CT}^{n-1}(C(B))\right) \\
& \left.\propto \operatorname{CT}^{1}\left(\operatorname{CT}^{n-1}(B)\right)\right) \\
& \propto x^{1}\left(C\left(C^{n-1}(B)\right)\right) \\
& \propto x^{1}\left(C^{n-1}(C(B))\right) \\
& \propto x^{1}\left(x^{n-1}(C(B))\right. \\
& =x^{n}(C(B)) .
\end{aligned}
$$

This proves the proposition in dimension n. $\nabla$

Taking $B$ to be the category of pointed sets gives us an equivalence between $n$-fold categories in $C$ and $n$-fold crossed modules in C.

## EXAMPLES OF CROSSED n-CUBES

O. INTRODUCTION

In the last chapter we introduced the notion of a crossed $n$-cube in an arbitrary category $C$ of $\Omega$-groups. Because of the generality in which we worked, the combinatorial nature of a crossed $n$-cube was lost. In this chapter we shall give, for specific choices of $C$, a more combinatorial version of certain low dimensional crossed n-cubes. In $\$ 1,2,3,4$ we shall take $C$ respectively to be the category of groups, Lie algebras, commutative algebras, and associative algebras.

## 1. CROSSED n-CUBES IN GROUPS

Let C be the category of groups. All groups will be written multiplicatively with identity e. Terms such as "C-n-action", "crossed n-cube in C" will be replaced by "group n-action", "crossed n-cube in groups" etc.

Let $M, P$ be groups. A group action of $P$ on $M$ (see example 1.3.2)) is a map $P \times M \rightarrow M,(p, m) \rightarrow P_{m}$ satisfying: $e_{m}=m$, $P\left(m m^{\prime}\right)=\left(P_{m}\right)\left(P_{m}\right)$,
$p p^{\prime}(m)=p\left(p^{\prime} m\right)$, for all $m, m^{\prime} \in M, p, p^{\prime} \in P$.

A crossed module in groups (see definition (1.3.3)) is a group homomorphism $a: M \rightarrow P$ together with a group action of

P on M which satisfies:
(i) $a\left(p_{m}\right)=p(\partial m) p^{-1}$,
(ii) $\partial m_{m}{ }^{\prime}=m m^{\prime} m^{-1}$,
for all $m, m^{\prime} \in M, p \in P$.

Suppose given: four groups L, M,N,P; group actions of $P$ on $L, M, N$, of $M$ on $L$, and of $N$ on $L$; and a function $h: M \times N \rightarrow$ L. Then

PROPOSITION(2.1.1) This structure is a group 2-action (see definition (1.3.8)) if and only if:
(i) $p_{m}\left(p_{1}\right)=p\left(m_{1}\right)$,
$P_{n(1)}=P\left(n_{1}\right) ;$
(ii) $h\left(m m^{\prime}, n\right)=m_{h}\left(m^{\prime}, n\right) h(m, n)$,
$h\left(m, n n^{\prime}\right)=h(m, n) n_{h\left(m, n^{\prime}\right)} ;$
(iii) $\mathrm{Ph}_{\mathrm{h}}(\mathrm{m}, \mathrm{n})=\mathrm{h}\left(\mathrm{P}_{\mathrm{m}}, \mathrm{P}_{\mathrm{n}}\right)$;

for all $l \in L, m, m^{\prime} \in M, n, n^{\prime} \in N, p \in P$.
PROOF Strictly speaking a group 2-action should involve three functions $h^{+}, h^{*}, h^{* 0}: M \times N \rightarrow L$. However $h^{*}, h^{* \circ}$ are always trivial. Form the semi-direct products LXM, NXP and let $\alpha:(N \underline{X} P) \times\left(L_{X M}\right) \rightarrow(L \underline{X} M)$ be the function

$$
\alpha(n, p, 1, m)=\left(n^{n}\left(p_{1}\right) h\left(p_{m}, n\right)^{-1}, p_{m}\right)
$$

It follows from proposition (1.3.9) that the structure is a group 2-action if $\alpha$ is a group action of NXP on LXM. That is, if

$$
\begin{aligned}
& \alpha(e, e, l, m)=(l, m), \\
& \alpha\left((n, p)\left(n^{\prime}, p^{\prime}\right), l, m\right)=\alpha\left(n, p, \alpha\left(n^{\prime}, p^{\prime}, 1, m\right)\right), \\
& \alpha\left(n, p,(1, m)\left(l^{\prime}, m^{\prime}\right)\right)=\alpha(n, p, 1, m) \alpha\left(n, p, l^{\prime}, m^{\prime}\right) .
\end{aligned}
$$

This is verified in Appendix [I, verification (1). commutation, and by interpreting each group action as congugation, we see that every group 2-action satisfies these rules. $\nabla$

Now suppose given a commutative diagram of groups

$$
\begin{array}{rlll}
L & \rightarrow^{\lambda} & N \\
\lambda^{\prime}+ & & \downarrow & \delta^{\prime} \\
M & \rightarrow \sigma^{\delta} & P
\end{array}
$$

with group actions of $P$ on $L, M$ and $N$ (hence there are group actions of $M$ on $L$ and $N$ via $B$, and group actions of $N$ on $L$ and $M$ via $\left.\delta^{\prime}\right)$, and a function $h: M \times N \rightarrow L$. Then PROPOSITION(2.1.2) This structure is a crossed square in groups (see definition (1.3.10)) if and only if:
(i) each of the maps $\lambda, \lambda^{\prime}, 0^{\prime} \delta^{\prime}$ and the composite $\delta^{\prime} \lambda$ is a crossed module;
(ii) the maps $\lambda, \lambda$ ' preserve the actions of $P$;
(iii) $h\left(m m^{\prime}, n\right)=m_{h\left(m^{\prime}, n\right) h(m, n), ~}^{n}$,
$h\left(m, n n^{\prime}\right)=h(m, n) n_{h}\left(m, n^{\prime}\right) ;$
(iv) $P_{h}(m, n)=h\left(P_{m}, P_{n}\right)$;
(v) $\lambda h(m, n)=m_{n n-1}$,

$$
\lambda \cdot h(m, n)=m n_{m}-1
$$

(vi) $h(m, \lambda I)=m_{11}-1$,
$h(\lambda \cdot 1, n)=1^{n-1}$;
for all $l \in L, m, m^{\prime} \in M, n, n^{\prime} \in \mathbb{N}, p \in P$.
PROOF From definition (1.3.10) and proposition (2.1.1) we have that the above structure is a crossed square if and
only if rules (2.1.1,i,iv), (2.1.2,i to vi) hold. But rules. (2.I.l,i,iv) are redundant (see Appendix II, verifications (2) and (3)) and so we are done. $\nabla$

Suppose now we have a commutative diagram of groups

in which there is a group action of $S$ on each of the other seven groups (hence the eight groups act on each other via the action of $S$ ), and there are six functions

$$
\begin{aligned}
& h: Q \times L \rightarrow K, \\
& h: P \times M \rightarrow K, \\
& h: N \times R \rightarrow K, \\
& h: P \times R \rightarrow L_{r} \\
& h: Q \times R \rightarrow M, \\
& h: P \times Q \rightarrow N .
\end{aligned}
$$

Then
groups (see definition 1.3.10)) if and only if:
(i) each of the nine squares

is a crossed square; for the last three squares the functions $h: L \times M \rightarrow K, h: N \times L \rightarrow K, h: N \times M \rightarrow K$ are respectively given by $h(1, m)=h\left(\nu_{p} l, m\right), h(n, l)=h\left(n, \nu_{R} l\right), h(n, m)=$ $h\left(n, \nu_{R} m\right)$;
(ii) $h\left(\left(\nu_{\mathrm{p}} n\right)\left(\nu_{\mathrm{P}} \mathrm{l}\right), \mathrm{m}\right) \mathrm{h}\left(\left(\nu_{\mathrm{Q}} \mathrm{m}\right)\left(\nu_{\mathrm{Q}} \mathrm{n}\right), 1\right)=\mathrm{h}\left(\mathrm{n},\left(\nu_{\mathrm{R}} \mathrm{l}\right)\left(\nu_{\mathrm{R}} \mathrm{m}\right)\right)$;

(iv) $\lambda_{L} h(p, m)=h\left(p, \nu_{R} m\right)$,
$\lambda_{L} h(n, r)=h\left(\nu_{P} n, r\right)$, $\lambda_{M h}(q, 1)=h\left(q, \nu_{R} 1\right)$, $\lambda_{M h}(n, r)=h\left(\nu_{Q} n, r\right)$, $\lambda_{\mathrm{N}} h(\mathrm{p}, \mathrm{m})=h\left(\mathrm{p}, \nu_{\mathrm{Q}} \mathrm{m}\right)$, $\lambda_{\mathrm{N}} h(q, 1)=h\left(\nu_{p} 1, q\right)^{-1} ;$
(v) $h\left(\nu_{Q} m, 1\right)=h\left(\nu_{p} 1, m\right)^{-1}$;
$h\left(n, \nu_{R} l\right)=h\left(\nu_{Q} n, 1\right) ;$
$h\left(n, \nu_{R} m\right)=h(\nu p n, m) ;$
for all $1 \in \mathrm{~L}, \mathrm{~m} \in \mathrm{M}, \mathrm{n} \in \mathrm{N}, \mathrm{p} \in \mathrm{P}, \mathrm{q} \in \mathrm{Q}, \mathrm{r} \in \mathrm{R}$.
PROOF A crossed 3-cube in groups is defined in terms of a contravariant functor $\underline{\underline{y}}$ from 「l3l to groups, so it will
help to rewrite the above cubical diagram:


Note that, in addition to the six "h" functions listed above, a crossed 3-cube involves three functions $h: L \times M \rightarrow K$, $h: N \times L \rightarrow K ; N \times M \rightarrow K$. These extra functions are defined as in rule (2.1.3.1)

The rules (2.l.3.i to $v$ ) are clearly necessary if the structure is to be a crossed 3-cube. The proof that these rules are sufficient to give us a crossed 3-cube boils down that to a proof $/$ the rules ensure the existence of a group 3-action.

Form the semi-direct products KXN, LXP, MXQ, RXS. Given an arbitrary element ( $u, v, x, y$ ) in any one of the direct
 $(L \underline{X} P) \times(K \underline{X} N),(M \underline{Q}) \times(K \underline{X} N)$, we obtain five group actions by setting

$$
(u, v)(x, y)=\left(u v_{x} h\left(v_{Y}, u\right)^{-1}, v_{Y}\right)
$$

(Here, and in future, we write $u v_{x}$ instead of $u\left(v_{x}\right)$. This
abuse of notation is unlikely to calase confusion.) Let $h^{\prime}:(L \underline{X} P) \times(M \times Q) \rightarrow K \times N$ be the function $h^{\prime}(1, p, m, q)=(k, n)$ where
$k=$
 and

$$
n=h(p, q)
$$

Then, by proposition (1.3.9), we have to check that the folur semi-direct products together with the given group actions and function $h^{\prime}$, constitute a group 2-action. By proposition (2.1.1) we see that we must check
$(r, s)(m, q)((r, s)(k, n))=(r, s)((m, q)(k, n))$,
$(r, s)(1, p)((r, s)(k, n))=(r, s)((1, p)(k, n))$,
$h^{\prime}\left((1, p)\left(l^{\prime}, p^{\prime}\right), m, q\right)=(1, p) h^{\prime}\left(l^{\prime}, p^{\prime}, m, q\right) h^{\prime}(1, p, m, q)$,
$h^{\prime}\left(1, p,(m, q)\left(m^{\prime}, q^{\prime}\right)\right)=h^{\prime}(1, p, m, q)(m, q) h^{\prime}\left(1, p, m^{\prime}, q^{\prime}\right)$,
$(r, s) h^{\prime}(1, p, m, q)=h^{\prime}((r, s)(1, p),(r, s)(m, q))$,
$(1, p)((m, q)(k, n)) h^{\prime}(1, p, m, q)=$

$$
n^{\prime}(1, p, m, q)(m, q)((1, p)(k, n))
$$

These equations are checked in Appendix $I$ I, verifications (4),(5),(6),(7). $\quad$ -
2. CROSSED n-CUBES IN L[E ALGEBRAS

Fix a commutative ring A (with unit). Recall that a Lie algebra over $A$ is an A-module $M$ together with an A-bilinear map $[]:, M \times M \rightarrow M$ which satisfies

$$
\begin{aligned}
& {[x, x]=0,} \\
& {[[x, y], z]+[[y, z], x]+[[z, x], y]=0,}
\end{aligned}
$$

for all $x, y, z \in M$. We shall assume all Lie algebras to be over A: Let $C$ be the category of Lie algebras. Terms such "Lie n-action", "crossed n-cube in Lie algebras", etc.

Let $M, P$ be Lie algebras. Suppose given a map $P \times M \rightarrow M$, $(\mathrm{p}, \mathrm{m}) \rightarrow \mathrm{P}_{\mathrm{m}}$.

PROPOSITION(2. $\mathbf{2}^{\prime} .1$ ). This map is a Lie action of $P$ on $M$ if and only if:
(i) (ap)m $=P(a m)=a\left(p_{m}\right)$;
(ii) $P\left(m+m^{\prime}\right)=P_{m}+P_{m}^{\prime \prime}$;
(iii) $\left(p+p^{\prime}\right)_{m}=p_{m}+p^{\prime} m$;
(iv) $\left[p, p^{\prime}\right]_{m}=p\left(p^{\prime} m\right)-p^{\prime}\left(p_{m}\right)$;

for all a $\in A, m, m^{\prime} \in M, p, p^{\prime} \in P$.
PROOF Strictly speaking a Lie action should consist of three maps $\alpha^{+}, \alpha[],, \alpha[r]^{\circ}: P \times M \rightarrow M$. However, since we will always have $\alpha^{+}(p, m)=m$ and $\alpha[],(p, m)=-\alpha[,]^{\circ}(p, m)$, we can take a Lie action to consist of just one map. It is clear that every Lie action satisfies rules (2.2.1,i to v). Conversely, to show that these rules are sufficient to give us a Lie action, we must check that the semi-direct product MXP is a Lie algebra (see proposition (1.3.1)). This check is routine and we omit it. $\quad \nabla$

A crossed module in Lie algebras (see definition (1.3.3)) is a Lie homomorphism $a: M \rightarrow P$ with a Lie action of $P$ on $M$ such that
(i) $a\left(\mathrm{P}_{\mathrm{m}}\right)=[\mathrm{p}, \partial \mathrm{m}]$,
(ii) $\partial m_{m}{ }^{\prime}=\left[m, m^{\prime}\right]$, for all $m, m^{\prime} \in M, p \in P$.

Suppose given: four Lie algebras L,M,N,P; Lie actions of $P$ on $L, M$ and $N$, and of $M$ on $L$, and of $N$ on $L$; and a function $h: M \times N \rightarrow L$. Then

PROPOSTTION(2.2.2) This structure is a Lie 2-action (see definition (1.3.8)) if and only if:
(i) $\left.\quad P\left(m_{1}\right)=\left(p_{m}\right) 1+m_{(1)}\right)_{1}$.
$\left.p\left(n_{1}\right)=\left(P_{n}\right)_{1}+n_{(1)}\right) ;$
(ii) $a h(m, n)=h(a m, n)=h(m, a n) ;$
(iii) $h\left(m+m^{\prime}, n\right)=h(m, n)+h\left(m^{\prime}, n\right)$, $h\left(m, n+n^{\prime}\right)=h(m, n)+h\left(n, m^{\prime}\right) ;$
(iv) $h\left(\left[m, m^{\prime}\right], n\right)=m_{h}\left(m^{\prime}, n\right)-m^{\prime} h(m, n)$,
$h\left(m,\left[n, n^{\prime}\right]\right)=n_{h}\left(m, n^{\prime}\right)-n^{\prime} h(m, n) ;$
(v) $\quad P_{h}(m, n)=h\left(p_{m, n}\right)+h\left(m, p_{n}\right)$;
(v) $\quad n\left(m_{l}\right)=m\left(n_{l}\right)+[l, h(m, n)] ;$
for all a $\in A, \mathcal{l} \in L, m, m^{\prime} \in M, n_{n} n^{\prime} \in \mathbb{N}, p \in P$.
PROOF Strictly speaking, a Lie 2-action should involve three functions $h^{+}, h[r], h[r]^{\circ}: M \times N \rightarrow$. However, since $h^{+}(m, n)=0$ and $h[],(m, n)=-h[,]^{\circ}(m, n)$, we can take a Lie 2-action to involve just one function $h=h[$,$] . It is$ clear that every Lie 2-action satisfies the above rules.

In order to show that the above rules are sufficient to give us a Lie 2-action we define an action of the semi-direct product $N X P$ on the semi-direct product LXM by

$$
(n, p)(1, m)=\left(n_{1}+p_{1}-h(m, n), p_{m}\right)
$$

By propositions (1.3.9) and (2.2.1) we have to check that

$$
\begin{aligned}
& (a n, a p)(1, m)=(n, p)(a l, a m)=a((n, p)(1, m)), \\
& (n, p)\left((1, m)+\left(l^{\prime}, m^{\prime}\right)\right)=(n, p)(1, m)+(n, p)\left(l^{\prime}, m^{\prime}\right), \\
& \left((n, p)+\left(n^{\prime}, p^{\prime}\right)\right)(1, m)=(n, p)(1, m)+\left(n^{\prime}, p^{\prime}\right)(1, m), \\
& {\left[(n, p),\left(n^{\prime}, p^{\prime}\right)\right](1, m)=}
\end{aligned}
$$

$$
\begin{aligned}
& (n, p)\left(\left(n^{\prime}, p^{\prime}\right)(1, m)\right)-\left(n^{\prime}, p^{\prime}\right)((n, p)(1, m)), \\
& (n, p)\left[(1, m),\left(l^{\prime}, m^{\prime}\right)\right]= \\
& {\left[(n, p)(1, m),\left(l^{\prime}, m^{\prime}\right)\right]+\left[(1, m),(n, p)\left(l^{\prime}, m^{\prime}\right)\right] .}
\end{aligned}
$$

These equations are checked in Appendix II, verifications (8),(9),(10),(11),(12). $\quad \nabla$

Suppose now we have a commutative diagram of Lie algebras

$$
\begin{array}{rlll}
L & \rightarrow^{\lambda} & \mathrm{N} \\
\lambda^{\prime+} & & { }^{1} \delta^{\prime} \\
M & \rightarrow \delta & \mathrm{P}
\end{array}
$$

in which there are Lie actions of $P$ on $L, M$ and $N$ (hence there are Lie actions of M on L and N via $\delta$, and of N on $L$ and M via $\left.\mathrm{\delta}^{\prime}\right)$, and a function $\mathrm{h}: \mathrm{M} \times \mathrm{N} \rightarrow \mathrm{L}$. Then PROPOSITION(2.2.3) This structure is a crossed square in Lie algebras (see definition (1.3.10)) if and only if:
(i) each of the maps $\lambda, \lambda^{\prime}, \delta, \delta^{\prime}$ and the composite $\delta^{\prime} \lambda$ is a crossed module;
(ii) the maps $\lambda, \lambda$ ' preserve the actions of $P$;
(iii) $a h(m, n)=h(a m, n)=h(m, a n) ;$
(iv) $h\left(m+m^{\prime}, n\right)=h(m, n)+h\left(m^{\prime}, n\right)$,
$h\left(m, n+n^{\prime}\right)=h(m, n)+h\left(m, n^{\prime}\right) ;$
(v)
$h\left(\left[m^{\prime} m^{\prime}\right], n\right)=m_{h}\left(m^{\prime}, n\right)-m^{\prime} h(m, n)$,
$h\left(m,\left[n, n^{\prime}\right]\right)=n_{h\left(m, n^{\prime}\right)}-n^{\prime} h(m, n) ;$
(vi) $\mathrm{P}_{\mathrm{h}}(\mathrm{m}, \mathrm{n})=\mathrm{h}\left(\mathrm{P}_{\mathrm{m}}, \mathrm{n}\right)+\mathrm{h}\left(\mathrm{m}, \mathrm{P}_{\mathrm{n}}\right)$;
(vii) $\lambda h(m, n)=m_{n}$,
$\lambda^{\prime} h(m, n)=-n_{m} ;$
(viii) $h(m, \lambda l)=m_{1}$,

$$
h(\lambda \cdot 1, n)=-n_{1} ;
$$

for all $1 \in L, m, m^{\prime} \in M, n, n^{\prime} \in N, p \in P$.
PROOF The proof boils down to checking that rules (2.2.3.1 to viii) imply rules (2.2.2,i,vi). We check this in Appendix II, verifications (13),(14). $\nabla$

Suppose now we have a commutative diagram of Lie algebras

in which there is a Lie action of $S$ on each of the other seven algebras (hence all eight algebras act on each other via the actions of $S$ ), and there are six functions
$h: Q \times L \rightarrow K$,
$h:$ PXM $\rightarrow K$,
$\mathrm{h}: \mathrm{N} \times \mathrm{R} \rightarrow \mathrm{K}$,
$h: P \times R \rightarrow L$,
$\mathrm{h}: \mathrm{Q} \times \mathrm{R} \rightarrow \mathrm{M}$,
$h: P \times Q \rightarrow N$.

Then
PROPOSITION(2.2.4) This structure is a crossed 3-cube in Lie algebras (see definition (1.3.10)) if and only if: (i) each of the squares

| $\mathrm{K} \rightarrow \mathrm{L}$ | $\mathrm{K} \rightarrow \mathrm{M}$ | $\mathrm{K} \rightarrow \mathrm{R}$ | $L \rightarrow R$ | $\mathrm{M} \rightarrow \mathrm{R}$ | $N \rightarrow Q$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\downarrow \quad 1$ | 11 | $\downarrow 1$ | $1 \quad 1$ | 11 | $\downarrow$ |
| $Q \rightarrow S$ | $P \rightarrow S$ | $N \rightarrow S$ | $P \rightarrow S$ | $Q \rightarrow 5$ | $P \rightarrow S$ |
| $K \rightarrow M$ | $K \rightarrow L$ | $K \rightarrow M$ |  |  |  |
| $\downarrow \quad 1$ | 1 1 | $\downarrow \quad \downarrow$ |  |  |  |
| $L \rightarrow R$ | $N \rightarrow P$ | $N+\mathrm{Q}$ |  |  |  |

is a crossed square; for the last three squares the functions $h: L \times M \rightarrow K, h: N \times L \rightarrow K, h: N \times M \rightarrow K$ are respectively given by $h(1, m)=h\left(\nu_{P} l, m\right), h(n, l)=h\left(n, \nu_{R} l\right), h(n, m)=$ $h\left(n, \nu_{R} m\right) ;$
(ii) $\mathrm{Ph}_{\mathrm{h}}(1, m)=h\left(\mathrm{p}_{1}, \mathrm{~m}\right)+\mathrm{l}_{\mathrm{h}}(\mathrm{p}, \mathrm{m})$;
(iii) $h(p, h(q, r))=h(h(p, q), r)+h(q, h(p, r)) ;$
(iv) $\lambda_{[ } h(p, m)=h\left(p, \nu_{R} m\right)$,
$\lambda_{L} h(n, r)=h(\nu p n, r)$,
$\lambda_{M h}(q, 1)=h\left(q, \nu_{R} l\right)$,
$\lambda_{M h}(n, r)=h\left(\nu_{Q} n, r\right)$,
$\lambda_{N h}(p, m)=h\left(p, \nu_{Q} m\right)$,
$\lambda_{N} h(q, 1)=-h\left(\nu_{p} l, q\right) ;$
(v) $h\left(\nu_{Q}, I\right)=-h\left(\nu_{\mathrm{p}} \mathrm{l}, \mathrm{m}\right)$;
$h\left(n, \nu_{R} I\right)=h\left(\nu_{Q}, 1\right) ;$
$h\left(n, \nu_{R} m\right)=h(\nu \rho n, m) ;$
for all $1 \in L_{1}, m \in M, n \in N, p \in P, q \in Q, r \in R$.

