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Crossed modules and their higher dimensional analogues

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CROSSED MODULES AND THEIR HIGHER DIMENSIONAL ANALOGUES

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DECLARATION

The work of this thesis has been carried out by the candidate and contains the results of his own investigations. The work has not already been accepted in substance for any degree, and is not being concurrently submitted in candidature for any degree. All sources of information have been acknowledged in the text.

DIRECTOR OF STUDIES CANDIDATE

I Claire. Hebddi hi, ni fyddai'r thesis hwn wedi cael ei gwblhau.

CONTENTS

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SUMMARY

For a fairly general algebraic category C (possible interpretations of C include the categories of groups, rings (associative, commutative), algebras (associative, commutative, Lie or Jordan)) we give various alternative descriptions of an n-fold category internal to C. One of these descriptions we call a "crossed n-cube in C". Crossed 1-cubes are better known as "crossed modules" (this latter term being due to Whitehead [W1]). Crossed 2-cubes in the category of groups are originally due to Loday [L].

We give a combinatorial description of crossed n-cubes for n = 1,2,3 and C equal to the category of groups, Lie algebras, commutative algebras and (n = 1,2) associative algebras.

The study of certain universal crossed 2-cubes leads us to notions of non-abelian tensor, exterior and anti-symmetric products of groups and of Lie and commutative algebras. The tensor and exterior products of groups are originally due to Brown and Loday [B-L]. We also look at the crossed 3-cube analogue of the tensor product of groups.

We study the relevance of crossed modules and crossed 2-cubes to the homology of groups and Lie algebras. In particular we prove

THEOREM *If* $\partial : M \to P$ is a *projective crossed* P*-module (of groups) with im* $\partial = N$, *then* $H_2(N) \cong$ *ker* $\partial \cap [M,M]$.

THEOREM *If* M,N *are normal subgroups of a group* G *such that* $G = MN$, *then there is an exact sequence*

$$\pi_3(M \wedge N) \to H_2(G) \to H_2(G/M) \oplus H_2(G/N) \to M \cap N/[M,N] \to$$
$$\to H_1(G) \to H_1(G/M) \oplus H_1(G/N) \to 1$$

where $\pi_3(M \wedge N)$ *is the kernel of a map* $M \wedge N \to M$ *from the exterior product of* M *and* N.

THEOREM *If, in the preceding theorem,* $G = N$, *then we can extend the exact sequence by two terms:*

$$H_3(G) \to H_3(G/M) \to \pi_3(M \wedge N).$$

The second two theorems are originally due to Brown and Loday [B-L] who obtained them as a corollary to their van Kampen type theorem for squares of maps. The proofs in this thesis are purely algebraic.

We give the analogue of the second theorem in which $H_2(G)$ is replaced by the group $\underline{H}_2(G)$, this group $\underline{H}_2(G)$ being the one introduced by Dennis [D] as a kind of "second homology group suitable for algebraic K-theory".

We give Lie algebra versions of the first two theorems.

INTRODUCTION

For many common algebraic categories there exists a useful theory of crossed modules. In this thesis we introduce several equivalent notions of a higher dimensional crossed module, and we develop certain aspects of the resulting higher dimensional theory. This work is motivated by a recent result of R. Brown and J.-L. Loday, and also by the theory of crossed modules themselves, as we shall now explain. In parts, this introduction relies heavily on [B].

Recall that a group homomorphism $\partial: M \to P$ is said to be a *crossed P-module (in groups)* if there is an action of P on M, $(p,m) \to {}^{p}m$, which satisfies $\partial({}^{p}m) = p(\partial m)p^{-1}$, ${}^{\partial m}m' = mm'm^{-1}$ for $m,m' \in M$, $p \in P$. Standard examples of crossed modules are:

(i) the inclusion $N \to P$ of a normal subgroup N of the group P, with the action of P on N given by conjugation (throughout this thesis we shall keep to the convention that, if x,y are elements of some group then, the congugate of x by y is the elementt yxy^{-1});

(ii) the zero morphism $0: M \to P$ in which M is a P-module in the usual sense;

(iii) the morphism $\chi: M \to \mathrm{Aut}\, M$ from M to the group of automorphisms of M in which χm is the inner automorphism determined by $m \in M$, together with the standard action of Aut M on M;

(iv) the boundary map $\partial: \pi_2(X,Y,x_0) \to \pi_1(Y,x_0)$ from the second relative homotopy group to the fundamental group, with the standard action of $\pi_1(Y,x_0)$ on $\pi_2(X,Y,x_0)$.

As this last example suggests, crossed modules can be used to model certain homotopy types. In particular there is a functor B:(crossed modules) $\to$ (CW-complexes) such that if $\partial:M \to P$ is a crossed P-module then $B(M \to P)$ has fundamental group coker ∂ and second homotopy group ker ∂ [B-H1,L]. Further, any pointed, connected CW-complex X with $\pi_i(X) = 0$ for $i > 2$ is of the homotopy type of some $B(M \to P)$ [ML-W].

Also, a van Kampen type theorem for $\pi_2(X,Y,x_0)$ considered as a crossed $\pi_1(Y,x_0)$-module has been found [B-H2].

From the standpoint of homotopy theory, crossed modules should perhaps be viewed as 2-dimensional groups. It is reasonable to ask then, what are the higher-dimensional groups (or crossed modules)? J.H.C. Whitehead gave a partial answer to this by introducing what he called "homotopy systems", but what are now called *crossed complexes*. These gadgets consist of a sequence of groups

$$\to C_n \xrightarrow{\partial_n} \cdots C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1$$

in which:

(i) C_n is abelian for $n \geqslant 3$;

(ii) $\partial_{n-1}\partial_n = 0$;

(iii) C_1 acts on C_n, $n \geqslant 2$, and $\partial_2 C_2$ acts trivially on C_n, $n \geqslant 3$;

(iv) ∂_2 is a crossed module, and each ∂_n is an equivariant map.

The standard example of a crossed complex is obtained from a pointed filtered space $X \supset \ldots X_n \supset \ldots X_2 \supset X_1 \supset \{x_0\}$ by setting $C_1 = \pi_1(X_1, x_0)$, $C_n = \pi_n(X_n, X_{n-1}, x_0)$ and taking each ∂_n to be the boundary operator. Crossed complexes give certain partial generalisations to the homotopy theoretic results mentioned above involving crossed modules. However, the abelian nature of crossed complexes is a bar to the obvious full generalisations.

Note that crossed complexes arise in the cohomology of groups [ML] since, if P is a group and M is a P-module, then $H^{n+1}(P;M)$ can be obtained as equivalence classes of n-dimensional crossed complexes in which $\ker \partial_n = M$, $\text{coker } \partial_2 = P$, $\ker \partial_i / \text{im } \partial_{i+1} = 0$ for $2 \leq i \leq n-1$. It seems reasonable to expect that other notions of higher dimensional crossed modules might also be of relevance to (co-)homology.

A more recent and important reformulation of the fact that $\partial : \pi_2(X,Y,x_0) \to \pi_1(Y,x_0)$ has a crossed module structure is that, if $F \to E \to B$ is a fibration, then the induced map $\pi_1 F \to \pi_1 E$ is a crossed module. This is one of the reasons for the use of crossed modules in algebraic K-theory [L,GW-L]. Recall that if Λ is a ring (with unit) then $F\Lambda$ is defined as the homotopy fibre of the inclusion $BGL\Lambda \to (BGL\Lambda)^+$. Now $\pi_1 F$ is the Steinberg group $St\Lambda$, and $\pi_1 BGL\Lambda = GL\Lambda$; thus we have a crossed module $St\Lambda \to GL\Lambda$. The study of bi-relative Steinberg groups has led to the definition of a type of 2-dimensional crossed module, which is called a "crossed square" [GW-L]. With a few formal modifications this definition states that a *crossed square* consists of a commutative diagram of groups

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\delta'} \\ M & \xrightarrow{\delta} & P \end{array}$$

together with actions of P on L,M and N (hence M acts on L and N via δ, and N acts on L and M via δ'), and a function $h:M\times N \to L$ such that:

(i) each of the maps $\lambda,\lambda',\delta,\delta'$ and the composite $\delta\lambda'$ are crossed modules;

(ii) the maps λ,λ' preserve the actions of P;

(iii) $h(mm',n) = {}^{m}h(m',n)h(m,n)$,

$h(m,nn') = h(m,n)\ {}^{n}h(m,n')$;

(iv) ${}^{p}h(m,n) = h({}^{p}m,{}^{p}n)$;

(v) $\lambda h(m,n) = {}^{m}n\ n^{-1}$, $\lambda' h(m,n) = m\ {}^{n}m^{-1}$;

(vi) $h(m,\lambda l) = {}^{m}l\ l^{-1}$, $h(\lambda' l,n) = l\ {}^{n}l^{-1}$;

for all $l \in L$, $m,m' \in M$, $n,n' \in N$, $p \in P$.

The standard examples of crossed modules (see above) can be extended to examples of crossed squares:

(i) if M,N are normal subgroups of the group P, then the diagram of inclusions

$$\begin{array}{ccc} M\cap N & \rightarrow & N \\ \downarrow & & \downarrow \\ M & \rightarrow & P \end{array}$$

together with the actions of P on M,N and $M\cap N$ given by conjugation, and the function $h:M \times N \to M \cap N$, $(m,n) \to [m,n]$, is a crossed square (throuought this thesis we shall keep to the convention that, if x,y are elements of some group then, the comutator $[x,y]$ is the element $xyx^{-1}y^{-1}$);

(ii) if M,N are ordinary P-modules and A is an arbitrary abelian group on which P is assumed to act trivially, then the diagram

$$\begin{array}{ccc} A & \to & N \\ \downarrow & & \downarrow \\ M & \to & P \end{array}$$

in which each map is a zero map, together with the zero map $0:M \times N \to A$, is a crossed square;

(iii) the diagram

$$\begin{array}{ccc} M & \xrightarrow{\chi} & \mathrm{Inn}\, M \\ \chi\downarrow & & \downarrow\iota \\ \mathrm{Inn}\, M & \xrightarrow{\iota} & \mathrm{Aut}\, M \end{array}$$

where χm is the inner automorphism determined by $m \in M$ and where ι is the inclusion of the inner automorphism subgroup, together with the standard actions and the function $h: \mathrm{Inn}\, M \times \mathrm{Inn}\, M \to M$, $(\chi m, \chi m') \to [m,m']$, is a crossed square;

(iv) [B-L] if U,V are subspaces of X with a point x_0 in common, then the diagram of boundary maps

$$\begin{array}{ccc} \pi_3(X;U,V,x_0) & \to & \pi_2(V,U\cap V,x_0) \\ \downarrow & & \downarrow \\ \pi_2(U,U\cap V,x_0) & \to & \pi_1(U\cap V,x_0) \end{array}$$

in which $\pi_3(X;U,V,x_0)$ is the triad homotopy group, together with the standard actions and the triad Whitehead product $h:\pi_2(U,U\cap V,x_0) \times \pi_2(V,U\cap V,x_0) \to \pi_3(X;U,V,x_0)$, is a crossed square.

It is worth noting that the crossed complexes of length 3 are the crossed squares of the form

$$\begin{array}{ccc} C_3 & \to & 0 \\ \downarrow & & \downarrow \\ C_2 & \to & C_1 \end{array}$$

In this thesis we shall be very much concerned with crossed squares and their higher dimensional counterparts.

Let $\partial:M \to P$ be a crossed module. Since P acts on M we may form the semi-direct product $M\underline{\times}P$. Let $s,b:M\underline{\times}P \to P$ be given respectively by $(m,p) \to p$, $(m,p) \to (\partial m)p$. The group $M\underline{\times}P$ acquires a category structure, with s,b the source and target maps, and with category composition given by $(m,p) \circ (m',(\partial m)p) = (mm',p)$. The crossed module axioms are equivalent to this category structure making $M\underline{\times}P$ a category internal to the category of groups (a result noted by several people and published in [B-S]).

This suggests how to define a crossed module internal to other algebraic categories: consider an internal category object C with source and target maps $s,b:C \to P$; the associated "crossed module" is the restriction of b to $\ker s \to P$. This process is analysed in [L-R,P1] and in Chapter I of this thesis. In his work on deformation theory, Gerstenhaber [G] developes a cohomology based on crossed modules. Also, Lue [Lu1,2] (developing the work of Gerstenhaber and work of Frohlich [F]) uses "crossed modules" in varieties of algebras. The commutative algebra version of crossed modules has been used in essence rather than in name in [L-S], and has recently been shown to be closely related to Kozul complexes [P2].

In view of the widespread use of crossed modules in other algebraic categories, it is reasonable to expect that notions of higher dimensional crossed modules might also find use in these other categories.

The equivalence between crossed modules and categories internal to the category of groups suggests, as a possible notion of an n-dimensional crossed module, an n-fold category internal to the category of groups. Indeed, such n-fold categories have been introduced by Loday [L] as a model of truncated homotopy types. Loday gives them (or more precisely, a slightly reformulated version of them) the name "n-cat-group"; however, we shall follow the more recent [B-L] and use the more accurate term *cat^n-group*.

Given a cat^n-group G one can form its iterated nerve, an (n+1)-simplicial set, whose geometric realisation BG is called the *classifying space* of of G. Conversely, any pointed, connected CW-complex X with $\pi_i X = 0$ for $i > n+1$ is itself of the homotopy type of some BG [L].

Recently a van Kampen type theorem has been found [B-L] for the "fundamental cat^n-group of an n-cube of spaces". Here an *n-cube of spaces* is just a functor, from the n-fold product of the category associated with the ordered set $0 < 1$, to the category of pointed topological spaces. Clearly cat^n-groups are a reasonable generalisation of crossed modules.

In order to apply the n-dimensional van Kampen type theorem, one needs to compute colimits of cat^n-groups. For such computations a more combinatorial version of cat^n-groups is required. For $n = 1$ crossed modules prove to be sufficiently combinatorial. It turns out that cat^2-groups are equivalent to crossed squares [L], and that crossed squares are just the version needed for applications of the 2-dimensional theorem. For higher dimensions a notion of a "crossed n-cube" is clearly needed.

A striking fact about the algebraic theory of crossed modules is that many results on crossed modules in groups, for instance the crossed complex description of cohomology (see above), carry over to other algebraic categories. (In fact, the crossed complex description of cohomology was first given for varieties of algebras [Lul], and then rediscovered for the case of groups.) It is likely that (topologically motivated) results on "crossed n-cubes in groups" will also carry over to other algebraic categories, provided that the various algebraic versions of "crossed n-cubes" exist.

In Chapter I of this thesis we give several equivalent notions of a higher dimensional crossed module. Because of the many different algebraic categories in which these notions are likely to be of interest, we adapt P.J. Higgin's definition [H] of a category of groups with multiple operators, to obtain a fairly general algebraic category C which we call a "category of Ω-groups". We work in C throughout the chapter. Possible interpretations of C include the categories of groups, rings (associative or commutative), and algebras (associative, commutative, Lie or Jordan). The notions of higher dimensional crossed modules which we introduce, and prove equivalences between, are:

(i) n-fold categories internal to C;

(ii) cat^n-objects in C;

(iii) crossed n-cubes in C;

(iv) n-simplicial objects in C whose normal complexes are of length 1;

(v) n-fold crossed modules in C.

Crossed n-cubes will be of most interest to us. For n = 1 they are just crossed modules; for n = 2 (and C the category of groups) they are crossed squares (see above). The observation that simplicial groups whose normal complex is of length 1 are equivalent to categories internal to the category of groups, is well known and has led Conduché [C] to the definition of a "crossed module of length 2"; such a 'crossed module' being equivalent to a simplicial group whose normal complex is of length 2. It turns out that there is a functor from crossed squares to crossed modules of length 2.

In Chapter II we give detailed descriptions of some low dimensional crossed n-cubes for C equal to the category of groups, Lie algebras, commutative algebras, and associative algebras.

In Chapter III we look at certain colimits of crossed 2-cubes, and obtain non-abelian generalisations of some standard constructions: let M,N be groups which act on each other (and on themselves by conjugation); following [B-L] we obtain a *non abelian tensor product* $M \otimes N$, which is the group generated by elements $m \otimes n$ for $m \in M$, $n \in N$, subject to the relations

$$mm' \otimes n = ({}^{m}m' \otimes {}^{m}n)(m \otimes n),$$

$$m \otimes nn' = (m \otimes n)({}^{n}m \otimes {}^{n}n').$$

We obtain a non-abelian exterior product $M \wedge N$ (again originally due to [B-L]), and a non-abelian anti-symmetric product $M \underline{\wedge} N$ (a special case of which has been used in [D]). The Lie and commutative algebra versions of these constructions are also given. We consider a certain colimit of crossed 3-cubes (in groups) which leads us to the definition of a "cubical tensor product". In addition the chapter contains various exact sequences involving the non abelian constructions.

The relevance of crossed modules to cohomology has been mentioned above. Surprisingly, little work has been done on the dual situation of crossed modules in homology. In Chapter IV we show that if N is a group and $\partial : M \to P$ is a projective crossed module with $\mathrm{im}\ \partial = N$, then $H_2(N) \cong \ker \partial \cap [M,M]$ (this is joint work with T.Porter [E-P]).

This formula should perhaps be seen as a crossed version of Hopf's formula for $H_2(N)$. We give a weaker version of the formula for the case of Lie algebras. It is worth noting that our methods give a new and simpler proof of the key lemma 2.1 of [R]. We go on to investigate the link between crossed squares and homology. Let $R \to F \to G$ be a free presentation of a group G. We obtain, by algebraic means, two isomorphisms $H_2(G) \cong \ker(G \wedge G \to G)$, $H_3(G) \cong \ker(F \wedge R \to F)$. We combine these new descriptions of $H_2(G)$, $H_3(G)$ with certain of the exact sequences of Chapter III to obtain:

THEOREM *If* M,N *are normal subgroups of a group* G *such that* $G = MN$, *then there is an exact sequence*

$$\pi_3(M \wedge N) \to H_2(G) \to H_2(G/M) \oplus H_2(G/N) \to M \cap N/[M,N] \to$$
$$\to H_1(G) \to H_1(G/M) + H_1(G/N) \to 1$$

where $\pi_3(M \wedge N)$ *is the kernel of a canonical map* $M \wedge N \to M$.

THEOREM *If, in the preceding theorem,* $G = N$, *then we can extend the exact sequence by two terms:*

$$H_3(G) \to H_3(G/M) \to \pi_3(M \wedge N).$$

These two theorems are originally due to Brown and Loday [B-L] who obtained them as a corollary to their 3-dimensional van Kampen type theorem. Our proofs are purely algebraic, and consequently we are able to give the Lie algebra version of the first theorem. We also give an analogue of the first theorem in which $H_2(G)$ is replaced by the group $\underline{H}_2(G)$: the group $\underline{H}_2(G)$ being the group introduced by Dennis [D] as a kind of "second homology group suitable for algebraic K-theory".

Chapter V is a collection of miscellaneous comments.

CHAPTER I

VARIOUS ALTERNATIVE DESCRIPTIONS OF INTERNAL n-FOLD CATEGORIES

0. INTRODUCTION

We begin this chapter by defining a "category of Ω-groups" C. Interpretations of C include the categories of groups, rings (associative, commutative), and algebras (associative, commutative, Lie and Jordan). Thus the theory of Ω-groups provides a convenient setting in which to work. In §2,3,5,6 we introduce, and prove equivalences between:

(i) n-fold categories internal to C;

(ii) cat^n-objects in C;

(iii) crossed n-cubes in C;

(iv) n-simplicial objects in C whose normal complexes are of length 1.

(v) n-fold crossed modules in C.

In §4 we give a result on colimits of crossed n-cubes in C.

1. CATEGORIES OF Ω-GROUPS

Our definition of a "category of Ω-groups" is adapted from [H].

A pointed set X is said to *admit a set Ω of finitary operations* if to each $\omega \in \Omega$ is attached a non-negative integer $n = n(\omega)$ called its *weight* and, for this n, there is a pointed map of sets $X^n \to X$ from the n-fold product of X to X.

A pointed set X which admits a set Ω of finitary operations is called an *Ω-group* if the following five axioms hold:

(i) the set Ω contains no operations whose weights are greater than 2; there is precisely one operation (written 0) of weight 0, and precisely two operations (written +, *) of weight 2; there is a prefered operation (written -) of weight 1;

(ii) the operations 0, -, + satisfy the axioms of a (non abelian) group;

(iii) for all $x,y,z \in X$, and unitary operations ω, we have

$$\omega(x * y) = \omega x * y = x * \omega y$$

(iv) and, provided ω is not the prefered unitary operator -,

$$\omega(x + y) = \omega x + \omega y;$$

(v) and $(y + z) * x = (y * x) + (z * x)$,

$$x * (y + z) = (x * y) + (x * z).$$

A morphism of Ω-groups is a set map which preserves the operations. Any category whose objects are Ω-groups for some fixed Ω, and whose morphisms are precisely the morphisms of Ω-groups, will be called a *category of Ω-groups*.

EXAMPLE(1.1.1) Let $\Omega = \{0,-,+,*\}$. Then the category of groups is a category of Ω-groups in which the operation * is trivial (i.e. has constant value 0). The category of rings is a category of Ω-groups in which the operation * is non trivial.

EXAMPLE(1.1.2) Let A be a commutative ring (with unit) and let $\Omega = \{0,-,+,*\} \cup \{a \in A\}$. Then the categories of associative, commutative, Lie and Jordan algebras over A are categories of Ω-groups. in which $a \in A$ is scalar multiplication.

EXAMPLE(1.1.3) A category of interest (in the sense of Orzech [O]) which has only two binary operations is a category of Ω-groups. We could equally well work with a notion of Ω-groups which allows more than two binary operations, but have no examples to motivate this generalisation.

EXAMPLE(1.1.4) We note that an Ω-group has precisely one operation of weight 0. Thus, for instance, the category of associative rings with unit is not a category of Ω-groups.

For the remainder of this chapter we fix a category **C** of Ω-groups.

2. CAT^n-OBJECTS IN C

Recall that a *category internal to* **C** consists of: a pair of objects G,P in **C**; and four morphisms $s:G \to P$, $b:G \to P$, $i:P \to G$, $o:G\times_P G \to G$ (here $G\times_P G = \{(x,y) \in G\times_P G : bx = sy\}$) such that;

(i) $si = bi =$ identity;

(ii) $(isx) \circ x = x$, $x \circ (ibx) = x$;

(iii) $s(x \circ y) = sx$, $b(x \circ y) = by$;

(iv) $x \circ (y \circ z) = (x \circ y) \circ z$;

(whenever these last two equations are defined).

A map of categories internal to **C** is a pair of structure preserving morphisms $\phi:G \to G'$, $\psi:P \to P'$.

Note that, since the category composition o is a morphism in C, for $(u,v), (x,y) \in G \times_P G$ we have

(1) $(u \circ v) + (x \circ y) = (u + x) \circ (v + y)$,

(2) $(u \circ v) * (x \circ y) = (u * x) \circ (v * y)$.

Also we can write the category composition in terms of the group structure on G, since

$$
\begin{aligned}
(3) \quad u \circ v &= (ibu - ibu + u) \circ (v + 0) \\
&= ((ibu) \circ v) + ((-ibu + u) \circ 0) \qquad \text{(by 1)} \\
&= v - ibu + u.
\end{aligned}
$$

Suppose now we are given an arbitrary triple of morphisms $s:G \to P$, $b:G \to P$, $i:P \to G$ in C which satisfy $si = bi =$ identity. For $(u,v) \in G \times_P G$ we can define $u \circ v = v - ibu + u$. It is readily seen that this partial operation, together with the three morphisms, constitute a category internal to C if and only if equations (1) and (2) hold. But we have:

equation (1)

$\equiv (v - ibu + u) + (y - ibx + x)$

$= v + y - ibx - ibu + u + x$

$\equiv (-ibu + u) + (y - isy)$

$= (y - isy) + (-ibu + u)$.

Let ker s, ker b be the kernels of s,b, and denote by [ker b, ker s] the subobject of G generated by the commutator elements $p + q - p - q$ with $p \in$ ker b, $q \in$ ker s. Then equation (1) states precisely that [ker b,ker s] = 0. Under the assumption that [ker b, ker s] = 0, we also have:

equation (2)

$\equiv$ $(v - ibu + u) * (y - ibx + x)$

$=$ $(v * x) - (ibu * ibx) + (u * x)$

$\equiv$ $((v - isv) * (x - ibx)) + ((u - ibu) * (y - isy))$

$=$ 0.

Denote by $\langle \ker b, \ker s\rangle_*$ the subobject of G generated by the elements $p * q$, $q * p$ with $p \in \ker b$, $q \in \ker s$. Then equation (2) states precisely that $\langle \ker b, \ker s\rangle_* = 0$.

We are thus led to

DEFINITION(1.2.1) A *cat^1-object* $\underline{G}$ *in* C consists of: an object G in C and a subobject P of G; and two morphisms $s,b:G \to P$ such that;

(i) $s|_P = b|_P =$ identity;

(ii) and $[\ker b, \ker s] = 0$, $\langle \ker b, \ker s\rangle_* = 0$.

A map of cat^1-objects is a morphism $\phi:G \to G'$ such that $\phi s = s'\phi$ and $\phi b = b'\phi$.

We have immediately

PROPOSITION(1.2.2) *There is an equivalence of categories,* (*cat^1-objects in* C) $\simeq$ (*categories internal to* C).

We now aim to generalise this proposition.

DEFINITION(1.2.2) A *cat^n-object* $\underline{G}$ *in* C consists of a family of cat^1-objects in C, $s_i,b_i:G \to P_i$, $1 \leqslant i \leqslant n$, such that

(i) $s_is_j = s_js_i$, $b_ib_j = b_jb_i$, and $s_ib_j = b_js_i$ $(i \neq j)$.

(Here s_is_j is the composition of the map s_j with the map obtained by restricting s_i to P_j.)

A map of cat^n-objects is a morphism $\phi: G \to G'$ such that $\phi s_i = s_i'\phi$ and $\phi b_i = b_i'\phi$.

The notion of a cat^n-group is due to Loday [L], although he used the term "n-cat-group". The term "cat^n-group" is used in [B-L].

The definition of a category internal to C can be restated in purely categorical language. More precisely, axioms (ii),(iii),(iv) can be replaced by

(ii)' $so = s\pi_1$, $bo = b\pi_2$ (where $\pi_i: G \times_P G \to G$ is the i^{th} projection);

(iii)' $o(1 \times o) = o(o \times 1): G \times_P G \times_P G \to G$;

(iv)' $o(1 \times ib) = o(is \times 1) = 1_G$

Thus for an arbitrary category C' we can form the category $CT^1(C')$ of *categories internal to* C'. Inductively we define the category $CT^n(C')$ of *n-fold categories internal to* C' to be the category $CT^n(C') = CT^1(CT^{n-1}(C'))$.

PROPOSITION(1.2.3) *There is an equivalence of categories, (cat^n-objects in C) $\simeq$ (n-fold categories internal to C).*

PROOF We have already proved the proposition for $n = 1$. Assume it is true for a particular value n. Then there is an equivalence, $CT^{n+1}(C) \simeq CT^1(\mathrm{cat}^n\text{-objects in } C)$. So we need to prove an equivalence between $CT^1(\mathrm{cat}^n$-objects in C) and (cat^{n+1}-objects in C).

Suppose given a category internal to the category of cat^n-objects in C. Thus we have four maps $s,b:\underline{G} \to \underline{P}$, $i:\underline{P} \to \underline{G}$, $o:\underline{G} \times_{\underline{P}} \underline{G} \to \underline{G}$ of cat^n-objects. Suppose that the cat^n-object $\underline{G}$ consists of maps $s_i, b_i: G \to P_i$, $1 \leqslant i \leqslant n$. We obtain a cat^{n+1}-object from $\underline{G}$ by setting $P_{i+1} = P$ the underlying group of $\underline{P}$, and setting $s_{i+1} = s$, $b_{i+1} = b$.

Conversely suppose given a cat^{n+1}-object which consists of the maps $s_i, b_i : G \to P_i$, $1 \leq i \leq n+1$. Let $\underline{G}'$ be the cat^n-object consisting of the maps $s_i, b_i : G \to P_i$, $1 \leq i \leq n$. Let $\underline{P}'$ be the cat^n-object consisting of the restricted maps $s_i|P_{n+1}, b_i|P_{n+1} : P_{n+1} \to P_i \cap P_{n+1}$, $1 \leq i \leq n$. Then the maps $s_{n+1}, b_{n+1} : \underline{G}' \to \underline{P}'$, together with the inclusion $\underline{P}' \to \underline{G}'$ and the map $\underline{G}' \times_{\underline{P}'} \underline{G}' \to \underline{G}'$ given by $(u,v) \to v - b_{n+1}u + u$, constitute a category internal to the category of cat^n-objects in C.

This correspondence between categories internal to the category of cat^n-objects in C, and cat^{n+1}-objects in C, gives rise to a pair of quasi-inverse functors. ∇

3. CROSSED n-CUBES IN C

Our aim now is a definition of a "crossed module in C" (we shall also use the term "crossed 1-cube in C"), and a proof that such a gadget is equivalent to a cat^1-object in C.

Suppose that we are given a split, short exact sequence $M \xrightarrow{i_0} G \underset{s}{\overset{i}{\leftrightarrows}} P$ in C. Thus M,G,P are Ω-groups; the maps i_0, i are injective (and so we consider M,P to be subobjects of G); and $M = \ker s$, and $si = $ identity. Let $\alpha^+, \alpha^*, \alpha^{*\circ} : P \times M \to M$ be, respectively, the functions

$$(p,m) \to (^{p+})m = p + m - p,$$

$$(p,m) \to (^{p*})m = p * m,$$

$$(p,m) \to (^{p*\circ})m = m * p.$$

A triple of functions obtained in this manner will be called a *C-action* of P on M.

Suppose now we have an arbitrary triple of functions $P \times M \to P$, denoted by $\alpha^{+}, \alpha^{*}, \alpha^{*\circ}$ (we are not assuming that these functions are a C-action). The underlying set of $M \times P$ can be made to admit the set Ω of finitary operations (recall that C is a category of Ω-groups) by defining:

$$(m,p) + (m',p') = (m + {}^{(p+)}m', p + p'),$$

$$(m,p) * (m',p') = ((m * m') + {}^{(p^{*})}m' + {}^{(p'^{*\circ})}m, p * p'),$$

$$-(m,p) = (-{}^{(-p+)}m, -p),$$

$$\omega(m,p) = (\omega m, \omega p) \quad \text{for each unitary operation } \omega \text{ except } -.$$

The resulting Ω-group is the *semi-direct product* of M with P and will be denoted $M \underline{\times} P$.

PROPOSITION(1.3.1) *Let M,P be objects in C and let* $\alpha^{+}, \alpha^{*}, \alpha^{*\circ} : P \times M \to M$ *be three functions. These functions are a C-action if and only if the semi-direct product* $M \underline{\times} P$ *is an object in C.*

PROOF If the functions are a C-action then they are derived from a split, short exact sequence $M \to G \leftrightarrows P$, and G is isomorphic to $M \underline{\times} P$. Conversely, if $M \underline{\times} P$ is an object in C, then the functions are a C-action since they are derived from the sequence $M \xrightarrow{i_0} M \underline{\times} P \underset{s}{\overset{i}{\leftrightarrows}} P$ where $i_0 m = (m,0)$, $ip = (0,p)$, $s(m,p) = p$. ∇

This proposition is essentially due to Orzech [O]. As an application we give

EXAMPLE(1.3.2) Let C be the category of multiplicative

groups with identity e, let M,P be groups, and let $\alpha^{+}: P\times M \to$
$(p,m) \to {}^{p}m$
M be a function (we assume that both $\alpha^{*}, \alpha^{*\circ}: P\times M \to M$ are the zero map). In the semi-direct product $M \underline{\times} P$ multiplication is given by $(m,p)(m',p') = (m\,{}^{p}m', pp')$, the identity is (e,e), and $(m,p)^{-1} = ({}^{p^{-1}}m^{-1}, p^{-1})$. It is routine to check that $M\underline{\times}N$ is a group if and only if

(i) ${}^{e}m = m$,

(ii) ${}^{p}({}^{p'}m) = {}^{(pp')}m$,

(ii) ${}^{p}(mm') = ({}^{p}m)({}^{p}m')$,

for all $m,m' \in M$, $p \in P$. Thus in this case a C-action coincides with the usual notion of a group action.

We can now state the crucial

DEFINITION(1.3.3) A *crossed module in* **C** (or a *crossed 1-cube in* **C**) consists of: a pair of objects M,P in **C**; a morphism $\partial: M \to P$; and a **C**-action of P on M, such that;

(i) $\partial({}^{(p+)}m) = p + \partial m - p$,

$\partial({}^{(p*)}m) = p * \partial m$,

$\partial({}^{(p*\circ)}m) = \partial m * p$;

(ii) ${}^{(\partial m+)}m' = m + m' - m$,

${}^{(\partial m*)}m' = m * m'$,

${}^{(\partial m*\circ)}m' = m' * m$;

for $m,m' \in M$, $p \in P$.

A map of crossed modules is a pair of morphisms $\psi: M \to M'$, $\phi: P \to P'$ such that $\partial'\psi = \phi\partial$ and $\psi({}^{(p\omega)}m) = {}^{(\phi p\omega)}\psi m$ for $\omega = +, *, *\circ$.

When C is the category of groups this definition reduces to the classical definition of a crossed module [W1]. The definition also reduces to give the algebraic cases of a crossed module defined in [L-R]. Our general notion of a crossed module is essentially the same as the one given in [P3].

PROPOSITION(1.3.4) *There is an equivalence of categories, (crossed modules in* **C**) $\simeq$ *(cat*^{1}*-objects in* **C**).

PROOF Suppose given a cat^1-object $s,b:G \to P$. Conjugation and multiplication in G give rise to a **C**-action of P on ker s. The restriction of b to ker $s \to P$ clearly satisfies axiom (1.3.3.i). We must check axiom (1.3.3.ii).

Suppose now we are given a crossed module $\partial:M \to P$. We can construct two maps $s',b':M\underline{\times}P \to P$ by defining $s'(m,p) = p$, $b'(m,p) = (\partial m)p$. Axiom (1.3.3.i) ensures that b' is a morphism in **C**. The maps s',b' clearly satisfy axiom (1.2.1.i). We must check axiom (1.2.1.ii).

We shall show that axiom (1.3.3.ii) is equivalent to axiom (1.2.1.ii). Let $x \in \ker s'$, $y \in \ker b'$; then $x = (m,0)$, $y = (-n,\partial n)$ for some $m,n \in M$. We have:

axiom (1.2.1.ii)

$$\begin{array}{ll} & x + y = y + x \\ \equiv & x * y = 0 \\ & y * x = 0 \end{array}$$

$$\begin{array}{ll} & (m - n,\partial n) = (-n + {}^{(\partial n+)}m,\partial n) \\ \equiv & (-(m * n) + {}^{(\partial n*^{\circ})}m,0) = (0,0) \\ & (-(n * m) + {}^{(\partial n*)}m,0) = (0,0) \end{array}$$

$$n + m - n = (\partial n+)m$$

$$\equiv \quad m * n = (\partial n^{*\circ})m$$

$$n * m = (\partial n^{*})m$$

$\equiv$ axiom (1.3.3.ii).

We have thus given a correspondence between crossed modules in C and cat^1-objects in C. This correspondence gives rise to an equivalence of categories. ∇

We can combine propositions (1.2.2) and (1.3.4) to obtain an equivalence between categories internal to C and crossed modules in C. For C equal to the category of groups this equivalence has been known for some time; it first seems to have appeared in print in [B-S]. The equivalence is given in [L-R] for C equal to various common algebraic categories. Porter [P3] gives the equivalence in the general setting of a category of groups with multiple operations.

We now aim for a definition of a "crossed n-cube in C" and a proof that such a gadget is equivalent to a cat^n-object in C. We begin by generalising the notion of a C-action.

DEFINITION(1.3.5) An *n-fold split short exact sequence in* C (abbreviated to *n.s.s.e.s*) is an object G in C together with n subobjects P_i and endomorphisms $s_i : G \to P_i$, $1 \leq i \leq n$, such that $s_{i|P_i}$ = [illegible] identity, and $s_i s_j = s_j s_i$

So a 0.s.s.e.s. is just an object in C; a 1.s.s.e.s. is the standard notion of a split, short exact sequence.

In order to handle n.s.s.e.s.'s we introduce some notation. Let $|n|$ denote the set $\{1,2,\ldots,n\}$, and let $\Gamma|n|$ be the poset consisting of the subsets of $|n|$. For each $\gamma \in \Gamma|n|$, let $\gamma' = |n| \setminus \gamma$. Let the largest number in γ be denoted max γ. Let $\iota_0, \iota_n : \Gamma|n-1| \to \Gamma|n|$ be the poset maps given respectively by $\gamma \to \gamma$, $\gamma \to \gamma\cup\{n\}$.

Suppose we are given an n.s.s.e.s. as above. For each $\gamma \in \Gamma|n|$ construct the multiple intersection

$Y\gamma = (\cap_{i\epsilon\gamma}\ker s_i) \cap (\cap_{i\epsilon\gamma'}P_i)$.

Thus if $n = 1$, we have $Y\{1\} = \ker s_1$, $Y\emptyset = P_1$. If $n = 2$, we have $Y\{1,2\} = \ker s_1 \cap \ker s_2$, $Y\{1\} = \ker s_1 \cap P_2$, $Y\{2\} = \ker s_2 \cap P_1$, $Y\emptyset = P_1 \cap P_2$.

PROPOSITION(1.3.6) *If $\gamma_1 \subset \gamma_2 \in \Gamma|n|$, then there is a C-action of $Y\gamma_1$ on $Y\gamma_2$.*

PROOF For each $x \in Y\gamma_1$, $y \in Y\gamma_2$ define $^{(x+)}y = x + y - x$, $^{(x^*)}y = x * y$, $^{(x^{*\circ})}y = y * x$. It is readily seen that $^{(x+)}y$, $^{(x^*)}y$, $^{(x^{*\circ})}y \in Y\gamma_2$. ∇

We also have

PROPOSITION(1.3.7) *Let $\gamma_1,\gamma_2 \in \Gamma|n|$ be such that $\gamma_1 \not\subset \gamma_2$, $\gamma_2 \not\subset \gamma_1$. Then there are three functions $h^+,h^*,h^{*\circ}: Y\gamma_1 \times Y\gamma_2 \to Y(\gamma_1\cup\gamma_2)$ given, respectively, by $(x,y) \to x + y - x - y$, $(x,y) \to x * y$, $(x,y) \to y * x$.* ∇

Note that there are also functions $h^{+\prime}, h^{*\prime}, h^{*\circ\prime}$: $Y\gamma_2 \times Y\gamma_1 \to Y(\gamma_2 \cup \gamma_1)$, and that $h^+(x,y) = -h^{+\prime}(y,x)$, $h^*(x,y) = h^{*\circ\prime}(y,x)$, $h^{*\circ}(x,y) = h^{*\prime}(y,x)$.

In the light of propositions (1.3.6),(1.3.7), and this last observation, we make the following

DEFINITION(1.3.8) An *n-action* $\underline{Y}$, $n \geq 0$, consists of:

(a) an object $Y\gamma$ in **C** for each $\gamma \in \Gamma|n|$;

(b) three functions $\alpha^+, \alpha^*, \alpha^{*\circ}: Y\gamma_1 \times Y\gamma_2 \to Y\gamma_2$ for each $\gamma_1 \subset \gamma_2 \in \Gamma|n|$;

(c) three functions $h^+, h^*, h^{*\circ}: Y\gamma_3 \times Y\gamma_4 \to Y(\gamma_3 \cup \gamma_4)$ for each $\gamma_3, \gamma_4 \in \Gamma|n|$ such that $\gamma_3 \not\subset \gamma_4$, $\gamma_4 \not\subset \gamma_3$ and $\max(\gamma_3 \backslash (\gamma_3 \cap \gamma_4)) < \max(\gamma_4 \backslash (\gamma_3 \cap \gamma_4))$;

An n-action which is derived from an n.s.s.e.s. will be called a C-*n-action*.

For $n = 0$, a C-0-action is just an object in C. For $n = 1$, a C-1-action is just a C-action. For $n = 2$, a 2-action consists of four objects $Y\{1,2\}$, $Y\{1\}$, $Y\{2\}$, $Y\emptyset$ together with three functions $\alpha^+, \alpha^*, \alpha^{*\circ}: Y\gamma_1 \times Y\gamma_2 \to Y\gamma_2$ for each $\gamma_1 \subset \gamma_2$, and precisely three functions $h^+, h^*, h^{*\circ}: Y\{1\} \times Y\{2\} \to Y\{1,2\}$.

Given a 1-action $\alpha^+, \alpha^*, \alpha^{*\circ}: P \times M \to M$ we can construct a 0-action by forming the semi-direct product $M \underline{\times} P$. The 1-action is easily retrieved from the 0-action. More generally, given an n-action $\underline{Y}$ we can construct an (n-1)-action $\underline{RY}$ without loosing information. The details are as follows.

Suppose given an n-action $\underline{Y}$. Recall the poset maps $\iota_0, \iota_n : \Gamma|n-1| \to \Gamma|n|$. For each $\gamma \in \Gamma|n-1|$ define

$$RY\gamma = Y\iota_n\gamma \underline{\times} Y\iota_0\gamma.$$

For each $\gamma_1 \subset \gamma_2 \in \Gamma|n-1|$ we can construct three maps $\alpha^+, \alpha^*, \alpha^{*\circ}: RY\gamma_1 \times RY\gamma_2 \to RY\gamma_2$ as follows: set

$$L = Y\iota_n\gamma_2,$$

$$M = Y\iota_0\gamma_2,$$

$$N = Y\iota_n\gamma_1,$$

$$P = Y\iota_0\gamma_1;$$

thus $RY\gamma_1 = N\underline{\times}P$, $RY\gamma_2 = L\underline{\times}M$; it is readily seen that there are three functions $h^+, h^*, h^{*\circ}: M\times N \to L$ and that, for each $(n,p,l,m) \in RY\gamma_1 \times RY\gamma_2$, we can define

$$\alpha^+(n,p,l,m) = (({}^{(n+)}(({}^{(p+)}l) - h^+(({}^{(p+)}m, n), {}^{(p+)}m),$$

$$\alpha^*(n,p,l,m) = ({}^{(n^*)}l + {}^{(p^*)}l + h^{*\circ}(m,n), {}^{(p^*)}m),$$

$$\alpha^{*\circ}(n,p,l,m) = ({}^{(n^{*\circ})}l + {}^{(p^{*\circ})}l + h^*(m,n), {}^{(p^{*\circ})}m).$$

For each $\gamma_3, \gamma_4 \in \Gamma|n-1|$ such that $\gamma_3 \not\subset \gamma_4$, $\gamma_4 \not\subset \gamma_3$, $\max(\gamma_3\backslash(\gamma_3\cap\gamma_4)) < \max(\gamma_4\backslash(\gamma_3\cap\gamma_4))$, we can construct three functions $h^+, h^*, h^{*\circ}: RY\gamma_3 \times RY\gamma_4 \to RY(\gamma_3\cup\gamma_4)$ as follows: set

$$K = Y\iota_n(\gamma_3\cup\gamma_4),$$

$$L = Y\iota_n\gamma_3,$$

$$M = Y\iota_n\gamma_4,$$

$$N = Y\iota_0(\gamma_3\cup\gamma_4),$$

$$P = Y\iota_0\gamma_3,$$

$$Q = Y\iota_0\gamma_4;$$

thus $RY\gamma_3 = L\underline{\times}P$, $RY\gamma_4 = M\underline{\times}Q$, $RY(\gamma_3\cup\gamma_4) = K\underline{\times}N$; it is readily checked that there are six triples of functions

$$h^+, h^*, h^{*\circ}: Q\times L \to K,$$

$$h^+, h^*, h^{*\circ}: P\times M \to K,$$

$$h^+, h^*, h^{*\circ}: L\times M \to K,$$

$h^+, h^*, h^{*\circ} : N \times L \to K,$

$h^+, h^*, h^{*\circ} : N \times M \to K,$

$h^+, h^*, h^{*\circ} : P \times Q \to N,$

and that for each $(l,p,m,q) \in RY\gamma_3 \times RY\gamma_4$ we can define

$$h^+(l,p,m,q) =$$
$$({}^{(l+)}h^+(p,m) + h^+(l,m) - {}^{(m+)}h^+(h^+(p,q),l) - h^+(h^+(p,q),m) - {}^{(h^+(p,q)+)}({}^{(m+)}h^+(q,l)),\ h^+(p,q)),$$

$$h^*(l,p,m,q) =$$
$$(h^*(l,m) + h^*(p,m) + h^{*\circ}(q,l),\ h^*(p,q)),$$

$$h^{*\circ}(l,p,m,q) =$$
$$(h^{*\circ}(l,m) + h^{*\circ}(p,m) + h^*(q,l),\ h^{*\circ}(p,q)).$$

We have thus completed the construction of <u>RY</u>.

PROPOSITION(1.3.9) *An n-action* <u>Y</u>*, n > 1, is a* C-*n-action if and only if* <u>RY</u> *is a* C-*(n-1)-action.*

PROOF Suppose <u>Y</u> is a C-n-action. Then <u>Y</u> is derived from some n.s.s.e.s. which consists of maps $s_i : G \to P_i$, $1 \leq i \leq n$, say. It is routine to check that <u>RY</u> is derived from the (n-1).s.s.e.s. consisting of the maps $s_i : G \to P_i$, $1 \leq i \leq n-1$.

Conversely, suppose <u>RY</u> is a C-(n-1)-action which is derived from a (n-1).s.s.e.s. consisting of the maps $s_i : G \to P_i$, $1 \leq i \leq n-1$. For each $\gamma \in \Gamma|n-1|$ the object $RY\gamma$ is a semi-direct product; these semi-direct products give rise to a semi-direct product structure on G, say $G = G_n \underline{\times} G_0$. By setting $s_n:G \to G_0$ equal to the canonical projection we obtain an n.s.s.e.s. from which $\underline{Y}$ is derived. ∇

In Chapter II we use this proposition to obtain explicit descriptions of C-n-actions for specific choices of C and low values of n.

We are now in a position to make the main definition of this section.

DEFINITION(1.3.10). A *crossed n-cube in* C consists of a contravariant functor $\underline{Y}:\Gamma|n| \to C$ and an n-action structure on the set $\{Y\gamma : \gamma \in \Gamma|n|\}$ (where $Y\gamma$ denotes the image of γ under $\underline{Y}$), such that:

(i) the n-action is a C-n-action;

(ii) for each $\gamma_1 \subset \gamma_2 \in \Gamma|n|$ the C-action of $Y\gamma_1$ on $Y\gamma_2$ is via the map $Y\gamma_1 \to Y\emptyset$ (in fact we shall assume that for any $\gamma,\gamma' \in \Gamma|n|$, there is a C-action of $Y\gamma$ on $Y\gamma'$ via the map $Y\gamma \to Y\emptyset$);

(iii) for each $\gamma_1 \subset \gamma_2 \in \Gamma|n|$, the map $Y\gamma_2 \to Y\gamma_1$ is a crossed module, and it preserves the actions of $Y\emptyset$;

(iv) let $\gamma_3,\gamma_4,\gamma_5,\gamma_6 \in \Gamma|n|$ be such that $\gamma_3 \supset \gamma_5$, $\gamma_4 \supset \gamma_6$, $\gamma_3 \not\subset \gamma_4$, $\gamma_4 \not\subset \gamma_3$, $\gamma_5 \not\subset \gamma_6$, $\gamma_6 \not\subset \gamma_5$ and $\max(\gamma_3\backslash\gamma_3\cap\gamma_4) < \max(\gamma_4\backslash\gamma_3\cap\gamma_4)$; thus we have maps $\delta_3:Y\gamma_3 \to Y\gamma_5$, $\delta_4:Y\gamma_4 \to Y\gamma_6$, $\delta:Y(\gamma_3\cup\gamma_4) \to Y(\gamma_5\cup\gamma_6)$ (we are allowing the possibility that certain of these maps may be identity maps) and functions $h^+,h^*,h^{*\circ}:Y\gamma_3\times Y\gamma_4 \to Y(\gamma_3\cup\gamma_4)$; then

$$\left.\begin{aligned} \delta h^+(x,y) &= h^+(\delta_3x,\delta_4y)\\ \delta h^*(x,y) &= h^*(\delta_3x,\delta_4y)\\ \delta h^{*\circ}(x,y) &= h^{*\circ}(\delta_3x,\delta_4y)\end{aligned}\right\} \text{ if } \max(\gamma_5\backslash\gamma_5\cap\gamma_6) < \max(\gamma_6\backslash\gamma_5\cap\gamma_6),$$

$$\left.\begin{array}{l}\delta h^{+}(x,y) = -\,h^{+}(\delta_4 y,\delta_3 x)\\ \delta h^{*}(x,y) = h^{*\circ}(\delta_4 y,\delta_3 x)\\ \delta h^{*\circ}(x,y) = h^{*}(\delta_4 y,\delta_3 x)\end{array}\right\}\ \text{if } \max(\gamma_6\backslash\gamma_5\cap\gamma_6) < \max(\gamma_5\backslash\gamma_5\cap\gamma_6);$$

(v) for each $\gamma_3,\gamma_4 \in \Gamma\langle n\rangle$ such that $\gamma_3 \not\subset \gamma_4$, $\gamma_4 \not\subset \gamma_3$ and $\max(\gamma_3\backslash\gamma_3\cap\gamma_4) < \max(\gamma_4\backslash\gamma_3\cap\gamma_4)$, the functions $h^{+},h^{*},h^{*\circ}:Y\gamma_3 \times Y\gamma_4 \to Y(\gamma_3\cup\gamma_4)$ and the morphisms $\delta:Y(\gamma_3\cup\gamma_4) \to Y\gamma_3$, $\delta':Y(\gamma_3\cup\gamma_4) \to Y\gamma_4$ satisfy

$$\begin{aligned}
\delta h^{+}(x,y) &= x - ({}^{y+})x,\\
\delta h^{*}(x,y) &= ({}^{y^{*\circ}})x,\\
\delta h^{*\circ}(x,y) &= ({}^{y^{*}})x,\\
\delta' h^{+}(x,y) &= ({}^{x+})y - y,\\
\delta' h^{*}(x,y) &= ({}^{x^{*}})y,\\
\delta' h^{*\circ}(x,y) &= ({}^{x^{*\circ}})y;
\end{aligned}$$

(vi) and also

$$\begin{aligned}
h^{+}(\delta z,y) &= z - ({}^{y+})z,\\
h^{*}(\delta z,y) &= ({}^{y^{*\circ}})z,\\
h^{*\circ}(\delta z,y) &= ({}^{y^{*}})z,\\
h^{+}(x,\delta' z) &= ({}^{x+})z - z,\\
h^{*}(x,\delta' z) &= ({}^{x^{*}})z,\\
h^{*\circ}(x,\delta' z) &= ({}^{x^{*\circ}})z;
\end{aligned}$$

for all $x \in Y\gamma_3$, $y \in Y\gamma_4$, $z \in Y(\gamma_3\cup\gamma_4)$.

A map $\Phi:\underline{Y} \to \underline{Y}'$ of crossed n-cubes is a family of structure preserving morphisms $\{\phi_\gamma:Y\gamma \to Y'\gamma\}$.

We shall frequently use the term *crossed module* instead of *crossed 1-cube*, and *crossed square* instead of *crossed 2-cube*.

For many practical purposes axiom (1.3.10.i) is not combinatorial enough. The fact that $\underline{Y}$ is a C-n-action needs to be expressed in terms of rules governing the various "h" functions (cf. the description of a crossed square of groups, i.e. a crossed 2-cube in the category of groups, which is given in the introduction). In chapter II we will give such a combinatorial description of crossed n-cubes for various particular choices of **C** and low values of n.

PROPOSITION(1.3.11) *There is an equivalence of categories,* *(crossed n-cubes in* **C**) $\simeq$ *(cat*^{n}*-objects in* **C**).

PROOF The proposition is trivial for $n = 0$, so let $n \geq 1$.

Suppose given a cat^n-object consisting of the maps $s_i, b_i : G \to P_i$, $1 \leq i \leq n$. The maps s_i, $1 \leq i \leq n$, constitute an n.s.s.e.s. and thus give rise to a C-n-action $\underline{Y}$. Given $\gamma_1 \subset \gamma_2 \in \Gamma|n|$ satisfying $\gamma_2 \backslash \gamma_1 = \{k\}$ for some k, axiom (1.2.1.i) and the commutativity conditions of a cat^n-object ensure that the morphism $b_k : G \to P_k$ restricts to a map $Y\gamma_2 \to Y\gamma_1$, and that the resulting diagram is commutative. Thus $\underline{Y}$ is a contravariant functor from $\Gamma|n|$ to **C**. It is readily verified that $\underline{Y}$ is a crossed n-cube.

Conversely, suppose given a crossed n-cube $\underline{Y}$. Considering $\underline{Y}$ as a C-n-action, we can construct a C-(n-1)-action $\underline{RY}$ (see above). This construction can be extended to give a contravariant functor $\underline{RY} : \Gamma|n-1| \to$ **C** since, for each $\gamma_1 \subset \gamma_2 \in \Gamma|n-1|$, the maps $\delta_n : Y\iota_n\gamma_2 \to Y\iota_n\gamma_1$, $\delta_0 : Y\iota_0\gamma_2 \to Y\iota_0\gamma_1$ in the image of $\underline{Y}$ give rise to a morphism $RY\gamma_2 \to RY\gamma_1$, $(x,y) \to (\delta_n x, \delta_0 y)$.

CLAIM *The functor* $\underline{RY}$ *is a crossed (n-1)-cube.*

The verification of this claim is routine and we omit it. Note that by iterating m times $, 1 \leq m \leq n$, the construction $\underline{R}$ we obtain a crossed (n-m)-cube $\underline{R^mY}$ say.

For $1 \leq i \leq n$ let $\theta_i':\{1,\ldots,n\} \to \{1,\ldots,n\}$ be the set map which interchanges i with n and leaves the other elements of $\{1,\ldots,n\}$ unchanged. This map θ_i' induces an endofunctor $\theta_i:\Gamma|n| \to \Gamma|n|$. The composite functor $\underline{Y}\theta_i$ is clearly a crossed n-cube. We can thus construct the crossed 1-cube $\underline{R^{n-1}Y}\theta_i$, or the equivalent cat^1-object $s_i,b_i:R^{n-1}Y\theta_i\{1\} \underline{\times} R^{n-1}Y\theta_i\emptyset \to R^{n-1}Y\theta_i\emptyset$. For convenience let $G_i = R^{n-1}Y\theta_i\{1\} \underline{\times} R^{n-1}Y\theta_i\emptyset$, and let $P_i = R^{n-1}Y\theta_i\emptyset$. We shall show how the n such cat^1-objects combine to form a cat^n-object.

For each $\gamma \in \Gamma|n|$ there is a canonical inclusion $Y\gamma \to G_i$. There is also an isomorphism $\psi_i:G_n \to G_i$ whose restriction to $Y\gamma \to Y\gamma$ is the identity. Let $P_i' = \psi_i^{-1}P_i$ and let $s_i',b_i':G_n \to P_i'$ be the composite maps $s_i' = \psi_i^{-1}s_i\psi_i$, $b_i' = \psi_i^{-1}b_i\psi_i$. For each $x \in Y\gamma$ we have $s_i's_j'x = s_j's_i'x$, $b_i'b_j'x = b_j'b_i'x$, $b_i's_j'x = s_j'b_i'x$ $(i \neq j)$. Hence the maps $s_i',b_i':G \to P_i'$, $1 \leq i \leq n$, constitute a cat^n-object.

We have thus given a correspondence between cat^n-objects and crossed n-cubes. This correspondence gives rise to an equivalence of categories, completing the proof. ∇

As an illustration of the equivalence between crossed n-cubes and cat^n-objects we give the following

EXAMPLE(1.3.12) Suppose given a crossed square in groups (as described in the Introduction):

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\delta'} \\ M & \xrightarrow{\delta} & P \end{array}$$

with $h:M\times N \to L$. Form the semi-direct products $L\underline{\times}M$, $N\underline{\times}P$ and define a group action of $N\underline{\times}P$ on $L\underline{\times}M$ by ${}^{(n,p)}(l,m) = ({}^{n}({}^{p}l)\, h({}^{p}m,n)^{-1}, {}^{p}m)$. The maps $s_2,b_2:(L\underline{\times}M)\underline{\times}(N\underline{\times}P) \to N\underline{\times}P$ given respectively by $(l,m,n,p) \to (n,p)$, $(l,m,n,p) \to (\lambda l\ {}^{m}n, (\delta m)p)$ constitute a cat^1-group.

Now the above diagram of groups together with the function $h':N\times M \to L$, $h'(n,m) = h(m,n)^{-1}$, is also a crossed square. Thus we can similarly form a cat^1-group $s_1,b_1:(L\underline{\times}N)\underline{\times}(M\underline{\times}P) \to M\underline{\times}P$.

Let $G_2 = (L\underline{\times}M)\underline{\times}(N\underline{\times}P)$, $G_1 = (L\underline{\times}N)\underline{\times}(M\underline{\times}P)$, $P_2' = N\underline{\times}P$, $P_1' = M\underline{\times}P$ (we consider P_1',P_2' as subgroups of G_2). There is an isomorphism $\psi_1:G_2 \to G_1$, $(l,m,n,p) \to (lh(m,n),n,m,p)$. Let $s_1',b_1':G_2 \to P_1'$ be the composite maps $s_1' = \psi_1^{-1}s_1\psi_1$, $b_1' = \psi_1^{-1}b_1\psi_1$. The four maps s_1',b_1',s_2,b_2 constitute a cat^2-group.

4. ON n-PUSHOUTS OF CROSSED n-CUBES

The computation of colimits of crossed n-cubes (in the category of groups) is of relevance to the higher dimensional van Kampen theorem of [B-L]. In this section we look at one particular type of colimit which, following Brown and Loday, we call an "n-pushout".

The following definitions will appear in [B-L2] (see also [Wa]). Let C' be an arbitrary category. An *n-cube in*

C' is a contravariant functor $\underline{Y}'$ from $\Gamma|n|$ to C'. An *n-corner in* C' is a contravariant functor $\underline{Y}$ from $\Gamma|n|\backslash\emptyset$ to C'. (Here $\Gamma|n|\backslash\emptyset$ denotes the poset $\Gamma|n|$ with the empty set removed.) Suppose colim $\underline{Y}$ of such a corner exists. Then $\underline{Y}$ and the natural transformation $\underline{Y} \to$ colim $\underline{Y}$ define an n-cube $\underline{Y}'$ in C'. The object colim $\underline{Y}$ is called the *pushout of the corner* $\underline{Y}$, and $\underline{Y}'$ is called an *n-pushout in* C'.