PROOF Form the semi-direct products KXXN, LXP, MXQ, RXS. Given an element ( $u, v, x, y$ ) in any one of the direct
 (LXPP) $\times(K \underline{X} N),(M \underline{Q}) \times(K \underline{X} N)$, we obtain five Lie actions by setting

$$
(u, v)(x, y)=\left(u_{x}+v_{Y}-h(y, u), v_{Y}\right) .
$$

Let $h^{\prime}:(L \underline{X} P) \times(M \underline{Q}) \rightarrow K \underline{X} N$ be the function
$h^{\prime}(1, p, m, q)=(h(1, m)+h(p, m)-h(q, l), h(p, q))$.
The proof boils down to checking that h' satisfies the rules for a Lie 2-action, i.e. we must check that

$$
\begin{aligned}
& a h^{\prime}(1, p, m, q)=h^{\prime}(a l, a p, m, q)=h^{\prime}(1, p, a m, a q), \\
& (r, s)((1, p)(k, n)) \\
& =((r, s)(1, p))(k, n)+(1, p)((x, s)(k, n)), \\
& (r, s)((m, q)(k, n)) \\
& =((r, s)(m, q))(k, n)+(m, q)((x, s)(k, n)), \\
& h^{\prime}\left((1, p)+\left(l^{\prime}, p^{\prime}\right), m, q\right)=h^{\prime}(1, p, m, q)+h^{\prime}\left(l^{\prime}, p^{\prime}, m, q\right), \\
& h^{\prime}\left(1, p,(m, q)+\left(m^{\prime}, q^{\prime}\right)\right)=h^{\prime}(1, p, m, q)+h^{\prime}\left(1, p, m^{\prime}, q^{\prime}\right), \\
& h^{\prime}\left(\left[(1, p),\left(1 l^{\prime}, p^{\prime}\right)\right], m, q\right) \\
& =(1, p)_{h^{\prime}}\left(1 l^{\prime}, p^{\prime}, m, q\right)-\left(l^{\prime}, p^{\prime}\right) h^{\prime}(1, p, m, q), \\
& h^{\prime}\left(1, p,\left[(m, q),\left(m^{\prime}, q^{\prime}\right)\right]\right) \\
& =(m, q)_{h^{\prime}}\left(1, p, m^{\prime}, q^{\prime}\right)-\left(m^{\prime}, q^{\prime}\right) h^{\prime}((1, p, m, q), \\
& (r, s) h^{\prime}(1, p, m, q) \\
& =h^{\prime}((x, s)(1, p), m, q)+h^{\prime}(1, p,(r, s)(m, q)), \\
& (m, q)((1, p)(k, n)) \\
& =(1, p)((m, q)(k, n))-\left[(k, n), h^{\prime}(1, p, m, q)\right] .
\end{aligned}
$$

This check is routine and we omit it. $\nabla$
3. CROSSED n-CUBES [N COMMUTATIVE ALGEBRAS

Fix a commutative ring $A$ (with unit). Recall that a commutative algebra over $A$ is an A-module $M$ with an A-bilinear map $M \times M \rightarrow M,\left(m, m^{\prime}\right) \rightarrow m^{\prime}$, satisfying:

$$
\begin{aligned}
& m m^{\prime}=m^{\prime} m, \\
& \left(m^{\prime}\right) m^{\prime \prime}=m\left(m^{\prime} m^{\prime}{ }^{\prime}\right),
\end{aligned}
$$

for all $m, m^{\prime}, m^{\prime \prime} \in M$. We shall assume all commutative algebras to be over $A$. Let $C$ be the category of commutative algebras. We shall replace terms such as "C-n-action", "crossed n-cube in C" with "commutative action", "crossed n-cube in commutative algebras", etc.

The proofs of the propositions in this section are similar to (and simpler than) the corresponding proofs of the previous section. For this reason we shall omit all proofs.

Let $M, P$ be commutative algebras. Suppose given a map $P \times M \rightarrow M,(p, m) \rightarrow P_{m}$.

PROPOSITION(2.3.1) This map is a commutative action if and only if:
(i) $a\left(P_{m}\right)=(a p)_{m}=P(a m)$;
(ii) $P\left(m+m^{\prime}\right)=P_{m}+P_{m}$; ;
(iii) $\left(p+p^{\prime}\right)_{m}=p_{m}+p^{\prime} m ;$
(iv) $p\left(m m^{\prime}\right)=\left(P_{m}\right) m^{\prime}$;
(v) $\quad\left(p p^{\prime}\right)_{m}=P\left(p^{\prime} m\right) ;$
for all a $\in A, m_{1} m^{\prime} \in M, p, p^{\prime} \in P . \quad \nabla$

A crossed module in commutative algebras (see definition
(1.3.3)) is a commutative algebra homomorphism $\partial: M \rightarrow P$ with
a commatative action of $P$ on $M$ such that:
(i) $a\left(p_{m}\right)=p a(m)$;
(ii) $\partial m_{m}{ }^{\prime}=m m^{\prime}$;
for all $m, m^{\prime} \in M, p \in P$.

Suppose given a commutative diagram of commutative algebras

$$
\begin{array}{rll}
L & \rightarrow \lambda & N \\
\lambda^{\prime} \downarrow & & \downarrow^{\prime \prime} \\
M & \rightarrow B^{\prime} & \mathrm{P}
\end{array}
$$

in which there are commative actions of $P$ on $L_{1} M$ and $N$ (hence there are commutative actions of $M$ on $L$ and $N$, and of $N$ on $L$ and $M$, all via the actions of $P$ ), and a function $h: M \times N \rightarrow$ L. Then

PROPOSITION(2.3.2) This structure is a crossed square in commutative algebras (see definition (1.3.10)) if and only if:
(i) each of the maps $\lambda, \lambda^{\prime}, \delta, \delta^{\prime}$ and the composite $\delta^{\prime} \lambda$ is a crossed module;
(ii) the maps $\lambda, \lambda^{\prime}$ preserve the actions of $P$;
(iii) $a h(m, n)=h(a m, n)=h(m, a n) ;$
(iv) $h\left(m+m^{\prime}, n\right)=h(m, n)+h\left(m^{\prime}, n\right)$,
$h\left(m, n+n^{\prime}\right)=h(m, n)+h\left(m, n^{\prime}\right) ;$
(v) $\quad P_{h}(m, n)=h\left(p_{m, n}\right)=h\left(m, D_{n}\right) ;$
(vi) $\lambda h(m, n)=m_{n}$,
$\lambda^{\prime} h(m, n)=-n_{m} ;$
(vii) $h(m, \lambda l)=m_{1}$,

$$
\begin{aligned}
& h\left(\lambda^{\prime} l, n\right)=-n_{l} ; \\
& \text { for all a } \in A, l \in L, m, m^{\prime} \in M, n_{1} n^{\prime} \in N, p \in P . \quad \nabla
\end{aligned}
$$

Suppose now we have a commutative diagram of commutative algebras

in which there is a commutative action of $s$ on each of the other seven algebras (hence all eight algebras act on each other via the actions of $S$ ), and there are six functions
$h: Q \times L \rightarrow K$,
$h: P \times M \rightarrow K$,
$h: N \times R \rightarrow K$,
$h: P \times R \rightarrow L$,
$h: Q \times R \rightarrow M$,
$h: P \times Q \rightarrow N$.
Then
PROPOSITION(2.3.3) This structure is a crossed 3-cube in
commutative algebras (see definition (1.3.10)) if and only if:
(i) each of the squares

is a crossed square; for the last three squares the functions $h: L \times M \rightarrow K, h: N \times L \rightarrow K, h: N \times M \rightarrow K$ are respectively given by $h(l, m)=h\left(\nu_{p} l, m\right), h(n, l)=h\left(n, \nu_{R} l\right), h(n, m)=$ $h\left(n, \nu_{R} m\right) ;$
(ii) $h(h(p, q), r)=h(p, h(q, r))=h(q, h(p, r))$;
(iii) $\lambda_{L} h(p, m)=h\left(p, \nu_{R} m\right)$,
$\lambda_{L h}(n, r)=h(\nu p n, r)$,
$\lambda_{M h}(q, l)=h\left(q, \nu_{R} l\right)$,
$\lambda_{M h}(n, r)=h\left(\nu_{Q} n, r\right)$,
$\lambda_{N h}(p, m)=h\left(p, \nu_{Q} m\right)$,
$\lambda_{N h}(q, I)=h\left(\nu_{p l}, q\right) ;$
(iv) $h\left(\nu_{Q} m, I\right)=h\left(\nu_{\mathrm{P}} \mathrm{l}, \mathrm{m}\right)$;
$h\left(n, \nu_{R} l\right)=h\left(\nu_{Q} n, I\right) ;$
$h\left(n, \nu_{R^{m}}\right)=h(\nu p n, m) ;$
for all $1 \in L, m \in M, n \in N, p \in P, q \in Q, r \in R . \quad \nabla$
4. CROSSED n-CUBES IN ASSOCIATIVE ALGEBRAS

Fix a commutative ring A (with unit). Recall that an associative algebra over $A$ is an A-module $M$ with an A-bilinear map $M \times M \rightarrow M,\left(m, m^{\prime}\right) \rightarrow m m '$ such that:

$$
m\left(m^{\prime} m^{\prime \prime}\right)=\left(m m^{\prime}\right) m^{\prime \prime} ;
$$

for all mim', $\mathrm{m}^{\prime \prime} \in \mathrm{M}$. We shall assume all associative algebras to be over $A$. Let $C$ be the category of associative algebras. We shall replace terms such as "C-n-action", "crossed n-cube in $C "$ by "associative n-action", "crossed n-cube in associative algebras", etc. In this section we shall again omit all proofs.

Let $M, P$ be associative algebras. Suppose given two maps $(p, m) \rightarrow p_{m},(p, m) \rightarrow p^{\circ} m$ from $P \times M$ to $M$. Then

PROPOSITION (2.4.1) These maps constitute an associative action of $P$ on $M$ if and only if:
(i) $a\left(P_{m}\right)=(a p)_{m}=P(a m)$,

$$
a\left(p^{\circ} m\right)=\left(a p^{0}\right)_{m}=p^{\circ}(a m) ;
$$

$$
\begin{align*}
& \left(p+p^{\prime}\right)_{m}=p_{m}+p^{\prime} m  \tag{ii}\\
& \left(p+p^{\prime}\right)^{\circ} m=p^{\circ} m+p^{\prime 0} m
\end{align*}
$$

(iii) $D^{\prime}\left(m+m^{\prime}\right)=P_{m}+p_{m}^{\prime}$, $p^{\circ}\left(m+m^{\prime}\right)=p^{\circ} m+p^{\circ} m^{\prime} ;$
(iv) ( Pm ) $\mathrm{m}^{\prime}=\mathrm{P}\left(m \mathrm{~m}^{\prime}\right)$,
$m\left(p^{\circ} m^{\prime}\right)=p^{\circ}\left(m m^{\prime}\right) ;$
(v) $p\left(p^{\prime} m\right)=\left(p p^{\prime}\right)_{m}$, $p^{\circ}\left(p^{\prime 0} m\right)=\left(p p^{i}\right)^{\circ} m ;$
(vi) $p^{0}\left(p^{\prime} m\right)=p^{\prime}\left(p^{0} m\right)$;
(vii) ( $\left.\mathrm{p}^{\circ} \mathrm{m}\right) \mathrm{m}^{\prime}=\mathrm{m}\left(\mathrm{P}_{\mathrm{m}}{ }^{\prime}\right)$;
for all a $\in A, m, m^{\prime} \in M, p, p^{\prime} \in \mathbb{R}$.

A crossed module in associative algebras (see definition (1.3.3)) is a morphism of associative algebras $a: M \rightarrow P$ with an associative action of $P$ on $M$ such that:
(i) $a\left(p_{m}\right)=p(\partial m)$,
$a\left(p^{\circ} m\right)=(\partial m) p ;$
(ii) $\partial m_{m}{ }^{\prime}=m m^{\prime}$, $\partial m^{\circ} \mathrm{m}^{\prime}=\mathrm{m}^{\prime} \mathrm{m}^{\prime}$
for all $m, m^{\prime} \in M, p \in P$.

Suppose given a commutative diagram of associative algebras

$$
\begin{array}{rll}
L & \rightarrow \lambda & N \\
\lambda^{\prime}+1 & & { }^{\prime} \delta^{\prime} \\
M & \rightarrow \delta & P
\end{array}
$$

in which there are associative actions of $P$ on $L, M$ and $N$ (hence there are associative actions of $M$ on $L$ and $N$, and of $N$ on $L$ and $M$, all via the actions of $P$ ), and two functions $h, h^{\circ}: M \times N \rightarrow L . \quad$ Then PROPOSITION(2.3.3) This structure is a crossed square in associative algebras (see definition (1.3.10)) if and only if:
(i) each of the maps $\lambda_{r} \lambda^{\prime}, \delta^{\prime} \delta^{\prime \prime}$ and the composite $\delta^{\prime \prime} \lambda$ is a crossed module:
(ii) the maps $\lambda, \lambda^{\prime}$ preserve the actions of $P$;
(iii) $a h(m, n)=h(a m, n)=h(m, a n)$,
$a h^{\circ}(m, n)=h^{\circ}(a m, n)=h^{\circ}(m, a n) ;$
(iv) $h\left(m+m^{\prime}, n\right)=h(m, n)+h\left(m^{\prime}, n\right)$,

$$
\begin{aligned}
& h\left(m, n+n^{\prime}\right)=h(m, n)+h\left(m, n^{\prime}\right), \\
& h^{\circ}\left(m+m^{\prime}, n\right)=h^{0}(m, n)+h^{0}\left(m^{\prime}, n\right), \\
& h^{\circ}\left(m, n+n^{\prime}\right)=h^{\circ}(m, n)+h^{\circ}\left(m, n^{\prime}\right) ; \\
& \text { (v) } \quad P_{h}(m, n)=h\left(P_{m}, n\right) \text {, } \\
& p^{\circ} h^{\circ}(m, n)=h^{\circ}\left(p^{\circ} m, n\right), \\
& p^{0} h(m, n)=h\left(m, p^{0} n\right), \\
& \mathrm{Ph}^{\circ}(\mathrm{m}, \mathrm{n})=h^{\circ}\left(\mathrm{m}, \mathrm{P}_{\mathrm{n}}\right), \\
& h\left(p^{0} m, n\right)=h\left(m, p_{n}\right), \\
& h^{\circ}\left(P_{m, n}\right)=h^{\circ}\left(m, p^{\circ} n\right) ; \\
& \text { (vi) } \quad m^{\prime} h^{\circ}(m, n)=m^{\circ} h\left(m^{\prime}, n\right) \text {, } \\
& n^{\prime} h(m, n)=n^{0} h^{0}\left(m, n^{\prime}\right) ; \\
& \text { (vii) } \lambda h(m, n)=m_{n} \text {, } \\
& \lambda h^{\circ}(m, n)=m^{0} n, \\
& \lambda^{\prime} h(m, n)=n^{0} m, \\
& \lambda^{\prime} h^{\circ}(m, n)=n_{m} ; \\
& \text { (viii) } h(m, \lambda 1)=m_{1} \text {, } \\
& h^{0}(m, \lambda 1)=m^{0} 1, \\
& h(\lambda \cdot 1, n)=n^{0} 1 \text {, } \\
& h^{\circ}(\lambda \cdot 1, n)=n_{1} ; \\
& \text { for ala } \in A, 1 \in L, m, m^{\prime} \in M, n, n^{\prime} \in N, p \in P . \quad \nabla
\end{aligned}
$$

We shall not bother to work out the axioms for a crossed 3-cube in associative algebras!

## O. INTRODUCTION

In this chapter we investigate cetrain universal crossed squares and crossed 3-cubes in groups (\$1,2), Lie algebras (\$3), and commutative algebras (\$4). The universal crossed squares involve notions of non-abelian tensor, antisymmetric, and exterior products. We obtain various exact sequences involving these non-abelian constructions, certain of which will be used in Chapter IV when we look at the relevance of crossed squares to homology.

## 1. CERTAIN UNIVERSAL CROSSED SQUARES IN GROUPS

Let $M, N$ be groups such that there is a group action of $M$ (resp. N) on $N$ (resp. M). Assume each group acts on itself by conjugation.

Following [B-L] we define the tensor product of $M$ with $N$ to be the group $M \otimes N$ which is generated by the elements $m \otimes n$, for $(m, n) \in M \times N$, subject to the relations
(1) $m m^{\prime} \otimes n=\left(m_{m}{ }^{\prime} \otimes m_{n}\right)(m \otimes n)$,
(2) $m \otimes n n^{\prime}=(m \otimes n)\left(n_{m} \otimes n_{n}\right)^{\prime}$.

Note that, when the actions of $M$ and $N$ are trivial, then $M \otimes N$ is just the standard tensor product $M^{a b} \otimes_{Z} N^{a b}$ of abelian groups.

For $m \in M, n \in N, x \in M$ or $N$ we shall write (mn) $x$ instead of $m\left(n_{x}\right)$, and $(n m) x$ instead of $n^{\left(m_{x}\right)}$. We can

Consider $m n$ and $n m$ as elements of the free product $M * N$. This abuse of notation is unlikely to cause confusion. The tensor product $M \otimes N$ is of particular interest when the actions of $M$ and $N$ are compatible; i.e. when

$$
\left(n_{m}\right)_{x}=\left(n m n^{-1}\right)_{x} \text { and }\left(m_{n}\right)_{x}=\left(m n m^{-1}\right) x
$$

'In this case we can interpret $m \otimes n$ as a commutator and we can interpret the actions of $M$ and $N$ as conjugation: anything which looks like a universal commutator relation is then actually a relation in the tensor product $M \otimes N . '$ Since much of this section is concerned with obtaining consequences of relations (1) and (2), it will be helpful to have a precise version of this statement.

The tensor product $M \otimes N$ admits a group action of $N$, given by

$$
n^{\prime}(m \otimes n)=n^{\prime} m \otimes n^{\prime} m
$$

The resulting semi-direct product $(M \otimes N) \times N$ admits a group action of $M$, given by

$$
m(1, n)=\left(m_{1}(m \otimes n), n\right)
$$

where $m \in M, n \in N, l \in M \otimes N$. To see this we note that:
(i) $m^{\prime}\{m(1, n)\}$

$$
\begin{aligned}
& =\left(m^{\prime} m_{1} m^{\prime}(m \otimes n)\left(m^{\prime} \otimes n\right), n\right) \\
& =\left(m^{\prime} m_{1}\left(m^{\prime} m \otimes n\right), n\right) \\
& =m^{\prime} m^{(1, n)} ;
\end{aligned}
$$

(ii) by expanding (mm' $\otimes n^{\prime}$ ') in two different ways we obtain the identity

$$
m n\left(m^{\prime} \otimes n^{\prime}\right)(m \otimes n)=(m \otimes n)^{n m}\left(m^{\prime} \otimes n^{\prime}\right)
$$

and hence, for any $l \in M \otimes N$, the identity

$$
m n_{l}(m \otimes n)=(m \otimes n) n m_{1} ;
$$

it follows that

$$
\begin{aligned}
& m(1, n) m^{m}\left(l^{\prime}, n^{\prime}\right) \\
& =\left(m_{1}(m \otimes n) m_{1} n^{n}\left(m \otimes n^{\prime}\right), n n^{\prime}\right) \\
& =\left(m_{1} m n_{1}(m \otimes n) n^{\prime}\left(m \otimes n^{\prime}\right), n n^{\prime}\right) \\
& \left.=\left(m_{\{1} n_{1}\right\}\left(m \otimes n n^{\prime}\right), n n^{\prime}\right) \\
& \left.=m^{\prime}(1, n)\left(l^{\prime}, n^{\prime}\right)\right\} .
\end{aligned}
$$

Now the natural inclusions $M \rightarrow((M \otimes N) \times N) \times M), N \rightarrow$ $((M \otimes N) \underline{X} N) \underline{X} M$ ）induce a map $\rangle: M * N \rightarrow((M \otimes N) \underline{X} N) \times M$ ．For $p \in M^{*} N$ we shall denote by $\langle p$ 〉 the image of $p$ under 〈〉． For $m \in M$ we shall denote by $\langle m$ 〉 the image of $m$ under the composite map $M \rightarrow i N * N \rightarrow\rangle((M \otimes N) \underline{N}) \underline{M}$ where $i$ is the canonical inclusion．Similarly we shall write $\langle n\rangle$ for $n \in$ $N$ ．Let $l: M \otimes N \rightarrow((M \otimes N) \underline{X} N) \times M$ be the natural inclusion． As we noted above，the group actions $M \rightarrow$ Aut $M, N \rightarrow$ Aut $M$ give rise to an action $M^{*} N \rightarrow$ Aut M．Likewise，there is an action of $M^{*} N$ on $N$ and on $M \otimes N$ ．

Suppose we have an arbitrary generator $x$ of $M \otimes N$ ，which is of the form

$$
x=p_{1 m_{1}} p_{2 m_{2}} \ldots p_{\ell m_{l}} \otimes q_{1 n_{1}} q_{2 n_{2}} \ldots q_{\ell} n_{\ell} \prime
$$

with $m_{i} \in M, n_{i} \in N, p_{i}, q_{i} \in M^{*} N$ ．Then

LEMMA（3．1．1）If the actions of $M$ and $N$ are compatible，we have

$$
\iota x=\left[\left\langle p_{1} m_{1} p_{1}^{-1}\right\rangle \ldots\left\langle p_{\ell} m_{l} p_{l}^{-1}\right\rangle,\left\langle q_{1} n_{1} q_{1}^{-1}\right\rangle \ldots\left\langle q_{l} n_{l} \cdot q_{l} \cdot\right\rangle\right]
$$