It will be convenient to have some functors from crossed n-cubes to crossed m-cubes $m \neq n$.

For $1 \leq i \leq n$ let $|n|_i = \{1,2,\ldots,n\}\backslash\{i\}$ and let $\Gamma|n|_i$ be the poset consisting of the subsets of $|n|_i$. The map $|n-1| \to |n|_i$ which takes j to j for $1 \leq j \leq i-1$, and j to j+1 for $i \leq j \leq n-1$, induces an isomorphism of posets $\hbar:\Gamma|n-1| \to \Gamma|n|_i$. There is a canonical inclusion $\zeta:\Gamma|n|_i \to \Gamma|n|$. Given a crossed n-cube $\underline{Y}:\Gamma|n| \to$ C, we can restrict to obtain a crossed (n-1)-cube $\partial^i\underline{Y} = \underline{Y}\zeta\hbar:\Gamma|n-1| \to$ C.

Given a crossed (n-1)-cube $\underline{Y}$ in C, let $\tau^i\underline{Y}$ be the unique crossed n-cube such that $\partial^i\tau^i\underline{Y} = \underline{Y}$ and $Y\gamma = 0$ whenever $i \in \gamma \in \Gamma|n|$. Let $\xi^i\underline{Y}$ be the unique crossed n-cube with $\partial^i\xi^i\underline{Y} = \underline{Y}$ and $Y\gamma = Y(\gamma\backslash\{i\})$ whenever $i \in \gamma \in \Gamma|n|$ and where: the map $Y\gamma \to Y(\gamma\backslash\{i\})$ is the identity; and whenever $i \in \gamma_1 \subset \gamma_2 \in \Gamma|n|$ the map $Y\gamma_2 \to Y\gamma_1$ is the same as the map $Y(\gamma_2\backslash\{i\}) \to Y(\gamma_1\backslash\{i\})$.

The constructions ∂^i,τ^i,ξ^i are functorial.

Let $\underline{Y}$ be an n-corner in the category of crossed n-cubes. So for each $\beta \in \Gamma|n|\backslash\emptyset$ we have a crossed n-cube

$\underline{Y}\beta$. For $\gamma \in \Gamma\langle n\rangle$ we shall denote by $Y\beta\gamma$ the image of γ under $\underline{Y}\beta$.

Let us suppose that $Y\beta\gamma$ is trivial whenever the intersection $\beta \cap \gamma$ is non empty. For all $\beta_1, \beta_2 \in \Gamma\langle n\rangle$ such that the intersections $\beta_1 \cap \gamma$, $\beta_2 \cap \gamma$ are both empty, let us suppose that $Y\beta_1\gamma = Y\beta_2\gamma$ and that the maps from $Y\beta_1\gamma$ are the same as those from $Y\beta_2\gamma$.

Let us also suppose that all maps of crossed n-cubes in the corner $\underline{Y}$ are the canonical inclusions. Then

PROPOSITION(1.4.1) *Let $\underline{Y}:\Gamma\langle n\rangle \to C$ be a crossed n-cube. The following conditions on $\underline{Y}$ are equivalent:*

(i) $\underline{Y}$ = colim $\underline{Y}$ is the pushout of the corner $\underline{Y}$;

(ii) for $1 \leqslant i \leqslant n$ we have

(a) $\partial\{i\}\underline{Y} = \partial\{i\}\underline{Y}\{i\}$ and,

(b) *given any other crossed n-cube $\underline{Y}'$ satisfying $\partial\{i\}\underline{Y}' = \partial\{i\}Y\{i\}$, the identity maps $Y\gamma \to Y'\gamma$, for $\langle n\rangle \neq \gamma \in \Gamma\langle n\rangle$, extend uniquely to a map of crossed n-cubes $\underline{Y} \to \underline{Y}'$.*

PROOF For $1 \leqslant i \leqslant n$ the functor ∂^i has as left and right adjoints the functors τ^i, ξ^i. Hence ∂^i preserves colimits, and it follows that (i) implies (ii.a). The rest of the proof is trivial. ▽

For C the category of groups and n = 2, this proposition is given in [B-L]. Our proof is just a generalisation of the one given there.

To illustrate the proposition, take n = 2. Then the following two statements are equivalent:

(i) the diagram of crossed squares in C (i.e. crossed

2-cubes in C)

$$
\begin{array}{ccc}
\begin{array}{ccc} 0 & \to & 0 \\ \downarrow & & \downarrow \\ 0 & \to & P \end{array} & \xrightarrow{\iota} & \begin{array}{ccc} 0 & \to & N \\ \downarrow & & \downarrow{\scriptstyle \delta'} \\ 0 & \to & P \end{array} \\
\downarrow{\scriptstyle \iota'} & & \downarrow \\
\begin{array}{ccc} 0 & \to & 0 \\ \downarrow & & \downarrow \\ M & \xrightarrow{\delta} & P \end{array} & \to & \begin{array}{ccc} L & \to & N' \\ \downarrow & & \downarrow{\scriptstyle \delta'} \\ M' & \xrightarrow{\delta} & P' \end{array}
\end{array}
$$

in which ι, ι' are the canonical inclusions, is a pushout;
(ii) $M' = M$, $N' = N$, $P' = P$ and, given any crossed square

$$
\begin{array}{ccc} L' & \to & N \\ \downarrow & & \downarrow{\scriptstyle \delta'} \\ M & \xrightarrow{\delta} & P \end{array}
$$

there is a unique morphism $\alpha : L \to L'$ such that the quadruple $(\alpha, 1_M, 1_N, 1_P)$ is a map of crossed squares.

5. n-SIMPLICIAL OBJECTS IN C

We shall now show that cat^n-objects in C are equivalent to certain types of n-fold simplicial objects in C, thus highlighting the fact that simplicial techniques are applicable to the theory of higher dimensional crossed modules.

Let C' be any category. Recall that a *simplicial object* $K_\#$ *in* C' consists of a family K_n, $n \geq 0$, of objects in C' and morphisms $d_i : K_n \to K_{n-1}$, $v_i : K_n \to K_{n+1}$, $0 \leq i \leq n$, satisfying:

(i) $d_i d_j = d_{j-1} d_i \qquad 0 \leq i < j \leq n+1,$

(ii) $v_i v_j = v_{j+1} v_i \qquad 0 \leq i \leq j \leq n-1,$

(iii) $d_i v_j = v_{j-1} d_i \qquad i < j,$

$d_i v_j = v_j d_{i-1} \qquad i > j+1,$

$d_i v_j = \text{identity} \qquad i = j \text{ or } j+1.$

A map $\psi : K_\# \to K_\#'$ of simplicial objects is a family of structure preserving morphisms $\psi_n : K_n \to K_n'$. Suppose that the category C' has kernels. Then recall that the *normal complex* of a simplicial object $K_\#$ is obtained by taking for each n the subobject $\bigcap_{i=1}^{n} \ker d_i$ of K_n; the restriction of d_0 to this subobject is the differential of the complex. The complex is said to be of *length* r if it is non-trivial in dimension r and trivial in each dimension greater than r.

We shall denote by SMP^1C' the category of simplicial objects in C' whose normal complexes are of length 1. Inductively we define $SMP^nC' = SMP^1(SMP^{n-1}C')$. An object in SMP^nC' will be called an *n-simplicial object in* C' *whose normal complexes are of length 1.*

PROPOSITION(1.5.1) *There is an equivalence of categories, (n-simplicial objects in C whose normal complexes are of length 1)* $\simeq$ *(cat^n-objects in C).*

PROOF We shall first consider the case $n = 1$. Suppose given a simplicial object $K_\#$ in C whose normal complex is of length 1. By restricting to dimensions 0,1 we obtain an inclusion $K_0 \to K_1$ and two maps $d_0, d_1 : K_1 \to K_0$. Axiom (1.2.1.i) is clearly satisfied. In order to verify axiom (1.2.1.ii) let $x \in \ker d_1$, $y \in \ker d_0$. Then

$$x + y - x - y = d_0(v_0x + v_0y - v_1y - v_0x + v_1y - v_0y),$$

$$x * y = d_0(v_0x * (v_0y - v_1y)),$$

$$y * x = d_0((v_0y - v_1y) * v_0x).$$

It is now a simple matter to check that the elements $x + y - x - y$, $x * y$, $y * x$ all lie in the image of the intersection $\ker d_1 \cap \ker d_2$ and are hence trivial. Thus $d_0, d_1 : K_1 \to K_0$ is a cat^1-object.

Conversely suppose given a cat^1-object $s, b : G \to P$. By taking the nerve of the associated category we obtain a simplicial object $\mathrm{Ner}(G,P)_\#$. That is $\mathrm{Ner}(G,P)_0 = P$, $\mathrm{Ner}(G,P)_1 = G$, $\mathrm{Ner}(G,P)_n = \{(g_1,\ldots,g_n) \in G^n : bg_i = sg_{i+1}\}$, and the maps are:

$$d_i(g_1,\ldots,g_n) = \begin{cases} bg_1, & i = 0,\ n = 1 \\ sg_1, & i = 1,\ n = 1 \\ (g_2,\ldots,g_n), & i = 1,\ n > 1 \\ (g_1,\ldots,g_{i-1},g_{i+1} - bg_i + g_i, g_{i+2},\ldots,g_n), & 1 < i < n \\ (g_1,\ldots g_{n-1}), & i = n,\ n > 1 \end{cases}$$

$$v_i(g_1,\ldots,g_n) = \begin{cases} (sg_1, g_1,\ldots,g_n), & i = 0 \\ (g_1,\ldots g_i, bg_i, g_{i+1},\ldots,g_n), & i > n. \end{cases}$$

(Recall that $g_{i+1} - bg_i + g_i$ is the category composition $g_i \circ g_{i+1}$.) The normal complex of $\mathrm{Ner}(G,P)_\#$ is of length 1.

We now consider the case $n > 1$. Suppose given an n-simplicial object in **C** whose normal complexes are of length 1. By restricting to dimensions 0,1 we have, for "each of the n directions", an associated cat^1-object. The n such cat^1-objects clearly satisfy the commutativity conditions of definition (1.2.2) and thus constitute a cat^n-object.

Conversely suppose given a cat^n-object consisting of the maps $s_i, b_i : G \to P_i$, $1 \leq i \leq n$. By taking the nerve of the category associated to the cat^1-object $s_1, b_1 : G \to P_1$ we obtain a simplicial object $Ner(G,P_1)_\#$ whose normal complex is of length 1. Now the cat^1-object $s_2, b_2, : G \to P_2$ induces a category structure on $Ner(G,P_1)_\#$. By taking the nerve of this induced category structure we obtain a 2-simplicial object whose normal complexes are of length 1. Iterating the process we obtain an n-simplicial object whose normal complexes are of length 1.

This correspondence between cat^n-objects in **C** and n-simplicial objects in **C** whose normal complexes are of length 1, gives rise to an equivalence of categories. ∇

Proposition (1.5.1), for **C** the category of groups and n = 1, is given in [L].

6. n-FOLD CROSSED MODULES

A crossed n-cube is, in some sense, equivalent to an "n-fold crossed module", i.e. a "crossed module in the category of (n-1)-fold crossed modules". A precise definition of an "n-fold crossed module" is desirable since it will enable inductive arguments to be applied to crossed n-cubes.

In this section we give a definition of an "n-fold crossed module in C^n, and we outline a proof that such an entity is equivalent to an n-fold category internal to C.

Let C' be an arbitrary category with kernels and a null object 0.

DEFINITION(1.6.1) A *1-fold crossed module in* C' consists of:

(a) an object E in C' with two subobjects M,P;

(b) four morphisms $\partial:M \to P$, $s:E \to P$, $\nu:E \to P\times P$, $\eta:M\times M \to E$ such that;

(i) $M = \ker s$ and the restriction of s to P is the identity;

(ii) the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\partial"'] & E \arrow[d, "\nu"] & P \arrow[l] \arrow[d, equal] \\ P \arrow[r, "(1_P{,}0)"] & P\times P & P \arrow[l, "(1_P{,}1_P)"'] \end{tikzcd}$$

is commutative and;

(iii) the diagram

$$\begin{tikzcd} M \arrow[r, "(1_M{,}0)"] \arrow[d, equal] & M\times M \arrow[d, "\eta"] & M \arrow[l, "(1_M{,}1_M)"'] \arrow[d, "\partial"] \\ M \arrow[r] & E & P \arrow[l] \end{tikzcd}$$

is commutative.

A map of 1-fold crossed modules is a structure preserving morphism $\psi:E \to E'$.

Essentially this definition is given in [P1] for C' the category of groups, and in [AG] for C' a category of interest.

PROPOSITION(1.6.2) *There is an equivalence of categories, (crossed modules in* C) $\simeq$ *(1-fold crossed modules in* C).

PROOF Suppose given a crossed module $\partial:M \to P$ in C. Then set $E = M\underline{\times}P$, let $s:M\underline{\times}P \to P$ be the map $s(m,p) = p$, let $\nu:M\underline{\times}P \to P\times P$ be the map $\nu(m,p) = (\partial mp,p)$, and let $\eta:M\times M \to M\underline{\times}P$ be the map $\nu(m,m') = (mm'^{-1},\partial m')$.

Conversely suppose given a 1-fold crossed module. Then there is a map $\partial:M \to P$, and the split short exact sequence $M \to E \rightleftarrows P$ gives rise to a C-action of P on M.

It is readily seen that axioms (1.6.1,ii,iii) are equivalent, respectively, to axioms (1.3.3,i,ii). So we have a correspondence between crossed modules in C and 1-fold crossed modules in C; this gives rise to the required equivalence of categories. ∇

DEFINITION(1.6.3) An *n-fold crossed module in* C', $n > 1$, is a 1-fold crossed module in the category of (n-1)-fold crossed modules in C.

In order to prove the n-dimensional version of proposition (1.6.2) let us digress for a moment, and recall some results on algebraic theories. A general reference for this digression is [S]

The category C of Ω-groups is an algebraic category and so, for any category B with a terminal object, we can form the category C(B) of Ω-groups over B. For example, when C is the category of groups and B is the category of pointed topological spaces, then C(B) is the category of topological groups.

Suppose B has kernels and a null object.
Let us denote by $X^n(C(B))$, $CT^n(C(B))$ respectively the category of n-fold crossed modules in C(B) and the category of n-fold categories internal to C(B). The objects of both of these categories are many sorted, partial algebraic theories over B.

Let A(B) be an arbitrary category of many sorted, partial algebraic theories over B. Let $\hat{B}$ be the category whose objects are the functors from B^{opp} to the category of sets, and whose morphisms are the natural transformations of such functors. The Yoneda embedding induces an embedding $\iota:A(B) \to A(\hat{B})$.

There is a canonical equivalence $\kappa:A(\hat{B}) \simeq (A(\text{sets}))^{B^{opp}}$, where $(A(\text{sets}))^{B^{opp}}$ is the category of functors from B^{opp} to the category A(sets), and natural transformations of such functors.

We can now prove

PROPOSITION(1.6.4) *There is an equivalence of categories, (n-fold categories in C(B))* $\simeq$ *(n-fold crossed modules in C(B)).*

PROOF First let us consider the case n = 1. There is a diagram of functors

$$\begin{array}{ccccc} CT^1(C(B)) & \xrightarrow{\iota} & CT^1(C(\hat{B})) & \xrightarrow{\kappa} & (CT^1(C))^{B^{opp}} \\ & & & & \downarrow \lambda \\ X^1(C(B)) & \xrightarrow{\iota'} & X^1(C(\hat{B})) & \xrightarrow{\kappa'} & (X^1(C))^{B^{opp}} \end{array}$$

in which ι, ι' are the Yoneda embeddings, κ, κ' are the canonical equivalences, and λ is the equivalence induced by the equivalence $CT^1(C) \simeq X^1(C)$. The image of the composite functor $\lambda\kappa\iota$ is equivalent to the image of the composite functor $\kappa'\iota'$. This proves the proposition for n = 1.

Let us assume that the proposition has been proved in dimension n-1. Then we have a sequence of equivalences

$$\begin{aligned} CT^n(C(B)) &= CT^1(CT^{n-1}(C(B))) \\ &\simeq CT^1(C(CT^{n-1}(B))) \\ &\simeq X^1(C(CT^{n-1}(B))) \\ &\simeq X^1(CT^{n-1}(C(B))) \\ &\simeq X^1(X^{n-1}(C(B)) \\ &= X^n(C(B)). \end{aligned}$$

This proves the proposition in dimension n. ∇

Taking B to be the category of pointed sets gives us an equivalence between n-fold categories in C and n-fold crossed modules in C.

CHAPTER II

EXAMPLES OF CROSSED n-CUBES

0. INTRODUCTION

In the last chapter we introduced the notion of a crossed n-cube in an arbitrary category C of Ω-groups. Because of the generality in which we worked, the combinatorial nature of a crossed n-cube was lost. In this chapter we shall give, for specific choices of C, a more combinatorial version of certain low dimensional crossed n-cubes. In §1,2,3,4 we shall take C respectively to be the category of groups, Lie algebras, commutative algebras, and associative algebras.

1. CROSSED n-CUBES IN GROUPS

Let C be the category of groups. All groups will be written multiplicatively with identity e. Terms such as "C-n-action", "crossed n-cube in C" will be replaced by "group n-action", "crossed n-cube in groups" etc.

Let M,P be groups. A *group action* of P on M (see example 1.3.2)) is a map $P \times M \to M$, $(p,m) \to {}^pm$ satisfying:

$$ {}^em = m, $$
$$ {}^p(mm') = ({}^pm)({}^pm'), $$
$$ {}^{pp'}(m) = {}^p({}^{p'}m), $$

for all $m,m' \in M$, $p,p' \in P$.

A *crossed module in groups* (see definition (1.3.3)) is a group homomorphism $\partial : M \to P$ together with a group action of

P on M which satisfies:

(i) $\partial({}^{p}m) = p(\partial m)p^{-1}$,

(ii) ${}^{\partial m}m' = mm'm^{-1}$,

for all $m,m' \in M$, $p \in P$.

Suppose given: four groups L,M,N,P; group actions of P on L,M,N, of M on L, and of N on L; and a function $h:M\times N \to L$. Then

PROPOSITION(2.1.1) *This structure is a group 2-action (see definition (1.3.8)) if and only if:*

(i) $\quad {}^{{}^{p}m}({}^{p}l) = {}^{p}({}^{m}l)$,

$\quad\quad {}^{{}^{p}n}({}^{p}l) = {}^{p}({}^{n}l)$;

(ii) $\quad h(mm',n) = {}^{m}h(m',n)h(m,n)$,

$\quad\quad h(m,nn') = h(m,n)\,{}^{n}h(m,n')$;

(iii) ${}^{p}h(m,n) = h({}^{p}m,{}^{p}n)$;

(iv) ${}^{m}({}^{n}l)h(m,n) = h(m,n)\,{}^{n}({}^{m}l)$;

for all $l \in L$, $m,m' \in M$, $n,n' \in N$, $p \in P$.

PROOF Strictly speaking a group 2-action should involve three functions $h^{+},h^{*},h^{*\circ}:M\times N \to L$. However $h^{*},h^{*\circ}$ are always trivial. Form the semi-direct products $L\underline{\times}M$, $N\underline{\times}P$ and let $\alpha:(N\underline{\times}P)\times(L\underline{\times}M) \to (L\underline{\times}M)$ be the function

$$\alpha(n,p,l,m) = ({}^{n}({}^{p}l)\ h({}^{p}m,n)^{-1},{}^{p}m).$$

It follows from proposition (1.3.9) that the structure is a group 2-action if α is a group action of $N\underline{\times}P$ on $L\underline{\times}M$. That is, if

$$\alpha(e,e,l,m) = (l,m),$$

$$\alpha((n,p)(n',p'),l,m) = \alpha(n,p,\alpha(n',p',l,m)),$$

$$\alpha(n,p,(l,m)(l',m')) = \alpha(n,p,l,m)\ \alpha(n,p,l',m').$$

This is verified in Appendix II, verification (1).

Conversely, by interpreting the function h as commutation, and by interpreting each group action as congugation, we see that every group 2-action satisfies these rules. ∇

Now suppose given a commutative diagram of groups

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\delta'} \\ M & \xrightarrow{\delta} & P \end{array}$$

with group actions of P on L,M and N (hence there are group actions of M on L and N via δ, and group actions of N on L and M via δ'), and a function $h:M\times N \to L$. Then

PROPOSITION(2.1.2) *This structure is a crossed square in groups (see definition (1.3.10)) if and only if:*

(*i*) *each of the maps* $\lambda,\lambda',\delta,\delta'$ *and the composite* $\delta'\lambda$ *is a crossed module;*

(*ii*) *the maps* λ,λ' *preserve the actions of* P*;*

(*iii*) $h(mm',n) = {}^{m}h(m',n)h(m,n)$,

$h(m,nn') = h(m,n)\ {}^{n}h(m,n')$;

(*iv*) ${}^{p}h(m,n) = h({}^{p}m,{}^{p}n)$;

(*v*) $\lambda h(m,n) = {}^{m}n n^{-1}$,

$\lambda' h(m,n) = m\ {}^{n}m^{-1}$;

(*vi*) $h(m,\lambda l) = {}^{m}l l^{-1}$,

$h(\lambda' l,n) = l\ {}^{n}l^{-1}$;

for all $l \in L$, $m,m' \in M$, $n,n' \in N$, $p \in P$.

PROOF From definition (1.3.10) and proposition (2.1.1) we have that the above structure is a crossed square if and only if rules (2.1.1,i,iv), (2.1.2,i to vi) hold. But rules (2.1.1,i,iv) are redundant (see Appendix II, verifications (2) and (3)) and so we are done. ∇

Suppose now we have a commutative diagram of groups

$$\begin{tikzcd}
& N \arrow[rr, "\nu_Q"] \arrow[dd, "\nu_P"'] & & Q \arrow[dd, "\delta"] \\
K \arrow[ur, "\lambda_N"] \arrow[rr, "\lambda_M"] \arrow[dd, "\lambda_L"'] & & M \arrow[ur, "\nu_Q"] \arrow[dd, "\nu_R"] & \\
& P \arrow[rr, "\delta"] & & S \\
L \arrow[ur, "\nu_P"] \arrow[rr, "\nu_R"] & & R \arrow[ur, "\delta"] &
\end{tikzcd}$$

in which there is a group action of S on each of the other seven groups (hence the eight groups act on each other via the action of S), and there are six functions

$h: Q \times L \to K$,

$h: P \times M \to K$,

$h: N \times R \to K$,

$h: P \times R \to L$,

$h: Q \times R \to M$,

$h: P \times Q \to N$.

Then

PROPOSITION(2.1.3) *This structure is a crossed 3-cube in* *groups (see definition 1.3.10)) if and only if:*

(*i*) *each of the nine squares*

$$\begin{array}{ccc} K & \to & L \\ \downarrow & & \downarrow \\ Q & \to & S \end{array} \quad \begin{array}{ccc} K & \to & M \\ \downarrow & & \downarrow \\ P & \to & S \end{array} \quad \begin{array}{ccc} K & \to & R \\ \downarrow & & \downarrow \\ N & \to & S \end{array} \quad \begin{array}{ccc} L & \to & R \\ \downarrow & & \downarrow \\ P & \to & S \end{array} \quad \begin{array}{ccc} M & \to & R \\ \downarrow & & \downarrow \\ Q & \to & S \end{array} \quad \begin{array}{ccc} N & \to & Q \\ \downarrow & & \downarrow \\ P & \to & S \end{array}$$

$$\begin{array}{ccc} K & \to & M \\ \downarrow & & \downarrow \\ L & \to & R \end{array} \quad \begin{array}{ccc} K & \to & L \\ \downarrow & & \downarrow \\ N & \to & P \end{array} \quad \begin{array}{ccc} K & \to & M \\ \downarrow & & \downarrow \\ N & \to & Q \end{array}$$

is a crossed square; for the last three squares the functions $h:L\times M \to K$, $h:N\times L \to K$, $h:N\times M \to K$ *are respectively given by* $h(l,m) = h(\nu_P l,m)$, $h(n,l) = h(n,\nu_R l)$, $h(n,m) = h(n,\nu_R m)$;

(*ii*) $h((\nu_P n)(\nu_P l),m)\ h((\nu_Q m)(\nu_Q n),l) = h(n,(\nu_R l)(\nu_R m))$;

(*iii*) ${}^{q}h(h(p,q^{-1})^{-1},r) = {}^{p}h(q,h(p^{-1},r))\ {}^{r}h(p,h(q,r^{-1})^{-1})$;

(*iv*) $\lambda_L h(p,m) = h(p,\nu_R m)$,

$\lambda_L h(n,r) = h(\nu_P n,r)$,

$\lambda_M h(q,l) = h(q, \nu_R l)$,

$\lambda_M h(n,r) = h(\nu_Q n,r)$,

$\lambda_N h(p,m) = h(p,\nu_Q m)$,

$\lambda_N h(q,l) = h(\nu_P l,q)^{-1}$;

(*v*) $h(\nu_Q m,l) = h(\nu_P l,m)^{-1}$;

$h(n,\nu_R l) = h(\nu_Q n,l)$;

$h(n,\nu_R m) = h(\nu_P n,m)$;

for all $l \in L$, $m \in M$, $n \in N$, $p \in P$, $q \in Q$, $r \in R$.

PROOF A crossed 3-cube in groups is defined in terms of a contravariant functor $\underline{Y}$ from $\Gamma|3|$ to groups, so it will help to rewrite the above cubical diagram:

$$
\begin{tikzcd}
& N = Y\{1,2\} \arrow[rr] \arrow[dd] & & Q = Y\{2\} \arrow[dd] \\
K = Y\{1,2,3\} \arrow[ur] \arrow[rr] \arrow[dd] & & M = Y\{2,3\} \arrow[ur] \arrow[dd] & \\
& P = Y\{1\} \arrow[rr] & & S = Y0 \\
L = Y\{1,3\} \arrow[ur] \arrow[rr] & & R = Y\{3\} \arrow[ur] &
\end{tikzcd}
$$

Note that, in addition to the six "h" functions listed above, a crossed 3-cube involves three functions $h: L \times M \to K$, $h: N \times L \to K$, $N \times M \to K$. These extra functions are defined as in rule (2.1.3.i)

The rules (2.1.3.i to v) are clearly necessary if the structure is to be a crossed 3-cube. The proof that these rules are sufficient to give us a crossed 3-cube boils down to a proof that the rules ensure the existence of a group 3-action.

Form the semi-direct products $K \underline{\times} N$, $L \underline{\times} P$, $M \underline{\times} Q$, $R \underline{\times} S$. Given an arbitrary element (u,v,x,y) in any one of the direct products $(R \underline{\times} S) \times (K \underline{\times} N)$, $(R \underline{\times} S) \times (M \underline{\times} Q)$, $(R \underline{\times} S) \times (L \underline{\times} P)$, $(L \underline{\times} P) \times (K \underline{\times} N)$, $(M \underline{\times} Q) \times (K \underline{\times} N)$, we obtain five group actions by setting

$$^{(u,v)}(x,y) = ({}^{uv}x \; h({}^{v}y,u)^{-1}, {}^{v}y).$$

(Here, and in future, we write ${}^{uv}x$ instead of ${}^{u}({}^{v}x)$. This abuse of notation is unlikely to cause confusion.) Let $h':(L\underline{\times}P)\times(M\underline{\times}Q) \to K\underline{\times}N$ be the function $h'(l,p,m,q) = (k,n)$ where

$$k =$$

$$^{l}h(p,m)h(l,m)\ ^{m}h(h(p,q),l)^{-1}\ h(h(p,q),m)^{-1}\ ^{h(p,q)m}h(q,l)^{-1},$$

and

$$n = h(p,q).$$

Then, by proposition (1.3.9), we have to check that the four semi-direct products together with the given group actions and function h', constitute a group 2-action. By proposition (2.1.1) we see that we must check

$$^{(r,s)(m,q)}(^{(r,s)}(k,n)) = {}^{(r,s)}(^{(m,q)}(k,n)),$$

$$^{(r,s)(l,p)}(^{(r,s)}(k,n)) = {}^{(r,s)}(^{(l,p)}(k,n)),$$

$$h'((l,p)(l',p'),m,q) = {}^{(l,p)}h'(l',p',m,q)\ h'(l,p,m,q),$$

$$h'(l,p,(m,q)(m',q')) = h'(l,p,m,q)\ ^{(m,q)}h'(l,p,m',q'),$$

$$^{(r,s)}h'(l,p,m,q) = h'(^{(r,s)}(l,p),{}^{(r,s)}(m,q)),$$

$$^{(l,p)}(^{(m,q)}(k,n))\ h'(l,p,m,q) = h'(l,p,m,q)\ ^{(m,q)}(^{(l,p)}(k,n)).$$

These equations are checked in Appendix II, verifications (4),(5),(6),(7). ∇

2. CROSSED n-CUBES IN LIE ALGEBRAS

Fix a commutative ring A (with unit). Recall that a *Lie algebra* over A is an A-module M together with an A-bilinear map $[,]:M\times M \to M$ which satisfies

$$[x,x] = 0,$$

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0,$$

for all $x,y,z \in M$. We shall assume all Lie algebras to be over A. Let C be the category of Lie algebras. Terms such as "C-n-action", "crossed n-cube in C" will be replaced by "Lie n-action", "crossed n-cube in Lie algebras", etc.

Let M,P be Lie algebras. Suppose given a map $P \times M \to M$, $(p,m) \to {}^{p}m$.

PROPOSITION(2.2.1) *This map is a Lie action of P on M if and only if:*

(*i*) $\quad ({}^{a}p)m = {}^{p}(am) - a({}^{p}m);$

(*ii*) $\quad {}^{p}(m + m') = {}^{p}m + {}^{p}m';$

(*iii*) $\quad {}^{(p + p')}m = {}^{p}m + {}^{p'}m;$

(*iv*) $\quad {}^{[p,p']}m = {}^{p}({}^{p'}m) - {}^{p'}({}^{p}m);$

(*v*) $\quad {}^{p}[m,m'] = [{}^{p}m,m'] + [m,{}^{p}m'];$

for all $a \in A$, $m,m' \in M$, $p,p' \in P$.

PROOF Strictly speaking a Lie action should consist of three maps $\alpha^{+},\alpha^{[,]},\alpha^{[,]^{\circ}}:P \times M \to M$. However, since we will always have $\alpha^{+}(p,m) = m$ and $\alpha^{[,]}(p,m) = -\alpha^{[,]^{\circ}}(p,m)$, we can take a Lie action to consist of just one map. It is clear that every Lie action satisfies rules (2.2.1,i to v). Conversely, to show that these rules are sufficient to give us a Lie action, we must check that the semi-direct product $M \underline{\times} P$ is a Lie algebra (see proposition (1.3.1)). This check is routine and we omit it. ∇

A *crossed module in Lie algebras* (see definition (1.3.3)) is a Lie homomorphism $\partial:M \to P$ with a Lie action of P on M such that

(i) $\quad \partial({}^{p}m) = [p,\partial m]$,

(ii) $\quad {}^{\partial m}m' = [m,m']$,

for all $m,m' \in M$, $p \in P$.

Suppose given: four Lie algebras L,M,N,P; Lie actions of P on L,M and N, and of M on L, and of N on L; and a function $h: M \times N \to L$. Then

PROPOSITION(2.2.2) *This structure is a Lie 2-action (see definition (1.3.8)) if and only if:*

(*i*) $\quad {}^{p}({}^{m}l) = {}^{({}^{p}m)}l + {}^{m}({}^{p}l),$

$\quad\quad {}^{p}({}^{n}l) = {}^{({}^{p}n)}l + {}^{n}({}^{p}l);$

(*ii*) $\quad ah(m,n) = h(am,n) = h(m,an);$

(*iii*) $h(m + m',n) = h(m,n) + h(m',n),$

$\quad\quad h(m,n + n') = h(m,n) + h(n,m');$

(*iv*) $\quad h([m,m'],n) = {}^{m}h(m',n) - {}^{m'}h(m,n),$

$\quad\quad h(m,[n,n']) = {}^{n}h(m,n') - {}^{n'}h(m,n);$

(*v*) $\quad {}^{p}h(m,n) = h({}^{p}m,n) + h(m,{}^{p}n);$

(*v*) $\quad {}^{n}({}^{m}l) = {}^{m}({}^{n}l) + [l,h(m,n)];$

for all $a \in A$, $l \in L$, $m,m' \in M$, $n,n' \in N$, $p \in P$.

PROOF Strictly speaking, a Lie 2-action should involve three functions $h^{+}, h^{[,]}, h^{[,]^\circ}: M \times N \to L$. However, since $h^{+}(m,n) = 0$ and $h^{[,]}(m,n) = -h^{[,]^\circ}(m,n)$, we can take a Lie 2-action to involve just one function $h = h^{[,]}$. It is clear that every Lie 2-action satisfies the above rules.

In order to show that the above rules are sufficient to give us a Lie 2-action we define an action of the semi-direct product $N \underline{\times} P$ on the semi-direct product $L \underline{\times} M$ by

$$^{(n,p)}(l,m) = ({}^{n}l + {}^{p}l - h(m,n), {}^{p}m).$$

By propositions (1.3.9) and (2.2.1) we have to check that

$$^{(an,ap)}(l,m) = {}^{(n,p)}(al,am) = a({}^{(n,p)}(l,m)),$$

$$^{(n,p)}((l,m) + (l',m')) = {}^{(n,p)}(l,m) + {}^{(n,p)}(l',m'),$$

$$^{((n,p) + (n',p'))}(l,m) = {}^{(n,p)}(l,m) + {}^{(n',p')}(l,m),$$

$$^{[(n,p),(n',p')]}(l,m) =$$

$$^{(n,p)}((n',p')(l,m)) - {}^{(n',p')}((n,p)(l,m)),$$

$$^{(n,p)}[(l,m),(l',m')] =$$

$$[^{(n,p)}(l,m),(l',m')] + [(l,m),{}^{(n,p)}(l',m')].$$

These equations are checked in Appendix II, verifications (8),(9),(10),(11),(12). ∇

Suppose now we have a commutative diagram of Lie algebras

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\delta'} \\ M & \xrightarrow{\delta} & P \end{array}$$

in which there are Lie actions of P on L,M and N (hence there are Lie actions of M on L and N via δ, and of N on L and M via δ'), and a function $h: M \times N \to L$. Then

PROPOSITION(2.2.3) *This structure is a crossed square in Lie algebras (see definition (1.3.10)) if and only if:*

(*i*) *each of the maps* $\lambda,\lambda',\delta,\delta'$ *and the composite* $\delta'\lambda$ *is a crossed module;*

(*ii*) *the maps* λ,λ' *preserve the actions of* P*;*

(*iii*) $ah(m,n) = h(am,n) = h(m,an)$;

(*iv*) $h(m + m',n) = h(m,n) + h(m',n)$,
$h(m,n + n') = h(m,n) + h(m,n')$;

(*v*) $h([m,m'],n) = {}^{m}h(m',n) - {}^{m'}h(m,n)$,
$h(m,[n,n']) = {}^{n}h(m,n') - {}^{n'}h(m,n)$;

(*vi*) ${}^{p}h(m,n) = h({}^{p}m,n) + h(m,{}^{p}n)$;

(*vii*) $\lambda h(m,n) = {}^{m}n$,
$\lambda' h(m,n) = -{}^{n}m$;

(*viii*) $h(m,\lambda l) = {}^{m}l$,

$$h(\lambda' l,n) = -\,{}^{n}l;$$

for all $l \in L$, $m,m' \in M$, $n,n' \in N$, $p \in P$.

PROOF The proof boils down to checking that rules (2.2.3,i to viii) imply rules (2.2.2,i,vi). We check this in Appendix II, verifications (13),(14). ▽

Suppose now we have a commutative diagram of Lie algebras

$$\begin{tikzcd}
 & N \arrow[rr, "\nu_Q'"] \arrow[dd, "\nu_P"] & & Q \arrow[dd, "\delta"] \\
K \arrow[ur, "\lambda_N"] \arrow[rr, "\lambda_M"] \arrow[dd, "\lambda_L"'] & & M \arrow[ur, "\nu_Q"] \arrow[dd, "\nu_R"] & \\
 & P \arrow[rr, "\delta"] & & S \\
L \arrow[ur, "\nu_P"] \arrow[rr, "\nu_R"] & & R \arrow[ur, "\delta"] &
\end{tikzcd}$$

in which there is a Lie action of S on each of the other seven algebras (hence all eight algebras act on each other via the actions of S), and there are six functions

$h:Q\times L \to K$,

$h:P\times M \to K$,

$h:N\times R \to K$,

$h:P\times R \to L$,

$h:Q\times R \to M$,

$h:P\times Q \to N$.

Then

PROPOSITION(2.2.4) *This structure is a crossed 3-cube in Lie algebras (see definition (1.3.10)) if and only if:*

(*i*) *each of the squares*

$$\begin{array}{cccccc} K \to L & K \to M & K \to R & L \to R & M \to R & N \to Q \\ \downarrow \quad \downarrow & \downarrow \quad \downarrow & \downarrow \quad \downarrow & \downarrow \quad \downarrow & \downarrow \quad \downarrow & \downarrow \quad \downarrow \\ Q \to S & P \to S & N \to S & P \to S & Q \to S & P \to S \end{array}$$

$$\begin{array}{ccc} K \to M & K \to L & K \to M \\ \downarrow \quad \downarrow & \downarrow \quad \downarrow & \downarrow \quad \downarrow \\ L \to R & N \to P & N \to Q \end{array}$$

is a crossed square; for the last three squares the functions $h:L\times M \to K$, $h:N\times L \to K$, $h:N\times M \to K$ *are respectively given by* $h(l,m) = h(\nu_P l,m)$, $h(n,l) = h(n,\nu_R l)$, $h(n,m) = h(n,\nu_R m)$;

(*ii*) $^{p}h(l,m) = h(^{p}l,m) + {}^{l}h(p,m)$;

(*iii*) $h(p,h(q,r)) = h(h(p,q),r) + h(q,h(p,r))$;

(*iv*) $\lambda_L h(p,m) = h(p,\nu_R m)$,

$\lambda_L h(n,r) = h(\nu_P n,r)$,

$\lambda_M h(q,l) = h(q, \nu_R l)$,

$\lambda_M h(n,r) = h(\nu_Q n,r)$,

$\lambda_N h(p,m) = h(p,\nu_Q m)$,

$\lambda_N h(q,l) = - h(\nu_P l,q)$;

(*v*) $h(\nu_Q m,l) = - h(\nu_P l,m)$;

$h(n,\nu_R l) = h(\nu_Q n,l)$;

$h(n,\nu_R m) = h(\nu_P n,m)$;

for all $l \in L$, $m \in M$, $n \in N$, $p \in P$, $q \in Q$, $r \in R$.

PROOF Form the semi-direct products $K\underline{\times}N$, $L\underline{\times}P$, $M\underline{\times}Q$, $R\underline{\times}S$. Given an element (u,v,x,y) in any one of the direct products $(R\underline{\times}S)\times(K\underline{\times}N)$, $(R\underline{\times}S)\times(M\underline{\times}Q)$, $(R\underline{\times}S)\times(L\underline{\times}P)$, $(L\underline{\times}P)\times(K\underline{\times}N)$, $(M\underline{\times}Q)\times(K\underline{\times}N)$, we obtain five Lie actions by setting

$$^{(u,v)}(x,y) = ({}^{u}x + {}^{v}y - h(y,u), {}^{v}y).$$

Let $h':(L\underline{\times}P)\times(M\underline{\times}Q) \to K\underline{\times}N$ be the function

$$h'(l,p,m,q) = (h(l,m) + h(p,m) - h(q,l),\ h(p,q)).$$

The proof boils down to checking that h' satisfies the rules for a Lie 2-action, i.e. we must check that

$$ah'(l,p,m,q) = h'(al,ap,m,q) = h'(l,p,am,aq),$$

$$^{(r,s)}(^{(l,p)}(k,n))$$

$$= {}^{(^{(r,s)}(l,p))}(k,n) + {}^{(l,p)}(^{(r,s)}(k,n)),$$

$$^{(r,s)}(^{(m,q)}(k,n))$$

$$= {}^{(^{(r,s)}(m,q))}(k,n) + {}^{(m,q)}(^{(r,s)}(k,n)),$$

$$h'((l,p) + (l',p'),m,q) = h'(l,p,m,q) + h'(l',p',m,q),$$

$$h'(l,p,(m,q) + (m',q')) = h'(l,p,m,q) + h'(l,p,m',q'),$$

$$h'([(l,p),(l',p')],m,q)$$

$$= {}^{(l,p)}h'(l',p',m,q) - {}^{(l',p')}h'(l,p,m,q),$$

$$h'(l,p,[(m,q),(m',q')])$$

$$= {}^{(m,q)}h'(l,p,m',q') - {}^{(m',q')}h'((l,p,m,q),$$

$$^{(r,s)}h'(l,p,m,q)$$

$$= h'(^{(r,s)}(l,p),m,q) + h'(l,p,{}^{(r,s)}(m,q)),$$

$$^{(m,q)}(^{(l,p)}(k,n))$$

$$= {}^{(l,p)}(^{(m,q)}(k,n)) - [(k,n),h'(l,p,m,q)].$$

This check is routine and we omit it. ∇

3. CROSSED n-CUBES IN COMMUTATIVE ALGEBRAS

Fix a commutative ring A (with unit). Recall that a *commutative algebra* over A is an A-module M with an A-bilinear map $M \times M \to M$, $(m,m') \to mm'$, satisfying:

$$mm' = m'm,$$

$$(mm')m'' = m(m'm''),$$

for all $m,m',m'' \in M$. We shall assume all commutative algebras to be over A. Let C be the category of commutative algebras. We shall replace terms such as "C-n-action", "crossed n-cube in C" with "commutative action", "crossed n-cube in commutative algebras", etc.

The proofs of the propositions in this section are similar to (and simpler than) the corresponding proofs of the previous section. For this reason we shall omit all proofs.

Let M,P be commutative algebras. Suppose given a map $P \times M \to M$, $(p,m) \to {}^{p}m$.

PROPOSITION(2.3.1) *This map is a commutative action if and only if:*

(*i*) $a({}^{p}m) = {}^{(ap)}m = {}^{p}(am)$;

(*ii*) ${}^{p}(m + m') = {}^{p}m + {}^{p}m'$;

(*iii*) ${}^{(p + p')}m = {}^{p}m + {}^{p'}m$;

(*iv*) ${}^{p}(mm') = ({}^{p}m)m'$;

(*v*) ${}^{(pp')}m = {}^{p}({}^{p'}m)$;

for all $a \in A$, $m,m' \in M$, $p,p' \in P$. ∇

A *crossed module in commutative algebras* (see definition (1.3.3)) is a commutative algebra homomorphism $\partial : M \to P$ with a commutative action of P on M such that:

(i) $\partial({}^{p}m) = p\partial(m)$;

(ii) ${}^{\partial m}m' = mm'$;

for all $m,m' \in M$, $p \in P$.

Suppose given a commutative diagram of commutative algebras

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \delta'} \\ M & \xrightarrow{\delta} & P \end{array}$$

in which there are commutative actions of P on L,M and N (hence there are commutative actions of M on L and N, and of N on L and M, all via the actions of P), and a function $h: M \times N \to L$. Then

PROPOSITION(2.3.2) *This structure is a crossed square in commutative algebras (see definition (1.3.10)) if and only if:*

(i) *each of the maps $\lambda, \lambda', \delta, \delta'$ and the composite $\delta'\lambda$ is a crossed module;*

(ii) *the maps λ, λ' preserve the actions of P;*

(iii) $ah(m,n) = h(am,n) = h(m,an)$;

(iv) $h(m + m',n) = h(m,n) + h(m',n)$,

$h(m,n + n') = h(m,n) + h(m,n')$;

(v) ${}^{p}h(m,n) = h({}^{p}m,n) = h(m,{}^{p}n)$;

(vi) $\lambda h(m,n) = {}^{m}n$,

$\lambda' h(m,n) = -{}^{n}m$;

(vii) $h(m,\lambda l) = {}^{m}l$,

$$h(\lambda' l,n) = -{}^{n}l;$$

for all $a \in A$, $l \in L$, $m,m' \in M$, $n,n' \in N$, $p \in P$. ∇

Suppose now we have a commutative diagram of commutative algebras

$$\begin{tikzcd}
& N \arrow[rr, "\nu_Q"] \arrow[dd, "\nu_P"'] & & Q \arrow[dd, "\delta"] \\
K \arrow[ur, "\lambda_N"] \arrow[rr, "\lambda_M"] \arrow[dd, "\lambda_L"'] & & M \arrow[ur, "\nu_Q"] \arrow[dd, "\nu_R"] & \\
& P \arrow[rr, "\delta"] & & S \\
L \arrow[ur, "\nu_P"] \arrow[rr, "\nu_R"] & & R \arrow[ur, "\delta"] &
\end{tikzcd}$$

in which there is a commutative action of S on each of the other seven algebras (hence all eight algebras act on each other via the actions of S), and there are six functions

$h: Q \times L \to K$,

$h: P \times M \to K$,

$h: N \times R \to K$,

$h: P \times R \to L$,

$h: Q \times R \to M$,

$h: P \times Q \to N$.

Then

PROPOSITION(2.3.3) *This structure is a crossed 3-cube in* commutative algebras (see definition (1.3.10)) if and only if:

(i) each of the squares

$$\begin{array}{ccc} K & \to & L \\ \downarrow & & \downarrow \\ Q & \to & S \end{array} \quad \begin{array}{ccc} K & \to & M \\ \downarrow & & \downarrow \\ P & \to & S \end{array} \quad \begin{array}{ccc} K & \to & R \\ \downarrow & & \downarrow \\ N & \to & S \end{array} \quad \begin{array}{ccc} L & \to & R \\ \downarrow & & \downarrow \\ P & \to & S \end{array} \quad \begin{array}{ccc} M & \to & R \\ \downarrow & & \downarrow \\ Q & \to & S \end{array} \quad \begin{array}{ccc} N & \to & Q \\ \downarrow & & \downarrow \\ P & \to & S \end{array}$$

$$\begin{array}{ccc} K & \to & M \\ \downarrow & & \downarrow \\ L & \to & R \end{array} \quad \begin{array}{ccc} K & \to & L \\ \downarrow & & \downarrow \\ N & \to & P \end{array} \quad \begin{array}{ccc} K & \to & M \\ \downarrow & & \downarrow \\ N & \to & Q \end{array}$$

is a crossed square; for the last three squares the functions $h:L\times M \to K$, $h:N\times L \to K$, $h:N\times M \to K$ are respectively given by $h(l,m) = h(\nu_P l,m)$, $h(n,l) = h(n,\nu_R l)$, $h(n,m) = h(n,\nu_R m)$;

(ii) $h(h(p,q),r) = h(p,h(q,r)) = h(q,h(p,r))$;

(iii) $\lambda_L h(p,m) = h(p,\nu_R m)$,

$\lambda_L h(n,r) = h(\nu_P n,r)$,

$\lambda_M h(q,l) = h(q,\nu_R l)$,

$\lambda_M h(n,r) = h(\nu_Q n,r)$,

$\lambda_N h(p,m) = h(p,\nu_Q m)$,

$\lambda_N h(q,l) = h(\nu_P l,q)$;

(iv) $h(\nu_Q m,l) = h(\nu_P l,m)$;

$h(n,\nu_R l) = h(\nu_Q n,l)$;

$h(n,\nu_R m) = h(\nu_P n,m)$;

for all $l \in L$, $m \in M$, $n \in N$, $p \in P$, $q \in Q$, $r \in R$. ∇

4. CROSSED n-CUBES IN ASSOCIATIVE ALGEBRAS

Fix a commutative ring A (with unit). Recall that an *associative algebra over A* is an A-module M with an A-bilinear map $M \times M \to M$, $(m,m') \to mm'$, such that:

$$m(m'm'') = (mm')m'';$$

for all $m,m',m'' \in M$. We shall assume all associative algebras to be over A. Let **C** be the category of associative algebras. We shall replace terms such as "**C**-n-action", "crossed n-cube in **C**" by "associative n-action", "crossed n-cube in associative algebras", etc.

In this section we shall again omit all proofs.

Let M,P be associative algebras. Suppose given two maps $(p,m) \to {}^{p}m$, $(p,m) \to p^{\circ}m$ from $P \times M$ to M. Then

PROPOSITION(2.4.1) *These maps constitute an associative action of P on M if and only if:*

(*i*) $a({}^{p}m) = {}^{(ap)}m = {}^{p}(am)$,

$a(p^{\circ}m) = (ap)^{\circ}m = p^{\circ}(am)$;

(*ii*) ${}^{(p + p')}m = {}^{p}m + {}^{p'}m$,

$(p + p')^{\circ}m = p^{\circ}m + p'^{\circ}m$;

(*iii*) ${}^{p}(m + m') = {}^{p}m + {}^{p}m'$,

$p^{\circ}(m + m') = p^{\circ}m + p^{\circ}m'$;

(*iv*) $({}^{p}m)m' = {}^{p}(mm')$,

$m(p^{\circ}m') = p^{\circ}(mm')$;

(*v*) ${}^{p}({}^{p'}m) = {}^{(pp')}m$,

$p^{\circ}(p'^{\circ}m) = (p'p)^{\circ}m$;

(*vi*) $p^{\circ}({}^{p'}m) = {}^{p'}(p^{\circ}m)$;

(*vii*) $(p^{\circ}m)m' = m({}^{p}m')$;

for all $a \in A$, $m,m' \in M$, $p,p' \in P$. ∇

A *crossed module in associative algebras* (see definition (1.3.3)) is a morphism of associative algebras $\partial : M \to P$ with an associative action of P on M such that:

(i) $\quad \partial({}^{p}m) \;=\; p(\partial m),$

$\quad\quad \partial({}^{p^\circ}m) \;=\; (\partial m)p;$

(ii) $\quad {}^{\partial m}m' \;=\; mm',$

$\quad\quad {}^{\partial m^\circ}m' \;=\; m'm;$

for all $m, m' \in M$, $p \in P$.

Suppose given a commutative diagram of associative algebras

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & N \arrow[d, "\delta'"] \\ M \arrow[r, "\delta"] & P \end{tikzcd}$$

in which there are associative actions of P on L,M and N (hence there are associative actions of M on L and N, and of N on L and M, all via the actions of P), and two functions $h, h^\circ : M \times N \to L$. Then

PROPOSITION(2.3.3) *This structure is a crossed square in associative algebras (see definition (1.3.10)) if and only if:*

(i) *each of the maps* $\lambda, \lambda', \delta, \delta'$ *and the composite* $\delta'\lambda$ *is a crossed module;*

(ii) *the maps* λ, λ' *preserve the actions of P;*

(iii) $\quad ah(m,n) \;=\; h(am,n) \;=\; h(m,an),$

$\quad\quad ah^\circ(m,n) \;=\; h^\circ(am,n) \;=\; h^\circ(m,an);$

(iv) $\quad h(m + m',n) \;=\; h(m,n) + h(m',n),$

$$h(m,n + n') = h(m,n) + h(m,n'),$$

$$h^\circ(m + m',n) = h^\circ(m,n) + h^\circ(m',n),$$

$$h^\circ(m,n + n') = h^\circ(m,n) + h^\circ(m,n');$$

(v) $$^{p}h(m,n) = h(^{p}m,n),$$

$$p^\circ h^\circ(m,n) = h^\circ(p^\circ m,n),$$

$$p^\circ h(m,n) = h(m,p^\circ n),$$

$$^{p}h^\circ(m,n) = h^\circ(m,{}^{p}n),$$

$$h(p^\circ m,n) = h(m,{}^{p}n),$$

$$h^\circ(^{p}m,n) = h^\circ(m,p^\circ n);$$

(vi) $$m' h^\circ(m,n) = m^\circ h(m',n),$$

$$n' h(m,n) = n^\circ h^\circ(m,n');$$

(vii) $$\lambda h(m,n) = {}^{m}n,$$

$$\lambda h^\circ(m,n) = m^\circ n,$$

$$\lambda' h(m,n) = n^\circ m,$$

$$\lambda' h^\circ(m,n) = {}^{n}m;$$

(viii) $$h(m,\lambda l) = {}^{m}l,$$

$$h^\circ(m,\lambda l) = m^\circ l,$$

$$h(\lambda' l,n) = n^\circ l,$$

$$h^\circ(\lambda' l,n) = {}^{n}l;$$

for all $a \in A$, $l \in L$, $m,m' \in M$, $n,n' \in N$, $p \in P$. ∇

We shall not bother to work out the axioms for a crossed 3-cube in associative algebras!

CHAPTER III

UNIVERSAL CROSSED n-CUBES

0. INTRODUCTION

In this chapter we investigate cetrain universal crossed squares and crossed 3-cubes in groups (§1,2), Lie algebras (§3), and commutative algebras (§4). The universal crossed squares involve notions of non-abelian tensor, anti-symmetric, and exterior products. We obtain various exact sequences involving these non-abelian constructions, certain of which will be used in Chapter IV when we look at the relevance of crossed squares to homology.

1. CERTAIN UNIVERSAL CROSSED SQUARES IN GROUPS

Let M,N be groups such that there is a group action of M (resp. N) on N (resp. M). Assume each group acts on itself by conjugation.

Following [B-L] we define the *tensor product* of M with N to be the group $M \otimes N$ which is generated by the elements $m \otimes n$, for $(m,n) \in M\times N$, subject to the relations

(1) $\quad mm' \otimes n = ({}^{m}m' \otimes {}^{m}n)(m \otimes n),$

(2) $\quad m \otimes nn' = (m \otimes n)({}^{n}m \otimes {}^{n}n').$

Note that, when the actions of M and N are trivial, then $M \otimes N$ is just the standard tensor product $M^{ab} \otimes_{\mathbb{Z}} N^{ab}$ of abelian groups.

For $m \in M$, $n \in N$, $x \in M$ or N we shall write ${}^{(mn)}x$ instead of ${}^{m}({}^{n}x)$, and ${}^{(nm)}x$ instead of ${}^{n}({}^{m}x)$. We can consider mn and nm as elements of the free product $M*N$. This abuse of notation is unlikely to cause confusion. The tensor product $M \otimes N$ is of particular interest when the actions of M and N are *compatible*; i.e. when

$$({}^{n}m)_x = (nmn^{-1})_x \quad \text{and} \quad ({}^{m}n)_x = (mnm^{-1})_x.$$

'In this case we can interpret $m \otimes n$ as a commutator and we can interpret the actions of M and N as conjugation: anything which looks like a universal commutator relation is then actually a relation in the tensor product $M \otimes N$.' Since much of this section is concerned with obtaining consequences of relations (1) and (2), it will be helpful to have a precise version of this statement.

The tensor product $M \otimes N$ admits a group action of N, given by

$$^{n'}(m \otimes n) = {}^{n'}m \otimes {}^{n'}m.$$

The resulting semi-direct product $(M \otimes N)\underline{x}N$ admits a group action of M, given by

$$^{m}(1,n) = ({}^{m}1\ (m \otimes n),\ n)$$

where $m \in M$, $n \in N$, $1 \in M \otimes N$. To see this we note that:

(i) $m'\{{}^{m}(1,n)\}$

$$= ({}^{m'm}1\ {}^{m'}(m \otimes n)(m' \otimes n),\ n)$$

$$= ({}^{m'm}1\ (m'm \otimes n),\ n)$$

$$= {}^{m'm}(1,n);$$

(ii) by expanding $(mm' \otimes nn')$ in two different ways we obtain the identity

$$^{mn}(m' \otimes n')(m \otimes n) = (m \otimes n)\ {}^{nm}(m' \otimes n')$$

and hence, for any $1 \in M \otimes N$, the identity

$$^{mn}1(m \otimes n) = (m \otimes n)\ {}^{nm}1;$$

it follows that

$$
\begin{aligned}
&{}^{m}(l,n)\ {}^{m}(l',n') \\
&= ({}^{m}l\ (m \otimes n)\ {}^{nm}l'\ {}^{n}(m \otimes n'),\ nn') \\
&= ({}^{m}l\ {}^{mn}l'\ (m \otimes n)\ {}^{n}(m \otimes n'),\ nn') \\
&= ({}^{m}\{l\ {}^{n}l'\}\ (m \otimes nn'),\ nn') \\
&= {}^{m}\{(l,n)(l',n')\}.
\end{aligned}
$$

Now the natural inclusions $M \to ((M \otimes N)\underline{\times}N)\underline{\times}M)$, $N \to ((M \otimes N)\underline{\times}N)\underline{\times}M)$ induce a map $\langle\rangle : M*N \to ((M \otimes N)\underline{\times}N)\underline{\times}M$. For $p \in M*N$ we shall denote by $\langle p\rangle$ the image of p under $\langle\rangle$. For $m \in M$ we shall denote by $\langle m\rangle$ the image of m under the composite map $M \to^{i} M*N \to^{\langle\rangle} ((M \otimes N)\underline{\times}N)\underline{\times}M$ where i is the canonical inclusion. Similarly we shall write $\langle n\rangle$ for $n \in N$. Let $\iota : M \otimes N \to ((M \otimes N)\underline{\times}N)\underline{\times}M$ be the natural inclusion.

As we noted above, the group actions $M \to \mathrm{Aut}\ M$, $N \to \mathrm{Aut}\ M$ give rise to an action $M*N \to \mathrm{Aut}\ M$. Likewise, there is an action of $M*N$ on N and on $M \otimes N$.