PROOF The generator $x$ can be expanded into a product of terms of the form $p_{m_{1}} \otimes q_{n_{j}}$ with $p, q \in M^{*} N$ ，and it suffices to check the following identity：

$$
\begin{equation*}
\left(p_{m_{i}} \otimes q_{n_{j}}, e, e\right)=\left[\left\langle p m_{i} p^{-1}\right\rangle,\left\langle q n_{j} q^{-1}\right\rangle\right] \tag{a}
\end{equation*}
$$

Note that, for $m_{0} \in M, n_{0} \in N, l \in M \otimes N$, we have:

$$
\begin{aligned}
& \left\langle m_{0}\right\rangle(1, e, e)\left\langle m_{0}\right\rangle^{-1}=\left(m_{0}, e, e\right), \\
& \left\langle n_{0}\right\rangle(1, e, e)\left\langle n_{0}\right\rangle-1=\left(n_{0}, e, e\right),
\end{aligned}
$$

and thus, for $q \in M^{*} N$, we have

$$
\begin{equation*}
\langle q\rangle(1, e, e)\langle q\rangle^{-1}=\left(q_{1}, e, e\right) . \tag{b}
\end{equation*}
$$

This last identity, together with the identity
$\left(P_{m} \otimes n, e, e\right)=\left[\left\langle p m p^{-1}\right\rangle,\langle n\rangle\right]$
(where $m \in M, n \in N$ ) imply identity (a) since:
$\left(p_{m_{i}} \otimes q_{n_{j}}, e, e\right)$
$=\left(q\left\{q^{-1} p_{m_{i}} \otimes n_{j}\right\}, e, e\right)$
$=\langle q\rangle\left(q^{-1} p_{m_{i}} \otimes n_{j}, e, e\right)\langle q\rangle^{-1} \quad$ using (b)
$=\langle q\rangle\left[\left\langle q^{-1} p_{i} p^{-1} q\right\rangle,\left\langle n_{j}\right\rangle\right]\langle q\rangle^{-1}$ using (a)'
$=\left[\left\langle p m_{i} p^{-1}\right\rangle,\left\langle q n_{j} q^{-1}\right\rangle\right]$.
So it remains to verify (a)'.
It is routine to verify (a)' for $p$ equal to the identity element of $M^{*} N$. We shall say that $p$ is of length $\ell(\ell \geqslant 1)$ if we can write $p=m_{1} n_{2} \ldots m_{\ell-1} n_{\ell}$ or $p=n_{1} m_{2} \ldots n_{l-2} m_{l}$ with $m_{i} \in M, n_{i} \in N$. We shall say that $p$ is of length 0 if $p$ is the identity element. Fix $\ell \geqslant 0$, suppose that $p$ is of length $l$, and suppose that we have verified identity (a)' for the case when $p$ is of length $\ell$. For $m_{0} \in M$ we have

$$
\left[\left\langle m_{0} p^{p m p}-1 m_{0}-1\right\rangle,\langle n\rangle\right]
$$

$$
=\left\langle m_{0} p m p^{-1}\right\rangle\left[\left\langle m_{0}\right\rangle-1,\langle n\rangle\right]\left\langle m_{0}\right\rangle\left[\left\langle p m p^{-1}\right\rangle,\langle n\rangle\right]\left[\left\langle m_{0}\right\rangle,\langle n\rangle\right]
$$

$$
=\left(m_{0} p m p^{-1}\left(m_{0}^{-1} \otimes n\right), e, e\right)\left(m_{0} p_{m} \otimes n, e, e\right)
$$

$=\left(m_{0} p_{m} \otimes n, e, e\right) \quad$ (using compatibility).
For $n_{0} \in N$ we have

$$
\begin{aligned}
& {\left[\left\langle n_{0} p m p^{-1} n_{n_{0}}-1\right\rangle,\langle n\rangle\right]} \\
& =\left\langle n_{0}\right\rangle\left[\left\langlep_{\left.m p^{-1}\right\rangle,\left\langle n_{0}-1\right.}^{\left.\left.n n_{0}\right\rangle\right]\left\langle n_{0}\right\rangle-1}\right.\right. \\
& =\left\langle n_{0}\right\rangle\left(p_{m} \otimes n_{0}-1 n_{n}, e, e\right)\left\langle n_{0}\right\rangle^{-1}
\end{aligned}
$$

$=\left(n_{0} P_{m} \otimes n, e, e\right) . \quad$ (using (b))
Identity (a)', for an arbitrary $p$, follows by induction. $\nabla$

As a typical application of this lemma we give EXAMPLE(3.1.2) Suppose that the actions of $M$ and $N$ are compatible, and suppose that we wish to verify that the relation

$$
\left[m \otimes n, m^{\prime} \otimes n^{\prime}\right]=m n_{m}^{-1} \otimes m^{\prime} n^{\prime} n^{\prime-1}
$$

holds in the tensor product. Then using the lemma we note that

```
\(L\left[m \otimes n, m^{\prime} \otimes n^{\prime}\right]\)
\(=l\left(m n^{-1} \otimes m^{\prime} n^{\prime} n^{\prime-1}\right)\)
\(=\left[[(e, e, m),(e, n, e)],\left[\left(e, e, m^{\prime}\right),\left(e, n^{\prime}, \theta\right)\right]\right]\).
```

From now on let us suppose that the groups $M, N$ and their actions are obtained from two crossed modules $8: M \rightarrow P, \delta^{\prime}: N$ $\rightarrow$ P. It follows that the actions of $M$ and $N$ are compatible.

The tensor product $M \otimes N$ fits into a crossed square

| $M \otimes N$ | $\rightarrow \lambda$ | N |
| :---: | :---: | :---: |
| $\lambda{ }^{1}$ |  | ${ }^{1} 0^{\prime}$ |
| M | $\rightarrow 8$ | P |

in which: $P$ acts on $M \otimes N$ by $P(m \otimes n)=p_{m} \otimes P_{n}$; the maps $\lambda, \lambda^{\prime}$ are $\lambda(m \otimes n)=m_{n} n^{-1}, \lambda^{\prime}(m \otimes n)=m n^{-1} ;$ and the function $h: M \times N \rightarrow M \otimes N$ is given by $h(m, n)=m \otimes n$.

The fact that this structure is a crossed square is noted and verified in $[B-L]$. It is also noted that this
crossed square has a defining universal property, namely property (1.4.1.ii) for the case $n=2$ and $C$ equal to the category of groups.

Following [B-L] we also define the exterior product of $M$ with N to be the group $\mathrm{M} \triangle \mathrm{N}$ generated by the elements $m \wedge n, f o r(m, n) \in M \times N$, subject to the relations
(3) $m m^{\prime} \wedge n=\left(m_{m} \wedge m_{n}\right)(m \wedge n)$,
(4) $m \wedge n^{\prime}=(m \wedge n)\left(n_{m} \wedge n_{n}\right)^{\prime}$,
(5) $m \wedge n=e$ whenever $\delta \mathrm{m}=\mathrm{D}^{\prime} \mathrm{n}$.

Thus $M \triangle N$ is a quotient of $M \otimes N$. The quotient map $M \otimes N$ $\rightarrow M \wedge N$ preserves the crossed square which contains $M \otimes N$. That is to say, the exterior product $M \wedge N$ also fits into a crossed square, and this crossed square has a defining universal property.

It seems reasonable to define the anti-symmetric product of $M$ with $N$ to be the group $M \triangle N$ generated by the elements $m \triangle n$, for ( $m, n$ ) $\in M \times N$, subject to the relations (6) $\quad m^{\prime} \Lambda_{n}=\left(m_{m} \underline{\Lambda}^{m_{n}}\right)(m \underline{n})$,
(7) $m \triangle n^{\prime}=(m \triangle n)\left(n_{m} \Lambda^{n_{n}^{\prime}}\right)$,
(8) $m \underline{\Lambda} n^{\prime}=\left(m^{\prime} \underline{\Lambda}\right)^{-1}$ whenever $8 \mathrm{~m}=8^{\prime} \mathrm{n}$ and

$$
\delta^{\prime} \mathrm{m}^{\prime}=\delta^{\prime} \mathrm{m}^{\prime} .
$$

We shall abbreviate the term "anti-symmetric" to "asymmetric".

Clearly $M \triangle N$ is a quotient of $M \otimes N$. It can also be seen that $M \wedge N$ is a quotient of $M \triangle N$. Again, the quotient map $M \otimes N \rightarrow M \triangle N$ preserves the crossed square containing $\mathrm{M} \otimes \mathrm{N}$.

We note that there are isomorphisms
$M \otimes N \quad \approx \quad N \otimes M$,
$M \wedge N E N \wedge M$,
$M \underline{N} \approx N \underline{M}$.
If we are given an extra crossed module $0^{\prime \prime}: L \rightarrow P$, then there are two obvious ways of constructing a triple tensor product: in general there is NOT an isomorphism between $(L \otimes M) \otimes N$ and $L \otimes(M \otimes N)$. A similar remark applies to the exterior and asymmetric products.

It will be convenient to have the notion of an "induced crossed square". So in the following diagram of group homomorphisms

suppose that the back square is a crossed square, that the maps $\nu, \nu^{\prime}$ are crossed modules, and that the two pairs $(\alpha, \gamma),(\beta, \gamma)$ are maps of crossed modules. Then the crossed square

| $L^{\star}$ | $\rightarrow$ | $S$ |
| :--- | :--- | :--- |
| $\downarrow$ |  | $\iota^{\prime}$ |
| $R$ | $\rightarrow \nu$ | $T$ |

is said to be induced by the above diagram if:
(i) there is a homomorphism $\rho: L \rightarrow L^{*}$ such that the quadruple $(\alpha, \beta, \gamma, \rho)$ is a map of crossed squares;
(ii) any other map ( $\alpha, \beta, \gamma, \rho^{\prime}$ ) of crossed squares factors uniquely through $(\alpha, \beta, \gamma, \rho)$.

A routine argument using universal properties shows that if $L=M \otimes N \quad($ resp. $L=M \Lambda N, L=M \Lambda N)$, then $L^{*}=R \otimes S \quad\left(\right.$ resp. $L^{*}=R \Lambda S, L^{*}=R \Lambda S$ ).

This observation together with the following result will be used extensively in this section and in section 4.4.

$$
\text { if her } \gamma c(\delta \text { ker } \alpha)\left(\delta^{\prime} \text { ker } \beta\right) \text {, }
$$

PROPOSITION(3.1.3) If the maps $\alpha, \beta, \gamma$ are surjectiveland if the group $L$ is generated by the image of the function $h: M \times N$ $\rightarrow$ L, then the induced group $L^{*}$ is the quotient of L obtained by factoring out the subgroup generated by the elements $h(a, n), h(m, b)$, with $m \in M, n \in N$, a $\in \operatorname{ker} \alpha$, b $\in \operatorname{ker} \beta$.

PROOF The proof is a straightforward check which we omit. $\nabla$

In the remainder of this section we shall give results concerned with computing the above tensor, exterior and asymmetric products.

Suppose that the group $M$ contains two subgroups $M_{A}, M_{B}$,
and that the group $N$ contains two subgroups $N_{A}, N_{B}$, such that $\delta M_{A}=\delta^{\prime} N_{A}=A, \delta M_{B}=\delta^{\prime} N_{B}=B$ say. The crossed modules $\delta: M \rightarrow P, \delta^{\prime}: N \rightarrow P$ restrict to give us crossed modules $\delta_{A}: M_{A} \rightarrow A, \delta_{B}: M_{B} \rightarrow B, \delta^{\prime} A: N_{A} \rightarrow A, \delta^{\prime}{ }_{B}: N_{B} \rightarrow B$. We can thus construct the groups $M_{A} \otimes N_{A}, M_{B} \otimes N_{B}$. The inclusions $M_{A} \rightarrow M, N_{A} \rightarrow N$ induce a map $\iota_{A}: M_{A} \otimes N_{A} \rightarrow M \otimes N$; similarly there is a map $\iota_{B}: M_{B} \otimes N_{B} \rightarrow M \otimes N$.

Let $\left\langle M_{A}, N_{B}\right\rangle$ be the subgroup of $M \otimes N$ generated by the elements $a_{0} \otimes b_{1}, b_{0} \otimes a_{1}$ with $\left(a_{0}, b_{1}\right) \in M_{A} \times N_{B},\left(b_{0}, a_{1}\right) \in$ $M_{B} \times N_{A}$. Then we have the useful

LEMMA(3.1.4) An arbitrary element $x \in M \otimes N$ can be written as a product of elements

$$
x=u v w
$$

with $u \in \iota_{A}\left(M_{A} \otimes N_{A}\right), v \in\left\langle M_{A}, N_{B}\right\rangle, w \in \iota_{B}\left(M_{B} \otimes N_{B}\right)$. This assertion also holds if we replace the tensor product with either the exterior product or the asymmetric product. PROOF We shall just consider the tensor product case. Using lemma (3.1.1) we can see immediately that the following identities hold in $M \otimes N$ :
(9) $a\left(a_{0} \otimes a_{1}\right)=a_{a_{0}} \otimes a_{a_{1}}$,
(10) $b\left(a_{0} \otimes a_{1}\right)=\left(b \otimes a_{0} a_{1} a_{1}^{-1}\right)\left(a_{0} \otimes a_{1}\right)$,
(11) $a\left(a_{0} \otimes b_{1}\right)=\left(a a_{0} \otimes b_{1}\right)\left(a_{\otimes} \otimes b_{1}\right)^{-1}$,
(12) $b^{b}\left(a_{0} \otimes b_{1}\right)=\left(a_{0} \otimes b\right)^{-1}\left(a_{0} \otimes b b_{1}\right)$,
(13) $\left(b_{0} \otimes b_{1}\right)\left(a_{0} \otimes b_{1}^{\prime}\right)=$
$\left(a_{0} \otimes b_{0} b_{1} b_{1}^{-1}\right)^{-1}\left(a_{0} \otimes b_{0 b_{1}} b_{1}^{-1} b_{1}^{\prime}\right)\left(b_{0} \otimes b_{1}\right)$,

$$
\begin{equation*}
\left(b_{0} \otimes b_{1}\right)\left(b_{0}^{\prime} \otimes a_{1}\right)= \tag{14}
\end{equation*}
$$

$$
\begin{equation*}
\left(b_{0} b_{1} b_{0}-1 b_{0} \cdot \otimes a_{1}\right)\left(b_{0} b_{1} b_{0}^{-1} \otimes a_{1}\right)^{-1}\left(b_{0} \otimes b_{1}\right), \tag{15}
\end{equation*}
$$ $\left(b_{0} \otimes b_{1}\right)\left(a_{0} \otimes a_{1}\right)=$

$$
\left(b_{0} b_{1} b_{0}^{-1} \otimes a_{0} a_{1}^{-1} a_{1}\right)\left(a_{0} \otimes a_{1}\right)\left(b_{0} \otimes b_{1}\right),
$$

where $a_{0} \in M_{A}, b_{0}, b_{0} ' \in M_{B}, a_{1} \in N_{A}, b_{1}, b_{1} ' \in N_{B}$, and where a $\in M_{A}$ or $N_{A}, b \in M_{B}$ or $N_{B}$ as appropriate.

Let $x_{0} \otimes x_{1}$ be a generator of $M \otimes N$, with $x_{0}=$ $a_{0} b_{0} \ldots a_{0} b_{0} \prime^{\prime}, x_{1}=a_{1} b_{1} \ldots a_{1} b_{1}^{\prime}$. By repeated application of the product rules (1) and (2) we see that $x_{0} \otimes x_{1}$ can be written as a product of elements of the form $p\left(a_{0} \otimes a_{1}\right)$, $p\left(b_{0} \otimes b_{1}\right), p\left(a_{0} \otimes b_{1}\right), p\left(b_{0} \otimes a_{1}\right)$ with $p \in P$. Each element of this form can be broken down into a product of elements of the same type, without exponent $p$ appearing, by repeated use of rules (9),(10),(11),(12) and the duals of these rules. Thus any element $x$ of $M \otimes N$ can be written as a product of terms $\left(a_{0} \otimes a_{1}\right),\left(b_{0} \otimes b_{1}\right),\left(a_{0} \otimes b_{1}\right)$, $\left(b_{0} \otimes b_{1}\right)$.

Assume that $x$ is written as such a product. Now take each term $b_{0} \otimes b_{1}$ and commute it to the right (beginning with the farthermost right one and proceed one at a time) using the rules (13),(14),(15). When this has been done, take each term $a_{0} \otimes a_{1}$ and commute it to the left using the duals of the rules (13),(14),(15). This gives us x written as a product uvw as required. $\boldsymbol{\nabla}$

Let $\lambda: M \otimes N \rightarrow N, \lambda \prime: M \otimes N \rightarrow M$ be the maps $m \otimes n \rightarrow$ $m_{n} n^{-1}, m \otimes n \rightarrow m_{m}-1$ and define $\pi_{3}(M \otimes N)=\operatorname{ker}(\lambda: M \otimes N \rightarrow N) n \operatorname{ker}(\lambda 1: M \otimes N \rightarrow M)$. Similarly define $\pi_{3}(M \wedge N)=\operatorname{ker}(\lambda: M \Delta N \rightarrow N) \cap \operatorname{ker}\left(\lambda^{\prime}: M \Lambda N \rightarrow M\right)$, $\pi_{3}(M \triangle N)=\operatorname{ker}(\lambda: M \triangle N \rightarrow N) \cap \operatorname{ker}\left(\lambda^{\prime}: M \triangle N \rightarrow M\right)$. (This notation is in keeping with the topological
significance of crossed squares since, if a space $X$ is the classifying space (see [L]) of a crossed square

$$
\begin{array}{rll}
L & \rightarrow^{\lambda} & N \\
\lambda \cdot 1 & & 1^{\prime} \delta^{\prime} \\
M & \rightarrow B^{8} & P
\end{array}
$$

then $\pi_{3} X \cong \operatorname{ker} \lambda \cap \operatorname{ker} \lambda^{\prime}$. )

PROPOSITION(3.1.5) The function

$$
\left(M_{A} \otimes N_{A}\right) \times\left(M_{B} \otimes N_{B}\right) \rightarrow M \otimes N,(X, Y) \rightarrow(\iota A X)(\iota B Y)
$$

Induces homomorphisms
(i) $\quad \pi_{3}\left(M_{A} \otimes N_{A}\right) \times \pi_{3}\left(M_{B} \otimes N_{B}\right) \rightarrow \pi_{3}(M \otimes N)$,
(ii) $\quad \pi_{3}\left(M_{A} \wedge N_{A}\right) \times \pi_{3}\left(M_{B} \wedge N_{B}\right) \rightarrow \pi_{3}(M \wedge N)$,
(iii) $\quad \pi_{3}\left(M_{A} \Lambda N_{A}\right) \times \pi_{3}\left(M_{B} \Lambda N_{B}\right) \rightarrow \pi_{3}(M \underline{N})$.

PROOF if $(x, y)$ is an element of $\pi_{3}\left(M_{A} \otimes N_{A}\right) \times \pi_{3}\left(M_{B} \otimes N_{B}\right)$ then clearly $\left(\iota_{A} x\right)\left(\iota_{B Y}\right)$ is an element of $\pi_{3}(M \otimes N)$. Now $\pi_{3}(M \otimes N)$ lies in the kernel of a crossed module, and is therefore abelian. It follows that map (i) (and similarly maps (ii) and (iii)) is a homomorphism. $\nabla$

For an arbitrary group $G$ we shall denote by $G \otimes G$, $G \Lambda G, G \Omega G$ the tensor, exterior and asymmetric products belonging to the crossed squares

| $\mathbf{G} \otimes \mathbf{G}$ | $\rightarrow \mathbf{G}$ | $\mathbf{G} \Lambda \mathbf{G}$ | $\rightarrow \mathbf{G}$ | $\mathbf{G} \triangle \mathbf{G}$ | $\rightarrow$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |  |
| $\mathbf{L}$ | $=\mathbf{G}$ | $\mathbf{G}$ | $=\mathbf{G}$ | $\mathbf{G}$ | $=\mathbf{G}$ |

PROPOSITION(3.1.6) Let $G$ be an arbitrary group and let $i_{A}: A \rightarrow G, i_{B}: B \rightarrow G, \alpha: G \rightarrow A, \beta: G \rightarrow B$ be homomorphisms satisfying: $\quad \alpha i_{A}=1_{A}, \beta i_{B}=1_{B}, \alpha i_{B}=0, \beta i_{A}=0$.

Then the three maps of proposition (3.1.5)
(i) $\quad \pi_{3}(A \otimes A) \times \pi_{3}(B \otimes B) \rightarrow \pi_{3}(G \otimes G)$,
(ii) $\quad \pi_{3}(A \wedge A) \times \pi_{3}(B \wedge B) \rightarrow \pi_{3}(G \wedge G)$,
(iii) $\pi_{3}(A \wedge A) \times \pi_{3}(B \wedge B) \rightarrow \pi_{3}(G \wedge G)$,
are injective.
PROOF We shall just consider case (i). The map $\alpha$ induces a map $\alpha_{\#}: \pi_{3}\left(G^{\prime} \otimes G\right) \rightarrow \pi_{3}(A \otimes A)$, and the map $\beta$ induces a map $\beta_{\#}: \pi_{3}(G \otimes G) \rightarrow \pi_{3}(B \otimes B)$. The map $\left(\alpha_{\#}, \beta_{\#}\right): \pi_{3}(G \otimes G) \rightarrow$ $\pi_{3}(A \otimes A) \times \pi_{3}(B \otimes B)$ is readily seen to be a splitting of map (i). $\nabla$

PROPOSITION(3.1.7) Let $G=A * B$ be the free product of groups $A$ and $B$. Then the three maps of proposition (3.1.5)
(i) $\quad \pi_{3}(A \otimes A) \times \pi_{3}(B \otimes B) \rightarrow \pi_{3}(G \otimes G)$,
(ii) $\pi_{3}(A \wedge A) \times \pi_{3}(B \wedge B) \rightarrow \pi_{3}(G \wedge G)$,
(iii) $\pi_{3}(A \triangle A) \times \pi_{3}(B \triangle B) \rightarrow \pi_{3}(G \wedge G)$.
are injective. Maps (ii) and (iii) are surjective.
PROOF The injectivity of these maps follows from
proposition (3.1.6). We shall prove map (iii) surjective. Let $x \in \pi_{3}(G \triangle G)$. By lemma (3.1.4) we can assume that $x=$ uvw with $u \in \iota_{A}(A \cap A), w \in \iota_{B}(B \triangle B)$, and (using rule (8)) with $v$ a product of terms $\left(a_{0} \Lambda b_{1}\right)^{ \pm 1}$. It follows that $u \in$ $\left.{ }^{\prime} A^{\prime} \pi_{3}(A \triangle A)\right), w \in \iota_{B} \pi_{3}(B \triangle B)$, and that the image of $v$ in $G$ is trivial. Since the subgroup $[A, B] \subset G$ is free on the commutators $[a, b]$ with $a \in A \backslash\{e\}, b \in B \backslash\{e\}$, it follows that v must be trivial. Hence map (iii) is surjective.

To prove map (ii) surjective it suffices to note that rule (8) holds in the exterior product, and hence that the preceeding argument is valid for the exterior product. $\nabla$

EXAMPLE(3.1.8) Let $\mathrm{F}_{1}$ be the free group of rank 1 . It is easy to see that in this case the exterior product $\mathrm{F}_{1} \wedge \mathrm{~F}_{1}$ is trivial. Let $F_{n}$ be the free group of rank $n$. That is $F_{n}$ is the $n$-fold free product of $n$ copies of $F_{1}$. It Eollows from proposition (3.1.7) that $\operatorname{ker}\left(F_{n} \Lambda F_{n} \rightarrow \lambda F_{n}\right)$ is trivial, and hence that $k e r(F \wedge F \rightarrow \lambda F)$ is trivial for an arbitrary free group F. Thus there is an isomorphism F $\wedge$ F $\cong[F, F]$. This presentation of the commutator subgroup of a free group is essentially the presentation given by C. Miller [M] (see also [Ho]). (Indeed, the arguments used In the proofs of lemma (3.1.4) and proposition (3.1.7) are modifications of Miller's arguments.) This presentation is also obtained in $[B-L]$ as a corollary to the van Kampen type theorem for squares of maps.

EXAMPLE(3.1.9) It is easy to see that the asymmetric product $F_{I} \wedge F_{1}$ is isomorphic to $Z_{2}$ the group of order 2. It follows from proposition (3.1.7) that $\operatorname{ker}\left(F_{n} \triangle F_{n} \rightarrow F_{n}\right)$ is the direct sum of $n$ copies of $Z_{2}$.

PROPOSITION(3.1.10) Let $G=A \times B$ be the direct product of groups A and B. Then there are isomorphisms
(i) $\pi_{3}(A \wedge A) \times \pi_{3}(B \wedge B) \times A^{a b} \otimes_{Z} B^{a b} \equiv \pi_{3}(G \wedge G)$, (ii) $\pi_{3}(A \wedge A) \times \pi_{3}(B \triangle B) \times A^{a b} \otimes_{Z} B^{a b} \simeq \pi_{3}(G \wedge G)$. PROOF We shall just consider case (i). Note that by
proposition (3.1.7) there is an injection $\psi: \pi_{3}(A \wedge A) \times$ $\pi_{3}(B \wedge B) \rightarrow \pi_{3}(G \wedge G)$. There is a homomorphism $\phi: A^{a b} \otimes_{Z} B^{a b} \rightarrow \pi_{3}(G \Lambda G),(a, b) \rightarrow a \Lambda b$. For $(x, y, z) \epsilon$ $\pi_{3}(A \wedge A) \times \pi_{3}(B \wedge B) \times A^{a b} \otimes_{Z} B^{a b}, \operatorname{set} \theta(x, y, z)=$ $\psi(x, y) \phi z$. Clearly $\theta$ is a homomorphism. Let $\alpha_{\#}: \pi_{3}(G \Lambda G) \rightarrow$ $\pi_{3}(A \wedge A)$ be the map induced by the projection $G \rightarrow A$, and let $\beta_{\#}: \pi_{3}(G \wedge G) \rightarrow \pi_{3}(B \wedge B)$ be the map induced by the projection $G \rightarrow B$. Let $\gamma: G \wedge G \rightarrow A^{a b} \otimes_{Z} B^{a b}$ be the map $(a, b) \wedge\left(a^{\prime}, b^{\prime}\right) \rightarrow a[A, A] \otimes b[B, B]$, and let $\gamma_{\#}: \pi_{3}(G \wedge G) \rightarrow$ $A^{a b} \Omega_{2} B^{a b}$ be the restriction of $\gamma$. The map $\left(\alpha_{\#}, \beta_{\#}, \gamma_{\#}\right)$ is a splitting of the map $\theta$. Hence $\theta$ is injective. To show that $\theta$ is surjective suppose we have an arbitrary element $x$ $\epsilon \pi_{3}(G \wedge G)$. By lemma (3.1.4) we can write $x$ as a product $x=$ uvw with $u \in \mathcal{L}_{A}(A \wedge A), W \in \mathcal{L}_{B}(B \wedge B)$ and $v \in$ $\phi\left(A^{a b} \otimes_{Z} B^{a b}\right)$. But clearly uw is in the image of the map W. It follows that $\theta$ is surjective. $\nabla$

We shall now investigate the kernels of the quotient maps $M \otimes N \rightarrow M \Delta N, M \wedge N \rightarrow M \Lambda N$.

## Let

$$
\begin{array}{lll}
M \times P N & \rightarrow{ }^{\pi} 1 & N \\
\pi_{0}^{!} & & { }^{\prime} \delta^{\prime} \\
M & \rightarrow 0 & P
\end{array}
$$

be the pullback square. (It is interesting, but not of relevance here, to note that this square has a natural structure of a crossed square.) Let (M,N\} be the subgroup of $M \times P N$ generated by the elements $\left(m n_{m-1}, m_{n} n^{-h}\right)$ with $m \in$
$M, n \in N$. This subgroup is normal.
We can define a function $M X_{P N} \rightarrow M \otimes N,(m, n) \rightarrow m \otimes n$. One can check that this function induces a function from $M \times P N /\{M, N\}$ to $M \otimes N$ (see [B-L]). There is also, therefore, a function from $M X_{P} N /\{M, N\}$ to $M \triangle N$.

In order to analyse the kernel of $M \otimes N \rightarrow M \wedge N / r e c a l l$ the definition of Whitehead's $\Gamma$-functor [W2], which is the "universal quadratic functor" from abelian groups to abelian groups. Let $A$ be an abelian group. Then $\Gamma A$ is the abelian group with generators ra, for a $\epsilon A$, and the following relations:
(i) $\gamma(-a)=\gamma a$,
(ii) if $\beta(a, b)=\gamma(a+b)-\gamma a-\gamma b$, for $a, b \in B$, then $\beta: A \times A \rightarrow \Gamma A$ is biadditive.

PROPOSITION(3.1.11) [B-L] The quotient group $M \times p N /\{M, N\}$ is abelian, and there is an exact sequence
$\Gamma(M \times P N /\{M, N\}) \rightarrow M \otimes N \rightarrow M \wedge N \rightarrow 1$
where $\psi(\gamma(m, n))=m \otimes n$. Also, $\psi$ has central image.
PROOF The proof, which is a straightforward algebraic one, is given in [B-L]. $\nabla$

PROPOSITION(3.1.12) There is an exact sequence

$$
M x_{P} N /\{M, N\} \otimes_{Z} Z_{2} \rightarrow \psi^{\prime} M \triangle N \rightarrow M \Lambda N \rightarrow 1
$$

where $\psi^{\prime}(m, n)=(m \wedge n)$.
PROOF We have already noted that the function $M \times P N /\{M, N\} \rightarrow$ $M \triangle N,(m, n) \rightarrow m \otimes n$ is well defined. It is a homomorphism since

```
\(\left(m m^{\prime} \wedge n n^{\prime}\right)=\left(m^{\prime} \wedge n\right)\left(m \Lambda n^{\prime}\right)(m \Lambda n)\left(m^{\prime} \Lambda n^{\prime}\right)\)
    \(=(m \underline{n})\left(m^{\prime} \underline{\wedge} n^{\prime}\right)\)
```

for $(m, n),\left(m^{\prime}, n^{\prime}\right) \in M \times p N$. When $m=m^{\prime}$ and $n=n^{\prime}$ we see that
$(m m \wedge n n)=e$.
It follows that $\psi^{\prime}$ is a homomorphism. Clearly $\psi^{\prime}$ maps onto the kernel of the quotient map $M \wedge N \rightarrow M \wedge N$. $\nabla$

EXAMPLE(3.1.13) There is an exact sequence
$G^{a b} \otimes_{Z} Z_{2} \rightarrow \psi^{\prime} G \wedge G \rightarrow G \wedge G \rightarrow 1$.
This particular case is given in [D]. For this case it is also shown in [D] that the map $\psi^{\prime}$ is injective and has a splitting. In general there is no reason to expect $\psi^{\prime}$ to be injective.

We shall end this section with two exact sequences, one involving the exterior product, the other involving the asymmetric product. The sequence involving the exterior product has been obtained previously as a consequence of the van Kampen theorem for squares of maps [B-L] and is of relevance to the homology of groups. In Chapter IV we shall explain this relevance using purely algebraic techniques.

THEOREM(3.1.14) Let M,N be normal subgroups of a group $G$ such that $G=\mathbb{N}$ the group product of the subgroups. Then there is an exact sequence
$\pi_{3}(M \wedge N) \rightarrow \psi_{5} \pi_{3}(G \wedge G) \boldsymbol{H}_{4} \quad \pi_{3}(G / M \wedge G / M) \times \pi_{3}(G / N \wedge G / N)$ $\rightarrow \psi_{3} \quad M \cap N /[M, N] \rightarrow \Psi_{2} \quad G^{a b} \rightarrow \Psi_{1} \quad(G / M)^{a b} \times(G / N)^{a b} \rightarrow 1$.

Also, by replacing the exterior product sign $\Lambda$ with the asymmetric product $\Lambda$, we obtain another exact sequence. PROOF We shall just consider the case of the exterior product. It will be convenient to have the following notation: if $H$ is a group with two normal subgroups $H_{0}, H_{1}$ then we shall denote by $\left\langle H_{0}, H_{1}\right\rangle_{H}$ the subgroup of $H \Lambda H$ generated by the elements $h_{0} \Lambda h_{1}$ with $h_{0} \in H_{0}, h_{1} \in H_{1}$. We must define the maps $\psi_{i}$. The map $\psi_{1}$. Let $\tau_{1}: G^{a b} \rightarrow(G / M)^{a b}, \tau_{2}: G^{a b} \rightarrow(G / N)^{a b}$ be the quotient maps, and define $\psi_{1}=\left(\tau_{1}, \tau_{2}\right)$.

The map $\psi_{2}$. This map is induced by the inclusion $M \cap N \rightarrow G$. The map $\psi_{3}$. Note that $G / M$ is isomorphic to $N / M \cap N$. It follows from proposition (3.1.3) that $G / M \Lambda G / M$ is isomorphic to $N \Lambda N /\langle N, N \cap M\rangle N$. There is a commutative diagram
$\downarrow$

$$
\text { ker } c^{\prime} \cong \pi_{3}(G / M \wedge G / M)
$$

$\downarrow$
$1 \rightarrow\langle N, N \cap M\rangle_{N} \rightarrow N \Lambda N \rightarrow N \wedge N /\langle N, N \cap M\rangle_{N} \rightarrow 1$
เc" bc bc'
$1 \rightarrow \mathrm{MnN} \rightarrow \mathrm{N} \rightarrow \mathrm{N} / \mathrm{MnN} \rightarrow 1$
in which $c\left(n \wedge n^{\prime}\right)=\left[n, n^{\prime}\right]$, and in which the rows and columns are exact. The diagram gives rise to a map $T_{3}: \pi_{3}(G / M \wedge G / M) \rightarrow[N, N] \cap M /[N, M \cap N]$. Similarly there is a $\operatorname{map} \tau_{4}: \pi_{3}(G / N \Lambda G / N) \rightarrow[M, M] \cap N /[M, M \cap N]$. Let $l:[N, N] \cap M /[N, M \cap N] \rightarrow M n N /[M, N]$ be the map induced by the
inclusion $[N, N] \cap M \rightarrow M N N$, let $l^{\prime}:[M, M] \cap N /[M, M \cap N] \rightarrow M \cap N /[M, N]$ be induced by the inclusion $[M, M] \cap N \rightarrow M N N$, and define $\psi_{3}$ to be the map $\psi_{3}(x, Y)=\left(\iota \tau_{3} x\right)\left(\iota ' \tau_{4} Y\right)$.

The map $\psi_{4} \cdot$ Let $\tau_{5}: \pi_{3}(G \wedge G) \rightarrow \pi_{3}(G / M \wedge G / M)$ be the map induced by the quotient map $G \rightarrow G / M$, and let $\tau_{6}: \pi_{3}(G \wedge G) \rightarrow$ $\pi_{3}(G / N \Lambda G / N)$ be induced by the quotient map $G \rightarrow G / N$.

Define $\Psi_{4}=\left(\tau_{5}, \tau_{6}\right)$
The map $\psi_{5}$. This map is induced by the inclusions $M \rightarrow G$ and $N \rightarrow G$.

We must now check exactness.
Let $x=\left(g M[G, G], g^{\prime} N[G, G]\right)$ be an arbitrary element of $(G / M)^{a b} \times(G / N)^{a b}$. Since $G=M N$ we can assume that $g=m n$, $g^{\prime}=m^{\prime} n^{\prime}$ for some $m, m^{\prime} \in M, n, n^{\prime} \in \mathbb{N}$. The element $n m^{\prime}[G, G]$ in $G^{a b}$ is mapped onto $x$ by $\psi_{1}$. Thus $\psi_{1}$ is surjective.

The kernel of $\psi_{1}$ is $M \cap N /[G, G] \cap M \cap N$. Certainly the image of $\psi_{2}$ is equal to the kernel of $\psi_{1}$.

The kernel of $\psi_{2}$ is $[G, G] \cap M O N /[M, N]$. Denote by $K$ the group $([M, M] \cap N)([N, N] \cap M) /[M, M][N, N] \cap[M, N]$. To see that the image of $\psi_{3}$ is equal to the kernel of $\psi_{2}$ we have only to show that there is a commutative diagram of groups

| $\pi_{3}(G / M \wedge G / M)$ | $\times \pi_{3}(G / N \wedge G / N)$ | $\rightarrow W_{3}$ | $M \cap N /[M, N]$ |
| :---: | :--- | :---: | :---: |
|  | $\downarrow \alpha$ |  | $\uparrow i$ |
| $K$ | $\rightarrow B$ | ker $\psi_{2}$ |  |

in which $\alpha$ and $\beta$ are surjective and $i$ is the inclusion. Let $\rho:[N, N] \cap M /[N, M \cap N] \rightarrow K$ be the map induced by the inclusion $[N, N] \cap M \rightarrow([M, M] \cap N)([N, N] \cap M)$, and let
$\rho^{\prime}:[M, M] \cap N /[M, M \cap N] \rightarrow K$ be induced by the inclusion $[M, M] \cap N$ $\rightarrow([M, M] \cap N)([N, N] \cap M)$. Then $\alpha$ is defined by $\alpha(x, y)=$ ( $\rho \tau_{3} \mathrm{X}$ ) ( $\rho^{\prime} \tau_{4} y$ ) where $\tau_{3}, \tau_{4}$ are given above. The map $\alpha$ is surjective since the maps $r_{3}, \tau_{4}, \rho, \rho^{\prime}$ are all surjective. The map $\beta$ is induced by the inclusion $([M, M] \cap N)([N, N] \cap M) \rightarrow$ [G,G]nMnN. To see that $\beta$ is surjective it suffices to note that $[G, G]=[M, M][N, N][M, N]$, and that $[M, M][N, N] \cap M \cap N=$ ( $[M, M] \cap N)([N, N] \cap M)$.

We now aim to show that the kernel of $\psi_{3}$ is equal to the image of $\Psi_{4}$. The following commutative diagram of canonical maps

(where $\phi$ is induced by the map $G \Lambda G \rightarrow G, g \Lambda g^{\prime} \rightarrow\left[g^{\prime} g^{\prime}\right]$ ) gives rise to a map $\tau_{7}: \operatorname{ker} \phi \rightarrow \pi_{3}(G / M \Lambda G / M)$. Similarly we have a map $\tau_{8}: \operatorname{ker} \phi \rightarrow \pi_{3}(G / N \Lambda G / N)$.

Consider the commutative diagram

where $\gamma$ is induced by the quotient map $G \Lambda G \rightarrow$ $G \wedge G /\langle M, M\rangle_{G}$, and $l$ is the inclusion. It is readily seen
that $\gamma$ is surjective. To prove that the kernel of $\psi_{3}$ is equal to the image of $\psi_{4}$ it suffices to show that the map ( $\tau_{7}, \tau_{8}$ ) is surjective: let ( $\mathrm{x}, \mathrm{y}$ ) be an arbitrary element of $\pi_{3}(G / M \wedge G / M) \times \pi_{3}(G / N \wedge G / N) ;$ since $G=M N$ we can assume that $x$ is a product $x=\left(n_{l} M \wedge n_{l}{ }^{\prime} M\right) \ldots\left(n_{\ell} M \wedge n_{\ell}{ }^{\prime M}\right)$ with $n_{i}, n_{i}^{\prime} \in N$, and that $y$ is a product $y=\left(m_{1} N \wedge m_{l}{ }^{\prime} N\right) \ldots$ $\cdots\left(m_{\ell} N \wedge m_{\ell}{ }^{\prime} N\right)$ with $m_{i}, m_{i}{ }^{\prime} \in M$; the element $\left(n_{l} \wedge n_{l}^{\prime}\right) \ldots\left(n_{l} \wedge n_{l}{ }^{\prime}\right)\left(m_{l} \wedge m_{l}{ }^{\prime}\right) \ldots\left(m_{l} \wedge m_{l}{ }^{\prime}\right)\langle M, N\rangle_{G}$ of ker $\phi$ is mapped onto $(x, y)$ by $\left(\tau_{7}, \tau_{8}\right)$. Thus $\left(\tau_{7}, \tau_{8}\right)$ is surjective.

We now want to show that the image of $\psi_{5}$ is equal to the kernel of $\psi_{4}$. By proposition (3.1.3) there are isomorphisms $G / M \wedge G / M \cong G \Lambda G /\langle G, M\rangle_{G}$ and $G / N \Lambda G / N \cong$ $G \Lambda G /\langle G, N\rangle_{G}$. It is thus clear that the kernel of $\psi_{4}$ is the intersection $\langle G, M\rangle_{G} \cap\langle G, N\rangle_{G} \cap \pi_{3}(G \wedge G)$. We shall show that $\langle G, M\rangle_{G} \cap\langle G, N\rangle_{G}=\langle M, N\rangle_{G}$, and it will then be clear that the image of $\psi_{5}$ is the kernel of $\psi_{4}$.

Certainly $\langle G, M\rangle_{G} \cap\langle G, N\rangle_{G} \supset\langle M, N\rangle_{G}$. Suppose that $x \in$ $\langle G, M\rangle_{G} \cap\langle G, N\rangle_{G}$. Since $G=\mathbb{M}$ and since $x \in\langle G, M\rangle_{G}$, we can write $x$ as a product $x=x_{0} x_{1}$ with $x_{0} \in\langle M, M\rangle_{G} \cap\langle G, N\rangle_{G}$ and $x_{1} \in\langle M, N\rangle_{G}$. Note that $x_{0}$ is in the kernel of the canonical $\operatorname{map}\langle M, M\rangle_{G} \rightarrow G \Lambda G \rightarrow G \Lambda G /\langle G, N\rangle_{G} \approx M \Lambda M /\langle M, M \cap N\rangle_{M}$. Thus $x_{0} \in\left\langle M_{M} M \cap N\right\rangle_{G}$ and hence $x \in\langle M, N\rangle_{G}$. $\nabla$

Suppose in this last theorem that we have $N=G$. We can extend the exact sequence(s) by two terms since, if we have a commutative diagram of groups

|  |  | 1 |  | 1 |
| :---: | :---: | :---: | :---: | :---: |
|  |  | $\downarrow$ |  | $\downarrow$ |
|  |  | R | $=$ | R |
|  |  | $\downarrow$ |  | 1 |
| 1 | $\rightarrow$ | S | $\rightarrow$ | F |
|  |  | $\downarrow$ |  | 1 |
| 1 | $\rightarrow$ | M | $\rightarrow$ | G |
|  |  | 1 |  | $\downarrow$ |
|  |  | 1 |  | 1 |

in which the rows and columns are exact, and the group $F$ is free, then

THEOREM(3.1.15) There is an exact sequence $\pi_{3}(F \wedge R) \rightarrow \psi_{7} \quad \pi_{3}(F \wedge S) \rightarrow \psi_{6} \quad \pi_{3}(G \wedge M) \quad \psi_{5} \quad \pi_{3}(G \wedge G) \quad$. where $\psi_{5}$ is as in the preceding proposition. Again, by replacing the exterior product sign $\Lambda$ with the asymmetric product $\Lambda$, we obtain another exact sequence. PROOF Using proposition (3.1.3) we see that we have a commutative diagram of canonical maps

in which the columns are exact, in which $\langle F, R\rangle$ denotes the subgroup of $F \Lambda S$ generated by the elements $f \Lambda x$ with $f \in$ $F, r \in R$, and in which $c$ is the map $f \Lambda s \rightarrow[f, s]$. This diagram induces the maps $\psi_{5}, \psi_{6}, \psi_{7}$. We must check exactness.

Suppose $x$ is an element in the kernel of $\psi_{5}$. We can represent $x$ by an element $\langle x\rangle$ of $F \wedge S$, and we have $c\langle x\rangle \epsilon$ [F,R]. There exists an $\langle x\rangle ' \in\langle F, R\rangle$ such that $c\langle x\rangle$ ' = $c\langle x\rangle^{-1}$. The product $\langle x\rangle\langle x\rangle$ ' $\in \pi_{3}(F \Lambda S)$ also represents $x$. Thus $\psi_{6}$ maps onto the kernel of $\psi_{5}$.

Clearly the image of $\psi_{7}$ is equal to the kernel of $\psi_{6}$. $\nabla$
2. A UNIVERSAL CROSSED 3-CUBE IN GROUPS

We shall now look at the "3-dimensional" analogue of the tensor product of groups of the preceding section.

Suppose given a diagram of groups

(*)
in which $S$ acts on each of the other groups in such a way that each of the three faces has a crossed square structure. We want to construct a group $T$ and maps $\lambda_{L}: T \rightarrow$ $L, \lambda_{M}: T \rightarrow M, \lambda_{N}: T \rightarrow N$, such that the resulting cubical diagram is a crossed 3-cube with the following defining universal property (see proposition (1.4.1)): for any other group $T^{\prime}$ and maps $\lambda_{L}{ }^{\prime}, \lambda_{M^{\prime}}, \lambda_{N}{ }^{\prime}$ such that the resulting cubical diagram is a crossed 3-cube, there is a unique map $T \rightarrow T^{\prime}$ of crossed 3-cubes. We shall call $T$ the cubical tensor product of the above diagram.