Suppose we have an arbitrary generator x of $M \otimes N$, which is of the form

$$x = {}^{p_1}m_1\ {}^{p_2}m_2\ \ldots\ {}^{p_\ell}m_\ell \otimes {}^{q_1}n_1\ {}^{q_2}n_2\ \ldots\ {}^{q_{\ell'}}n_{\ell'}$$

with $m_i \in M$, $n_i \in N$, $p_i, q_i \in M*N$. Then

LEMMA(3.1.1) *If the actions of* M *and* N *are compatible, we have*

$$\iota x = [\langle p_1 m_1 p_1^{-1}\rangle \ldots \langle p_\ell m_\ell p_\ell^{-1}\rangle,\ \langle q_1 n_1 q_1^{-1}\rangle \ldots \langle q_{\ell'} n_{\ell'} q_{\ell'}\rangle].$$

PROOF The generator x can be expanded into a product of terms of the form ${}^{p}m_i \otimes {}^{q}n_j$ with $p, q \in M*N$, and it suffices to check the following identity:

$$({}^{p}m_i \otimes {}^{q}n_j,\ e,\ e) = [\langle p m_i p^{-1}\rangle,\ \langle q n_j q^{-1}\rangle]. \qquad \text{(a)}$$

Note that, for $m_0 \in M$, $n_0 \in N$, $l \in M \otimes N$, we have:

$$\langle m_0\rangle(l,e,e)\langle m_0\rangle^{-1} = ({}^{m_0}l,e,e),$$

$$\langle n_0\rangle(l,e,e)\langle n_0\rangle^{-1} = ({}^{n_0}l,e,e),$$

and thus, for $q \in M*N$, we have

$$\langle q\rangle(l,e,e)\langle q\rangle^{-1} = ({}^{q}l,e,e). \qquad \text{(b)}$$

This last identity, together with the identity

$$({}^{p}m \otimes n,\ e,\ e) = [\langle pmp^{-1}\rangle,\ \langle n\rangle] \qquad \text{(a)'}$$

(where $m \in M$, $n \in N$) imply identity (a) since:

$$\begin{aligned}
&({}^{p}m_i \otimes {}^{q}n_j,\ e,\ e)\\
&= ({}^{q}\{{}^{q^{-1}p}m_i \otimes n_j\},\ e,\ e)\\
&= \langle q\rangle({}^{q^{-1}p}m_i \otimes n_j,\ e,\ e)\langle q\rangle^{-1} && \text{using (b)}\\
&= \langle q\rangle[\langle q^{-1}pm_ip^{-1}q\rangle,\ \langle n_j\rangle]\langle q\rangle^{-1} && \text{using (a)'}\\
&= [\langle pm_ip^{-1}\rangle,\ \langle qn_jq^{-1}\rangle].
\end{aligned}$$

So it remains to verify (a)'.

It is routine to verify (a)' for p equal to the identity element of $M*N$. We shall say that p is of *length* ℓ ($\ell \geq 1$) if we can write $p = m_1n_2\ldots m_{\ell-1}n_\ell$ or $p = n_1m_2\ldots n_{\ell-1}m_\ell$ with $m_i \in M$, $n_i \in N$. We shall say that p is of *length* 0 if p is the identity element. Fix $\ell \geq 0$, suppose that p is of length ℓ, and suppose that we have verified identity (a)' for the case when p is of length ℓ. For $m_0 \in M$ we have

$$\begin{aligned}
&[\langle m_0pmp^{-1}m_0{}^{-1}\rangle,\ \langle n\rangle]\\
&= {}^{\langle m_0pmp^{-1}\rangle}[\langle m_0\rangle^{-1},\ \langle n\rangle]\ {}^{\langle m_0\rangle}[\langle pmp^{-1}\rangle,\ \langle n\rangle][\langle m_0\rangle,\ \langle n\rangle]\\
&= ({}^{m_0pmp^{-1}}(m_0{}^{-1} \otimes n),\ e,\ e)({}^{m_0}\,{}^{p}m \otimes n,\ e,\ e)\\
&= ({}^{m_0p}m \otimes n,\ e,\ e) \qquad \text{(using compatibility).}
\end{aligned}$$

For $n_0 \in N$ we have

$$\begin{aligned}
&[\langle n_0pmp^{-1}n_0{}^{-1}\rangle,\ \langle n\rangle]\\
&= \langle n_0\rangle[\langle pmp^{-1}\rangle,\ \langle n_0{}^{-1}nn_0\rangle]\langle n_0\rangle^{-1}\\
&= \langle n_0\rangle({}^{p}m \otimes n_0{}^{-1}nn_0,\ e,\ e)\langle n_0\rangle^{-1}
\end{aligned}$$

$= ({}^{n_0p}m \otimes n,\ e,\ e).$ (using (b))

Identity (a)', for an arbitrary p, follows by induction. ∇

As a typical application of this lemma we give

EXAMPLE(3.1.2) Suppose that the actions of M and N are compatible, and suppose that we wish to verify that the relation

$$[m \otimes n, m' \otimes n'] = m\, {}^{n}m^{-1} \otimes {}^{m'}n'\, n'^{-1}$$

holds in the tensor product. Then using the lemma we note that

$$\iota[m \otimes n, m' \otimes n']$$
$$= \iota(m\, {}^{n}m^{-1} \otimes {}^{m'}n'\, n'^{-1})$$
$$= [[(e,e,m),(e,n,e)],[(e,e,m'),(e,n',e)]].$$

From now on let us suppose that the groups M,N and their actions are obtained from two crossed modules $\partial:M \to P$, $\partial':N \to P$. It follows that the actions of M and N are compatible.

The tensor product $M \otimes N$ fits into a crossed square

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\lambda} & N \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\partial'} \\ M & \xrightarrow{\partial} & P \end{array}$$

in which: P acts on $M \otimes N$ by ${}^{p}(m \otimes n) = {}^{p}m \otimes {}^{p}n$; the maps λ, λ' are $\lambda(m \otimes n) = {}^{m}n\, n^{-1}$, $\lambda'(m \otimes n) = m\, {}^{n}m^{-1}$; and the function $h:M \times N \to M \otimes N$ is given by $h(m,n) = m \otimes n$.

The fact that this structure is a crossed square is noted and verified in [B-L]. It is also noted that this crossed square has a defining universal property, namely property (1.4.1.ii) for the case $n = 2$ and C equal to the category of groups.

Following [B-L] we also define the *exterior product* of M with N to be the group $M \wedge N$ generated by the elements $m \wedge n$, for $(m,n) \in M{\times}N$, subject to the relations

(3) $\quad mm' \wedge n \;=\; ({}^{m}m' \wedge {}^{m}n)(m \wedge n),$

(4) $\quad m \wedge nn' \;=\; (m \wedge n)({}^{n}m \wedge {}^{n}n'),$

(5) $\quad m \wedge n \;=\; e \quad$ whenever $\delta m = \delta' n$.

Thus $M \wedge N$ is a quotient of $M \otimes N$. The quotient map $M \otimes N \to M \wedge N$ preserves the crossed square which contains $M \otimes N$. That is to say, the exterior product $M \wedge N$ also fits into a crossed square, and this crossed square has a defining universal property.

It seems reasonable to define the *anti-symmetric product* of M with N to be the group $M \underline{\wedge} N$ generated by the elements $m \underline{\wedge} n$, for $(m,n) \in M{\times}N$, subject to the relations

(6) $\quad mm' \underline{\wedge} n \;=\; ({}^{m}m' \underline{\wedge} {}^{m}n)(m \underline{\wedge} n),$

(7) $\quad m \underline{\wedge} nn' \;=\; (m \underline{\wedge} n)({}^{n}m \underline{\wedge} {}^{n}n'),$

(8) $\quad m \underline{\wedge} n' \;=\; (m' \underline{\wedge} n)^{-1} \quad$ whenever $\delta m = \delta' n$ and $\delta m' = \delta' m'$.

We shall abbreviate the term "anti-symmetric" to "asymmetric".

Clearly $M \underline{\wedge} N$ is a quotient of $M \otimes N$. It can also be seen that $M \wedge N$ is a quotient of $M \underline{\wedge} N$. Again, the quotient map $M \otimes N \to M \underline{\wedge} N$ preserves the crossed square containing $M \otimes N$.

We note that there are isomorphisms

$$M \otimes N \cong N \otimes M,$$

$$M \wedge N \cong N \wedge M,$$

$$M \underline{\wedge} N \cong N \underline{\wedge} M.$$

If we are given an extra crossed module $\delta'':L \to P$, then there are two obvious ways of constructing a triple tensor product: in general there is NOT an isomorphism between $(L \otimes M) \otimes N$ and $L \otimes (M \otimes N)$. A similar remark applies to the exterior and asymmetric products.

It will be convenient to have the notion of an "induced crossed square". So in the following diagram of group homomorphisms

$$\begin{tikzcd}
L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & N \arrow[d, "\delta'"'] \arrow[ddr, "\beta"] & \\
M \arrow[r, "\delta"] \arrow[ddr, "\alpha"'] & P \arrow[ddr, "\gamma"'] & \\
& & S \arrow[d, "\nu'"] \\
& R \arrow[r, "\nu"] & T
\end{tikzcd}$$

suppose that the back square is a crossed square, that the maps ν, ν' are crossed modules, and that the two pairs (α, γ), (β, γ) are maps of crossed modules. Then the crossed square

$$\begin{array}{ccc} L^* & \rightarrow & S \\ \downarrow & & \downarrow \nu' \\ R & \xrightarrow{\nu} & T \end{array}$$

is said to be *induced* by the above diagram if:

(i) there is a homomorphism $\rho : L \to L^*$ such that the quadruple $(\alpha,\beta,\gamma,\rho)$ is a map of crossed squares;

(ii) any other map $(\alpha,\beta,\gamma,\rho')$ of crossed squares factors uniquely through $(\alpha,\beta,\gamma,\rho)$.

A routine argument using universal properties shows that if $L = M \otimes N$ (resp. $L = M \wedge N$, $L = M \underline{\wedge} N$), then $L^* = R \otimes S$ (resp. $L^* = R \wedge S$, $L^* = R \underline{\wedge} S$).

This observation together with the following result will be used extensively in this section and in section 4.4.

PROPOSITION(3.1.3) *If the maps α,β,γ are surjective*, if $\ker\gamma \subset (\delta \ker\alpha)(\delta' \ker\beta)$, *and if the group L is generated by the image of the function $h : M\times N \to L$, then the induced group L^* is the quotient of L obtained by factoring out the subgroup generated by the elements $h(a,n)$, $h(m,b)$, with $m \in M$, $n \in N$, $a \in \ker\alpha$, $b \in \ker\beta$.*

PROOF The proof is a straightforward check which we omit. ∇

In the remainder of this section we shall give results concerned with computing the above tensor, exterior and asymmetric products.

Suppose that the group M contains two subgroups M_A, M_B, and that the group N contains two subgroups N_A, N_B, such that $\delta M_A = \delta' N_A = A$, $\delta M_B = \delta' N_B = B$ say. The crossed modules $\delta: M \to P$, $\delta': N \to P$ restrict to give us crossed modules $\delta_A: M_A \to A$, $\delta_B: M_B \to B$, $\delta'_A: N_A \to A$, $\delta'_B: N_B \to B$. We can thus construct the groups $M_A \otimes N_A$, $M_B \otimes N_B$. The inclusions $M_A \to M$, $N_A \to N$ induce a map $\iota_A: M_A \otimes N_A \to M \otimes N$; similarly there is a map $\iota_B: M_B \otimes N_B \to M \otimes N$.

Let $\langle M_A, N_B \rangle$ be the subgroup of $M \otimes N$ generated by the elements $a_0 \otimes b_1$, $b_0 \otimes a_1$ with $(a_0, b_1) \in M_A \times N_B$, $(b_0, a_1) \in M_B \times N_A$. Then we have the useful

LEMMA(3.1.4) *An arbitrary element $x \in M \otimes N$ can be written as a product of elements*

$$x = uvw$$

with $u \in \iota_A(M_A \otimes N_A)$, $v \in \langle M_A, N_B \rangle$, $w \in \iota_B(M_B \otimes N_B)$. This assertion also holds if we replace the tensor product with either the exterior product or the asymmetric product.

PROOF We shall just consider the tensor product case. Using lemma (3.1.1) we can see immediately that the following identities hold in $M \otimes N$:

(9) $\quad {}^{a}(a_0 \otimes a_1) = {}^{a}a_0 \otimes {}^{a}a_1$,

(10) $\quad {}^{b}(a_0 \otimes a_1) = (b \otimes {}^{a_0 a_1} a_1^{-1})(a_0 \otimes a_1)$,

(11) $\quad {}^{a}(a_0 \otimes b_1) = (aa_0 \otimes b_1)(a \otimes b_1)^{-1}$,

(12) $\quad {}^{b}(a_0 \otimes b_1) = (a_0 \otimes b)^{-1} (a_0 \otimes bb_1)$,

(13) $\quad (b_0 \otimes b_1)(a_0 \otimes b_1') =$

$$(a_0 \otimes {}^{b_0 b_1} b_1^{-1})^{-1} (a_0 \otimes {}^{b_0 b_1} b_1^{-1} b_1')(b_0 \otimes b_1),$$

(14) $\quad (b_0 \otimes b_1)(b_0' \otimes a_1) =$

$$({}^{b_0 b_1 b_0^{-1}} b_0' \otimes a_1)({}^{b_0 b_1 b_0^{-1}} \otimes a_1)^{-1}(b_0 \otimes b_1),$$

(15) $\quad (b_0 \otimes b_1)(a_0 \otimes a_1) =$

$$({}^{b_0}b_1{b_0'}^{-1} \otimes {}^{a_0}{a_1'}^{-1}a_1)(a_0 \otimes a_1)(b_0 \otimes b_1),$$

where $a_0 \in M_A$, $b_0, b_0' \in M_B$, $a_1 \in N_A$, $b_1, b_1' \in N_B$, and where $a \in M_A$ or N_A, $b \in M_B$ or N_B as appropriate.

Let $x_0 \otimes x_1$ be a generator of $M \otimes N$, with $x_0 = a_0b_0 \ldots a_0'b_0'$, $x_1 = a_1b_1 \ldots a_1'b_1'$. By repeated application of the product rules (1) and (2) we see that $x_0 \otimes x_1$ can be written as a product of elements of the form ${}^{p}(a_0 \otimes a_1)$, ${}^{p}(b_0 \otimes b_1)$, ${}^{p}(a_0 \otimes b_1)$, ${}^{p}(b_0 \otimes a_1)$ with $p \in P$. Each element of this form can be broken down into a product of elements of the same type, without exponent p appearing, by repeated use of rules (9),(10),(11),(12) and the duals of these rules. Thus any element x of $M \otimes N$ can be written as a product of terms $(a_0 \otimes a_1)$, $(b_0 \otimes b_1)$, $(a_0 \otimes b_1)$, $(b_0 \otimes b_1)$.

Assume that x is written as such a product. Now take each term $b_0 \otimes b_1$ and commute it to the right (beginning with the farthermost right one and proceed one at a time) using the rules (13),(14),(15). When this has been done, take each term $a_0 \otimes a_1$ and commute it to the left using the duals of the rules (13),(14),(15). This gives us x written as a product uvw as required. ∇

Let $\lambda: M \otimes N \to N$, $\lambda': M \otimes N \to M$ be the maps $m \otimes n \to {}^{m}n\, n^{-1}$, $m \otimes n \to m\, {}^{n}m^{-1}$ and define

$\pi_3(M \otimes N) = \ker(\lambda: M \otimes N \to N) \cap \ker(\lambda': M \otimes N \to M)$.

Similarly define

$\pi_3(M \wedge N) = \ker(\lambda: M \wedge N \to N) \cap \ker(\lambda': M \wedge N \to M)$,

$\pi_3(M \underline{\wedge} N) = \ker(\lambda: M \underline{\wedge} N \to N) \cap \ker(\lambda': M \underline{\wedge} N \to M)$.

(This notation is in keeping with the topological significance of crossed squares since, if a space X is the classifying space (see [L]) of a crossed square

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \delta'} \\ M & \xrightarrow{\delta} & P \end{array}$$

then $\pi_3 X \cong \ker \lambda \cap \ker \lambda'$.)

PROPOSITION(3.1.5) *The function*

$$(M_A \otimes N_A) \times (M_B \otimes N_B) \to M \otimes N,\ (x,y) \to (\iota_A x)(\iota_B y)$$

induces homomorphisms

(i) $\pi_3(M_A \otimes N_A) \times \pi_3(M_B \otimes N_B) \to \pi_3(M \otimes N)$,

(ii) $\pi_3(M_A \wedge N_A) \times \pi_3(M_B \wedge N_B) \to \pi_3(M \wedge N)$,

(iii) $\pi_3(M_A \,\underline{\wedge}\, N_A) \times \pi_3(M_B \,\underline{\wedge}\, N_B) \to \pi_3(M \,\underline{\wedge}\, N)$.

PROOF If (x,y) is an element of $\pi_3(M_A \otimes N_A) \times \pi_3(M_B \otimes N_B)$ then clearly $(\iota_A x)(\iota_B y)$ is an element of $\pi_3(M \otimes N)$. Now $\pi_3(M \otimes N)$ lies in the kernel of a crossed module, and is therefore abelian. It follows that map (i) (and similarly maps (ii) and (iii)) is a homomorphism. $\triangledown$

For an arbitrary group G we shall denote by $G \otimes G$, $G \wedge G$, $G \,\underline{\wedge}\, G$ the tensor, exterior and asymmetric products belonging to the crossed squares

$$\begin{array}{ccc} G \otimes G & \to & G \\ \downarrow & & \| \\ G & = & G \end{array} \qquad \begin{array}{ccc} G \wedge G & \to & G \\ \downarrow & & \| \\ G & = & G \end{array} \qquad \begin{array}{ccc} G \,\underline{\wedge}\, G & \to & G \\ \downarrow & & \downarrow \\ G & = & G \end{array}$$

PROPOSITION(3.1.6) *Let G be an arbitrary group and let* $i_A:A \to G$, $i_B:B \to G$, $\alpha:G \to A$, $\beta:G \to B$ *be homomorphisms satisfying:* $\alpha i_A = 1_A$, $\beta i_B = 1_B$, $\alpha i_B = 0$, $\beta i_A = 0$.
Then the three maps of proposition (3.1.5)

(*i*) $\pi_3(A \otimes A) \times \pi_3(B \otimes B) \to \pi_3(G \otimes G)$,

(*ii*) $\pi_3(A \wedge A) \times \pi_3(B \wedge B) \to \pi_3(G \wedge G)$,

(*iii*) $\pi_3(A \underline{\wedge} A) \times \pi_3(B \underline{\wedge} B) \to \pi_3(G \underline{\wedge} G)$,

are injective.

PROOF We shall just consider case (i). The map α induces a map $\alpha_\#:\pi_3(G \otimes G) \to \pi_3(A \otimes A)$, and the map β induces a map $\beta_\#:\pi_3(G \otimes G) \to \pi_3(B \otimes B)$. The map $(\alpha_\#,\beta_\#):\pi_3(G \otimes G) \to \pi_3(A \otimes A) \times \pi_3(B \otimes B)$ is readily seen to be a splitting of map (i). ∇

PROPOSITION(3.1.7) *Let* $G = A*B$ *be the free product of groups A and B. Then the three maps of proposition (3.1.5)*

(*i*) $\pi_3(A \otimes A) \times \pi_3(B \otimes B) \to \pi_3(G \otimes G)$,

(*ii*) $\pi_3(A \wedge A) \times \pi_3(B \wedge B) \to \pi_3(G \wedge G)$,

(*iii*) $\pi_3(A \underline{\wedge} A) \times \pi_3(B \underline{\wedge} B) \to \pi_3(G \underline{\wedge} G)$.

are injective. Maps (ii) and (iii) are surjective.

PROOF The injectivity of these maps follows from proposition (3.1.6). We shall prove map (iii) surjective. Let $x \in \pi_3(G \underline{\wedge} G)$. By lemma (3.1.4) we can assume that $x = uvw$ with $u \in \iota_A(A \underline{\wedge} A)$, $w \in \iota_B(B \underline{\wedge} B)$, and (using rule (8)) with v a product of terms $(a_0 \underline{\wedge} b_1)^{\pm 1}$. It follows that $u \in \iota_A\pi_3(A \underline{\wedge} A))$, $w \in \iota_B\pi_3(B \underline{\wedge} B)$, and that the image of v in G is trivial. Since the subgroup $[A,B] \subset G$ is free on the commutators $[a,b]$ with $a \in A\backslash\{e\}$, $b \in B\backslash\{e\}$, it follows that v must be trivial. Hence map (iii) is surjective.

To prove map (ii) surjective it suffices to note that rule (8) holds in the exterior product, and hence that the preceeding argument is valid for the exterior product. ∇

EXAMPLE(3.1.8) Let F_1 be the free group of rank 1. It is easy to see that in this case the exterior product $F_1 \wedge F_1$ is trivial. Let F_n be the free group of rank n. That is F_n is the n-fold free product of n copies of F_1. It follows from proposition (3.1.7) that $\ker(F_n \wedge F_n \to^{\lambda} F_n)$ is trivial, and hence that $\ker(F \wedge F \to^{\lambda} F)$ is trivial for an arbitrary free group F. Thus there is an isomorphism $F \wedge F \cong [F,F]$. This presentation of the commutator subgroup of a free group is essentially the presentation given by C. Miller [M] (see also [Ho]). (Indeed, the arguments used in the proofs of lemma (3.1.4) and proposition (3.1.7) are modifications of Miller's arguments.) This presentation is also obtained in [B-L] as a corollary to the van Kampen type theorem for squares of maps.

EXAMPLE(3.1.9) It is easy to see that the asymmetric product $F_1 \underline{\wedge} F_1$ is isomorphic to Z_2 the group of order 2. It follows from proposition (3.1.7) that $\ker(F_n \underline{\wedge} F_n \to F_n)$ is the direct sum of n copies of Z_2.

PROPOSITION(3.1.10) *Let* $G = A \times B$ *be the direct product of groups A and B. Then there are isomorphisms*

(i) $\pi_3(A \wedge A) \times \pi_3(B \wedge B) \times A^{ab} \otimes_Z B^{ab} \cong \pi_3(G \wedge G)$,

(ii) $\pi_3(A \underline{\wedge} A) \times \pi_3(B \underline{\wedge} B) \times A^{ab} \otimes_Z B^{ab} \cong \pi_3(G \underline{\wedge} G)$.

PROOF We shall just consider case (i). Note that by proposition (3.1.7) there is an injection $\psi:\pi_3(A \wedge A) \times \pi_3(B \wedge B) \to \pi_3(G \wedge G)$. There is a homomorphism $\phi:A^{ab} \otimes_Z B^{ab} \to \pi_3(G \wedge G)$, $(a,b) \to a \wedge b$. For $(x,y,z) \in \pi_3(A \wedge A) \times \pi_3(B \wedge B) \times A^{ab} \otimes_Z B^{ab}$, set $\theta(x,y,z) = \psi(x,y)\phi z$. Clearly θ is a homomorphism. Let $\alpha_{\#}:\pi_3(G \wedge G) \to \pi_3(A \wedge A)$ be the map induced by the projection $G \to A$, and let $\beta_{\#}:\pi_3(G \wedge G) \to \pi_3(B \wedge B)$ be the map induced by the projection $G \to B$. Let $\gamma:G \wedge G \to A^{ab} \otimes_Z B^{ab}$ be the map $(a,b) \wedge (a',b') \to a[A,A] \otimes b[B,B]$, and let $\gamma_{\#}:\pi_3(G \wedge G) \to A^{ab} \otimes_Z B^{ab}$ be the restriction of γ. The map $(\alpha_{\#},\beta_{\#},\gamma_{\#})$ is a splitting of the map θ. Hence θ is injective. To show that θ is surjective suppose we have an arbitrary element $x \in \pi_3(G \wedge G)$. By lemma (3.1.4) we can write x as a product $x = uvw$ with $u \in \iota_A(A \wedge A)$, $w \in \iota_B(B \wedge B)$ and $v \in \phi(A^{ab} \otimes_Z B^{ab})$. But clearly uw is in the image of the map ψ. It follows that θ is surjective. ∇

We shall now investigate the kernels of the quotient maps $M \otimes N \to M \wedge N$, $M \underline{\wedge} N \to M \wedge N$.

Let

$$\begin{tikzcd}
M \times_P N \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & N \arrow[d, "\delta'"] \\
M \arrow[r, "\delta"] & P
\end{tikzcd}$$

be the pullback square. (It is interesting, but not of relevance here, to note that this square has a natural structure of a crossed square.) Let $\{M,N\}$ be the subgroup of $M \times_P N$ generated by the elements $(m\,{}^{n}m^{-1}, {}^{m}n\,n^{-1})$ with $m \in$

M, $n \in N$. This subgroup is normal.

We can define a function $M \times pN \to M \otimes N$, $(m,n) \to m \otimes n$. One can check that this function induces a function from $M \times pN/\{M,N\}$ to $M \otimes N$ (see [B-L]). There is also, therefore, a function from $M \times pN/\{M,N\}$ to $M \underline{\wedge} N$.

we follow [B-L] and

In order to analyse the kernel of $M \otimes N \to M \wedge N$ recall the definition of Whitehead's Γ-functor [W2], which is the "universal quadratic functor" from abelian groups to abelian groups. Let A be an abelian group. Then ΓA is the abelian group with generators γa, for $a \in A$, and the following relations:

(i) $\gamma(-a) = \gamma a$,

(ii) if $\beta(a,b) = \gamma(a+b) - \gamma a - \gamma b$, for $a,b \in B$, then $\beta : A \times A \to \Gamma A$ is biadditive.

PROPOSITION(3.1.11) [B-L] *The quotient group $M \times pN/\{M,N\}$ is abelian, and there is an exact sequence*

$$\Gamma(M \times pN/\{M,N\}) \xrightarrow{\psi} M \otimes N \to M \wedge N \to 1$$

where $\psi(\gamma(m,n)) = m \otimes n$. Also, ψ has central image.

PROOF The proof, which is a straightforward algebraic one, is given in [B-L]. ∇

PROPOSITION(3.1.12) *There is an exact sequence*

$$M \times pN/\{M,N\} \otimes_Z Z_2 \xrightarrow{\psi'} M \underline{\wedge} N \to M \wedge N \to 1$$

where $\psi'(m,n) = (m \wedge n)$.

PROOF We have already noted that the function $M \times pN/\{M,N\} \to M \underline{\wedge} N$, $(m,n) \to m \otimes n$ is well defined. It is a homomorphism since

$$(mm' \underline{\wedge}\, nn') = (m' \underline{\wedge}\, n)(m \underline{\wedge}\, n')(m \underline{\wedge}\, n)(m' \underline{\wedge}\, n')$$
$$= (m \underline{\wedge}\, n)(m' \underline{\wedge}\, n')$$

for (m,n), $(m',n') \in M \times_p N$. When $m = m'$ and $n = n'$ we see that

$$(mm \underline{\wedge}\, nn) = e.$$

It follows that ψ' is a homomorphism. Clearly ψ' maps onto the kernel of the quotient map $M \underline{\wedge}\, N \to M \wedge N$. ∇

EXAMPLE(3.1.13) There is an exact sequence

$$G^{ab} \otimes_Z Z_2 \xrightarrow{\psi'} G \underline{\wedge}\, G \to G \wedge G \to 1.$$

This particular case is given in [D]. For this case it is also shown in [D] that the map ψ' is injective and has a splitting. In general there is no reason to expect ψ' to be injective.

We shall end this section with two exact sequences, one involving the exterior product, the other involving the asymmetric product. The sequence involving the exterior product has been obtained previously as a consequence of the van Kampen theorem for squares of maps [B-L] and is of relevance to the homology of groups. In Chapter IV we shall explain this relevance using purely algebraic techniques.

THEOREM(3.1.14) *Let M,N be normal subgroups of a group G such that G = MN the group product of the subgroups. Then there is an exact sequence*

$$\pi_3(M \wedge N) \xrightarrow{\psi_5} \pi_3(G \wedge G) \xrightarrow{\psi_4} \pi_3(G/M \wedge G/M) \times \pi_3(G/N \wedge G/N)$$
$$\xrightarrow{\psi_3} M \cap N/[M,N] \xrightarrow{\psi_2} G^{ab} \xrightarrow{\psi_1} (G/M)^{ab} \times (G/N)^{ab} \to 1 .$$

Also, by replacing the exterior product sign $\wedge$ with the asymmetric product $\underline{\wedge}$, we obtain another exact sequence.

PROOF We shall just consider the case of the exterior product. It will be convenient to have the following notation: if H is a group with two normal subgroups H_0, H_1 then we shall denote by $\langle H_0, H_1\rangle_H$ the subgroup of $H \wedge H$ generated by the elements $h_0 \wedge h_1$ with $h_0 \in H_0$, $h_1 \in H_1$.

We must define the maps ψ_i.

<u>The map ψ_1.</u> Let $\tau_1 : G^{ab} \to (G/M)^{ab}$, $\tau_2 : G^{ab} \to (G/N)^{ab}$ be the quotient maps, and define $\psi_1 = (\tau_1, \tau_2)$.

<u>The map ψ_2.</u> This map is induced by the inclusion $M \cap N \to G$.

<u>The map ψ_3.</u> Note that G/M is isomorphic to $N/M\cap N$. It follows from proposition (3.1.3) that $G/M \wedge G/M$ is isomorphic to $N \wedge N/\langle N, N\cap M\rangle_N$. There is a commutative diagram

$$\begin{tikzcd}
 & & & 1 \arrow[d] & \\
 & & & \ker c' \cong \pi_3(G/M \wedge G/M) \arrow[d] & \\
1 \arrow[r] & \langle N, N\cap M\rangle_N \arrow[r] \arrow[d, "c''"] & N \wedge N \arrow[r] \arrow[d, "c"] & N \wedge N/\langle N, N\cap M\rangle_N \arrow[r] \arrow[d, "c'"] & 1 \\
1 \arrow[r] & M\cap N \arrow[r] & N \arrow[r] & N/M\cap N \arrow[r] & 1
\end{tikzcd}$$

in which $c(n \wedge n') = [n, n']$, and in which the rows and columns are exact. The diagram gives rise to a map $\tau_3 : \pi_3(G/M \wedge G/M) \to [N,N]\cap M/[N, M\cap N]$. Similarly there is a map $\tau_4 : \pi_3(G/N \wedge G/N) \to [M,M]\cap N/[M, M\cap N]$. Let $\iota : [N,N]\cap M/[N, M\cap N] \to M\cap N/[M,N]$ be the map induced by the inclusion $[N,N]\cap M \to M\cap N$, let $\iota':[M,M]\cap N/[M,M\cap N] \to M\cap N/[M,N]$ be induced by the inclusion $[M,M]\cap N \to M\cap N$, and define ψ_3 to be the map $\psi_3(x,y) = (\iota\tau_3 x)(\iota'\tau_4 y)$.