Let $T_{0}$ be the group generated by the elements $q \otimes_{1} 1, p \otimes_{2} m, n \otimes_{3} r$
where $(q, I) \in Q \times I,(p, m) \in P \times M,(n, r) \in N \times R$, subject to the following relations: (All actions are assumed to be via $S$, and we write ${ }^{s}\left(x \otimes_{i} y\right)$ instead of $\left.s_{x} \otimes_{i} s_{y}.\right)$
(i) $q \otimes_{1} 11^{\prime}=\left(q \otimes_{1} 1\right)^{l}\left(q \otimes_{1} l^{\prime}\right)$,
$q q^{\prime} \otimes_{1} l=q\left(q^{\prime} \otimes_{1} I\right)\left(q \otimes_{1} I\right)$.
$p p^{\prime} \otimes_{2} m=p\left(p^{\prime} \otimes_{2} m\right)\left(p \otimes_{2} m\right)$,
$p \otimes_{2} \mathrm{~mm}^{\prime}=\left(p \otimes_{2} m\right)^{m}\left(p \otimes_{2} m^{\prime}\right)$,
$n n^{\prime} \otimes_{3} r=n^{\prime}\left(n^{\prime} \otimes_{3} r\right)\left(n \otimes_{3} r\right)$,
$n \otimes_{3} r r^{\prime}=\left(n \otimes_{3} r\right) r\left(n \otimes_{3} r^{\prime}\right) ;$
(ii) $\left(p \otimes_{2} m\right)\left(q \otimes_{1} 1\right)\left(p \otimes_{2} m\right)^{-1}=[p, m]\left(q \otimes_{1} 1\right)$, $\left(n \otimes_{3} r\right)\left(q \otimes_{1} 1\right)\left(n \otimes_{3} r\right)^{-1}=[n, r]\left(q \otimes_{1} 1\right)$, $\left(q \otimes_{1} 1\right)\left(p \otimes_{2} m\right)\left(q \otimes_{1} I\right)^{-1}=[q, 1]\left(p \otimes_{2} m\right)$, $\left(n \otimes_{3} r\right)\left(p \otimes_{2} m\right)\left(n \otimes_{3} r\right)^{-1}=[n, r]\left(p \otimes_{2} m\right)$, $\left(q \otimes_{1} 1\right)\left(n \otimes_{3} r\right)\left(q \otimes_{1} 1\right)^{-1}=[q, 1]\left(n \otimes_{3} r\right)$,
$\left(p \otimes_{2} m\right)\left(n \otimes_{3} r\right)\left(p \otimes_{2} m\right)^{-1}=[p, m]\left(n \otimes_{3} r\right)$,

$$
\begin{aligned}
& \text { (iii) } \nu_{Q}\left(p_{m} m^{-1}\right) \otimes_{1} I=\left(p \otimes_{2} m\right)^{1}\left(p \otimes_{2} m\right)^{-1} \text {, } \\
& \nu_{Q}\left(n r^{-1}\right) \otimes_{1} 1=\left(n \otimes_{3} r\right)^{1}\left(n \otimes_{3} r\right)^{-1} \text {, } \\
& \nu_{p}\left(q_{1} 1^{-1}\right) \otimes_{2} m=\left(q \otimes_{1} 1\right)^{m}\left(q \otimes_{1} 1\right)^{-1} \text {, } \\
& \nu_{p}\left(n r^{-1}\right) \otimes_{2} m=\left(n \otimes_{3} r\right)^{m}\left(n \otimes_{3} r\right)^{-1} \text {, } \\
& n \otimes_{3} \nu_{R}\left(p_{m} m^{-1}\right)={ }^{n}\left(p \otimes_{2} m\right)\left(p \otimes_{2} m\right)^{-1} \text {, } \\
& n \otimes_{3} \nu_{R}\left(q_{1} 1^{-1}\right)=n\left(q \otimes_{1} 1\right)\left(q \otimes_{1} 1\right)^{-1} \text {, } \\
& p \otimes_{2} h\left(q, \nu_{R} l\right)=p\left(q \otimes_{1} l\right)\left(q \otimes_{1} 1\right)^{-1} \text {, } \\
& p \otimes_{2} h(\nu Q n, r)=p\left(n \otimes_{3} r\right)\left(n \otimes_{3} r\right)^{-1}, \\
& q \otimes_{1} h\left(p, \nu_{R} m\right)=q\left(p \otimes_{2} m\right)\left(p \otimes_{2} m\right)^{-1}, \\
& q \otimes_{1} h(\nu p n, r)=q\left(n \otimes_{3} r\right)\left(n \otimes_{3} r\right)^{-1}, \\
& h\left(\nu_{p} l, q\right) \otimes_{3} r=\left(1 \otimes_{1} q\right)^{r}\left(1 \otimes_{1} q\right)^{-1}, \\
& h\left(p, \nu Q^{m}\right) \otimes_{3} r=\left(p \otimes_{2} m\right)^{r}\left(p \otimes_{2} m\right)^{-1} ;
\end{aligned}
$$

(iv) $\left(\left(\nu_{P n}\right)\left(\nu_{P} l\right) \otimes_{2} m\right)\left(\left(\nu_{Q} m\right)\left(\nu_{Q^{n}}\right) \otimes_{1} 1\right)$
$=\left(n \otimes_{3}\left(\nu_{R}\right)^{\prime}\left(\nu_{R} m\right)\right)$,
(v) $q\left(h\left(p, q^{-1}\right)^{-1} \otimes_{3} r\right)$
$=p\left(q \otimes_{1} h\left(p^{-1}, r\right)\right)^{r}\left(p \otimes_{2} h\left(q, r^{-1}\right)^{-1}\right) ;$
(vi) $\nu_{Q^{m}} \otimes_{1} 1=\left(\nu_{\mathrm{P} I} \otimes_{2} m\right)^{-1}$,
$\nu_{\mathrm{pn}} \otimes_{2} \mathrm{~m}=\mathrm{n} \otimes_{3} \nu_{\mathrm{R}} \mathrm{m}$,
$\nu_{Q}{ }^{n} \otimes_{1} 1=n \otimes_{3} \nu_{R}{ }^{1} ;$
for all $1, l^{\prime} \in L, m^{\prime} m^{\prime} \in M, n, n^{\prime} \in N, p, p^{\prime} \in P, q, q ' \in Q$, rr' $\in$ R.

Note that there is a group action of $S$ on $T_{0}$ given by

$$
\begin{aligned}
& { }^{s}\left(q \otimes_{1} 1\right)=s_{q} \otimes_{1} s_{1}, \\
& s^{s}\left(p \otimes_{2} m\right)=s_{p} \otimes_{2} s_{m}, \\
& { }^{3}\left(n \otimes_{3} r\right)=s_{n} \otimes_{3} s_{r} . \\
& \text { Define maps } \lambda_{L}: T_{0} \rightarrow L_{1}, \lambda_{M}: T_{0} \rightarrow M, \lambda_{N}: T_{0} \rightarrow N \text { on }
\end{aligned}
$$

generators by

$$
\begin{aligned}
& \lambda_{L_{1}}\left(q \otimes_{1} 1\right)=q 11^{-1}, \\
& \lambda_{L}\left(p \otimes_{2} m\right)=h\left(p, \nu_{R} m\right) \text {, } \\
& \lambda_{L}\left(n \otimes_{3} r\right)=h(\nu P n, r), \\
& \lambda_{M}\left(q \otimes_{1} 1\right)=h\left(q, \nu_{R} I\right) \text {, } \\
& \lambda_{M}\left(p \otimes_{2} m\right)=p_{m} m^{-1} \text {, } \\
& \lambda_{M}\left(n \otimes_{3} r\right)=h\left(\nu_{Q} n, r\right), \\
& \lambda_{N}\left(q \otimes_{1} 1\right)=h\left(\nu_{p} 1, q\right)^{-1} \text {, } \\
& \lambda_{N}\left(p \otimes_{2} m\right)=h\left(p, \nu_{Q} m\right), \\
& \lambda_{N}\left(n \theta_{3} r\right)=n r^{-1} \text {. }
\end{aligned}
$$

It is routine to check that these maps are well defined.
Let us define three functions
$h: Q \times L \rightarrow T_{0},(q, 1) \rightarrow q \otimes_{1} 1$,
$h: P \times M \rightarrow T_{0},(p, m) \rightarrow p \otimes_{2} m$,
$h: N \times R \rightarrow T_{0},(n, r) \rightarrow n \bigotimes_{3} r$.

PROPOSITION(3.2.1) The group $T_{0}$ is the cubical tensor product of diagram (*).

PROOF We must check that the above cubical structure is a crossed 3-cube.

Note that the cubical diagram is commutative, and that axiom (2.1.3.iv) holds. Axioms (2.1.3,ii,iii, v) follow respectively from the identities (iv,v,vi) above. It remains to check rule (2.1.3.i).


For $t \in T_{0}, r \in R$ we clearly have $\lambda_{L}\left(r_{t}\right)=r_{\lambda_{L}}(t)$. For t,t' $\in T_{0}$ we have (using the first two identities of (i), and the first two identities of (ii) above) that ( $\left.\lambda_{L} t\right) t^{\prime}=$ tt' $t^{-1}$. Thus $\lambda_{L}$ (and similarly $\lambda_{M}$ ) is a crossed module and preserves the action of $R$. The function $h: L \times M \rightarrow T_{0}$, $(1, m) \rightarrow v_{\mathrm{pl}} \otimes_{2} m$ certainly satisfies axioms (2.1.2,iii, iv, $v$, ). Axiom (2.l.2.vi) follows from (iii) above. Thus the square under consideration is a crossed square. Similarly the square

$$
\begin{array}{cc}
T_{0} \rightarrow \nu_{Q} \lambda_{M} & Q \\
\lambda_{I_{1}}{ }^{\downarrow} & \\
{\left[\downarrow_{0}\right.} & \rightarrow \delta \nu_{R}
\end{array} \quad S
$$

is a crossed square. By symmetry it follows that rule (2.1.3.i) holds. Hence the cubical structure under consideration is a crossed 3-cube. This crossed 3-cube clearly has the required universal property. $\nabla$

In certain special cases the cubical tensor product has a simpler presentation.

It follows that each of the groups $L, M, N, P, Q, R$ is abelian, and that all actions are trivial, and that the cubical tensor product $T$ is the quotient of the direct sum of (standard, abelian) tensor products
$T=\left(Q \otimes_{Z} L\right) \oplus\left(P \otimes_{Z} M\right) \oplus\left(N \otimes_{Z} R\right) / \sim$
obtained by factoring out the relations
$\left(h\left(p, q^{-1}\right)^{-1} \otimes r\right)=\left(q \otimes h\left(p^{-1}, r\right)\right)+\left(p \otimes h\left(q, r^{-1}\right)^{-1}\right)$, or equivalently the relations

$$
(h(p, q) \otimes r)=(p \otimes h(q, r))-(q \otimes h(p, r)),
$$

where $p \in P, q \in Q, r \in R$.

EXAMPLE(3.2.3) Suppose that all the maps in (*) are trivial, and that $L=P \otimes_{Z} R, M=Q \otimes_{Z} R, N=P \otimes_{Z} Q$ (where these are standard tensor products of abelian groups). From the preceding example we see that the cubical tensor product in this case is the quotient $T=\left(Q \otimes_{Z}\left(P \otimes_{Z} R\right)\right) \oplus\left(P \otimes_{Z}\left(Q \otimes_{Z} R\right)\right) \oplus\left(\left(P \otimes_{Z} Q\right) \otimes_{Z} R\right) / \sim$ obtained by factoring the relations

$$
((p \otimes q) \otimes r)=(p \otimes(q \otimes r))-(q \otimes(p \otimes r)),
$$

where $p \in P, q \in Q, r \in R$.
That is, the cubical tensor product $T$ is isomorphic to $\left(P \otimes_{Z} Q \otimes_{Z} R\right) \oplus\left(P \otimes_{Z} Q \otimes_{Z} R\right)$.

PROROSITION(3.2.4) Suppose that the maps $\nu_{P}, \nu_{Q}, \nu_{R}$ in diagram (*) are surjective, and let $T$ ' be the group with a presentation consisting of generators $q \otimes 1$ for $(q, 1) \in$ Q $\times 1$, and relations:
(i)

$$
\begin{aligned}
& q \otimes 11^{\prime}=(q \otimes 1) 1\left(q \otimes 1^{\prime}\right) \\
& q q^{\prime} \otimes 1=q\left(q^{\prime} \otimes 1\right)(q \otimes 1) ;
\end{aligned}
$$

(ii) $q \otimes I=q \otimes I^{\prime} \quad$ whenever $\nu_{R} l=\nu_{p l} l^{\prime}$ or $\nu_{R} l=\nu_{R} I^{\prime} ;$
(iii) $\nu_{\mathrm{Q}} \mathrm{m} \otimes 1=\left(\nu_{\mathrm{Q}} \otimes \mathrm{l}^{\prime}\right)^{-1} \quad$ whenever $\nu_{\mathrm{P}}=\nu_{\mathrm{P}}$ and

$$
\nu_{R^{m}}=\nu_{R^{\prime}} l^{\prime} ;
$$

for all $1,1^{\prime} \in L, m \in M, n \in N, q, q^{\prime} \in Q$. Then $T^{\prime}$ is the cubical tensor product of the diagram (*).

PROOF Let $T_{0}$ be the cubical tensor product with the presentation of proposition (3.2.1). There is certainly a homomorphism $\psi: T^{\prime} \rightarrow T_{0}$ given on generators by $q \otimes 1 \rightarrow$ $q \otimes_{1}$ 1. We must construct an inverse to $\psi$.

Let us construct a set map $\psi^{\prime}$ from the generators of $T_{0}$ to the group $T$ ' by defining

```
*'(q\otimes | 1) = q* 1,
\mp@subsup{\psi}{}{\prime}}(p\mp@subsup{\otimes}{2}{}m)=(\mp@subsup{\nu}{Q}{m}\otimes1\mp@subsup{)}{}{-1}\mathrm{ for I EL such that }\mp@subsup{\nu}{P}{}l=p\mathrm{ ,
```

 It is clear that $\psi^{\prime}$ is independent of any choices. It is also clear that if we extend $\psi^{\prime}$ to a map from the free group on the generators of $T_{0}$ to $T^{\prime}$, then $\psi^{\prime}$ annihilates the relations (i), (ii), (iii) and (vi) of the presentation of $T_{0}$. It remains to check that $\psi^{\prime}$ also annihilates relations (iv) and (v). That is, we must check that the following identities hold in $T$ ':

$$
\nu_{Q^{n}} \otimes 1 x=\left(\nu_{Q} m \otimes y 1\right)^{-1}\left(\left(\nu_{Q} m\right)\left(\nu_{Q} n\right) \otimes 1\right)
$$

for $x, y, 1 \in L, m \in M, n \in N$ with $\nu_{R} x=\nu_{R} m$ and $\nu_{P Y}=\nu_{p n}$; and

$$
\left.q\left(q^{-1} v_{q} \otimes u\right)=v\left(q \otimes\left[v^{-1}, u\right]\right) u_{\left(u^{-1}\right.} q q^{-1} \otimes v\right)^{-1}
$$

for $u, v \in L, q \in Q$.
The check is done in Appendix III, verifications (1) and (2). $\nabla$

EXAMPLE(3.2.5) Suppose that $L=M=N=P=Q=R=S$ ( $=G$ say), and that all maps in (*) are the identity. Then it follows from proposition (3.2.4) that the cubical tensor product in this case is isomorphic to the asymmetric product $G \triangle G$ introduced in the last section.
3. SOME UNIVERSAL CROSSED SQUARES IN LIE ALGEBRAS

Let us fix a commutative ring A (with unit), and assume all Lie algebras to be over A.

We shall begin this section by recalling the construction of a free Lie algebra.

A set $V$ with a map $V X V \rightarrow V,\left(V, V^{\prime}\right)+V V^{\prime}$, is called a magma. Given an arbitrary set $X$ we can construct a magma $V_{X}$ : let $\left\{X_{n}\right\}$ be the family of sets with $X_{1}=X$ and $X_{n}(n \geqslant$ 2) the disjoint union of the sets $X_{p} \times X_{q}$ such that $p+q=$ $n$; let $V_{X}$ be the disjoint union of the sets $X_{n}$, and let $V_{X} \times V_{X} \rightarrow V_{X}$ be the map induced by the canonical map $X_{p} \times X_{q} \rightarrow$ $X_{p+q} \subset V_{X}$. The magma $V_{X}$ is called the free magma on $X$.

Given an arbitrary magma $V$ we can construct the free A-algebra FV whose elements $\alpha$ are finite sums $\alpha=\Sigma a_{V} v$ with $a_{V} \in A, v \in V$; the multiplication in $F V$ extends the multiplication in $V$.

Let I be the two sided ideal of FV generated by the elements $\alpha \alpha,(\alpha \beta) \gamma+(\beta \gamma) \alpha+(\gamma \alpha) \beta$ with $\alpha, \beta, \gamma \in F V$. Let $L V$ be the quotient $F V / I . L V$ is a Lie algebra. The Lie algebra $\mathrm{r}_{\mathrm{X}}$ is called the free Lie algebra on X .

Let $M, N$ be Lie algebras such that there is a fie action of $M$ (resp. N) on $N(r e s p . ~ M)$. We define their tensor
product $M \otimes N$ to be the quotient algebra $L V(M \times N) / J$ where LV (MXN) is the free Lie algebra on the set of elements $m \otimes n$ with $(m, n) \in M \times N$, and where $J$ is the ideal generated by the relations
$a(m \otimes n)=a m \otimes n=m \otimes a n ;$
(ii) $\left(m+m^{\prime}\right) \otimes n=m \otimes n+m^{\prime} \otimes n$,
$m \otimes\left(n+n^{\prime}\right)=m \otimes n+m \otimes n^{\prime} ;$
(iii) $\left[m, m^{\prime}\right] \otimes n=m \otimes m^{\prime} n-m^{\prime} \otimes m_{n}$,
$\left.m \otimes\left[n, n^{\prime}\right]=n^{\prime} m \otimes n-n_{m} \otimes n^{\prime} ; i v\right)\left[m \otimes n, m^{\prime} \otimes n^{\prime}\right]=\left(-n_{m} \otimes m^{\prime} n^{\prime}\right) ;$
for a $\in A, m^{\prime} m^{\prime} \in M, n_{1} n^{\prime} \in N$.
Note that if the actions of $M$ and $N$ are trivial (i.e.
if $m_{n}=0, n_{m}=0$ for all $m \in M, n \in N$ ) then the tensor product $M \otimes N$ is just the standard tensor product of A-modules $M^{a b} \otimes_{A} N^{a b}$, where $M^{a b}=M /[M, M], N^{a b}=N /[N, N]$.

Suppose that the rie algebras $M, N$ and their actions are obtained from crossed modules $\delta: M \rightarrow P, \delta^{\prime}: N \rightarrow P$. Then it is routine to show that the tensor product $M \otimes N$ fits into a crossed square

$$
M \otimes N \quad \rightarrow^{\lambda} N
$$


in which: the action of $P$ on $M \otimes N$ is given by $P(m \otimes n)=p_{m} \otimes n+m \otimes n_{n} ;$
the maps $\lambda, \lambda^{\prime}$ are given respectively by $m \otimes n \rightarrow m_{n, m} \otimes n \rightarrow$ $-n_{m}$; the function $h: M \times N \rightarrow M \otimes N$ is $(m, n) \rightarrow m \otimes n$.

This crossed square has a defining universal property
(cf. proposition (1.4.1)).

We define the exterior product $M \wedge N$ to be the quotient of $M \otimes N$ obtained by factoring out the relations: (v) $m \otimes n=0$, whenever $\delta \mathrm{m}=\delta^{\prime} \mathrm{n}$.

We shall denote by $m \wedge n$ the element of $M \triangle N$ which is represented by the element $m \otimes n$ in $M \otimes N$.

The crossed square containing $M \otimes N$ is preserved by the quotient map $M \otimes N \rightarrow M \wedge N$. Thus the exterior product M $\mathrm{A} N$ also fits into a crossed square.

Given an arbitrary Lie algebra L, we shall denote by L $\wedge \mathrm{L}$ the exterior product belonging to the crossed square

| $L \Lambda L$ | $\rightarrow$ | $L$ |
| :---: | :---: | :---: |
| + |  |  |
| $L$ | $=$ | $L$ |

PROPOSITION(3.3.1) Let LVX be the free Lie algebra on some set X . Then there is a Lie isomorphism
$\left[\mathrm{LV}_{\mathrm{X}}, \mathrm{LV} \mathrm{X}_{\mathrm{X}}\right] \equiv \mathrm{LV} \mathrm{X}_{\mathrm{X}} \wedge \mathrm{LV}_{\mathrm{X}}$.
PROOP The universal property of the exterior product gives us a homomorphism $\psi: L V_{X} \wedge L V_{X} \rightarrow\left[L V_{X}, L V_{X}\right], \ell \wedge \ell \prime \rightarrow$ [ $\ell, \ell$ ']. We need to construct an inverse to $\psi$.

Let $\mathrm{V}_{\mathrm{X}}$ be the free magma on X and let $\mathrm{V}_{\mathrm{X}}{ }^{1}=\mathrm{V}_{\mathrm{X}} \backslash \mathrm{X}$ be the submagma obtained by excluding the set X . Let ( $\mathrm{FV} \mathrm{V}_{\mathrm{X}}$ ( $\mathrm{FV} \mathrm{X}_{\mathrm{X}}$ ) be the subalgebra of the free algebra $F V X$ generated ${ }^{b y}$ those elements $\alpha$ which can be written as a product $\alpha=\alpha_{0} \alpha_{1}$ with $\alpha_{0}, \alpha_{1} \in$
$F V_{X}$. There is a canonical isomorphism $\left(F V_{X}\right)\left(F V_{X}\right) \cong F V_{X}{ }^{2}$.
The ideal. [ c $F V_{X}$ which is generated by the elements $\alpha \alpha$, $(\alpha \beta) \gamma+(\beta \gamma) \alpha+(\gamma \alpha) \beta$ with $\alpha, \beta, \gamma \in F V_{X}$, is contained in $\left(F V_{X}\right)\left(F V_{X}\right)$. Let $I^{\prime} \subset F V_{X}$ be the isomorphic image of I. Then [ $\left.L V_{X}, L V_{X}\right]$ is isomorphic to $\mathrm{FV}_{\mathrm{X}^{1} / I^{\prime}}$.