<u>The map ψ_4.</u> Let $\tau_5:\pi_3(G \wedge G) \to \pi_3(G/M \wedge G/M)$ be the map induced by the quotient map $G \to G/M$, and let $\tau_6:\pi_3(G \wedge G) \to \pi_3(G/N \wedge G/N)$ be induced by the quotient map $G \to G/N$. Define $\psi_4 = (\tau_5,\tau_6)$

<u>The map ψ_5.</u> This map is induced by the inclusions $M \to G$ and $N \to G$.

We must now check exactness.

Let $x = (gM[G,G], g'N[G,G])$ be an arbitrary element of $(G/M)^{ab} \times (G/N)^{ab}$. Since $G = MN$ we can assume that $g = mn$, $g' = m'n'$ for some $m,m' \in M$, $n,n' \in N$. The element $nm'[G,G]$ in G^{ab} is mapped onto x by ψ_1. Thus ψ_1 is surjective.

The kernel of ψ_1 is $M\cap N/[G,G]\cap M\cap N$. Certainly the image of ψ_2 is equal to the kernel of ψ_1.

The kernel of ψ_2 is $[G,G]\cap M\cap N/[M,N]$. Denote by K the group $([M,M]\cap N)([N,N]\cap M)/[M,M][N,N]\cap[M,N]$. To see that the image of ψ_3 is equal to the kernel of ψ_2 we have only to show that there is a commutative diagram of groups

$$\begin{array}{ccc} \pi_3(G/M \wedge G/M) \times \pi_3(G/N \wedge G/N) & \xrightarrow{\psi_3} & M\cap N/[M,N] \\ \downarrow \alpha & & \uparrow i \\ K & \xrightarrow{\beta} & \ker \psi_2 \end{array}$$

in which α and β are surjective and i is the inclusion. Let $\rho:[N,N]\cap M/[N,M\cap N] \to K$ be the map induced by the inclusion $[N,N]\cap M \to ([M,M]\cap N)([N,N]\cap M)$, and let

$\rho' : [M,M]\cap N/[M,M\cap N] \to K$ be induced by the inclusion $[M,M]\cap N \to ([M,M]\cap N)([N,N]\cap M)$. Then α is defined by $\alpha(x,y) = (\rho\tau_3 x)(\rho'\tau_4 y)$ where τ_3, τ_4 are given above. The map α is surjective since the maps $\tau_3, \tau_4, \rho, \rho'$ are all surjective. The map β is induced by the inclusion $([M,M]\cap N)([N,N]\cap M) \to [G,G]\cap M\cap N$. To see that β is surjective it suffices to note that $[G,G] = [M,M][N,N][M,N]$, and that $[M,M][N,N]\cap M\cap N = ([M,M]\cap N)([N,N]\cap M)$.

We now aim to show that the kernel of ψ_3 is equal to the image of ψ_4. The following commutative diagram of canonical maps

$$\begin{tikzcd}
1 \arrow[r] & \ker\phi \arrow[r] & G \wedge G/\langle M,N\rangle_G \arrow[r, "\phi"] \arrow[d] & G/M\cap N \arrow[d] \\
 & & G \wedge G/\langle G,M\rangle_G \arrow[r] \arrow[d, "\cong"] & G/M \arrow[d, equal] \\
1 \arrow[r] & \pi_3(G/M \wedge G/M) \arrow[r] & G/M \wedge G/M \arrow[r] & G/M
\end{tikzcd}$$

(where ϕ is induced by the map $G \wedge G \to G$, $g \wedge g' \to [g,g']$) gives rise to a map $\tau_7 : \ker\phi \to \pi_3(G/M \wedge G/M)$. Similarly we have a map $\tau_8 : \ker\phi \to \pi_3(G/N \wedge G/N)$.

Consider the commutative diagram

$$\begin{tikzcd}
\pi_3(G \wedge G) \arrow[r, "\psi_4"] \arrow[d, "\gamma"] & \pi_3(G/M \wedge G/M) \times \pi_3(G/N \wedge G/N) \arrow[r, "\psi_3"] & M\cap N/[M,N] \arrow[d, equal] \\
\ker \psi_3(\tau_7,\tau_8) \arrow[r, "\iota"] & \ker\phi \arrow[u, "(\tau_7,\tau_8)"] \arrow[r] & M\cap N/[M,N]
\end{tikzcd}$$

where γ is induced by the quotient map $G \wedge G \to G \wedge G/\langle M,N\rangle_G$, and ι is the inclusion. It is readily seen that γ is surjective. To prove that the kernel of ψ_3 is equal to the image of ψ_4 it suffices to show that the map (τ_7,τ_8) is surjective: let (x,y) be an arbitrary element of $\pi_3(G/M \wedge G/M) \times \pi_3(G/N \wedge G/N)$; since $G = MN$ we can assume that x is a product $x = (n_1M \wedge n_1'M)\ldots(n_\ell M \wedge n_\ell'M)$ with $n_i,n_i' \in N$, and that y is a product $y = (m_1N \wedge m_1'N)\ldots$ $\ldots(m_\ell N \wedge m_\ell'N)$ with $m_i,m_i' \in M$; the element $(n_1 \wedge n_1')\ldots(n_\ell \wedge n_\ell')(m_1 \wedge m_1')\ldots(m_\ell \wedge m_\ell')\langle M,N\rangle_G$ of $\ker \phi$ is mapped onto (x,y) by (τ_7,τ_8). Thus (τ_7,τ_8) is surjective.

We now want to show that the image of ψ_5 is equal to the kernel of ψ_4. By proposition (3.1.3) there are isomorphisms $G/M \wedge G/M \cong G \wedge G/\langle G,M\rangle_G$ and $G/N \wedge G/N \cong G \wedge G/\langle G,N\rangle_G$. It is thus clear that the kernel of ψ_4 is the intersection $\langle G,M\rangle_G \cap \langle G,N\rangle_G \cap \pi_3(G \wedge G)$. We shall show that $\langle G,M\rangle_G \cap \langle G,N\rangle_G = \langle M,N\rangle_G$, and it will then be clear that the image of ψ_5 is the kernel of ψ_4.

Certainly $\langle G,M\rangle_G \cap \langle G,N\rangle_G \supset \langle M,N\rangle_G$. Suppose that $x \in \langle G,M\rangle_G \cap \langle G,N\rangle_G$. Since $G = MN$ and since $x \in \langle G,M\rangle_G$, we can write x as a product $x = x_0x_1$ with $x_0 \in \langle M,M\rangle_G \cap \langle G,N\rangle_G$ and $x_1 \in \langle M,N\rangle_G$. Note that x_0 is in the kernel of the canonical map $\langle M,M\rangle_G \to G \wedge G \to G \wedge G/\langle G,N\rangle_G \cong M \wedge M/\langle M,M\cap N\rangle_M$. Thus $x_0 \in \langle M,M\cap N\rangle_G$ and hence $x \in \langle M,N\rangle_G$. ∇

Suppose in this last theorem that we have $N = G$. We can extend the exact sequence(s) by two terms since, if we have a commutative diagram of groups

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] \\
 & R \arrow[r, equal] \arrow[d] & R \arrow[d] \\
1 \arrow[r] & S \arrow[r] \arrow[d] & F \arrow[d] \\
1 \arrow[r] & M \arrow[r] \arrow[d] & G \arrow[d] \\
 & 1 & 1
\end{tikzcd}
$$

in which the rows and columns are exact, and the group F is free, then

THEOREM(3.1.15) *There is an exact sequence*

$$\pi_3(F \wedge R) \xrightarrow{\psi_7} \pi_3(F \wedge S) \xrightarrow{\psi_6} \pi_3(G \wedge M) \xrightarrow{\psi_5} \pi_3(G \wedge G)\ .$$

where ψ_5 *is as in the preceding proposition. Again, by replacing the exterior product sign* $\wedge$ *with the asymmetric product* $\underline{\wedge}$, *we obtain another exact sequence.*

PROOF Using proposition (3.1.3) we see that we have a commutative diagram of canonical maps

$$
\begin{tikzcd}
1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] \\
\pi_3(F \wedge R) \arrow[d] & \pi_3(F \wedge S) \arrow[d] & \pi_3(G \wedge M) \arrow[d] & \pi_3(G \wedge G) \arrow[d] \\
F \wedge R \arrow[r] \arrow[d] & F \wedge S \arrow[r] \arrow[d, "c"] & F \wedge S/\langle F,R\rangle \arrow[r] \arrow[d] & F \wedge F/\langle F,R\rangle_F \arrow[d] \\
F \arrow[r, equal] & F \arrow[r] & F/R \arrow[r, equal] & F/R
\end{tikzcd}
$$

in which the columns are exact, in which $\langle F,R\rangle$ denotes the subgroup of $F \wedge S$ generated by the elements $f \wedge r$ with $f \in F$, $r \in R$, and in which c is the map $f \wedge s \to [f,s]$. This diagram induces the maps ψ_5, ψ_6, ψ_7. We must check exactness.

Suppose x is an element in the kernel of ψ_5. We can represent x by an element $\langle x\rangle$ of $F \wedge S$, and we have $c\langle x\rangle \in [F,R]$. There exists an $\langle x\rangle' \in \langle F,R\rangle$ such that $c\langle x\rangle' = c\langle x\rangle^{-1}$. The product $\langle x\rangle\langle x\rangle' \in \pi_3(F \wedge S)$ also represents x. Thus ψ_6 maps onto the kernel of ψ_5.

Clearly the image of ψ_7 is equal to the kernel of ψ_6. ∇

2. A UNIVERSAL CROSSED 3-CUBE IN GROUPS

We shall now look at the "3-dimensional" analogue of the tensor product of groups of the preceding section.

Suppose given a diagram of groups

$$\begin{tikzcd}
 & N \arrow[rrrrr, "\nu_Q"] \arrow[dd, "\nu_P"'] & & & & & Q \arrow[dd, "\delta"] \\
 & & & & & M \arrow[ur, "\nu_Q"] \arrow[dd, "\nu_R"] & \\
 & P \arrow[rrrrr, "\delta"] & & & & & S \\
L \arrow[ur, "\nu_P"] \arrow[rrrr, "\nu_R"] & & & & & R \arrow[ur, "\delta"] &
\end{tikzcd} \tag{*}$$

in which S acts on each of the other groups in such a way that each of the three faces has a crossed square structure. We want to construct a group T and maps $\lambda_L : T \to L$, $\lambda_M : T \to M$, $\lambda_N : T \to N$, such that the resulting cubical diagram is a crossed 3-cube with the following defining universal property (see proposition (1.4.1)): for any other group T' and maps λ_L', λ_M', λ_N' such that the resulting cubical diagram is a crossed 3-cube, there is a unique map $T \to T'$ of crossed 3-cubes. We shall call T the *cubical tensor product* of the above diagram.

Let T_0 be the group generated by the elements

$$q \otimes_1 l \;, \quad p \otimes_2 m \;, \quad n \otimes_3 r$$

where $(q,l) \in Q \times L$, $(p,m) \in P \times M$, $(n,r) \in N \times R$, subject to the following relations: (All actions are assumed to be via S, and we write ${}^{s}(x \otimes_i y)$ instead of ${}^{s}x \otimes_i {}^{s}y$.)

(i)
$$\begin{aligned}
q \otimes_1 ll' &= (q \otimes_1 l)\, {}^{l}(q \otimes_1 l'),\\
qq' \otimes_1 l &= {}^{q}(q' \otimes_1 l)(q \otimes_1 l),\\
pp' \otimes_2 m &= {}^{p}(p' \otimes_2 m)(p \otimes_2 m),\\
p \otimes_2 mm' &= (p \otimes_2 m)\, {}^{m}(p \otimes_2 m'),\\
nn' \otimes_3 r &= {}^{n}(n' \otimes_3 r)(n \otimes_3 r),\\
n \otimes_3 rr' &= (n \otimes_3 r)\, {}^{r}(n \otimes_3 r');
\end{aligned}$$

(ii)
$$\begin{aligned}
(p \otimes_2 m)(q \otimes_1 l)(p \otimes_2 m)^{-1} &= {}^{[p,m]}(q \otimes_1 l),\\
(n \otimes_3 r)(q \otimes_1 l)(n \otimes_3 r)^{-1} &= {}^{[n,r]}(q \otimes_1 l),\\
(q \otimes_1 l)(p \otimes_2 m)(q \otimes_1 l)^{-1} &= {}^{[q,l]}(p \otimes_2 m),\\
(n \otimes_3 r)(p \otimes_2 m)(n \otimes_3 r)^{-1} &= {}^{[n,r]}(p \otimes_2 m),\\
(q \otimes_1 l)(n \otimes_3 r)(q \otimes_1 l)^{-1} &= {}^{[q,l]}(n \otimes_3 r),
\end{aligned}$$

$$(p \otimes_2 m)(n \otimes_3 r)(p \otimes_2 m)^{-1} = [p,m](n \otimes_3 r),$$

(iii)
$$\nu_Q({}^{p}m\, m^{-1}) \otimes_1 l = (p \otimes_2 m)\, {}^{l}(p \otimes_2 m)^{-1},$$
$$\nu_Q(n\, {}^{r}n^{-1}) \otimes_1 l = (n \otimes_3 r)\, {}^{l}(n \otimes_3 r)^{-1},$$
$$\nu_P({}^{q}l\, l^{-1}) \otimes_2 m = (q \otimes_1 l)\, {}^{m}(q \otimes_1 l)^{-1},$$
$$\nu_P(n\, {}^{r}n^{-1}) \otimes_2 m = (n \otimes_3 r)\, {}^{m}(n \otimes_3 r)^{-1},$$
$$n \otimes_3 \nu_R({}^{p}m\, m^{-1}) = {}^{n}(p \otimes_2 m)(p \otimes_2 m)^{-1},$$
$$n \otimes_3 \nu_R({}^{q}l\, l^{-1}) = {}^{n}(q \otimes_1 l)(q \otimes_1 l)^{-1},$$
$$p \otimes_2 h(q, \nu_R l) = {}^{p}(q \otimes_1 l)(q \otimes_1 l)^{-1},$$
$$p \otimes_2 h(\nu_Q n, r) = {}^{p}(n \otimes_3 r)(n \otimes_3 r)^{-1},$$
$$q \otimes_1 h(p, \nu_R m) = {}^{q}(p \otimes_2 m)(p \otimes_2 m)^{-1},$$
$$q \otimes_1 h(\nu_P n, r) = {}^{q}(n \otimes_3 r)(n \otimes_3 r)^{-1},$$
$$h(\nu_P l, q) \otimes_3 r = (l \otimes_1 q)\, {}^{r}(l \otimes_1 q)^{-1},$$
$$h(p, \nu_Q m) \otimes_3 r = (p \otimes_2 m)\, {}^{r}(p \otimes_2 m)^{-1};$$

(iv)
$$((\nu_P n)(\nu_P l) \otimes_2 m)((\nu_Q m)(\nu_Q n) \otimes_1 l)$$
$$= (n \otimes_3 (\nu_R l)(\nu_R m)),$$

(v)
$$^{q}(h(p,q^{-1})^{-1} \otimes_3 r)$$
$$= {}^{p}(q \otimes_1 h(p^{-1},r))\, {}^{r}(p \otimes_2 h(q,r^{-1})^{-1});$$

(vi)
$$\nu_Q m \otimes_1 l = (\nu_P l \otimes_2 m)^{-1},$$
$$\nu_P n \otimes_2 m = n \otimes_3 \nu_R m,$$
$$\nu_Q n \otimes_1 l = n \otimes_3 \nu_R l;$$

for all $l,l' \in L$, $m,m' \in M$, $n,n' \in N$, $p,p' \in P$, $q,q' \in Q$, $r,r' \in R$.

Note that there is a group action of S on T_0 given by

$$
\begin{aligned}
{}^s(q \otimes_1 l) &= {}^sq \otimes_1 {}^sl,\\
{}^s(p \otimes_2 m) &= {}^sp \otimes_2 {}^sm,\\
{}^s(n \otimes_3 r) &= {}^sn \otimes_3 {}^sr.
\end{aligned}
$$

Define maps $\lambda_L:T_0 \to L$, $\lambda_M:T_0 \to M$, $\lambda_N:T_0 \to N$ on generators by

$$
\begin{aligned}
\lambda_L(q \otimes_1 l) &= {}^ql\, l^{-1},\\
\lambda_L(p \otimes_2 m) &= h(p,\nu_R m),\\
\lambda_L(n \otimes_3 r) &= h(\nu_P n,r),\\
\lambda_M(q \otimes_1 l) &= h(q,\nu_R l),\\
\lambda_M(p \otimes_2 m) &= {}^pm\, m^{-1},\\
\lambda_M(n \otimes_3 r) &= h(\nu_Q n,r),\\
\lambda_N(q \otimes_1 l) &= h(\nu_P l,q)^{-1},\\
\lambda_N(p \otimes_2 m) &= h(p,\nu_Q m),\\
\lambda_N(n \otimes_3 r) &= n\, {}^rn^{-1}.
\end{aligned}
$$

It is routine to check that these maps are well defined.

Let us define three functions

$$
\begin{aligned}
&h:Q\times L \to T_0, \quad (q,l) \to q \otimes_1 l,\\
&h:P\times M \to T_0, \quad (p,m) \to p \otimes_2 m,\\
&h:N\times R \to T_0, \quad (n,r) \to n \otimes_3 r.
\end{aligned}
$$

PROPOSITION(3.2.1) *The group T_0 is the cubical tensor product of diagram* (*).

PROOF We must check that the above cubical structure is a crossed 3-cube.

Note that the cubical diagram is commutative, and that axiom (2.1.3.iv) holds. Axioms (2.1.3,ii,iii,v) follow respectively from the identities (iv,v,vi) above. It remains to check rule (2.1.3.i).

Consider the square

$$\begin{array}{ccc} T_0 & \xrightarrow{\lambda_M} & M \\ {\scriptstyle\lambda_L}\downarrow & & \downarrow{\scriptstyle\nu_R} \\ L & \xrightarrow{\nu_R} & R \end{array}$$

For $t \in T_0$, $r \in R$ we clearly have $\lambda_L({}^r t) = {}^r\lambda_L(t)$. For $t,t' \in T_0$ we have (using the first two identities of (i), and the first two identities of (ii) above) that ${}^{(\lambda_L t)}t' = tt'\,t^{-1}$. Thus λ_L (and similarly λ_M) is a crossed module and preserves the action of R. The function $h{:}L\times M \to T_0$, $(l,m) \to \nu pl \otimes_2 m$ certainly satisfies axioms (2.1.2,iii, iv,v,). Axiom (2.1.2.vi) follows from (iii) above. Thus the square under consideration is a crossed square.

Similarly the square

$$\begin{array}{ccc} T_0 & \xrightarrow{\nu_Q\lambda_M} & Q \\ {\scriptstyle\lambda_L}\downarrow & & \downarrow{\scriptstyle\delta} \\ L & \xrightarrow{\delta\nu_R} & S \end{array}$$

is a crossed square. By symmetry it follows that rule (2.1.3.i) holds. Hence the cubical structure under consideration is a crossed 3-cube. This crossed 3-cube clearly has the required universal property. ∇

In certain special cases the cubical tensor product has a simpler presentation.

EXAMPLE(3.2.2) Suppose that all maps in (*) are trivial.

It follows that each of the groups L, M, N, P, Q, R is abelian, and that all actions are trivial, and that the cubical tensor product T is the quotient of the direct sum of (standard, abelian) tensor products

$$T = (Q \otimes_Z L) \oplus (P \otimes_Z M) \oplus (N \otimes_Z R) / \sim$$

obtained by factoring out the relations

$$(h(p,q^{-1})^{-1} \otimes r) = (q \otimes h(p^{-1},r)) + (p \otimes h(q,r^{-1})^{-1}),$$

or equivalently the relations

$$(h(p,q) \otimes r) = (p \otimes h(q,r)) - (q \otimes h(p,r)),$$

where $p \in P$, $q \in Q$, $r \in R$.

EXAMPLE(3.2.3) Suppose that all the maps in (*) are trivial, and that $L = P \otimes_Z R$, $M = Q \otimes_Z R$, $N = P \otimes_Z Q$ (where these are standard tensor products of abelian groups). From the preceding example we see that the cubical tensor product in this case is the quotient

$$T = (Q \otimes_Z (P \otimes_Z R)) \oplus (P \otimes_Z (Q \otimes_Z R)) \oplus ((P \otimes_Z Q) \otimes_Z R) / \sim$$

obtained by factoring the relations

$$((p \otimes q) \otimes r) = (p \otimes (q \otimes r)) - (q \otimes (p \otimes r)),$$

where $p \in P$, $q \in Q$, $r \in R$.

That is, the cubical tensor product T is isomorphic to $(P \otimes_Z Q \otimes_Z R) \oplus (P \otimes_Z Q \otimes_Z R)$.

PROPOSITION(3.2.4) *Suppose that the maps* ν_P, ν_Q, ν_R *in diagram (*) are surjective, and let* T' *be the group with a presentation consisting of generators* $q \otimes l$ *for* $(q,l) \in Q \times L$, *and relations:*

(i) $\quad q \otimes ll' = (q \otimes l)\, {}^{l}(q \otimes l')$

$\quad\quad qq' \otimes l = {}^{q}(q' \otimes l)(q \otimes l);$

(ii) $q \otimes l = q \otimes l'$ *whenever* $\nu_P l = \nu_P l'$ *or* $\nu_R l = \nu_R l'$;

(iii) $\nu_Q m \otimes l = (\nu_Q n \otimes l')^{-1}$ *whenever* $\nu_P l = \nu_P n$ *and* $\nu_R m = \nu_R l'$;

for all $l, l' \in L$, $m \in M$, $n \in N$, $q, q' \in Q$. *Then* T' *is the cubical tensor product of the diagram* (*).

PROOF Let T_0 be the cubical tensor product with the presentation of proposition (3.2.1). There is certainly a homomorphism $\psi: T' \to T_0$ given on generators by $q \otimes l \to q \otimes_1 l$. We must construct an inverse to ψ.

Let us construct a set map ψ' from the generators of T_0 to the group T' by defining

$\psi'(q \otimes_1 l) = q \otimes l,$

$\psi'(p \otimes_2 m) = (\nu_Q m \otimes l)^{-1}$ for $l \in L$ such that $\nu_P l = p$,

$\psi'(n \otimes_3 r) = \nu_Q n \otimes l$ for $l \in L$ such that $\nu_R l = r$.

It is clear that ψ' is independent of any choices. It is also clear that if we extend ψ' to a map from the free group on the generators of T_0 to T', then ψ' annihilates the relations (i),(ii),(iii) and (vi) of the presentation of T_0. It remains to check that ψ' also annihilates relations (iv) and (v). That is, we must check that the following identities hold in T':

$$\nu_Q n \otimes lx = (\nu_Q m \otimes yl)^{-1} ((\nu_Q m)(\nu_Q n) \otimes l)$$

for $x, y, l \in L$, $m \in M$, $n \in N$ with $\nu_R x = \nu_R m$ and $\nu_P y = \nu_P n$; and

$$^{q}(q^{-1}\, {}^{v}q \otimes u) = {}^{v}(q \otimes [v^{-1}, u])\; {}^{u}({}^{u^{-1}}q\, q^{-1} \otimes v)^{-1}$$

for $u, v \in L$, $q \in Q$.

The check is done in Appendix III, verifications (1) and (2). ∇

EXAMPLE(3.2.5) Suppose that $L = M = N = P = Q = R = S$ ($= G$ say), and that all maps in (*) are the identity. Then it follows from proposition (3.2.4) that the cubical tensor product in this case is isomorphic to the asymmetric product $G \underline{\Lambda} G$ introduced in the last section.

3. SOME UNIVERSAL CROSSED SQUARES IN LIE ALGEBRAS

Let us fix a commutative ring A (with unit), and assume all Lie algebras to be over A.

We shall begin this section by recalling the construction of a free Lie algebra.

A set V with a map $V \times V \to V$, $(v,v') \to vv'$, is called a *magma*. Given an arbitrary set X we can construct a magma V_X: let $\{X_n\}$ be the family of sets with $X_1 = X$ and X_n ($n \geq 2$) the disjoint union of the sets $X_p \times X_q$ such that $p + q = n$; let V_X be the disjoint union of the sets X_n, and let $V_X \times V_X \to V_X$ be the map induced by the canonical map $X_p \times X_q \to X_{p+q} \subset V_X$. The magma V_X is called the *free magma on X*.

Given an arbitrary magma V we can construct the free A-algebra FV whose elements α are finite sums $\alpha = \Sigma a_v v$ with $a_v \in A$, $v \in V$; the multiplication in FV extends the multiplication in V.

Let I be the two sided ideal of FV generated by the elements $\alpha\alpha$, $(\alpha\beta)\gamma + (\beta\gamma)\alpha + (\gamma\alpha)\beta$ with $\alpha,\beta,\gamma \in FV$. Let LV be the quotient FV/I. LV is a Lie algebra. The Lie algebra LV_X is called the *free* Lie algebra on X.

Let M,N be Lie algebras such that there is a Lie action of M (resp. N) on N (resp. M). We define their tensor product $M \otimes N$ to be the quotient algebra $LV_{(M\times N)}/J$ where $LV_{(M\times N)}$ is the free Lie algebra on the set of elements $m \otimes n$ with $(m,n) \in M\times N$, and where J is the ideal generated by the relations

(i) $a(m \otimes n) = am \otimes n = m \otimes an;$

(ii) $(m + m') \otimes n = m \otimes n + m' \otimes n,$

$m \otimes (n + n') = m \otimes n + m \otimes n';$

(iii) $[m,m'] \otimes n = m \otimes {}^{m'}n - m' \otimes {}^{m}n,$

$m \otimes [n,n'] = {}^{n'}m \otimes n - {}^{n}m \otimes n';$ iv) $[m\otimes n, m'\otimes n'] = (-{}^{n}m \otimes {}^{m'}n');$

for $a \in A$, $m,m' \in M$, $n,n' \in N$.

Note that if the actions of M and N are trivial (i.e. if ${}^{m}n = 0$, ${}^{n}m = 0$ for all $m \in M$, $n \in N$) then the tensor product $M \otimes N$ is just the standard tensor product of A-modules $M^{ab} \otimes_A N^{ab}$, where $M^{ab} = M/[M,M]$, $N^{ab} = N/[N,N]$.

Suppose that the Lie algebras M,N and their actions are obtained from crossed modules $\delta : M \to P$, $\delta' : N \to P$. Then it is routine to show that the tensor product $M \otimes N$ fits into a crossed square

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\lambda} & N \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\delta'} \\ M & \xrightarrow{\delta} & P \end{array}$$

in which: the action of P on $M \otimes N$ is given by

${}^{p}(m \otimes n) = {}^{p}m \otimes n + m \otimes {}^{p}n;$

the maps λ, λ' are given respectively by $m \otimes n \to {}^{m}n$, $m \otimes n \to - {}^{n}m$; the function $h : M\times N \to M \otimes N$ is $(m,n) \to m \otimes n$.

This crossed square has a defining universal property (cf. proposition (1.4.1)).

We define the *exterior product* $M \wedge N$ to be the quotient of $M \otimes N$ obtained by factoring out the relations:

(v) $m \otimes n = 0$, whenever $\delta m = \delta' n$.

We shall denote by $m \wedge n$ the element of $M \wedge N$ which is represented by the element $m \otimes n$ in $M \otimes N$.

The crossed square containing $M \otimes N$ is preserved by the quotient map $M \otimes N \to M \wedge N$. Thus the exterior product $M \wedge N$ also fits into a crossed square.

Given an arbitrary Lie algebra L, we shall denote by $L \wedge L$ the exterior product belonging to the crossed square

$$\begin{array}{ccc} L \wedge L & \to & L \\ \downarrow & & \| \\ L & = & L \end{array}$$

PROPOSITION(3.3.1) *Let LV_X be the free Lie algebra on some set X. Then there is a Lie isomorphism*

$$[LV_X, LV_X] \cong LV_X \wedge LV_X.$$

PROOF The universal property of the exterior product gives us a homomorphism $\psi: LV_X \wedge LV_X \to [LV_X, LV_X]$, $\ell \wedge \ell' \to [\ell, \ell']$. We need to construct an inverse to ψ.

Let V_X be the free magma on X and let $V_{X1} = V_X \backslash X$ be the submagma obtained by excluding the set X. Let $(FV_X)(FV_X)$ be the subalgebra of the free algebra FV_X generated by those elements α which can be written as a product $\alpha = \alpha_0 \alpha_1$ with $\alpha_0, \alpha_1 \in$

FV_X. There is a canonical isomorphism $(FV_X)(FV_X) \cong FV_{X^1}$.