Each element $w \in V^{1}$ can be expressed uniquely as a product $w=u v$ with $u, v \in V_{X}$. The set map $V_{X}{ }^{1} \rightarrow \operatorname{LV} V_{X} \Lambda L V_{X}$, $W \rightarrow u \Lambda v$ extends to $a{ }^{*}$ homomorphism $\phi^{\prime}: F V X X^{l} \rightarrow$ $L V_{X} \wedge L V_{X}$. It is readily verified that $\phi^{\prime}$ induces a homomorphism $\phi:\left[L V_{X}, L V_{X}\right] \cong F V_{X 1 /} I^{\prime} \rightarrow L V_{X} \wedge L V_{X} . \quad$ The homomorphism $\phi$ is the inverse of $\psi$. $\nabla$

Note that the Lie algebra analogue of proposition (3.1.3) can easily be proved. Also, a straight forward translation of our group theoretic arguments gives us

THEOREM(3.3.2) Let $G$ be a Lie algebra containing two ideals $M, N$ such that any element $g \in G$ can be written as a sum $g=m+n$ with $m \in M, n \in N$. Then there is an exact sequence:

$$
\begin{aligned}
& \pi_{3}(M \wedge N) \rightarrow \pi_{3}(G \wedge G) \rightarrow \pi_{3}(G / M \wedge G / M) \oplus \pi_{3}(G / N \wedge G / N) \rightarrow \\
& \rightarrow M \cap N /[M, N] \rightarrow G^{a b} \rightarrow(G / M)^{a b} \oplus(G / N)^{a b} \rightarrow 0
\end{aligned}
$$

where $G^{a b}=G /[G, G]$ and $\pi_{3}(M \wedge N)=\operatorname{ker}(M \wedge N \rightarrow M, m \wedge n \rightarrow$ $[m, n]$ etc. $\nabla$
4. A UNIVERSAL CROSSED SQUARE IN COMMUTATIVE ALGEBRAS

Let us $f i x$ a commutative ring $A$ (with unit), and assume all commutative algebras to be over A.

The free commutative algebra $C V X$ on a set $X$ is the quotient $F V_{X} / I$ where $F V_{X}$ is the free algebra on $X$ (see \$3) and $I$ is the two sided ideal generated by the relations $\alpha \beta$ $=\beta \alpha,(\alpha \beta) \gamma=\alpha(\beta \gamma)$ for $\alpha, \beta, \gamma \in F V_{X}$.

Suppose given two crossed modules in commutative algebras $\delta: M \rightarrow P, \delta^{\prime}: N \rightarrow P$. We define their tensor product $M \otimes N$ to be the quotient algebra $C V(M \times N) / J$ where $C V(M \times N)$ is the free commutative algebra on the set of elements $m \otimes n$ with $(m, n) \in M \times N$, and where $J$ is the ideal generated by the relations
(i) $a(m \otimes n)=a m \otimes n=m \otimes a n$;
(ii) $\left(m+m^{\prime}\right) \otimes n=m \otimes n+m^{\prime} \otimes n$, $m \otimes\left(n+n^{\prime}\right)=m \otimes n+m \otimes n^{\prime} ;$
(iii) $(m \otimes n)\left(m^{\prime} \otimes n^{\prime}\right)=\left(m m^{\prime} \otimes n n^{\prime}\right)$;
(iv) $p_{m} \otimes n=m \otimes p_{n}$;
for $m^{\prime} m^{\prime} \in M, n^{\prime} n^{\prime} \in N, p \in P$.

It is routine to show that the tensor product $M \otimes N$ fits into a crossed square

$$
\begin{array}{rll}
M \otimes N & \rightarrow \lambda & N \\
\lambda^{\prime} \downarrow & & \iota^{\prime} \\
M & \rightarrow \delta^{\prime} & P
\end{array}
$$

in which: the action of $P$ on $M \otimes N$ is given by

$$
p_{(m \otimes n)}=p_{m} \otimes n=m \otimes p_{n} ;
$$

the maps $\lambda, \lambda$ ' are given respectively by $m \otimes n \rightarrow m_{n}, m \otimes n \rightarrow$ $n_{m}$; the function $h: M \times N \rightarrow M \otimes N$ is $(m, n) \rightarrow m \otimes n$.

This crossed square has a defining universal property (cf. proposition (1.4.1)).

A possible notion of an exterior product $M \wedge N$ of $M$ with N is obtained as a quotient of $\mathrm{M} \otimes \mathrm{N}$ by factoring out the relations
(v) $m \otimes n=m^{\prime} \otimes n^{\prime}$ whenever $\delta m=\delta^{\prime} n^{\prime}$ and $\delta^{\prime} m^{\prime}=\delta^{\prime} n$.

## O. INTRODUCTION

Crossed modules and crossed complexes have been used for some time now to give interpretations of cohomology of groups and algebras [M,Lul,Lu2]. In this chapter we shall study the dual situation of homology. We shall study free and projective crossed modules (\$1) and show how projective crossed modules in groups (\$2) and Lie algebras (\$3) can be used to obtain information on the 2 nd homology. We shall use the exterior products of Chapter III to give interpretations of the 2 nd and 3rd homology of a group (\$4) and of the second homology of a Lie algebra (\$5). These interpretations combined with theorems (3.1.14),(3.1.15) and (3.3.2) will give us exact sequences in homology.

1. FREE AND PROJECTIVE CROSSED MODULES

In this section we work in an arbitrary category $C$ of ת-groups.

Let us begin by adapting some well known terminology [B-Hu,R] to the category C.

We shall use the term crossed P -module to mean a crossed module $a: M \rightarrow P$ with codomain $P$. By a map of crossed. P-modules we shall mean a crossed module map which is the identity on $P$. The category of such crossed modules and maps will be called the category of crossed p-modules.

A projective crossed $P$-module is a projective object in
the category of crossed P-modules.
A crossed $P$-module $a: C(f) \rightarrow P$ is said to be the free crossed P -module on a function $\mathrm{f}: \mathrm{X} \rightarrow \mathrm{P}$ from a set X to P if the following universal property is satisfied: the function $f$ is the composite of a with some function $v: X \rightarrow$ $\mathrm{C}(\mathrm{f})$; given any crossed P -module $\mathrm{B}: \mathrm{M} \rightarrow \mathrm{P}$ and function $\mathrm{w}: \mathrm{X}$ $\rightarrow$ M satisfying $\delta \mathbf{w}=\partial v$, there is a unique map $\psi: C(f) \rightarrow M$ of crossed P -modules which satisfies $\psi \mathrm{v}=\mathrm{W}$.

Clearly free crossed modules are defined uniquely up to isomorphism, and are particular examples of projective crossed modules.

We shall now give three elementary results on crossed modules which will be needed in the following sections.

For an arbitrary $\Omega$-group $M$, let $[M, M$ ] denote the subobject generated by the elements $m+m^{\prime}-m-m^{\prime}$, and let $\langle M, M\rangle$ * denote the subobject generated by the elements $m * m^{\prime}$ for $m, m^{\prime} \in M$.

PROPOSITION(4.1.1) Let $\partial M \rightarrow P$ be a crossed P-module. If the restricted map $\mathrm{a}^{\prime}: \mathrm{M} \rightarrow \mathrm{JM}$ has a section $\mathrm{s}: \mathrm{OM} \rightarrow \mathrm{M}$ (here s need not preserve the action of $P$ ), then both $[M, M] \cap$ ker $a$ and 〈 $M, M\rangle * n$ ker a are the trivial $\Omega$-group. PROOF Since $a^{\prime}$ has a section we have that $M$ is isomorphic to the semi-direct product $M \equiv$ ker $a \underline{x} \mathrm{M}$. But both [ker $a, k e r \quad \partial$ ] and 〈ker $a$, ker $\partial>*$ are trivial, and $\partial M$ acts trivially on ker a. The proposition follows. $\nabla$

For C equal to the category of groups, proposition (4.1.1) is given in $[B-H u]$ and is originally due to J.H.C. Whitehead.

PROPOSITION(4.1.2) Let $a: C \rightarrow P$ be a projective crossed p -module, let $\mathrm{\delta}: \mathrm{M} \rightarrow \mathrm{P}$ be an arbitrary crossed. P -module, and let $\psi: M \rightarrow C$ be a surjective map of crossed $P$-modules. Then W has a section s:C $\rightarrow$ M. PROOF The proof is straightforward. $\nabla$

PROPOSITION(4.1.3) (R. Brown) Let $a: M \rightarrow P, a^{\prime}: M^{\prime} \rightarrow P$ be crossed P-modules and let $\psi: M \rightarrow M^{\prime}$ be a map of crossed P-modules. Then $\psi$ is a crossed $\mathrm{M}^{\prime}$-module with $\mathrm{M}^{\prime}$ acting on M via ${ }^{\prime}$.

PROOF The proof is a straightforward check. $\nabla$
2. CROSSED MODULES AND THE SECOND HOMOLOGY OF A GROUP The contents of this section are joint work with T. Porter [E-P]. In this section we take $C$ to be the category of groups.

We shall need the construction of free crossed modules (cf. [B-Hu]). So suppose we are given a function $f: X \rightarrow P$ from a set $X$ to a group P. Let $E=F(P \times X)$ be the free group on the set $P \times X$, and let $P$ act on $E$ by

$$
p^{\prime}\left(p^{\prime}, x\right)=\left(p p^{\prime}, x\right) .
$$

The function $f$ induces a homomorphism $\theta: E \rightarrow P$ which is defined on generators by

$$
\theta(p, x)=p f(x) p^{-1} .
$$

The Peiffer group $Q$ is the subgroup of $E$ generated by the elements

$$
u v u^{-1}\left(\theta u_{v}\right)^{-1}
$$

where $u, v \in E$. The group $Q$ is normal, invariant under the action of $P$, and $\theta Q$ is trivial. Thus, setting $C(f)=E / Q$, we obtain a crossed module $\theta_{\#}: C(f) \rightarrow P ;$ this is the required free crossed P-module.

PROPOSITION(4.2.1) Let $\mathrm{f}: \mathrm{X} \rightarrow \mathrm{P}$ be a function from a set X to a group $P$ such that $P$ is generated by the image of $f$. Let FX be the free group on X and denote by RX the kernel of the induced map from $F X$ to $P$. Then the function $f$ induces a homomorphism $\mathrm{FX} /[\mathrm{FX}, \mathrm{RX}] \rightarrow \mathrm{P}$; this is the free crossed P -module on f .

PROOF This proposition is a special case of [ $\mathrm{B}-\mathrm{Hu}$, proposition 9]. A direct proof is easy. $\nabla$

PROPOSITION(4.2.2) If $a: C \rightarrow P$ is a free crossed p-module with $\partial C=N$ say, then the restricted map $\partial^{\prime}: C \rightarrow N$ is a free crossed $N$-module.

PROOF The crossed P-module $d: C \rightarrow P$ is free on some function $f: X \rightarrow$ P. Let $T$ be a transversal of $N$ in $P$ which contains the identity. The function $f$ induces a function $f^{\prime}: T X X \rightarrow N$ given by

$$
f^{\prime}(t, x)=t f(x) t^{-1}
$$

We shall show that the crossed $N$-module $\partial^{\prime}: C \rightarrow N$ satisfies the universal property of the free crossed $N$-module on $f^{\prime}$.

Let $\delta: M \rightarrow N$ be an arbitrary crossed $N$-module and let $w: T X X \rightarrow M$ be a function such that $\delta \mathbf{W}=f^{\prime}$. Recall the
above description of the free crossed P-module on $f$.
Define a homomorphism $w^{\prime}: E \rightarrow M$ on generators by

$$
W^{\prime}(p, x) \rightarrow n_{w}(t, x)
$$

where $p=n t$ with $n \in N, t \in T$.
The Peiffer group $Q$ is normally generated by the elements

$$
u_{v u^{-1}}\left(\theta u_{v}\right)^{-1}
$$

with $u, v \in \operatorname{PxX}($ see $[B-H u])$. Suppose that $u=(q, y), v=$ ( $p, x$ ) with $p=n t$ as before. Since $\theta u \in N$ we have

$$
\begin{aligned}
w^{\prime}\left(\theta u_{v}\right) & =(\theta u) n_{w}(t, x) \\
& =\theta u_{w} \cdot v .
\end{aligned}
$$

But $\theta u=8 w^{\prime \prime} u$, so

$$
w^{\prime}\left(\theta u_{v}\right)=\left(w^{\prime} u\right)\left(w^{\prime} v\right)\left(w^{\prime} u\right)^{-1}
$$

That is, w'P is trivial; $w^{\prime}$ thus induces a homomorphism $\psi: C \rightarrow M$ satisfying $\delta \psi=a$. A routine calculation shows that $\psi$ is $N$-equivariant, and thus that $\psi$ is the required map of crossed N-modules. $\nabla$

PROPOSITION(4.2.3) Let $0: C \rightarrow P$ be a projective crossed $P$-module with $\partial C=N$ say. Let $R \rightarrow F \rightarrow \lambda$ be an arbitrary free presentation of N . Then there is an isomorphism $[C, C] \stackrel{\cong}{\rightarrow}[F, F] /[F, R]$ given by $[C, d] \rightarrow[X, Y][F, R]$ where $c, d \in C, x, y \in F$ and $\partial c=\lambda x, \partial d=\lambda y$.

PROOF First let us suppose that $\partial: C \rightarrow P$ is a free P-module. It follows from proposition (4.2.2) that the restriction of $a$ to $a^{\prime}: C \rightarrow N$ is a free crossed $N$-module on some function $f: X \rightarrow N$. Let $F_{0}=F(X X P)$ be the free group on $X \times P$ and denote by $R_{0}$ the kernel of the homomorphism $F_{0} \rightarrow$ E given on generators by $(x, p) \rightarrow p(f x) p^{-1}$. Propositions (4.1.I),(4.1.2),(4.2.3) give us $[C, C] \cong\left[F_{0}, P_{0}\right] /\left[F_{0}, R_{0}\right]$. Now
$[F, F] /[F, R]$ is an invariant of $N$. (A proof of invariance is not difficult, see [Ba].) It follows that [C,C] $\cong$ [ $\mathrm{F}, \mathrm{F}] /[\mathrm{F}, \mathrm{R}]$.

Suppose now that $d: C \rightarrow P$ is a projective crossed $P$-module. Let $a^{\prime}: C(a) \rightarrow P$ be the free crossed $P$-module on the map $a: C \rightarrow P$. There is a surjective map of crossed P-modules $\psi: C(\partial) \rightarrow C$. It follows from proposition (4.1.2) that $\psi$ has a section s:C $\rightarrow \mathrm{C}(\partial)$. Hence by propositions (4.1.1) and (4.1.3) there is an isomorphism $[C(a), C(a)] \cong$ $[C, C]$. There is thus an isomorphism $[C, C] \cong[F, P] /[P, R]$. It is easily checked that this isomorphism is as described in the proposition. $\nabla$

We now come to the main result of this section. THEOREM(4.2.4) If N is a group and $\mathrm{d}: \mathrm{C} \rightarrow \mathrm{P}$ is a projective crossed P -module with $\mathrm{aC}=\mathrm{N}$, then

$$
\mathrm{H}_{2}(\mathrm{~N}) \cong \text { ker a } \cap[\mathrm{C}, \mathrm{C}] .
$$

PROOF This proposition follows immediately from proposition (4.2.3) and Hopf's isomorphism $\mathrm{H}_{2} \mathrm{~N} \cong$ $\operatorname{Rn}[F, F] /[F, R] . \quad \nabla$

REMARK(4.2.5) We could have used the key lemma 2.1 of Ratcliffe's paper [ $R$ ] to prove this last theorem. Instead, we will show that our methods give a new and simple proof of Ratcliffe's lemma which avoids the detailed elementwise manipulations of the original proof.

Let $a: C(f) \rightarrow P$ be the free crossed $P$-module on a function $\mathrm{f}: \mathrm{X} \rightarrow \mathrm{P}$. Recall that $\mathrm{C}(\mathrm{f})=\mathrm{E} / \mathrm{Q}$. There is an isomorphism

$$
[C(f), C(f)] \cong[E, E] / Q \cap[E, E] .
$$

Let $I$ be the kernel of the induced map from $E$ to $P$ ．Thus $\mathrm{I} \rightarrow \mathrm{E} \rightarrow \mathrm{N}$ is a free presentation of N and so，by proposition（4．2．3）

$$
[C(f), C(f)] \cong[E, E] /[E, I] .
$$

It follows that $[E, I]=Q \cap[E, E]$ ；this equality is lemma 2.1 of［R］．

3．CROSSED MODULES AND THE SECOND HOMOLOGY OF A LIE ALGEBRA In this section we take $\mathbf{C}$ to be the category of Lie algebras over a commutative ring $A$（with unit）．

PROPOSITION（4．3．1）Let $\mathrm{f}: \mathrm{X} \rightarrow \mathrm{P}$ be a function from a set X to a Lie algebra $P$ such that the image of $f$ generates $P$ as an algebra．Let $L V_{X}$ be the free Lie algebra on $X$ and denote by $\mathrm{RV}_{\mathrm{X}}$ the kernel of the induced map from LV to P ． Then the function $f$ induces a homomorphism $\mathrm{f}_{\#}: \mathrm{LV}_{\mathrm{X}} /\left[\mathrm{LV}_{\mathrm{X}}, \mathrm{RV} \mathrm{X}_{\mathrm{X}}\right]$
$\rightarrow \mathrm{P}$ ；this is the free crossed P －module on f ．
PROOF We have a short exact sequence $R V_{X} \rightarrow L V_{X} \rightarrow P$ ．For each $p$ in $P$ choose an element 〈 $p$ 〉 in $L V_{X}$ such that $\langle p$ 〉 maps down to $p$ ．The function $\mathrm{P} \times \mathrm{LV}_{\mathrm{X}} \rightarrow \mathrm{LV} \mathrm{X}_{\mathrm{X}},(\mathrm{p}, 1) \rightarrow[\langle\mathrm{p}\rangle, 1]$ induces a Lie action of $P$ on $L V_{X} /\left[L V_{X}, R V_{X}\right]$ ．It is routine to check that $f_{\#}$ ，together with this Lie action，satisfies the axioms of a crossed module and has the universal property of the free crossed P－module on $f . \nabla$

THEOREM（4．3．2）If $\mathrm{d}: \mathrm{C} \rightarrow \mathrm{P}$ is a projective crossed P－module with $\mathrm{aC}=\mathrm{P}$ ，then

$$
\mathrm{H}_{2}(\mathrm{P}) \approx \text { ker } \mathrm{a} \cap[\mathrm{C}, \mathrm{C}] .
$$

PROOF Let $C(a) \rightarrow P$ be the free crossed $P$-module on the function d. By proposition (4.3.1) we have
$C(a) \approx L V_{C} /\left[L V_{C}, R V_{C}\right]$ where $L V_{C}$ is the free Lie algebra on $C$ and $R V_{C}$ is the kernel of the induced map $E V_{C} \rightarrow P$. There is a surjective map $\psi: C(a) \rightarrow C$ of crossed $P$ modules. It follows from propositions (4.1.2) and (4.1.3) that $\psi$ is a crossed C-module with a section s:C $\rightarrow C(a)$. From proposition (4.1.1) we have $[C, C] \cong[C(a), C(a)]$. The proposition now follows from the Hopf type formula $\mathrm{H}_{2}(P) \approx$ $R V_{C} \cap\left[L V_{C}, L V_{C}\right] /\left[L V_{C}, R V_{C}\right]$ (see for example [H-S], Chapter VII, section 2). $\nabla$

In view of the group theoretic theorem (4.2.4) it is reasonable to conjecture that proposition (4.3.2) can be strengthened to the case where the image of $a$ is a proper ideal of $P$. In order to prove this we need the construction of the free crossed P-module (in Lie algebras) on an arbitrary function $\mathrm{f}: \mathrm{X} \rightarrow \mathrm{P}$. This construction is more complicated than its group theoretic analogue.

The commatative algebra version of theorem (4.2.4) is given in [P4].
4. CROSSED SQUARES AND THE SECOND AND THIRD HOMOLOGY OF A GROUP

In this section we take $C$ to be the category of groups. Throughout the section let $R \rightarrow F \rightarrow \lambda$ Ge a free presentation of a group $G$.

In example (3.1.8) we showed that the exterior product
$F \wedge F$ is isomorphic to the commutator subgroup [F,F]. By proposition (3.1.3) we have

PROPOSITION(4.4.1) There is a canonical isomorphism

$$
G \cap G \cong[F, F] /[F, R] . \quad \nabla
$$

Recall that $\pi_{3}(G \wedge G)$ is the kernel of the commutator $\operatorname{map} G \Lambda G \rightarrow G . \quad$ Proposition (4.4.1) together with the Hopf formula for $\mathrm{H}_{2}(\mathrm{G})$ gives us

THEOREM(4.4.2) There is an isomorphism

$$
H_{2}(G) \cong \pi_{3}(G \wedge G) \cdot \nabla
$$

This description of $\mathrm{H}_{2}(G)$ is obtained in [B-L]. It is also essentially the description given in [M].