The ideal $I \subset FV_X$ which is generated by the elements $\alpha\alpha$, $(\alpha\beta)\gamma + (\beta\gamma)\alpha + (\gamma\alpha)\beta$ with $\alpha,\beta,\gamma \in FV_X$, is contained in $(FV_X)(FV_X)$. Let $I' \subset FV_{X^1}$ be the isomorphic image of I. Then $[LV_X, LV_X]$ is isomorphic to FV_{X^1}/I'.

Each element $w \in V_{X^1}$ can be expressed uniquely as a product $w = uv$ with $u,v \in V_X$. The set map $V_{X^1} \to LV_X \wedge LV_X$, $w \to u \wedge v$ extends to a homomorphism $\phi' : FV_{X^1} \to LV_X \wedge LV_X$. It is readily verified that ϕ' induces a homomorphism $\phi : [LV_X, LV_X] \cong FV_{X^1}/I' \to LV_X \wedge LV_X$. The homomorphism ϕ is the inverse of ψ. ∇

Note that the Lie algebra analogue of proposition (3.1.3) can easily be proved. Also, a straight forward translation of our group theoretic arguments gives us

THEOREM(3.3.2) *Let G be a Lie algebra containing two ideals M,N such that any element $g \in G$ can be written as a sum $g = m + n$ with $m \in M$, $n \in N$. Then there is an exact sequence:*

$$\pi_3(M \wedge N) \to \pi_3(G \wedge G) \to \pi_3(G/M \wedge G/M) \oplus \pi_3(G/N \wedge G/N) \to$$
$$\to M \cap N/[M,N] \to G^{ab} \to (G/M)^{ab} \oplus (G/N)^{ab} \to 0$$

where $G^{ab} = G/[G,G]$ *and* $\pi_3(M \wedge N) = \ker(M \wedge N \to M,\ m \wedge n \to [m,n])$ *etc.* ∇

4. A UNIVERSAL CROSSED SQUARE IN COMMUTATIVE ALGEBRAS

Let us fix a commutative ring A (with unit), and assume all commutative algebras to be over A.

The *free* commutative algebra CV_X on a set X is the quotient FV_X/I where FV_X is the free algebra on X (see §3) and I is the two sided ideal generated by the relations $\alpha\beta = \beta\alpha$, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for $\alpha,\beta,\gamma \in FV_X$.

Suppose given two crossed modules in commutative algebras $\delta:M \to P$, $\delta':N \to P$. We define their *tensor product* $M \otimes N$ to be the quotient algebra $CV_{(M\times N)}/J$ where $CV_{(M\times N)}$ is the free commutative algebra on the set of elements $m \otimes n$ with $(m,n) \in M\times N$, and where J is the ideal generated by the relations

(i) $a(m \otimes n) = am \otimes n = m \otimes an$;

(ii) $(m + m') \otimes n = m \otimes n + m' \otimes n$,

$m \otimes (n + n') = m \otimes n + m \otimes n'$;

(iii) $(m \otimes n)(m' \otimes n') = (mm' \otimes nn')$;

(iv) ${}^{p}m \otimes n = m \otimes {}^{p}n$;

for $m,m' \in M$, $n,n' \in N$, $p \in P$.

It is routine to show that the tensor product $M \otimes N$ fits into a crossed square

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\lambda} & N \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\delta'} \\ M & \xrightarrow{\delta} & P \end{array}$$

in which: the action of P on $M \otimes N$ is given by

$${}^{p}(m \otimes n) = {}^{p}m \otimes n = m \otimes {}^{p}n;$$

the maps λ, λ' are given respectively by $m \otimes n \to {}^{m}n$, $m \otimes n \to {}^{n}m$; the function $h: M \times N \to M \otimes N$ is $(m,n) \to m \otimes n$.

This crossed square has a defining universal property (cf. proposition (1.4.1)).

A possible notion of an *exterior product* $M \wedge N$ of M with N is obtained as a quotient of $M \otimes N$ by factoring out the relations

(v) $\quad m \otimes n = m' \otimes n'$ whenever $\delta m = \delta' n'$ and $\delta m' = \delta' n$.

CHAPTER IV

CROSSED MODULES, CROSSED SQUARES AND HOMOLOGY

0. INTRODUCTION

Crossed modules and crossed complexes have been used for some time now to give interpretations of cohomology of groups and algebras [ML,Lu1,Lu2]. In this chapter we shall study the dual situation of homology. We shall study free and projective crossed modules (§1) and show how projective crossed modules in groups (§2) and Lie algebras (§3) can be used to obtain information on the 2nd homology. We shall use the exterior products of Chapter III to give interpretations of the 2nd and 3rd homology of a group (§4) and of the second homology of a Lie algebra (§5). These interpretations combined with theorems (3.1.14),(3.1.15) and (3.3.2) will give us exact sequences in homology.

1. FREE AND PROJECTIVE CROSSED MODULES

In this section we work in an arbitrary category C of Ω-groups.

Let us begin by adapting some well known terminology [B-Hu,R] to the category C.

We shall use the term *crossed P-module* to mean a crossed module $\partial : M \to P$ with codomain P. By a *map of crossed P-modules* we shall mean a crossed module map which is the identity on P. The category of such crossed modules and maps will be called the *category of crossed P-modules*.

A *projective* crossed P-module is a projective object in the category of crossed P-modules.

A crossed P-module $\partial:C(f) \to P$ is said to be the *free* crossed P-module on a function $f:X \to P$ from a set X to P if the following universal property is satisfied: the function f is the composite of ∂ with some function $v:X \to C(f)$; given any crossed P-module $\delta:M \to P$ and function $w:X \to M$ satisfying $\delta w = \partial v$, there is a unique map $\psi:C(f) \to M$ of crossed P-modules which satisfies $\psi v = w$.

Clearly free crossed modules are defined uniquely up to isomorphism, and are particular examples of projective crossed modules.

We shall now give three elementary results on crossed modules which will be needed in the following sections.

For an arbitrary Ω-group M, let $[M,M]$ denote the subobject generated by the elements $m + m' - m - m'$, and let $\langle M,M\rangle *$ denote the subobject generated by the elements $m * m'$ for $m,m' \in M$.

PROPOSITION(4.1.1) *Let* $\partial M \to P$ *be a crossed* P-*module. If the restricted map* $\partial':M \to \partial M$ has a section $s:\partial M \to M$ (here s need not preserve the action of P), then both $[M,M] \cap \ker \partial$ and $\langle M,M\rangle * \cap \ker \partial$ are the trivial Ω-group.

PROOF Since ∂' has a section we have that M is isomorphic to the semi-direct product $M \cong \ker \partial \underline{\times} \partial M$. But both $[\ker \partial, \ker \partial]$ and $\langle \ker \partial, \ker \partial\rangle *$ are trivial, and ∂M acts trivially on $\ker \partial$. The proposition follows. ∇

For C equal to the category of groups, proposition (4.1.1) is given in [B-Hu] and is originally due to J.H.C. Whitehead.

PROPOSITION(4.1.2) *Let* $\partial:C \to P$ *be a projective crossed P-module, let* $\delta:M \to P$ *be an arbitrary crossed P-module, and let* $\psi:M \to C$ *be a surjective map of crossed P-modules. Then* ψ *has a section* $s:C \to M$.

PROOF The proof is straightforward. ∇

PROPOSITION(4.1.3) (R. Brown) *Let* $\partial:M \to P$, $\partial':M' \to P$ *be crossed P-modules and let* $\psi:M \to M'$ *be a map of crossed P-modules. Then* ψ *is a crossed M'-module with M' acting on M via* ∂'.

PROOF The proof is a straightforward check. ∇

2. CROSSED MODULES AND THE SECOND HOMOLOGY OF A GROUP

The contents of this section are joint work with T. Porter [E-P]. In this section we take C to be the category of groups.

We shall need the construction of free crossed modules (cf. [B-Hu]). So suppose we are given a function $f:X \to P$ from a set X to a group P. Let $E = F(P\times X)$ be the free group on the set $P\times X$, and let P act on E by

$$^{p}(p',x) = (pp',x).$$

The function f induces a homomorphism $\theta:E \to P$ which is defined on generators by

$$\theta(p,x) = pf(x)p^{-1}.$$

The *Peiffer group* Q is the subgroup of E generated by the elements

$$uvu^{-1}({}^{\theta u}v)^{-1}$$

where $u,v \in E$. The group Q is normal, invariant under the action of P, and θQ is trivial. Thus, setting $C(f) = E/Q$, we obtain a crossed module $\theta_{\#}:C(f) \to P$; this is the required free crossed P-module.

PROPOSITION(4.2.1) *Let $f:X \to P$ be a function from a set X to a group P such that P is generated by the image of f. Let FX be the free group on X and denote by RX the kernel of the induced map from FX to P. Then the function f induces a homomorphism $FX/[FX,RX] \to P$; this is the free crossed P-module on f.*

PROOF This proposition is a special case of [B-Hu, proposition 9]. A direct proof is easy. ▽

PROPOSITION(4.2.2) *If $\partial:C \to P$ is a free crossed P-module with $\partial C = N$ say, then the restricted map $\partial':C \to N$ is a free crossed N-module.*

PROOF The crossed P-module $\partial:C \to P$ is free on some function $f:X \to P$. Let T be a transversal of N in P which contains the identity. The function f induces a function $f':T\times X \to N$ given by

$$f'(t,x) = tf(x)t^{-1}.$$

We shall show that the crossed N-module $\partial':C \to N$ satisfies the universal property of the free crossed N-module on f'.

Let $\delta:M \to N$ be an arbitrary crossed N-module and let $w:T\times X \to M$ be a function such that $\delta w = f'$. Recall the above description of the free crossed P-module on f.

Define a homomorphism $w':E \to M$ on generators by

$$w'(p,x) \to {}^{n}w(t,x)$$

where $p = nt$ with $n \in N$, $t \in T$.

The Peiffer group Q is normally generated by the elements

$$uvu^{-1}({}^{\theta u}v)^{-1}$$

with $u,v \in P\times X$ (see [B-Hu]). Suppose that $u = (q,y)$, $v = (p,x)$ with $p = nt$ as before. Since $\theta u \in N$ we have

$$\begin{aligned} w'({}^{\theta u}v) &= {}^{(\theta u)n}w(t,x) \\ &= {}^{\theta u}w'v. \end{aligned}$$

But $\theta u = \delta w'u$, so

$$w'({}^{\theta u}v) = (w'u)(w'v)(w'u)^{-1}.$$

That is, $w'P$ is trivial; w' thus induces a homomorphism $\psi:C \to M$ satisfying $\delta\psi = \partial$. A routine calculation shows that ψ is N-equivariant, and thus that ψ is the required map of crossed N-modules. ∇

PROPOSITION(4.2.3) *Let $\partial:C \to P$ be a projective crossed P-module with $\partial C = N$ say. Let $R \rightarrowtail F \xrightarrow{\lambda} N$ be an arbitrary free presentation of N. Then there is an isomorphism* $[C,C] \xrightarrow{\cong} [F,F]/[F,R]$ given by $[c,d] \to [x,y][F,R]$ where $c,d \in C$, $x,y \in F$ and $\partial c = \lambda x$, $\partial d = \lambda y$.

PROOF First let us suppose that $\partial:C \to P$ is a free P-module. It follows from proposition (4.2.2) that the restriction of ∂ to $\partial':C \to N$ is a free crossed N-module on some function $f:X \to N$. Let $F_0 = F(X\times P)$ be the free group on $X\times P$ and denote by R_0 the kernel of the homomorphism $F_0 \to P$ given on generators by $(x,p) \to p(fx)p^{-1}$. Propositions (4.1.1),(4.1.2),(4.1.3) give us $[C,C] \cong [F_0,F_0]/[F_0,R_0]$. Now

$[F,F]/[F,R]$ is an invariant of N. (A proof of invariance is not difficult, see [Ba].) It follows that $[C,C] \cong [F,F]/[F,R]$.

Suppose now that $\partial:C \to P$ is a projective crossed P-module. Let $\partial':C(\partial) \to P$ be the free crossed P-module on the map $\partial:C \to P$. There is a surjective map of crossed P-modules $\psi:C(\partial) \to C$. It follows from proposition (4.1.2) that ψ has a section $s:C \to C(\partial)$. Hence by propositions (4.1.1) and (4.1.3) there is an isomorphism $[C(\partial),C(\partial)] \cong [C,C]$. There is thus an isomorphism $[C,C] \cong [F,F]/[F,R]$. It is easily checked that this isomorphism is as described in the proposition. ∇

We now come to the main result of this section.

THEOREM(4.2.4) *If N is a group and $\partial:C \to P$ is a projective crossed P-module with $\partial C = N$, then*

$$H_2(N) \cong \ker \partial \cap [C,C].$$

PROOF This proposition follows immediately from proposition (4.2.3) and Hopf's isomorphism $H_2N \cong R\cap[F,F]/[F,R]$. ∇

REMARK(4.2.5) We could have used the key lemma 2.1 of Ratcliffe's paper [R] to prove this last theorem. Instead, we will show that our methods give a new and simple proof of Ratcliffe's lemma which avoids the detailed elementwise manipulations of the original proof.

Let $\partial:C(f) \to P$ be the free crossed P-module on a function $f:X \to P$. Recall that $C(f) = E/Q$. There is an isomorphism

$$[C(f),C(f)] \cong [E,E]/Q\cap[E,E].$$

Let I be the kernel of the induced map from E to P. Thus $I \to E \to N$ is a free presentation of N and so, by proposition (4.2.3)

$$[C(f),C(f)] \cong [E,E]/[E,I].$$

It follows that $[E,I] = Q\cap[E,E]$; this equality is lemma 2.1 of [R].

3. CROSSED MODULES AND THE SECOND HOMOLOGY OF A LIE ALGEBRA

In this section we take C to be the category of Lie algebras over a commutative ring A (with unit).

PROPOSITION(4.3.1) *Let $f:X \to P$ be a function from a set X to a Lie algebra P such that the image of f generates P as an algebra. Let LV_X be the free Lie algebra on X and denote by RV_X the kernel of the induced map from LV_X to P. Then the function f induces a homomorphism $f_\#:LV_X/[LV_X,RV_X] \to P$; this is the free crossed P-module on f.*

PROOF We have a short exact sequence $RV_X \to LV_X \to P$. For each p in P choose an element $\langle p\rangle$ in LV_X such that $\langle p\rangle$ maps down to p. The function $P \times LV_X \to LV_X$, $(p,l) \to [\langle p\rangle,l]$ induces a Lie action of P on $LV_X/[LV_X,RV_X]$. It is routine to check that $f_\#$, together with this Lie action, satisfies the axioms of a crossed module and has the universal property of the free crossed P-module on f. ∇

THEOREM(4.3.2) *If $\partial:C \to P$ is a projective crossed P-module with $\partial C = P$, then*

$$H_2(P) \cong \ker \partial \cap [C,C].$$

PROOF Let $C(\partial) \to P$ be the free crossed P-module on the function ∂. By proposition (4.3.1) we have $C(\partial) \cong LV_C/[LV_C,RV_C]$ where LV_C is the free Lie algebra on C and RV_C is the kernel of the induced map $LV_C \to P$. There is a surjective map $\psi:C(\partial) \to C$ of crossed P-modules. It follows from propositions (4.1.2) and (4.1.3) that ψ is a crossed C-module with a section $s:C \to C(\partial)$. From proposition (4.1.1) we have $[C,C] \cong [C(\partial),C(\partial)]$. The proposition now follows from the Hopf type formula $H_2(P) \cong RV_C \cap [LV_C,LV_C]/[LV_C,RV_C]$ (see for example [H-S], Chapter VII, section 2). ∇

In view of the group theoretic theorem (4.2.4) it is reasonable to conjecture that proposition (4.3.2) can be strengthened to the case where the image of ∂ is a proper ideal of P. In order to prove this we need the construction of the free crossed P-module (in Lie algebras) on an arbitrary function $f:X \to P$. This construction is more complicated than its group theoretic analogue.

The commutative algebra version of theorem (4.2.4) is given in [P4].

4. CROSSED SQUARES AND THE SECOND AND THIRD HOMOLOGY OF A GROUP

In this section we take C to be the category of groups. Throughout the section let $R \rightarrowtail F \twoheadrightarrow^{\lambda} G$ be a free presentation of a group G.

In example (3.1.8) we showed that the exterior product

$F \wedge F$ is isomorphic to the commutator subgroup $[F,F]$. By proposition (3.1.3) we have

PROPOSITION(4.4.1) *There is a canonical isomorphism*

$$G \wedge G \cong [F,F]/[F,R]. \quad \nabla$$

Recall that $\pi_3(G \wedge G)$ is the kernel of the commutator map $G \wedge G \to G$. Proposition (4.4.1) together with the Hopf formula for $H_2(G)$ gives us

THEOREM(4.4.2) *There is an isomorphism*

$$H_2(G) \cong \pi_3(G \wedge G). \quad \nabla$$

This description of $H_2(G)$ is obtained in [B-L]. It is also essentially the description given in [M].

Suppose that the group G has two normal subgroups M and N. Since the inclusions $M \to G$, $N \to G$ are crossed modules, we can construct the group $\pi_3(M \wedge N)$. In some sense, $\pi_3(M \wedge N)$ is a "relative second homology group of M with respect to N". Note that theorems (3.1.14) and (4.4.2) give us

THEOREM(4.4.3) *If the normal subgroups M,N of G are such that $G = MN$, then there is an exact sequence*

$$\pi_3(M \wedge N) \to H_2(G) \to H_2(G/M) \oplus H_2(G/N) \to M \cap N/[M,N] \to$$
$$\to H_1(G) \to H_1(G/M) \oplus H_1(G/N) \to 1. \quad \nabla$$

This sequence is obtained in [B-L] as a consequence of the

3-dimensional van Kampen type theorem.

The group $\underline{H}_2(G) = \pi_3(G \underline{\wedge} G)$ has been considered by [D] as a kind of "second homology group suitable for algebraic K-theory". In some respects $\underline{H}_2(G)$ certainly behaves like a second homology group. For example, given two groups A,B then we have two isomorphisms

$$\underline{H}_2(A*B) \cong \underline{H}_2(A) \oplus \underline{H}_2(B),$$

$$\underline{H}_2(A\times B) \cong \underline{H}_2(A) \oplus \underline{H}_2(B) \oplus A^{ab} \otimes_{\mathbb{Z}} B^{ab},$$

as a consequence of propositions (3.1.7) and (3.1.10). Also, theorem (3.1.14) gives us

PROPOSITION(4.4.4) *If the normal subgroups M,N of G are such that G = MN, then there is an exact sequence*

$$\pi_3(M \underline{\wedge} N) \to \underline{H}_2(G) \to \underline{H}_2(G/M) \oplus \underline{H}_2(G/N) \to M\cap N/[M,N] \to$$
$$\to H_1(G) \to H_1(G/M) \oplus H_1(G/N) \to 1. \quad \nabla$$

We now aim for a description of $H_3(G)$ in terms of the exterior product.

Note that the identity map $F \to F$ and the inclusion $R \to F$ are both crossed modules, and that we can thus form the exterior product $F \wedge R$.

PROPOSITION(4.4.5) *There is a short exact sequence*

$$1 \to R \wedge R \xrightarrow{\psi_1} F \wedge R \xrightarrow{\psi_2} IG \otimes_G R^{ab} \to 1$$

where IG is the augmentation ideal of G, and $\otimes_G$ denotes the usual tensor product of G-modules.

PROOF The canonical map ψ_1 is injective since we have a commutative diagram of maps

$$\begin{array}{ccc} R \wedge R & \cong & [R,R] \\ \downarrow\psi_1 & & \downarrow \\ F \wedge R & \to & [F,R] \end{array}$$

The map ψ_2 is given on generators by

$$\psi_2(f \wedge r) = (\lambda f - 1) \otimes r[R,R] .$$

The map ψ_2 is a homomorphism since

$$\begin{aligned} & {\psi_2}^f(f' \wedge r)\ \psi_2(f \wedge r) \\ = \ & (\lambda(ff'f^{-1}) - 1) \otimes frf^{-1}[R,R] \ + \ (\lambda f - 1) \otimes r[R,R] \\ = \ & (\lambda(ff') - \lambda f) \otimes r[R,R] \ + \ (\lambda f - 1) \otimes r[R,R] \\ = \ & (\lambda(ff') - 1) \otimes r[R,R] \\ = \ & \psi_2(ff' \wedge r), \end{aligned}$$

similarly

$$\psi_2(f \wedge r)\ {\psi_2}^r(f \wedge r') = \psi_2(f \wedge rr'),$$

and

$$\psi_2(r \wedge r) = 0.$$

Clearly ψ_2 is surjective. Set $T = F \wedge R/\psi_1(R \wedge R)$. In order to show that the kernel of ψ_2 is equal to the image of ψ_1 it will suffice to construct an isomorphism $IG \otimes_G R^{ab} \to T$. Note that the quotient T is abelian since, working in $F \wedge R$, we have

$$[f \wedge r,\ f' \wedge r'] = [f,r] \wedge [f',r'] \quad \in \psi_1(R \wedge R)$$

(see example (3.1.2)). For each g in G let $\langle g\rangle$ be an element of F such that $\lambda\langle g\rangle = g$. The group T has a G-module structure given by setting

$$^g(f \wedge r) = {}^{\langle g\rangle}f \wedge {}^{\langle g\rangle}r.$$

This G-action is well defined since, for $r' \in R$, we have

$$^{r'}(f \wedge r)(f \wedge r)^{-1} = r' \wedge [f,r] \quad \in \psi_1(R \wedge R)$$

Suppose $(x,r[R,R])$ is an element of the direct product

$IG \times R^{ab}$. *The augmentation ideal IG is the free abelian group on the set* $\{g - 1 : 1 \neq g \in G\}$, and so x can be written uniquely as a sum $\mp(g_1 - 1) \mp \ldots \mp (g_n - 1)$. Set $\phi(x, r[R,R]) = (\langle g_1 \rangle \wedge r)^{\mp 1} \ldots (\langle g_n \rangle \wedge r)^{\mp 1} \in T$. A routine check shows that ϕ is a well defined G-bilinear map from $IG \times R^{ab}$ to T. It follows that ϕ induces a map $\phi' : IG \otimes_G R^{ab} \to T$. The map ψ_2 induces a map $\psi_2' : T \to IG \otimes_G R^{ab}$. The maps ϕ', ψ_2' are inverse to each other. ∇

THEOREM(4.4.6) *There is an isomorphism*

$$H_3(G) \cong \pi_3(F \wedge R).$$

PROOF Let $\beta : IG \otimes_G R^{ab} \to R^{ab}$ be the homomorphism

$$((g - 1) \otimes r[R,R]) \to \langle g \rangle r \langle g \rangle^{-1}[R,R].$$

Then

$$H_3(G) \cong H_1(G; R^{ab}) \cong \ker \beta.$$

(See for example [H-S], Chapter VI, sections 4 and 12.) We thus have a commutative diagram

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & 1 \arrow[d] & \pi_3(F \wedge R) \arrow[d] & H_3(G) \arrow[d] & \\
1 \arrow[r] & R \wedge R \arrow[r] \arrow[d, equal] & F \wedge R \arrow[r] \arrow[d] & IG \otimes_G R^{ab} \arrow[r] \arrow[d, "\beta"] & 1 \\
1 \arrow[r] & {[R,R]} \arrow[r] \arrow[d] & {[F,R]} \arrow[r] \arrow[d] & {[F,R]/[R,R]} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

in which the rows and columns are exact. The proposition follows. ∇

Theorems (3.1.15), (4.4.3) and (4.4.6) give us

THEOREM(4.4.7) *Given a short exact sequence of groups* $M \to G \to Q$, *then we have an eight term exact sequence in the homology of groups:*

$$H_3(G) \to H_3(Q) \to \pi_3(G \wedge M) \to H_2(G) \to H_2(Q) \to$$
$$\to M/[G,M] \to H_1(G) \to H_1(Q) \to 1. \qquad \nabla$$

This sequence is obtained in [B-L] as a consequence of the 3-dimensional van Kampen theorem.

It is tempting to define the group $\underline{H}_3(G) = \pi_3(F \underline{\wedge} R)$. However, $\pi_3(F \underline{\wedge} R)$ is dependent on the choice of presentation of G. To see this, consider the presentation of the trivial group

$$F_n \to F_n \to 1$$

where F_n denotes the free group of rank n. Then from example (3.1.9) we have that $\pi_3(F_n \underline{\wedge} F_n)$ is isomorphic to the direct sum of n copies of Z_2. That is, $\pi_3(F_n \underline{\wedge} F_n)$ depends on n.

5. CROSSED SQUARES AND THE SECOND HOMOLOGY OF A LIE ALGEBRA

In this section we take C to be the category of Lie algebras over a commutative ring A (with unit). Let $R \to F \to G$ be a short exact sequence of Lie algebras in which F is free.

By proposition (3.3.1) we have that the exterior product $F \wedge F$ is isomorphic to $[F,F]$. Using the Lie algebra version of proposition (3.1.3) we get

PROPOSITION(4.5.1) *There is an isomorphism*

$$G \wedge G \cong [F,F]/[F,R] . \qquad \nabla$$

Recall that $\pi_3(G \wedge G) = \ker(G \wedge G \to G,\ g \wedge g' \to [g,g'])$. The Hopf type formula for $H_2(G)$ now gives us

THEOREM(4.5.2) *There is an isomorphism*

$$H_2(G) \cong \pi_3(G \wedge G) . \qquad \nabla$$

Theorems (3.3.2) and (4.5.2) imply

THEOREM(4.5.3) *Let* G *be a Lie algebra containing two ideals* M,N *such that any element* $g \in G$ *can be written as a sum* $g = m + n$ *with* $m \in M$, $n \in N$. *Then there is an exact sequence in homology:*

$$\pi_3(M \wedge N) \to H_2(G) \to H_2(G/M) \oplus H_2(G/N) \to M \cap N/[M,N] \to$$
$$\to H_1(G) \to H_1(G/M) \oplus H_1(G/N) \to 0. \ \nabla$$

CHAPTER V

MISCELLANEOUS COMMENTS

1. In example (3.2.5) we showed that, in a particular instance, the cubical tensor product of groups is isomorphic to the asymmetric product $G \,\underline{\wedge}\, G$. R. Brown has suggested an alternative proof of this isomorphism as follows.

Let X be a space such that $\pi_1 X = G$, and let $T^3 G$ be the cubical tensor product of example (3.2.5). There is an exact sequence (analogous to the sequence in [B-L, Theorem 5.4])

$$\pi_2 X \to \pi_4 S^2 X \to T^3 G \to \pi_1 X \to (\pi_1 X)^{ab} \to 1.$$

On taking $X = K(G,1)$ we get an exact sequence

$$0 \to \pi_4 S^2 K(G,1) \to T^3 G \to [G,G] \to 1.$$

But there is an exact sequence [B-L, proposition 6.9]

$$0 \to \pi_4 S^2 K(G,1) \to G \,\underline{\wedge}\, G \to [G,G] \to 1.$$

Let $\psi : G \,\underline{\wedge}\, G \to T^3 G$ be the map $g \,\underline{\wedge}\, g' \to g \otimes_1 g'$. Then it is readily seen that we have a commutative diagram

$$\begin{tikzcd}
\ker(G \,\underline{\wedge}\, G \to G) \arrow[r] \arrow[d, "\psi'"] & G \,\underline{\wedge}\, G \arrow[r] \arrow[d, "\psi"] & {[G,G]} \arrow[d, equal] \\
\ker(T^3 G \to G) \arrow[r] & T^3 G \arrow[r] & {[G,G]}
\end{tikzcd}$$

in which ψ' is onto. One ought to be able to prove ψ' injective. Hence $T^3 G \cong G \,\underline{\wedge}\, G$.

2. In theorem (4.4.2) we gave, for a group G, an isomorphism $H_2(G) \cong \pi_3(G \wedge G)$. If G is an abelian group then $\pi_3(G \wedge G)$ is isomorphic to $G \wedge G$ (where $G \wedge G$ is now

the standard exterior product of abelian groups). More generally, for G abelian, there is an isomorphism $H_n(G) \cong \Lambda^n G$ (see for example [Br]). It is reasonable to expect this latter isomorphism to generalise to the case where G is non-abelian (and $n \geq 3$). Even for the case $n = 3$ there seems to be no obvious choice for a "non-abelian cubical exterior product".

3. The six term exact sequence of theorem (4.4.3) is extended, by the following two terms, in [B-L] as an application of the van Kampen theorem for squares of maps:

$$H_3(G) \to H_3(G/M) \oplus H_3(G/N) \to \pi_3(M \wedge N).$$

It ought to be possile to obtain this extension by purely algebraic means! It is reasonable to expect that both of the six term exact sequences of theorem (3.1.14) can be extended by two terms.

4. (R. Brown) If X is a connected CW-complex with $\pi_1 X = G$, then the following exact sequence is obtained as a consequence of the van Kampen theorem for squares of spaces [B-L, Theorem 5.4]

$$\pi_2 X \to \pi_3 SX \to G \otimes G \to [G,G] \to 1. \qquad (*)$$

This sequence together with Whitehead's Γ-sequence [W2] gives us an exact sequence

$$H_3 X \to \Gamma(G^{ab}) \to \ker(G \otimes G \to G) \to H_2 X \to 0. \qquad (**)$$

On taking $X = K(F,1)$ where F is a free group, (**) gives us an isomorphism

$$\Gamma(F^{ab}) \cong \ker(F \otimes F \to F), \qquad (***)$$

and so via (*) we recover π_3 of a wedge of 2-spheres. In deducing (**) from (*) we must use Whitehead's result $\Gamma_3 X \cong \Gamma\pi_2 X$, and the proof of this uses a description of π_3 of a wedge of 2-spheres. So we cannot use the isomorphism (***), so obtained, together with (*) to deduce a description of a wedge of 2-shperes. Clearly a purely algebraic proof of (***) is desirable.

5. It would be nice to have the Lie algebra version of theorem (4.4.6).

6. A description of the homology groups $H_n(G)$ of a group G in terms of crossed n-cubes is known only for n = 1,2,3. In contrast a description of the cohomology groups $H^n(G,A)$, where A is a G-module, in terms of cat^n-groups is known for all n [L].

To each cat^n-group $\underline{H}$ one can associate a complex of (non-abelian) groups

$$C_{\#}\underline{H}: \quad C_n\underline{H} \xrightarrow{\partial_n} \cdots \rightarrow C_1\underline{H} \xrightarrow{\partial_1} C_0\underline{H}$$

such that the image of ∂_{i+1} is normal in the kernel of ∂_i (thus the homology groups $H_i(C_{\#}\underline{H})$ can be formed). For n > 2 and some fixed integer k > 1, let $\Xi(G,A)_k$ be the set consisting of triples $(\underline{H},\phi,\psi)$ where $\underline{H}$ is a cat^{n-2}-group, ϕ is an isomorphism between $H_k(C_{\#}\underline{H})$ and G, and ψ is an isomorphism between $H_{n-1}(C_{\#}\underline{H})$ and A. Moreover, suppose that $H_i(C_{\#}\underline{H}) = 0$ if $i \neq k$ or n-1. There is a Yoneda equivalence on the set $\Xi(G,A)_k$ such that

THEOREM [L] *There is a one-to-one correspondence between the cohomology group* $H^n(K(G,k);A)$ and the set $\Xi(G,A)_k$/(Yoneda equivalence). ∇

The construction of the set $\Xi(G,A)$/(Yoneda equivalence) is easily extended to the case where G is a Lie algebra (commutative algebra etc.). It would be worthwhile having a purely algebraic proof of the above theorem in the case $n = 1$, since this proof would likely generalise to the case where G is a Lie algebra (etc.).

7. Suppose given a crossed square

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \delta'} \\ M & \xrightarrow{\delta} & P \end{array}$$

with classifying space X (see [L]). The homotopy groups $\pi_1 X$, $\pi_2 X$, $\pi_3 X$ are the homology groups of the complex of (non-abelian) groups

$$L \xrightarrow{\alpha} M \underline{\times} N \xrightarrow{\beta} P$$

where $\alpha \ell = (\lambda' \ell, \lambda \ell^{-1})$ and $\beta(m,n) = (\beta m)(\beta n)$. R. Brown has recently shown that the Whitehead product

$$\pi_2 X \times \pi_2 X \rightarrow \pi_3 X$$

is induced by the function

$$W: \ker \beta \times \ker \beta \rightarrow L,$$

$$((m,n),(m'.n')) \rightarrow h(m'^{-1},n)h(m^{-1},n').$$

It would be satisfying to be able to identify the various Whitehead products in a crossed n-cube of groups for $n > 2$.

8. An important result of [W2] is that if X,Y are connected CW-complexes, dim X $\leq$ n and Y is a J_n-complex (for example if $\pi_i Y = 0$ for $1 < i < n$) then the functor which takes the space X to its fundamental crossed complex $\pi\underline{X}$ (described in the Introduction) induces a bijection of homotopy classes

$$[X,Y] \cong [\pi\underline{X},\pi\underline{Y}].$$

Further, there is a bijection

$$[\pi\underline{X},\pi\underline{Y}] \cong [CX,CY]$$

where CX is the cellular chain complex of the universal cover of X (considered as a complex of $\pi_1 X$-modules).

These bijections enable certain homotopy theoretic calculations to be done purely algebraically [E]. At present no progress has been made on using crossed n-cubes to generalise these bijections; the main obstacle is the complicated nature of the functor from crossed n-cubes to CW-complexes which involves taking iterated nerves of the associated cat^n-groups.

Verification (1) of Proposition (2.1.1)

$\alpha(e,e,l,m)$

$= \; ({}^{e}\{{}^{e}l\}\ h({}^{e}m,e)^{-1},{}^{e}m)$

$= \; (l,m) \quad \nabla$

$\alpha((n,p)(n',p'),l,m)$

$= \; ((n\ {}^{p}n')\{{}^{pp'}l\}\ h({}^{pp'}m,n\ {}^{p}n')^{-1},{}^{pp'}m)$

$= \; ((n\ {}^{p}n')\{{}^{pp'}l\}\ {}^{n}h({}^{pp'}m,{}^{p}n')^{-1}\ h({}^{pp'}m,n)^{-1},\ {}^{pp'}m)$ (2.1.1.ii)

$= \; ({}^{n}\{({}^{p}n')\{{}^{pp'}l\}\ {}^{p}h({}^{p'}m,n')^{-1}\}\ h({}^{pp'}m,n)^{-1},{}^{pp'}m)$ (2.1.1.iii)

$= \; ({}^{n}\{{}^{p}\{{}^{n'}\{{}^{p'}l\}\ h({}^{p'}m,n')^{-1}\}\}\ h({}^{pp'}m,n)^{-1},{}^{pp'}m)$ (2.1.1.i)

$= \; \alpha(n,p,{}^{n'}\{{}^{p'}l\}\ h({}^{p'}m,n')^{-1},{}^{p'}m)$

$= \; \alpha(n,p,\alpha(n'p',l,m)) \quad \nabla$

$\alpha(n,p,(l,m)(l',m'))$

$= \; ({}^{n}\{{}^{p}\{l\ {}^{m}l'\}\}\ h({}^{p}\{mm'\},n)^{-1},{}^{p}\{mm'\})$

$= \; ({}^{n}\{{}^{p}l\}\ {}^{n}\{({}^{p}m)\{{}^{p}l'\}\}\ h({}^{p}m,n)^{-1}\ ({}^{p}m)h({}^{p}m',n)^{-1},{}^{p}\{mm'\})$ (2.1.1,i,ii)

$= \; ({}^{n}\{{}^{p}l\}\ h({}^{p}m,n)^{-1}\ ({}^{p}m)\{{}^{n}\{{}^{p}l'\}\ h({}^{p}m',n)^{-1}\},{}^{p}m\ {}^{p}m')$ (2.1.1.iv)

$= \; ({}^{n}\{{}^{p}l\}\ h({}^{p}m,n)^{-1},{}^{p}m)\ ({}^{n}\{{}^{p}l\}\ h({}^{p}m',n),{}^{p}m')$

$= \; \alpha(n,p,l,m)\ \alpha(n,p,l',m') \quad \nabla$

Verification (2) of Proposition (2.1.2)

${}^{p}m({}^{p}l)$

$= \; \delta({}^{p}m)({}^{p}l)$

$= \; ({}^{p}\delta m)_{l}$ (2.1.2.i)

$= \; {}^{p}({}^{\delta m}l)$

$= \; {}^{p}({}^{m}l) \quad \nabla$

Similarly $\,^{p}n(^{p}l) = {}^{p}(^{n}l)$ ∇

Verification (3) of Proposition (2.1.2)

$$
\begin{aligned}
&{}^{m}(^{n}l)\, h(m,n) \\
&= {}^{(\delta m)(\delta' n)}l\; h(m,n) \\
&= h(m,n)\, h(m,n)^{-1}\, {}^{(\delta m)(\delta' n)}l\; h(m,n) \\
&= h(m,n)\, {}^{(\delta\lambda' h(m,n)^{-1})(\delta m)(\delta' n)}l && (2.1.2.i) \\
&= h(m,n)\, {}^{(\delta' n)(\delta m)}l && (2.1.2.v) \\
&= h(m,n)\, {}^{n}(^{m}l) \quad \nabla
\end{aligned}
$$

Verification (4) of Proposition (2.1.3)

The map $M \underline{\times} Q \to R \underline{\times} S$, $(m,q) \to (\nu_R m, \delta q)$ is a crossed module and $M \underline{\times} Q$ acts on $K \underline{\times} N$ via this map. Therefore

$$
\begin{aligned}
&{}^{(r,s)}\{{}^{(m,q)}\{{}^{(r,s)}(k,n)\}\} \\
&= {}^{(r,s)(\nu_R m,\delta q)}(k,n) \\
&= {}^{(r,s)}\{{}^{(m,q)}(k,n)\} \quad \nabla
\end{aligned}
$$

Similarly

$$
{}^{(r,s)(l,p)}\{{}^{(r,s)}(k,n)\} = {}^{(r,s)}\{{}^{(l,p)}(k,n)\} \quad \nabla
$$

Verification (5) of Proposition (2.1.3)

The identity

$$
h'((l,p)(l',p'),m,q) = {}^{(l,p)}h'(l',p',m,q)\; h'(l,p,m,q)
$$

will follow from the four special cases

(i) $h'((e,p)(l',e),m,q) = {}^{(e,p)}h'(l',e,m,q)\; h'(e,p,m,q)$,

(ii) $h'((l,e)(e,p'),m,q) = {}^{(l,e)}h'(e,p',m,q)\; h'(l,e,m,q)$,

(iii) $h'((e,p)(e,p'),m,q) = {}^{(e,p)}h'(e,p',m,q)\; h'(e,p,m,q)$,

(iv) $h'((l,e)(l',e),m,q) = {}^{(l,e)}h'(l',e,m,q)\; h'(l,e,m,q)$,

since (i) and (iii) imply

$$
\begin{aligned}
& h'((e,p)(l',p'),m,q) \\
&= h((e,pp')(p'^{-1}l',e),m,q) && \text{(i)} \\
&= {}^{(e,pp')}h'(p'^{-1}l',e,m,q)\ h'(e,pp',m,q) && \text{(iii)} \\
&= {}^{(e,p)}\{{}^{(e,p')}h'(p'^{-1}l',e,m,q)\ h'(e,p',m,q)\}\ h'(e,p,m,q) && \text{(i)} \\
&= {}^{(e,p)}h'(l',p',m,q)\ h'(e,p,m,q),
\end{aligned}
$$

that is

$$
\text{(v)}\quad h'((e,p)(l',p'),m,q) = {}^{(e,p)}h'(l',p',m,q)h'(e,p,m,q),
$$

and similarly (ii) and (iv) imply

$$
\text{(vi)}\quad h'((l,e)(l',p'),m,q) = {}^{(l,e)}h'(l',p',m,q)h'(l,e,m,q),
$$

and hence

$$
\begin{aligned}
& h'((l,p)(l',p'),m,q) \\
&= h'((l,e)(e,p)(l',p'),m,q) && \text{(vi)} \\
&= {}^{(l,e)}h'((e,p)(l',p'),m,q)\ h'(l,e,m,q) && \text{(v)} \\
&= {}^{(l,p)}h'(l',p',m,q)\ {}^{(l,e)}h'(e,p,m,q)\ h'(l,e,m,q) && \text{(ii)} \\
&= {}^{(l,p)}h'(l',p',m,q)\ h'(l,p,m,q) \quad \triangledown
\end{aligned}
$$

It remains to verify the four special cases.

<u>Case (i)</u>

$$
\begin{aligned}
& h'((e,p)(l,e),m,q) \\
&= h'({}^{p}l,p,m,q) \\
&= (\ {}^{({}^{p}l)}h(p,m)h({}^{p}l,m)\ {}^{m}h(h(p,q),{}^{p}l)^{-1}\ h(h(p,q),m)^{-1} \\
&\qquad h(p,q){}^{m}h(q,{}^{p}l)^{-1},\ h(p,q)\ ) && \text{(2.1.1.iv)} \\
&= (\ {}^{({}^{p}l)}h(p,m)h({}^{p}l,m)\ {}^{m}h(h(p,q),{}^{p}l)^{-1}\ {}^{m}h(p,q)h(q,{}^{p}l)^{-1} \\
&\qquad h(h(p,q),m)^{-1},\ h(p,q)\ ) && \text{(2.1.2,iii,iv)} \\
&= (\ {}^{({}^{p}l)}h(p,m)h({}^{p}l,m)\ {}^{m}h(pq,{}^{p}l)^{-1}\ h(h(p,q),m)^{-1},\ h(p,q)\ ) && \text{(2.1.2.iv)} \\
&= (\ {}^{({}^{p}l)}h(p,m)h({}^{p}l,m)\ {}^{mp}h(q,l)^{-1}\ h(h(p,q),m)^{-1},\ h(p,q)\ ) && \text{(2.1.2.ii)} \\
&= (\ h(p\nu_p l,m)\ {}^{mp}h(q,l)^{-1}\ h(h(p,q),m)^{-1},\ h(p,q)\ ) && \text{(2.1.2.ii)}
\end{aligned}
$$

$= (\ {}^{p}h(1,m)\ h(p,m)\ {}^{mp}h(q,1)^{-1}\ h(h(p,q),m)^{-1},\ h(p,q)\ )$

(2.1.1.iv)

$= (\ {}^{p}\{h(1,m)\ {}^{m}h(q,1)^{-1}\}\ h(p,m)\ h(h(p,q),m)^{-1},\ h(p,q)\ )$

$= {}^{(e,p)}h'(1,e,m,q)\ h'(e,p,m,q)$ ∇

Case (ii)

$h'((1,e)(e,p),m,q)$

$= h'(1,p,m,q)$

$= (\ {}^{1}h(p,m)h(1,m)\ {}^{m}h(h(p,q),1)^{-1}\ h(h(p,q),m)^{-1}$

$h(p,q){}^{m}h(q,1)^{-1},\ h(p,q)\ )$

(2.1.3.v), (2.1.2.iii)

$= (\ {}^{1}h(p,m)\ h((\nu_Q m)(\nu_Q h(p,q),1)^{-1}\ h(h(p,q),m)^{-1}$

$h(p,q){}^{m}h(q,1)^{-1},\ h(p,q)\ )$

(2.1.3.ii)

$= (\ {}^{1}h(p,m)\ h(h(p,q),(\nu_R 1)(\nu_R m))^{-1}\ h((\nu_P h(p,q))(\nu_P 1),m)$

$h(h(p,q),m)^{-1}\ h(p,q){}^{m}h(q,1)^{-1},\ h(p,q)\ )$

(2.1.2.iii)

$= (\ {}^{1}h(p,m)\ h(h(p,q),(\nu_R 1)(\nu_R m))^{-1}\ h(p,q)h(\nu_P 1,m)$

$h(h(p,q),m)\ h(h(p,q)^{-1}\ h(p,q){}^{m}h(q,1)^{-1},\ h(p,q))$

(2.1.2.iii)

$= (\ {}^{1}h(p,m)\ {}^{1}h(h(p,q),m)^{-1}\ h(h(p,q),1)^{-1}\ h(p,q)\{h(1,m)$

${}^{m}h(q,1)^{-1}\},\ h(p,q)\ )$

$= {}^{(1,e)}h'(e,p,m,q)\ h'(1,e,m,q)$ ∇

Case (iii)

$h'((e,p)(e,p'),m,q)$

$= h(e,pp',m,q)$

$= (\ h(pp',m)h(h(pp',q),m)^{-1},\ h(pp',q)\ )$

(2.1.2.iii,v)

$= (\ h(pp',m)h(pp',m)^{-1}\ {}^{pp'}h({}^{q}(pp')^{-1},m)^{-1},\ h(pp',q)\ )$

(2.1.2.iii)

$= (\ {}^{pp'}h({}^{q}p'^{-1},m)^{-1}\ {}^{p}h(p',q)h({}^{q}p^{-1},m)^{-1},\ h(pp',q)$

$= (\ {}^{p}h(p',m)\ {}^{p}h(p',m)^{-1}\ {}^{pp'}h({}^{q}p'^{-1},m)^{-1}\ {}^{p}h(p',q)\{h(p,m)$

$h(p,m)^{-1}\ {}^{p}h({}^{q}p^{-1},m)^{-1}\},\ h(pp',q)\ )$

(2.1.2.iii)

$= {}^{p}h(p',m)\ {}^{p}h(h(p',q),m)^{-1}\ {}^{p}h(p',q)\{h(p,m)h(h(p,q),m)^{-1}\},$

${}^{p}h(p',q)h(p,q)\ )$

$= \; {}^{(e,p)}h'(e,p',m,q)\; h'(e,p,m,q)$ ∇

Case (iv)

$h'((l,e)(l',e),m,q)$

$= \; h'(ll',e,m,q)$

$= \; (h(ll',m)\; {}^{m}h(q,ll')^{-1},\; e)$

$= \; ({}^{l}h(l',m)h(l,m)\; {}^{ml}h(q,l')^{-1}\; h(q,l)^{-1},\; e)$

$= \; {}^{(l,e)}h'(l',e,m,q)\; h'(l,e,m,q)$ ∇

Verification (6) of Proposition (2.1.3)

The identity

$${}^{(r,s)}h'(l,p,m,q) \; = \; h'({}^{(r,s)}(l,p),{}^{(r,s)}(m,q))$$

follows from the four special cases

(i) ${}^{(r,e)}h'(e,p,e,q) \; = \; h'({}^{(r,e)}(e,p),{}^{(r,e)}(e,q))$,

(ii) ${}^{(e,s)}h'(e,p,e,q) \; = \; h'({}^{(e,s)}(e,p),{}^{(e,s)}(e,q))$,

(iii) ${}^{(r,s)}h'(l,e,e,q) \; = \; h'({}^{(r,s)}(l,e),{}^{(r,s)}(e,q))$,

(iv) ${}^{(r,s)}h'(l,e,m,e) \; = \; h'({}^{(r,s)}(l,e),{}^{(r,s)}(m,e))$,

since

$h'({}^{(r,s)}(l,p),{}^{(r,s)}(m,q))$

$= \; h'({}^{(r,s)}(l,e)\; {}^{(r,s)}(e,p),\; {}^{(r,s)}(m,e)\; {}^{(r,s)}(e,q))$

$= \; {}^{(r,s)(l,e)}\{h'({}^{(r,s)}(e,p),\; {}^{(r,s)}(m,e))$

${}^{(r,s)(m,e)}h'({}^{(r,s)}(e,p),{}^{(r,s)}(e,q))\}$

$h'({}^{(r,s)}(l,e),{}^{(r,s)}(m,e))$

${}^{(r,s)(m,e)}h'({}^{(r,s)}(l,e),{}^{(r,s)}(e,q))$

$\overset{(i),(ii),(iii),(iv)}{=} \; {}^{(r,s)}h'(l,p,m,q)$ ∇

Cases (ii),(iii),(iv) have one line verifications. It remains to check

Case (i)

In order to check this case we shall need the following three identities:

(v) $\quad {}^{n}h(1,m)^{-1}\, h(n,1)\, {}^{1}h(n,m) \;=\; h(n,m)\, {}^{m}h(n,1)\, h(1,m)^{-1},$

(vi) $\quad h(p,q)\, {}^{r^{-1}}h({}^{r}q,h(p,r)^{-1})\, h(h(p,q),{}^{p}r^{-1})$

$\quad\quad {}^{pr^{-1}}h({}^{q}p^{-1},h(q,r)^{-1}) \;=\; e$

(vii) $\quad h(h(p,q),h(q,r)^{-1})\, {}^{h(q,r)^{-1}}h(h(p,q),h(p,r)^{-1})$

$\quad\quad h(h(p,r)^{-1},h(q,r)^{-1})^{-1}$

$\quad\quad = \; h(p,q)\, {}^{h(q,r)^{-1}}h(q,h(p,r)^{-1})^{-1}\, h(h(p,q),r)$

$\quad\quad {}^{h(p,r)^{-1}}h(p,h(q,r)^{-1}),$

since then we have

$$
\begin{aligned}
& h'((r,e)(e,p),(r,e)(e,q)) \\
= \; & h'(h(p,r)^{-1},p,h(q,r)^{-1},q) \\
= \; & (\, {}^{h(p,r)^{-1}}h(p,h(q,r)^{-1})\; h(h(p,r)^{-1},h(q,r)^{-1}) \\
& {}^{h(q,r)^{-1}}h(h(p,q),h(p,r)^{-1})^{-1}\; h(h(p,q),h(q,r)^{-1})^{-1} \\
& h(p,q)\, {}^{h(q,r)^{-1}}h(q,h(p,r)^{-1})^{-1},\; h(p,q) \,) \\
= \; & (h(h(p,q),r)^{-1},\; h(p,q)) \qquad \text{(vii)} \\
= \; & (r,e)h'(e,p,e,q).
\end{aligned}
$$

So it remains to check (v),(vi),(vii).

Identity (v)

This identity follows from (2.1.2.iii) and (2.1.3.ii).

Identity (vi)

$$
\begin{aligned}
& h(p,q)\, {}^{r^{-1}}h({}^{r}q,h(p,r)^{-1})\; h(h(p,q),{}^{p}r^{-1})\; {}^{pr^{-1}}h({}^{q}p^{-1},h(q,r)^{-1}) \\
= \; & h(p,q)h(q,h(p,r^{-1}))\; {}^{p}h(h(p^{-1},q)^{-1},r^{-1}) \\
& {}^{pr^{-1}q}h(p^{-1},h(q^{-1},r)^{-1}) \\
= \; & {}^{pq}\{ {}^{p^{-1}}h(q^{-1},h(p,r^{-1}))^{-1}\; {}^{q^{-1}}h(h(p^{-1},q)^{-1},r^{-1}) \\
& {}^{r^{-1}}h(p^{-1},h(q^{-1},r)^{-1})^{-1} \} \qquad (2.1.3.iii) \\
= \; & e \quad \nabla
\end{aligned}
$$

Identity (vii)

$h(h(p,q),h(q,r)^{-1})\ h(q,r)^{-1}h(h(p,q),h(p,r)^{-1})$
$h(h(p,r)^{-1},h(q,r))^{-1}$

$$= h(p,q)h(h(p,r)^{-1},h(q,r)^{-1})^{-1}\ h(h(p,q),h(p,r)^{-1})\ h(p,r)^{-1}h(h(p,q),h(q,r)^{-1}) \quad \text{(v)}$$

$$= h(p,q)h(h(p,r)^{-1},h(q,r)^{-1})^{-1}\ h(h(p,q),h(p,r)^{-1})\ h(p,r)^{-1}\{{}^{p}h({}^{q}p^{-1},h(q,r)^{-1})h(p,h(q,r))^{-1}\}$$

$$= h(p,q)h(q,r)^{-1}h(q,h(p,r)^{-1})^{-1}\ h(p,q)rr^{-1}h({}^{r}q,h(p,r))\ h(h(p,q),r)\ {}^{r}h(h(p,q),{}^{p}r^{-1})\ {}^{rpr^{-1}}h({}^{q}p^{-1},h(q,r)^{-1})\ h(p,r)^{-1}h(p,h(q,r)^{-1})$$

$$= h(p,q)h(q,r)^{-1}h(q,h(p,r)^{-1})^{-1}\ h(h(p,q),r)\ h(p,r)^{-1}h(p,h(q,r)^{-1}) \quad \nabla$$

Verification (7) of Proposition (2.1.3)

The map $\nu: K\underline{x}N \to L\underline{x}P$, $(k,n) \to (\lambda_L k, \nu pn)$ satisfies

$$\nu h'(l,p,m,q) = (l,p)\,{}^{(m,q)}(l,p)^{-1},$$

and the verification is a straightforward copy of verification (3).

Verification (8) of Proposition (2.2.2)

$(an,ap)(l,m)$

$$= ({}^{an}l + {}^{ap}l - h(m,an),\ {}^{ap}m)$$
$$= a({}^{n}l + {}^{p}l - h(m,n),\ {}^{p}m) \quad (2.2.1.i),(2.2.2.ii)$$
$$= a\{(n,p)(l,m)\} \quad \nabla$$

Verification (9) of Proposition (2.2.2)

$(n,p)\{(l,m) + (l',m')\}$

$$= ({}^{n}\{l + l'\} + {}^{p}\{l + l'\} - h(m + m',n),\ {}^{p}m + {}^{p}m')$$
$$= (n,p)(l,m) + (n,p)(l',m') \quad \nabla$$

Verification (10) of Proposition (2.2.2)

$$((n,p) + (n',p'))(l,m)$$

$$= \; ((n + n')l + (p + p')l - h(m,n + n'),\; (p + p')m)$$

$$\overset{(2.2.1.iii),\,(2.2.2.iii)}{=} \; ({}^{n}l + {}^{p}l - h(m,n) + {}^{n'}l + {}^{p'}l - h(m,n'),\; {}^{p}m + {}^{p'}m)$$

$$= \; (n,p)(l,m) + (n',p')(l,m) \qquad \nabla$$

Verification (11) of Proposition (2.2.2)

$$[(n,p),(n',p')](l,m)$$

$$= \; ([n,n'] + {}^{p}n' - {}^{p'}n,\; [p,p'])(l,m)$$

$$\overset{(2.2.1.iii),(2.2.2.iii)}{=} \; ({}^{[n,n']}l + {}^{({}^{p}n')}l - {}^{({}^{p'}n)}l + {}^{[p,p']}l - h(m,[n,n'])$$

$$- h(m,{}^{p}n') + h(m,{}^{p'}n),\; {}^{[p,p']}m)$$

$$\overset{(2.2.2,i,iv,v)}{=} \; ({}^{n}({}^{n'}l) - {}^{n'}({}^{n}l) + {}^{p}({}^{n'}l) - {}^{n'}({}^{p}l) - {}^{p'}({}^{n}l) + {}^{n}({}^{p'}l)$$

$$+ {}^{p}({}^{p'}l) - {}^{p'}({}^{p}l) - {}^{n}h(m,n') + {}^{n'}h(m,n) - {}^{p}h(m,n')$$

$$+ h({}^{p}m,n') + {}^{p'}h(m,n) - h({}^{p'}m,n),\; {}^{p}({}^{p'}m) - {}^{p'}({}^{p}m) \;)$$

$$= \; (n,p)\{(n',p')(l,m)\} - (n',p')\{(n,p)(l,m)\} \qquad \nabla$$

Verification (12) of Proposition (2.2.2)

$$(n,p)[(l,m),(l',m')]$$

$$= \; ({}^{n}[l,l'] + {}^{n}({}^{m}l') - {}^{n}({}^{m'}l) + {}^{p}[l,l'] + {}^{p}({}^{m}l') - {}^{p}({}^{m'}l) -$$

$$h([m,m'],n),\; {}^{p}[m,m'])$$

$$= \; ([{}^{n}l,l'] + [{}^{p}l,l'] - [h(m,n),l'] + {}^{({}^{p}m)}l' - {}^{m'}({}^{n}l)$$

$$- {}^{m'}({}^{p}l) + {}^{m'}h(m,n),\; [{}^{p}m,m'])$$

$$+ \; ([l,{}^{n}l'] + [l,{}^{p}l'] - [l,h(m,n)] + {}^{m}({}^{n}l') + {}^{m}({}^{p}l')$$

$$- {}^{m}h(m',n) - {}^{({}^{p}m')}l,\; [m,{}^{p}m'])$$

$$= \; [(n,p)(l,m),\; (l',m')] + [(l,m),\; (n,p)(l',m')] \qquad \nabla$$

Verification (13) of Proposition (2.2.3)

$p(m1)$

$= p(\delta m1)$

$= [p,\delta m]1 - \delta m(p1)$ (2.2.1.iv)

$= (pm)1 - m(p1)$ ∇

Verification (14) of Proposition (2.2.3)

$n(m1)$

$= \delta' n(\delta m1)$

$= [\delta' n,\delta m]1 + \delta m(\delta' n1)$ (2.2.1.iv)

$= \delta m(\delta' n1) - \delta\lambda' h(m,n)1$

$= m(n1) + [1,h(m,n)]$ ∇

Verification (1) of proposition (3.2.4)

$$= (\nu_Q m \otimes y1)^{-1} ((\nu_Q m)(\nu_Q n) \otimes 1) \qquad \text{where } \nu_P y = \nu_P n$$

$$= {}^{n}(\nu_Q m \otimes 1)^{-1} (\nu_Q m \otimes y)^{-1}\, {}^{m}(\nu_Q n \otimes 1)\, (\nu_Q m \otimes 1)$$

$$= n[1,m]n \quad (\nu_Q n \otimes x)\, {}^{m}(\nu_Q n \otimes 1)\, (\nu_Q n \otimes [1,x])$$

$$\text{where } \nu_R x = \nu_R m$$

$$= n[1,x]n \quad (\nu_Q n \otimes x1)\, (\nu_Q n \otimes [1,x])$$

$$= \nu_Q n \otimes 1x \qquad \nabla$$

Verification (2) of proposition (3.2.4)

$${}^{q}(q^{-1}\, {}^{v}q \otimes u)\, {}^{u}(u^{-1}qq^{-1} \otimes v)$$

Lemma (3.1.1)

$$= {}^{q}\{(q^{-1} \otimes v)\, {}^{u}(q^{-1} \otimes v)^{-1}\}\, {}^{u}\{(q \otimes u^{-1})^{-1}\, {}^{v}(q \otimes u^{-1})\}$$

Lemma (3.1.1)

$$= {}^{v}(q \otimes [v^{-1},u]) \qquad \nabla$$

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