Suppose that the group G has two normal subgroups $M$ and N. Since the inclusions $M \rightarrow G, N \rightarrow G$ are crossed modules, we can construct the group $\pi_{3}(M \wedge N)$. In some sense, $\pi_{3}(M \wedge N)$ is a "relative second homology group of $M$ with respect to $\mathrm{N}^{\prime \prime}$. Note that theorems (3.1.14) and (4.4.2) give us

THEOREM(4.4.3) If the normal subgroups $M, N$ of $G$ are such that $G=\mathbb{M N}$, then there is an exact sequence

$$
\begin{aligned}
& \pi_{3}(M \wedge N) \rightarrow H_{2}(G) \rightarrow H_{2}(G / M) \oplus H_{2}(G / N) \rightarrow M \cap N /[M, N] \rightarrow \\
& \rightarrow H_{1}(G) \rightarrow H_{1}(G / M) \oplus H_{1}(G / N) \rightarrow 1 . \nabla
\end{aligned}
$$

This sequence is obtained in $[B-L]$ as a consequence of the

The group $\mathrm{H}_{2}(G)=\pi_{3}(G \underline{\Lambda})$ has been considered by [D] as a kind of "second homology group suitable for algebraic K-theory". In some respects $\mathrm{H}_{2}(\mathrm{G})$ certainly behaves like a second homology group. For example, given two groups $A, B$ then we have two isomorphisms
$\mathrm{H}_{2}(\mathrm{~A} * B) \cong \mathrm{H}_{2}(\mathrm{~A}) \oplus \mathrm{H}_{2}(B)$,
$\mathrm{H}_{2}(A \times B) \cong \underline{H}_{2}(A) \oplus \underline{H}_{2}(B) \oplus A^{a b} \otimes_{2} B^{a b}$,
as a consequence of propositions (3.1.7) and (3.1.10). Also, theorem (3.1.14) gives us

```
PROPOSITION(4.4.4) If the normal subgroups M,N of G are
such that G = MN, then there is an exact sequence
```



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->H
```

We now aim for a description of $\mathrm{H}_{3}(G)$ in terms of the exterior product.

Note that the identity map $F \rightarrow F$ and the inclusion $R \rightarrow F$ are both crossed modules, and that we can thus form the exterior product $F \wedge R$.

PROPOSITION(4.4.5) There is a short exact sequence $1 \rightarrow R \wedge R \rightarrow H_{1} F \wedge R \rightarrow \psi_{2} \quad I G \otimes_{G} R^{a b} \rightarrow 1$ where IG is the augmentation ideal of $G$, and $Q_{G}$ denotes the usual tensor product of $G$-modules.

PROOF The canonical map $\psi_{1}$ is injective since we have a commutative diagram of maps


The map $\psi_{2}$ is given on generators by

$$
\psi_{2}(f \Lambda r)=(\lambda f-1) \otimes r[R, R]
$$

The map $\psi_{2}$ is a homomorphism since

$$
\begin{aligned}
& \psi_{2} f\left(f^{\prime} \Lambda r\right) \psi_{2}(f \Lambda r) \\
& =\left(\lambda\left(f f^{\prime} f^{-1}\right)-1\right) \otimes f r f^{-1}[R, R]+(\lambda f-1) \otimes r[R, R] \\
& =\left(\lambda\left(f f^{\prime}\right)-\lambda f\right) \otimes r[R, R]+(\lambda f-1) \otimes r[R, R] \\
& =\left(\lambda\left(f f^{\prime}\right)-1\right) \otimes r[R, R] \\
& =\psi_{2}\left(f f^{\prime} \Lambda r\right),
\end{aligned}
$$

similarly

$$
\psi_{2}(f \wedge r) \psi_{2}^{r}\left(f \Delta r^{\prime}\right)=\psi_{2}\left(f \Lambda r r^{\prime}\right),
$$

and

$$
\psi_{2}(r \wedge r)=0
$$

Clearly $\psi_{2}$ is surjective. Set $T=F \Lambda R / \psi_{1}(R \wedge R)$. In order to show that the kernel of $\psi_{2}$ is equal to the image of $\psi_{1}$ it will suffice to construct an isomorphism IG $\otimes_{G} R^{a b} \rightarrow T$. Note that the quotient $T$ is abelian since, working in $F \wedge R$, we have

$$
\left[f \wedge r, f^{\prime} \wedge r^{\prime}\right]=[f, r] \Delta\left[f^{\prime}, r^{\prime}\right] \in \psi_{1}(R \wedge R)
$$

(see example (3.1.2)). For each $g$ in $G$ let $\langle g\rangle$ be an element of $F$ such that $\lambda\langle g\rangle=g$. The group $T$ has a G-module structure given by setting

$$
g(f \wedge r)=\langle g\rangle_{f} \wedge\langle g\rangle_{r}
$$

This G-action is well defined since, for $r^{\prime} \in R$, we have

$$
r^{\prime}(f \Lambda r)(f \Lambda r)^{-1}=r^{\prime} \Lambda[f, r] \in \psi_{1}(R \wedge R)
$$

Suppose ( $x, r[R, R]$ ) is an element of the direct product

IG $\times \dot{R}^{a b}$. The augmentation ideal IG is the free abelian group on the set $\{g-1: 1 \neq g \in G\}$, and so $x$ can be written uniquely as a sum $\mp\left(g_{1}-1\right) \mp \ldots \bar{\mp}\left(g_{n}-1\right)$. set $\phi(x, r[R, R])=\left(\left\langle g_{1}\right\rangle \Lambda r\right)^{\mp 1} \ldots\left(\left\langle g_{n}\right\rangle \Lambda r\right)^{\mp 1} \in T$. A routine check shows that $\phi$ is a well defined G-bilinear map from IG $\times R^{a b}$ to $T$. It follows that $\phi$ induces a map $\phi^{\prime}: I G Q Q^{a b} \rightarrow T . \quad$ The map $\psi_{2}$ induces a map $\psi_{2}^{\prime}: T \rightarrow$ IG $\otimes_{G}^{\prime} R^{a b}$. The maps $\phi^{\prime}, \psi_{2}^{\prime}$ are inverse to each other. $\nabla$

THEOREM(4.4.6) There is an isomorphism

$$
H_{3}(G) \cong \pi_{3}(F \wedge R)
$$

PROOF Let $\beta: I G Q_{G} R^{a b} \rightarrow R^{a b}$ be the homomorphism

$$
((g-1) \otimes r[R, R]) \rightarrow\langle g\rangle r\langle g\rangle^{-1}[R, R]
$$

Then
$H_{3}(G) \llbracket H_{l}\left(G ; R^{a b}\right) \approx \operatorname{ker} \beta$.
(See for example [H-S], Chapter VI, sections 4 and 12.) We thus have a commutative diagram

in which the rows and columns are exact. The proposition

THEOREM(4.4.7) Given a short exact sequence of groups $M \rightarrow$ $G \rightarrow Q$, then we have an eight term exact sequence in the homology of groups:

```
\(\mathrm{H}_{3}(\mathrm{G}) \rightarrow \mathrm{H}_{3}(\mathrm{Q}) \rightarrow \mathrm{T}_{3}(\mathrm{G} \wedge \mathrm{M}) \rightarrow \mathrm{H}_{2}(\mathrm{G}) \rightarrow \mathrm{H}_{2}(\mathrm{Q}) \rightarrow\)
    \(\rightarrow M /[G, M] \rightarrow H_{l}(G) \rightarrow H_{l}(Q) \rightarrow 1\).

This sequence is obtained in [B-L] as a consequence of the 3-dimensional van Kampen theorem.

It is tempting to define the group \(\underline{H}_{3}(G)=\pi_{3}(F \Lambda R)\). However, \(\pi_{3}(F \triangle R)\) is dependent on the choice of presentation of G. To see this, consider the presentation of the trivial group
\[
F_{n} \rightarrow P_{n} \rightarrow I
\]
where \(\mathrm{F}_{\mathrm{n}}\) denotes the free group of rank n . Then from example (3.1.9) we have that \(\pi_{3}\left(F_{n} \triangle F_{n}\right)\) is isomorphic to the direct sum of \(n\) copies of \(Z_{2}\). That is, \(\pi_{3}\left(F_{n} \wedge F_{n}\right)\) depends on \(n\).
5. CROSSED SQUARES AND THE SECOND HOMOLOGY OF A LIE ALGEbRA In this section we take \(C\) to be the category of Lie algebras over a commutative ring \(A\) (with unit). Let \(R \rightarrow F\) \(\rightarrow G\) be a short exact sequence of Lie algebras in which \(F\) is free.

By proposition (3.3.1) we have that the exterior product \(F \wedge F\) is isomorphic to \([F, F]\). Using the Lie algebra version of proposition (3.1.3) we get

PROPOSITION(4.5.1) There is an isomorphism \(G \wedge G \cong[F, F] /[F, R] . \quad \nabla\)

Recall that \(\pi_{3}(G \wedge G)=\operatorname{ker}\left(G \wedge G \rightarrow G, g \Lambda g^{\prime} \rightarrow\left[g, g^{\prime}\right]\right)\). The Hopf type formula for \(H_{2}(G)\) now gives us

THEOREM(4.5.2) There is an isomorphism
\[
H_{2}(G) \cong \pi_{3}(G \wedge G) . \quad \nabla
\]

Theorems (3.3.2) and (4.5.2) imply

THEOREM(4.5.3) Let \(G\) be a Lie algebra containing two ideals \(M, N\) such that any element \(g \in G\) can be written as a sum \(g=m+n\) with \(m \in M, n \in N\). Then there is an exact sequence in homology:
\(\pi_{3}(M \wedge N) \rightarrow \mathrm{H}_{2}(G) \rightarrow \mathrm{H}_{2}(G / M) \oplus \mathrm{H}_{2}(G / N) \rightarrow \mathrm{M} N \mathrm{~N} /[\mathrm{M}, \mathrm{N}] \rightarrow\)
\(\rightarrow \mathrm{H}_{\mathrm{I}}(\mathrm{G}) \rightarrow \mathrm{H}_{\mathrm{I}}(\mathrm{G} / \mathrm{M}) \oplus \mathrm{H}_{\mathrm{I}}(\mathrm{G} / \mathrm{N}) \rightarrow 0 . \nabla\)

\section*{MISCELLANEOUS COMMENTS}
1. In example (3.2.5) we showed that, in a particular instance, the cubical tensor product of groups is
isomorphic to the asymmetric product \(G \perp G\). R. Brown has suggested :3:s an alternative proof of this isomorphism as follows. Let \(X\) be a space such that \(\pi_{1} X=G\), and let \(T^{3} G\) be the cubical tensor product of example (3.2.5). There is an exact sequence (analogous to the sequence in [B-L, Theorem 5.41)
\[
\pi_{2} X \rightarrow \pi_{4} S^{2} X \rightarrow T^{3} G \rightarrow \pi_{1} X \rightarrow\left(\pi_{1} X\right)^{a b} \rightarrow 1
\]

On taking \(X=K(G, I)\) we get an exact sequence
\[
0 \rightarrow \pi_{4} S^{2} K(G, 1) \rightarrow T^{3} G \rightarrow[G, G] \rightarrow 1 .
\]

But there is an exact sequence [B-L, proposition 6.9]
\[
0 \rightarrow \pi S^{2} K(G, 1) \rightarrow G \underline{\Lambda} \rightarrow \quad[G, G] \rightarrow 1 .
\]

Let \(\psi: G \triangle G \rightarrow T^{3} G\) be the map \(g \Lambda g^{\prime} \rightarrow g \bigotimes_{1} g^{\prime}\). Then it is readily seen that we have a commutative diagram
\[
\begin{array}{ccc}
\operatorname{ker}(G \underline{\Lambda} \rightarrow G) & \rightarrow G \leq G & \rightarrow[G, G] \\
\downarrow \rightarrow \psi^{\prime} & \downarrow \psi & \\
\operatorname{ker}\left(T^{3} G \rightarrow G\right) & \rightarrow T^{3} G \rightarrow[G, G]
\end{array}
\]

One ought th be able to prove \(\psi^{\prime \prime}\) injective. in which \(\boldsymbol{h}^{\prime}\) ! is onto. G \(\underline{\triangle} \mathbf{G}\).
2. In theorem (4.4.2) we gave, for a group G, an isomorphism \(H_{2}(G) \cong \pi_{3}(G \Lambda G)\). If \(G\) is an abelian group then \(\pi_{3}(G \wedge G)\) is isomorphic to \(G \Lambda G\) (where \(G \Lambda G\) is now
the standard exterior product of abelian groups). More generally, for \(G\) abelian, there is an isomorphism \(H_{n}(G) \cong\) \(\Delta^{n_{G}}\) (see for example [ Br\(]\) ). It is reasonable to expect this latter isomorphism to generalise to the case where \(G\) is non-abelian (and \(n \geqslant 3\) ). Even for the case \(n=3\) there seems to be no obvious choice for a "non-abelian cubical exterior product".
3. The six term exact sequence of theorem (4.4.3) is extended, by the following two terms, in [B-L] as an application of the van Kampen theorem for squares of maps:
\(\mathrm{H}_{3}(\mathrm{G}) \rightarrow \mathrm{H}_{3}(\mathrm{G} / \mathrm{M}) \oplus \mathrm{H}_{3}(\mathrm{G} / \mathrm{N}) \rightarrow \pi_{3}(\mathrm{M} \Lambda N)\).
It ought to be possile to obtain this extension by purely algebraic means! it is reasonable to expect that both of the six term exact sequences of theorem (3.1.14) can be extended by two terms.
4. (R. Brown) If \(X\) is a connected CW-complex with \(\pi_{1} X=\) \(G\), then the following exact sequence is obtained as a consequence of the van Kampen theorem for squares of spaces [B-L, Theorem 5.4]
\[
\begin{equation*}
\pi_{2} X \rightarrow \pi_{3} S X \rightarrow G \otimes G \rightarrow[G, G] \rightarrow 1 \tag{*}
\end{equation*}
\]

This sequence together with Whitehead's \(\Gamma\)-sequence [W2] gives us an exact sequence
\(\mathrm{H}_{3} \mathrm{X} \rightarrow \Gamma\left(\mathrm{G}^{\mathrm{ab}}\right) \rightarrow \operatorname{ker}(\mathrm{G} \otimes \mathrm{G} \rightarrow \mathrm{G}) \rightarrow \mathrm{H}_{2} \mathrm{X} \rightarrow 0 .(* *)\)
On taking \(\mathrm{X}=\mathrm{K}(\mathrm{F}, 1)\) where F is a free group, (**) gives us an isomorphism
\[
\Gamma(F a b) \approx \operatorname{ker}(F \otimes F \rightarrow F), \quad(* * *)
\]
and so via (*) we recover \(\pi_{3}\) of a wedge of 2 -spheres. In
deducing (**) from (*) we must use Whitehead's result \(\Gamma_{3} \mathrm{X} \cong \Gamma_{2} \mathrm{X}\), and the proof of this uses a description of \(\pi_{3}\) of a wedge of 2 -spheres. So we cannot use the isomorphism (***), so obtained, together with (*) to deduce a description of a wedge of 2 -shperes. Clearly a purely algebraic proof of (***) is desirable.
5. It would be nice to have the Lie algebra version of theorem (4.4.6).
6. A description of the homology groups \(H_{n}(G)\) of a group \(G\) in terms of crossed \(n\)-cubes is known only for \(n=1,2,3\). In contrast a description of the cohomology groups \(H^{n}(G, A)\), where \(A\) is a G-module, in terms of catn-groups is known for all \(n\) [L].

To each cat \({ }^{n_{-g r o u p ~}^{\prime}} \underline{H}\) one can associate a complex of (non-abelian) groups
\(\mathrm{C}_{\#} \underline{H}: \quad \mathrm{C}_{n} \mathrm{H} \rightarrow \mathrm{a}_{\mathrm{n}} \ldots \rightarrow \mathrm{C}_{1} \mathrm{H} \rightarrow \mathrm{a}_{1} \quad \mathrm{C}_{0} H\)
such that the image of \(\partial_{i+1}\) is normal in the kernel of \(\partial_{i}\) (thus the homology groups \(H_{i}\left(C_{\# H}\right)\) can be formed). For \(n \geqslant\) 2 and some fixed integer \(k \geqslant 1\), let \(g(G, A)_{k}\) be the set consisting of triples ( \(\underline{H}, \phi, \psi\) ) where \(\underline{H}\) is a catn-2-group, \(\phi\) is an isomorphism between \(H_{k}\left(\mathrm{C}_{\#} H\right)\) and \(G\), and \(\psi\) is an isomorphism between \(H_{n-1}\left(\mathrm{C}_{\#} \underline{H}\right)\) and \(A\). Moreover, suppose that \(H_{i}\left(C_{\# H}\right)=0\) if \(i \neq k\) or \(n-1\). There is a Yoneda equivalence on the set \(g(G, A)_{k}\) such that

THEOREM [L] There is a one-to-one correspondence between the cohomology group \(H^{n}(\mathrm{~K}(\mathrm{G}, \mathrm{K}) ; \mathrm{A})\) and the set E(G,A) \({ }_{k} /(\) Yoneda equivalence). \(\nabla\)

The construction of the set \(\mathrm{Z}(\mathrm{G}, \mathrm{A}) /(\) Yoneda equivalence) is easily extended to the case where \(G\) is a Lie algebra (commutative algebra etc.). It would be worthwhile having a purely algebraic proof of the above theorem in the case \(\mathrm{n}=1\), since this proof would likely generalise to the case where \(G\) is a Lie algebra (etc.).
7. Suppose given a crossed square
\[
\begin{array}{lll}
L & \rightarrow \lambda & N \\
\lambda_{1} t^{\prime} & & 1^{\prime} \\
M & \rightarrow O^{\prime} & P
\end{array}
\]
with classifying space \(X\) (see [L]). The homotopy groups \(\pi_{1} \mathrm{X}, \pi_{2} \mathrm{X}, \pi_{3} \mathrm{X}\) are the homology groups of the complex of (non-abelian) groups
\[
\mathrm{L} \rightarrow^{\alpha} \operatorname{MXN} \rightarrow^{\beta} P
\]
where \(\alpha \ell=\left(\lambda^{\prime} \ell, \lambda \ell^{-1}\right)\) and \(\beta(m, n)=(\beta m)(\beta n)\). R. Brown has recently shown that the Whitehead product
\[
\pi_{2} \mathrm{X} \times \pi_{2} \mathrm{X} \rightarrow \pi_{3} \mathrm{X}
\]
is induced by the function
\(W\) : \(\operatorname{ker} \beta \times \operatorname{ker} \beta \rightarrow L\),
\[
\left((m, n),\left(m^{\prime}, n^{\prime}\right)\right) \rightarrow h\left(m^{\prime-1}, n\right) h\left(m^{-1}, n^{\prime}\right)
\]

It would be satisfying to be able to identify the various Whitehead products in a crossed n-cube of groups for \(n \geqslant 2\).
8. An important result of [ \(W 2\) ] is that if \(X, Y\) are connected CW-complexes, \(\operatorname{dim} X \leqslant n\) and \(Y\) is a \(J_{n}\)-complex (for example if \(\pi_{i} Y=0\) for \(1<i<n\) ) then the functor which takes the space X to its fundamental crossed complex \(\pi \underline{X}\) (described in the Introduction) induces a bijection of homotopy classes
\([\mathrm{X}, \mathrm{Y}] \cong[\pi \underline{\mathrm{X}}, \pi \underline{\mathrm{Y}}]\).
Further, there is a bijection
\([\pi \underline{X}, \pi \underline{Y}] \cong[C X, C Y]\)
where CX is the cellular chain complex of the universal cover of X (considered as a complex of \(\pi_{1} \mathrm{X}\)-modules).

These bijections enable certain homotopy theoretic calculations to be done purely algebraically [E]. At present no progress has been made on using crossed n-cubes to generalise these bijections; the main obstacle is the complicated nature of the functor from crossed \(n\)-cubes to CW-complexes which involves taking iterated nerves of the associated catn-groups.

Verification (1) of Proposition (2.1.1)
```

\alpha(e,e,l,m)
= (e{ell h(em,e)
= (1,m) \nabla

```
\(\alpha\left((n, p)\left(n^{\prime}, p^{\prime}\right), 1, m\right)\)
\(=\left(\left(n P^{\prime}\right)\left\{p p^{\prime} 1\right\} h\left(p p^{\prime} m, n p_{n}\right)^{\left.-1, p p^{\prime} m\right)}\right.\)
\(\left.=\left(\left(n p_{n^{\prime}}\right)\left\{p p^{\prime} 1\right\} n_{h\left(p p^{\prime}\right.} m_{1} p_{n^{\prime}}\right)^{-1} h\left(p p^{\prime} m, n\right)^{-1}, p p^{\prime} m\right)\)

\(=\left(n_{\{ }\left(p^{\prime}\right)\left\{p p^{\prime} 1\right\} \operatorname{Ph}\left(p^{\prime} m, n^{\prime}\right)^{-1}\right\} h\left(p p^{\prime} m, n\right)^{\left.-1, p p^{\prime} m\right)}\)
\(=\left(n\left\{p\left\{n^{\prime}\left\{p^{\prime} 1\right\} h\left(p^{\prime} m, n^{\prime}\right)^{-1}\right\}\right\} h\left(p p^{\prime} m, n\right)^{\left.-1, p p^{\prime} m\right)}\right.\)
\(=\alpha\left(n, p, n^{\prime}\left\{p^{\prime} 1\right\} h\left(p^{\prime} m, n^{\prime}\right)^{-1}, p^{\prime} m\right)\)
\(=\alpha\left(n, p, \alpha\left(n^{\prime} p^{\prime}, I, m\right)\right) \quad \nabla\)
\(\alpha\left(n, p,(1, m)\left(l^{\prime}, m^{\prime}\right)\right)\)
\(=\left(n\left\{p\left\{1 m^{\prime}\right\}\right\} h\left(p\left\{m m^{\prime}\right\}, n\right)^{-1}, p\left(m m^{\prime}\right\}\right)\)

\(\left.=\left(n\left\{P_{1}\right\} h\left(P_{m}, n\right)^{-1}\left(P_{m}\right)\left\{n_{\left\{P_{1}\right.}\right\} h\left(P_{m}, n\right)^{-1}\right\}, P_{m} P_{m}\right)^{\prime}\)
\(\left.=\left(n\left\{P_{1}\right\} h\left(P_{m}, n\right)^{-1}, p_{m}\right)\left(n_{\{P l}\right\} h\left(P_{m}, n\right), P_{m}{ }^{\prime}\right)\)
\(=\alpha(n, p, 1, m) \alpha\left(n, p, l^{\prime}, m^{\prime}\right) \quad \nabla\)

\section*{Verification (2) of Proposition (2.1.2)}

\section*{Pm( \(\mathrm{Pl}_{1}\) )}
\(=\sigma\left(\mathrm{P}_{\mathrm{m}}\right)\left(\mathrm{p}_{1}\right)\)
\(=(p \delta m)_{1}\)
\(=p(0 \mathrm{ml})\)
\(=p\left(m_{1}\right) \quad \nabla\)

Similariy \(\mathrm{Pn}_{(\mathrm{P} 1)}=\mathrm{P}_{\left(\mathrm{n}_{1}\right)} \quad \nabla\)

Verification (3) of Proposition (2.1.2)
\(m(n) h(m, n)\)
\(=(\delta m)\left(0^{\prime} n\right) 1 h(m, n)\)
\(=h(m, n) h(m, n)^{-1}(\delta m)\left(\delta^{\prime} n\right) I h(m, n)\)
\(=h(m, n)\left(\delta \lambda^{\prime} h(m, n)^{-1}\right)(\delta m)\left(\delta^{\prime} n\right) 1\)
\(=h(m, n)\left(\delta^{\prime} n\right)(\delta m) 1\)
\(=h(m, n) n\left(m_{1}\right) \quad \nabla\)

\section*{Verification (4) of Proposition (2.1.3)}

The map \(M x Q \rightarrow R X S,(m, q) \rightarrow\left(\nu_{R} m, \delta q\right)\) is a crossed module and MXQ acts on KXN via this map. Therefore \((r, s)(m, q)\{(r, s)(k, n)\}\)
\(=(x, s)\left(\nu_{R m}, \delta q\right)(k, n)\)
\(=(r, s)\{(m, q)(k, n) \quad \nabla\)

Similarly
\((r, s)(1, p)\{(r, s)(k, n)\}=(r, s)\{(1, p)(k, n)\} \quad \nabla\)

\section*{Verification (5) of Proposition (2.1.3)}

The identity
\[
n^{\prime}\left((1, p)\left(l^{\prime}, p^{\prime}\right), m, q\right)=(1, p) h^{\prime}\left(1^{\prime}, p^{\prime}, m, q\right) n^{\prime}(1, p, m, q)
\]
will follow from the four special cases
(i) \(\quad h^{\prime}\left((e, p)\left(l^{\prime}, e\right), m, q\right)=(e, p)^{\prime}\left(l^{\prime}, e, m, q\right) h^{\prime}(e, p, m, q)\),
(ii) \(h^{\prime}\left((l, e)\left(e, p^{\prime}\right), m, q\right)=(1, e) h^{\prime}\left(e, p^{\prime}, m, q\right) h^{\prime}(1, e, m, q)\),
(iii) \(h^{\prime}\left((e, p)\left(e, p^{\prime}\right), m, q\right)=(e, p)_{h^{\prime}}\left(e, p^{\prime}, m, q\right) h^{\prime}(e, p, m, q)\), (iv) \(h^{\prime}\left((1, e)\left(l^{\prime}, e\right), m, q\right)=(1, e) h^{\prime}\left(l^{\prime}, e, m, q\right) h^{\prime}(1, e, m, q)\), since (i) and (iii) imply
\(h^{\prime}\left((e, p)\left(l^{\prime}, p^{\prime}\right), m, q\right)\)
\(=h\left(\left(e, p p^{\prime}\right)\left(p^{\prime-1} 1^{\prime}, e\right), m, q\right)\)
\(=\left(e, p p^{\prime}\right) h^{\prime}\left(p^{\prime-1} l^{\prime}, e, m, q\right) h^{\prime}\left(e, p p^{\prime}, m, q\right)\)
\(=(e, p)\left\{\left(e, p^{\prime}\right) h^{\prime}\left(p^{\prime-1} l^{\prime}, e, m, q\right) h^{\prime}\left(e, p^{\prime}, m, q\right)\right\} h^{\prime}(e, p, m, q)\)
\(=(e, p) h^{\prime}\left(l^{\prime}, p^{\prime}, m, q\right) h^{\prime}(e, p, m, q)\),
that is
(v) \(h^{\prime}\left((e, p)\left(1^{\prime}, p^{\prime}\right), m, q\right)=(e, p) h^{\prime}\left(l^{\prime}, p^{\prime}, m, q\right) h^{\prime}(e, p, m, q)\),
and similarly (ii) and (iv) imply
(vi) \(h^{\prime}\left((1, e)\left(I^{\prime}, p^{\prime}\right), m, q\right)=(1, e) h^{\prime}\left(l^{\prime}, p^{\prime}, m, q\right) h^{\prime}(1, e, m, q)\),
and hence
\(h^{\prime}\left((1, p)\left(1^{\prime}, p^{\prime}\right), m, q\right)\)
\(=h^{\prime}\left((1, e)(e, p)\left(l^{\prime}, p^{\prime}\right), m, q\right)\)
\(=(1, e) h^{\prime}\left((e, p)\left(l^{\prime}, p^{\prime}\right), m, q\right) h^{\prime}(1, e, m, q)\)
\(=(1, p) h^{\prime}\left(1^{\prime}, p^{\prime}, m, q\right)(1, e)_{h^{\prime}}(e, p, m, q) h^{\prime}(1, e, m, q)\)
\(=(1, p) h^{\prime}\left(I^{\prime}, p^{\prime}, m, q\right) h^{\prime}(1, p, m, q) \quad \nabla\)

It remains to verify the four special cases.

\section*{Case (i)}
\(h^{\prime}((e, p)(1, e), m, q)\)
- \(h^{\prime}\left(P_{1}, p, m, q\right)\)
- ( \(\left(p_{1}\right)_{h}(p, m) h\left(p_{1}, m\right) m_{h}\left(h(p, q), p_{1}\right)^{-1} h(h(p, q), m)^{-1}\) \(\left.h(p, q) m_{h}(q, p l)^{-1}, h(p, q)\right)\)
(2.1.1.iv)
\(=\left(p_{1}\right) h(p, m) h\left(p_{1}, m\right) m_{h}\left(h(p, q), p_{1}\right)^{-1} m h(p, q) h\left(q, p_{1}\right)-1\) \(\left.h(h(p, q), m)^{-1}, h(p, q)\right)\)
(2.1.2,iii,iv)
\(=\left(P_{1} h_{h}(p, m) h\left(p_{1}, m\right) m_{h}\left(p_{q}, p_{1}\right)^{-1} h(h(p, q), m)^{-1}, h(p, q)\right)\)
- ( \(p 1\) ) \(h(p, m) h(p 1, ~(2.1 .2 . i v)\)
\(=\left((p l) h(p, m) h(p l, m) \operatorname{mph}(q, 1)^{-1} h(h(p, q), m)^{-1}, h(p, q)\right)\)
\(=\left(h\left(p v_{p} 1, m\right) m_{h}(q, 1)^{-1} h(h(p, q), m)^{-1}, h(p, q)\right)\)
\(=\left(\operatorname{ph}(1, m) h(p, m) \operatorname{mph}(q, 1)^{-1} h(h(p, q), m)^{-1}, h(p, q)\right)\)
\(\left.=\left(p_{(h(1, m)} m_{h}(q, 1)^{-1}\right\} h(p, m) h(h(p, q), m)^{-1}, h(p, q)\right)\)
\(=(e, p) h^{\prime}(1, e, m, q) h^{\prime}(e, p, m, q) \quad \nabla\)

\section*{Case (ii)}
\(h^{\prime}((1, e)(e, p), m, q)\)
- \(h^{\prime}(1, p, m, q)\)
 \(\left.h(p, q) m_{h}(q, 1)^{-1}, h(p, q)\right)\)
\(=\left(1_{h(p, m)} h\left(\left(\nu_{Q^{m}}\right)\left(\nu_{Q h}(p, q), 1\right)^{-1} h(h(p, q), m)^{(2.1 .3 .} 1\right.\right.\)
(2.1.2.iii) \(\left.h(p, q) m_{h}(q, 1)^{-1}, h(p, q)\right)\)
\(=\left(l_{h}(p, m) h\left(h(p, q),\left(\nu_{R} l\right)\left(\nu_{R} m\right)\right)^{-1} h\left(\left(\nu_{p h}(p, q)\right)\left(\nu_{p} i\right), m\right)\right.\) \(\left.h(h(p, q), m)^{-1} h(p, q) m_{h}(q, 1)^{-1}, h(p, q)\right)\)
\(=\left(l_{h}(p, m) h\left(h(p, q),\left(\nu_{R} l\right)\left(\nu_{R} m\right)\right)^{-1} h(p, q) h\left(\nu_{p} l, m\right)\right.\)
\(h(h(p, q), m) h\left(h(p, q)^{-1} h(p, q) m_{h}(q, 1)^{-1}, h(p, q)\right)\)
(2.1.2.1ii)
\(=\left(I_{h(p, m)} I_{h(h(p, q), m)^{-1} h(h(p, q), 1)^{-1} h(p, q)(h(1, m)}\right.\) \(\left.\left.m_{h}(q, 1)^{-1}\right\}, h(p, q)\right)\)
\(=(1, e) h^{\prime}(e, p, m, q) h^{\prime}(1, e, m, q) \quad \nabla\)

\section*{Case (iii)}
\(h^{\prime}\left((e, p)\left(e, p^{\prime}\right), m, q\right)\)
- h(e,pp',m,q)
\(=\left(h\left(p p^{\prime}, m\right) h\left(h\left(p p^{\prime}, q\right), m\right)^{-1}, h\left(p p^{\prime}, q\right)\right)\)
(2.1.2.iii, v)
\(=\left(h\left(p p^{\prime}, m\right) h\left(p p^{\prime}, m\right)^{-1} p p^{\prime} h\left(q\left(p p^{\prime}\right)^{-1, m}\right)^{-1}, h\left(p p^{\prime}\left(q^{\prime}\right)_{1}\right) 2.1 i i\right)\)
\(=\left(p p^{\prime} h\left(q p^{\prime-1}, m\right)^{-1} p h\left(p^{\prime}, q\right) h\left(q^{\prime}-1, m\right)^{-1}, h\left(p p^{\prime}, q\right)\right.\)
\(=\left(\operatorname{Ph}\left(p^{\prime}, m\right) \operatorname{Ph}\left(p^{\prime}, m\right)^{-1} p p^{\prime} h\left(q_{p}^{\prime-1}, m\right)^{-1} \operatorname{Ph}\left(p^{\prime}, q\right)(h(p, m)\right.\) \(\left.\left.h(p, m)^{-1} p h\left(q p^{-1}, m\right)^{-1}\right\}, h\left(p p^{\prime}, q\right)\right)\)
\(=\operatorname{ph}^{\prime}\left(p^{\prime}, m\right) \operatorname{Ph}\left(h\left(p^{\prime}, q\right), m\right)^{-1} \operatorname{Ph}\left(p^{\prime}, q\right)\left\{h(p, m) h(h(p, q), m)^{-1}\right\}^{i j}\), \(\left.\operatorname{Ph}\left(p^{\prime}, q\right) h(p, q)\right)\)
\(=(e, p)_{h^{\prime}}\left(e, p^{\prime}, m, q\right) h^{\prime}(e, p, m, q) \quad \nabla\)

\section*{Case (iv)}
\(h^{\prime}\left((1, e)\left(1^{\prime}, e\right), m, q\right)\)
\(=h^{\prime}\left(11 l^{\prime}, \theta, m, q\right)\)
- ( \(\left.\left.h\left(1 l^{\prime}, m\right)_{h\left(q, 1 l^{\prime}\right)}\right)^{-1}, e\right)\)
\(-\quad\left(l_{h(1 ', m) h(1, m)} m_{h}\left(q, I^{\prime}\right)^{-1} h(q, 1)^{-1, e)}\right.\)
\(=(1, e)_{h^{\prime}}\left(11^{\prime}, e, m, q\right) h^{\prime}(1, e, m, q) \quad \nabla\)

\section*{Verification (6) of Proposition (2.1.3)}

The identity
\[
(r, s)_{h^{\prime}}(1, p, m, q)=h^{\prime}((r, s)(1, p),(r, s)(m, q))
\]
follows from the four special cases
(iii) \((r, s)_{h}(1, e, e, q)=h^{\prime}((r, s)(1, e),(r, s)(e, q))\),
(iv) \((x, s)_{h^{\prime}(1, e, m, e)}=h^{\prime}((r, s)(1, e),(x, s)(m, e))\),
since
\(h^{\prime}((r, s)(1, p),(r, s)(m, q))\)
\(\left.=h^{\prime}(r, s)(l, e)(r, s)(e, p),(r, s)(m, e)(r, s)(e, q)\right)\)
\(=(r, s)(1, e)\left\{h^{\prime}((x, s)(e, p),(r, s)(m, e))\right.\)
\(\left.(r, s)(m, e) h^{\prime}((r, s)(e, p),(r, s)(e, q))\right\}\)
\(h^{\prime}((x, s)(1, e),(r, s)(m, e))\)
\(\left.(r, s)(m, e)_{h^{\prime}}(r, s)(l, e),(r, s)(e, q)\right)\)
\(=(r, s)_{h^{\prime}}(1, p, m, q) \quad \nabla\)
(i), (ii), (iii), (iv)

Cases (ii), (i.ii), (iv) have one line verifications. It
remains to check
Case (i)
In order to check this case we shall need the following
three identities:
(v) \(\quad n_{h(1, m)^{-1} h(n, 1)} l_{h(n, m)}=h(n, m) m_{h(n, 1)} h(1, m)^{-1}\),
(vi) \(\quad h(p, q) r^{-1} h\left(r_{q}, h(p, r)^{-1}\right) h\left(h(p, q), p_{r}^{-1}\right)\)
\[
p r^{-1} h\left(q p^{-1}, h(q, r)^{-1}\right)=e
\]
(vii) \(h\left(h(p, q), h(q, r)^{-1}\right) h(q, r)^{-1} h\left(h(p, q), h(p, r)^{-1}\right)\)
\[
h\left(h(p, r)^{-1}, h(q, r)^{-1}\right)^{-1}
\]
\[
=h(p, q) h(q, r)^{-l} h\left(q, h(p, r)^{-1}\right)^{-1} h(h(p, q), r)
\]
\[
h(p, r)^{-1} h\left(p, h(q, r)^{-1}\right)
\]
since then we have
\(h^{\prime}((r, e)(e, p),(r, e)(e, q))\)
\(=h^{\prime}\left(h(p, r)^{-1}, p, h(q, r)^{-1}, q\right)\)
\(=\left(h(p, r)^{-1} h\left(p, h(q, r)^{-1}\right) h\left(h(p, r)^{-1}, h(q, r)^{-1}\right)\right.\)
\[
h(q, r)^{-1} h\left(h(p, q), h(p, r)^{-1}\right)^{-1} h\left(h(p, q), h(q, r)^{-1}\right)^{-1}
\]
\(\left.h(p, q) h(q, r)^{-1} h\left(q, h(p, r)^{-1}\right)^{-1}, h(p, q)\right)\)
\(=\left(h(h(p, q), r)^{-1}, h(p, q)\right)\)
\(=(r, e) h^{\prime}(e, p, e, q)\).
So it remains to check (v),(vi),(vii).

\section*{Identity (v)}

This ident.ity follows from (2.1.2.iii) and (2.1.3.ii).

\section*{Identity (vi)}
\(h(p, q) r^{-1} h\left(r_{q, h}(p, r)^{-1}\right) h\left(h(p, q), p_{r-1}\right) p r^{-1} h\left(q_{p-1}, h(q, r)^{-1}\right)\)
\(=h(p, q) h\left(q, h\left(p, r^{-1}\right)\right) \operatorname{ph}\left(h\left(p^{-1}, q\right) \cdots, r^{-1}\right)\)
\[
p r^{-1} q h\left(p^{-1}, h\left(q^{-1}, r\right)^{-1}\right)
\]
\(-p q\left\{p^{-1} h\left(q^{-1}, h\left(p, r^{-1}\right)\right)^{-1} q^{-1} h\left(h\left(p^{-1}, q\right)^{-1}, r^{-1}\right)\right.\)
\[
\begin{equation*}
\left.r^{-1} h\left(p^{-1}, h\left(q^{-1}, x\right)^{-1}\right)^{-1}\right\} \tag{2.1.3.iii}
\end{equation*}
\]
\(=\theta \quad \nabla\)

\section*{Identity (vii)}
\[
\begin{aligned}
& h\left(h(p, q), h(q, r)^{-1}\right) h(q, r)^{-1} h\left(h(p, q), h(p, r)^{-1}\right) \\
& h\left(h(p, r)^{-1}, h(q, r)\right)^{-1} \\
& =h(p, q) h\left(h(p, r)^{-1}, h(q, r)^{-1}\right)^{-1} h\left(h(p, q), h(p, r)^{-1}\right) \\
& h(p, r)^{-1} h\left(h(p, q), h(q, r)^{-1}\right) \\
& =h(p, q) h\left(h(p, r)^{-1}, h(q, r)^{-1}\right)^{-1} h\left(h(p, q), h(p, r)^{-1}\right) \\
& h(p, r)^{-1}\left\{\operatorname{Ph}\left(q p^{-1}, h(q, r)^{-1}\right) h(p, h(q, r))^{-1}\right\} \\
& =h(p, q) h(q, r)^{-1} h\left(q, h(p, r)^{-1}\right)^{-1} h(p, q) r r^{-1} h\left(r_{q}, h(p, r)\right) \\
& h(h(p, q), r) r_{h\left(h(p, q), p_{r}-1\right)} r p r^{-1} h\left(q p^{-1}, h(q, r)^{-1}\right) \\
& h(p, r)^{-1} h\left(p, h(q, r)^{-1}\right) \\
& =h(p, q) h(q, r)^{-1} h\left(q, h(p, r)^{-1}\right)^{-1} h(h(p, q), r) \\
& h(p, r)^{-1} h\left(p, h(q, r)^{-1}\right) \quad \nabla
\end{aligned}
\]

Verification (7) of Proposition (2.1.3).
The map \(\nu: K \underline{X} N \rightarrow \operatorname{lixp},(k, n) \rightarrow\left(\lambda_{L} k, \nu_{P} n\right)\) satisfies \(\nu h^{\prime}(1, p, m, q)=(1, p)(m, q)(1, p)^{-1}\),
and the verification is a slraightforward copy of verification (3).

Verification (8) of Proposition (2.2.2)
(an, ap) \((1, m)\)
\(=\left(a n_{1}+a p_{1}-h(m, a n), a p_{m}\right)\)
\(=a\left(n_{1}+p l_{l}-h(m, n), p_{m}\right)\)
\(=a\{(n, p)(1, m)\} \quad \nabla\)

Verification (9) of Proposition (2.2.2)
\((n, p)\left\{(1, m)+\left(1, m^{\prime}\right)\right\}\)
\(=\left(n\left\{1+I^{\prime}\right\}+p\left\{1+I^{\prime}\right\}-h\left(m+m^{\prime}, n\right), p m+p_{m}^{\prime}\right)\)
\(=(n, p)(1, m)+(n, p)\left(1, m^{\prime}\right) \quad \nabla\)
\(\left((n, p)+\left(n^{\prime}, p^{\prime}\right)\right)(1, m)\)
\(=\left(\left(n+n^{\prime}\right) 1+\left(p+p^{\prime}\right) 1-n\left(m, n+n^{\prime}\right),\left(p+p^{\prime}\right) m\right)\)
\(=\left(n_{1}+p_{1}-n(m, n)+n^{\prime} 1+p^{\prime} 1-n\left(m, n^{\prime}\right), p_{m}+p^{\prime} m\right)\)
\(=(n, p)(1, m)+\left(n^{\prime}, p^{\prime}\right)(1, m) \quad \nabla\)

Verification (11) of Proposition (2.2.2)
\(\left[(n, p),\left(n^{\prime}, p^{\prime}\right)\right](1, m)\)
\(=\left(\left[n, n^{\prime}\right]+p_{n}^{\prime}-p^{\prime} n_{r}\left[p, p^{\prime}\right]\right)(1, m)\)
\(\left.=\left(\left[n, n^{\prime}\right] 1+\left(n^{\prime}\right)\right]-\left(p^{\prime} n\right) 1+\left[p, p^{\{ }\right]_{1}^{2}-1 . i i i\right),(2.2 .2 . i i i)\)
\(=\left(\left[n, n^{\prime}\right] 1+\left(n^{\prime}\right) 1-(p n) 1+[p, p] l-h\left(m,\left[n, n^{\prime}\right]\right)\right.\)
\(\left.-h\left(m, p_{n}\right)+h\left(m, p^{\prime} n\right),\left[p, p^{\prime}\right] m\right)\)
\(=\left(n_{\left(n^{\prime} 1\right)}-n^{\prime}\left(n_{1}\right)+p\left(n^{\prime} 1\right)-n^{\prime}\left(p_{1}\right)-p^{\prime}\left(n_{1}\right)^{\left(2.2 n^{2}\left(p^{\prime}, i v, v\right)\right.}\right.\)
\(+p\left(p^{\prime} 1\right)-p^{\prime}(p l)-n_{h}\left(m, n^{\prime}\right)+n^{\prime} h(m, n)-p_{h}\left(m, n^{\prime}\right)\)
\(\left.+h\left(p_{m, n^{\prime}}\right)+p^{\prime} h(m, n)-h\left(p^{\prime} m, n\right), p\left(p^{\prime} m\right)-p^{\prime}\left(p_{m}\right)\right)\)
\(=(n, p)\left\{\left(n^{\prime}, p^{\prime}\right)(1, m)\right\}-\left(n^{\prime}, p^{\prime}\right)\{(n, p)(1, m)\} \quad \nabla\)

Verification (12) of Proposition (2.2.2)
\((n, p)\left[(1, m),\left(I^{\prime}, m^{\prime}\right)\right]\)
\(=\left(n\left[1, I^{\prime}\right]+n\left(m_{1}^{\prime}\right)-n\left(m^{\prime} 1\right)+p\left[1, I^{\prime}\right]+p\left(m_{1}^{\prime \prime}\right)-p\left(m^{\prime} 1\right)-\right.\) \(\left.h\left(\left[m, m^{\prime}\right], n\right), p\left[m, m^{\prime}\right]\right)\)
\(=\left(\left[n_{1}, 1^{\prime}\right]+\left[p_{1} 1^{\prime}\right]-\left[h(m, n), 1^{\prime}\right]+\left(P_{m}\right) l^{\prime}-m^{\prime}\left(n_{1}\right)\right.\)
\(\left.-m^{\prime}\left(P_{1}\right)+m^{\prime} h(m, n),\left[P_{m}, m^{\prime}\right]\right)\)
\(+\left(\left[1, n_{1}^{\prime}\right]+\left[1, p_{1}\right]-[1, h(m, n)]+m\left(n_{1}^{\prime}\right)+m\left(p_{1}\right)\right.\)
\(\left.-m_{h}\left(m^{\prime}, n\right)-\left(p_{m^{\prime}}\right) 1,\left[m, p_{m^{\prime}}\right]\right)\)
\(=\left[(n, p)(l, m),\left(l^{\prime}, m^{\prime}\right)\right]+\left[(1, m),(n, p)\left(l^{\prime}, m^{\prime}\right)\right] \quad \nabla\)

Verification (13) of Proposition (2.2.3)
\(p\left(m_{1}\right)\)
\(=p\left(\delta_{l}\right)\)
\(=[p, 8 \mathrm{~m}] 1-\delta m(P 1)\)
(2.2.1.iv)
\(=(P m) 1-m(P 1) \quad \nabla\)

Verification (14) of Proposition (2.2.3)
\(n\left(m_{1}\right)\)
\(=\delta^{\prime} \mathrm{n}(8 \mathrm{ml})\)
\(=\left[\delta^{\prime} n, \delta m\right] 1+\delta m^{\prime}\left(\delta^{\prime} n_{1}\right)\)
\(\left.=o m^{\prime} \delta^{\prime} n_{1}\right)-o \lambda^{\prime} h(m, n) 1\)
\(=m(n 1)+[1, h(m, n)] \quad \nabla\)
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Verification (1) of proposition (3.2.4)

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$=n_{\left.\left(\nu^{m} \otimes 1\right)^{-1}\left(\nu_{Q^{m}} \otimes Y\right)^{-1} m_{\left(\nu_{Q}\right.} \otimes 1\right)\left(\nu_{Q^{m}} \otimes 1\right)}$
$=n[1, m] n\left(v_{Q} n \otimes x\right) m^{m}\left(\nu_{Q n} \otimes I\right)\left(\nu_{Q} \cap \otimes[1, x]\right)$
$=n[1, x] n\left(v_{Q} n \otimes x 1\right)\left(v_{Q} \cap \otimes[1, x]\right)$
$=\nu Q ⿴ 囗>\quad \nabla$

```
Verification (2) of proposition (3.2.4)
\(q\left(q^{-1} v q \otimes u\right) u\left(u^{-1} q_{q}-1 \otimes v\right)\)
\(=q\left\{\left(q^{-1} \otimes v\right)^{u}\left(q^{-1} \otimes v\right)^{-1}\right\} u\left\{\left(q \otimes u^{-1}\right)^{-1} v\left(q \otimes u^{\text {Lemma }}\right)\right\}^{(3}\)
\(=v\left(q \otimes\left[v^{-1}, u\right]\right) \quad \nabla\)
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