## Bangor University

## DOCTOR OF PHILOSOPHY

## The relationship between the local and global structure of semigroups

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Award date:
2001

Awarding institution:
University of Wales, Bangor

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# The Relationship between the Local and Global Structure of Semigroups 

Thesis submitted to the University of Wales for the degree of - Doctor of Philosophy

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October 11, 2001


## Dedication

This thesis is dedicated to the memory of my late father
MIRZA KHAN
who encouraged me to come to Britain to work on my doctorate.

## Summary of Thesis

The main aim of this thesis is to generalise McAlister's theory of locally inverse regular semigroups to the class of semigroups with local units in which the local submonoids have commuting idempotents. We prove that if such a semigroup has what we call a McAlister sandwich function then the semigroup can be covered by means of a Rees matrix semigroup over a semigroup with commuting idempotents. Examples of such semigroups are easily constructed. Indeed, if $T$ is a semigroup with local units having an idempotent $e$ such that $T=T e T$, and $e T e$ has commuting idempotents, then all the local submonoids of $T$ have commuting idempotents and $T$ is equipped with a McAlister sandwich function. We prove that the semigroups with local units having local submonoids with commuting idempotents $S$ which can be embedded in such a semigroup $T$ in such a way that $S=S T S$ are precisely the ones having a McAlister sandwich function.

Finally, in a different direction, we study variants of semigroups concentrating on the relationship between the local structure of a semigroup and the global structure of its variants.

## Acknowledgements

I first thank Almighty ALLAH who provided me with the opportunity to finish the work on my doctorate.

Thanks are also due to all members of staff of the School of Mathematics. In particular, my supervisor Dr Mark Verus Lawson who not only enabled me to submit this thesis on time but who also guided me in all aspects of my stay in Britain. I would also like to thank him for reading through this thesis and correcting my English.

Thanks are due to my family who provided both moral and financial support throughout my stay in Britain in circumstances which were often very difficult in my home of Kashmir.

I would also like to thank all of my friends in the School for their encouragement, and also all my Pakistani friends both home and abroad.

Finally, I would like to thank the Government of Pakistan and the Government of the State of Azad Jummun and Kashmir who supported my postgraduate studies.

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## Chapter 1

## Introduction

In this chapter, we outline the background material needed to understand this thesis. In Section 1, we recall the basic definitions and constructions which form the basis of semigroup theory as a whole, whereas in Section 2, we recall the basic definitions and constructions of regular semigroup theory. A good reference for this material is [11]. In Section 3, we describe McAlister's theory of locally inverse regular semigroups; this summarises the contents of the papers [19], [20], [21], and [22]. Finally, in Section 4, we summarise the small amount of category theory needed to read this thesis.

### 1.1 General semigroup theory

### 1.1.1 Basic definitions

Let $S$ be a set. A binary operation $\star$ on S is a function $\star: S \times S \rightarrow S$. We usually write $s \star t$ rather than $\star(s, t)$ and, generally, we follow the multiplicative convention and write st instead of $s \star t$. We write $(S, \star)$ to indicate that we wish to regard the set $S$ as being equipped with the binary operation $\star$.

Arbitrary binary operations are of little interest. We now single out properties which make them more interesting. We say that a
binary operation is associative if for all $a, b, c \in S$, we have

$$
a(b c)=(a b) c,
$$

and it is is commutative if for all $a, b \in S$ we have

$$
a b=b a .
$$

An idempotent is an element $e \in S$ such that $e^{2}=e e=e$. The set of idempotents in $S$ is denoted $E(S)$, and if $A \subseteq S$ then $E(A)=$ $A \cap E(S)$. An element $e \in S$ such that

$$
e a=a=a e
$$

for all $a \in S$ is called an identity in $S$. An element $z \in S$ such that $z a=z=a z$ for all $a \in S$ is called a zero in $S$. Both identities and zeros are idempotents. It is easy to check that if a binary operation has an identity (resp, a zero) then it is unique.

A set $S$ equipped with an associative binary operation is called a semigroup. A semigroup with an identity is called a monoid. A semigroup in which every element is an idempotent is called a band. Commutative bands are usually called semilattices. A subset $T$ of $S$ closed under the binary operation, in the sense that $a, b \in T$ implies $a b \in T$, is called a subsemigroup. A subsemigroup of a monoid containing the identity is called a submonoid. A semigroup $S$ is said to be orthodox if the idempotents in $S$ form a subsemigroup. A semigroup $S$ with a zero having at least two elements is called a semigroup with zero. An example of such a semigroup is the null semigroup in which all products are zero

If $S$ is any semigroup then $S^{1}$ is the semigroup with underlying set $S \cup\{1\}$ (where $1 \notin S$ ) and multiplication determined by the one in $S$ together with the conditions $1 s=s=s 1$ for all $s \in S$ and $11=1$. It is clear that $S^{1}$ is a monoid containing $S$ as a subsemigroup. The semigroup $S^{0}$ is defined in a similar way except that it is a semigroup with zero containing $S$ as a subsemigroup. If $G$ is a group then $G^{0}$ is called a 0 -group.

Let $S$ be a semigroup and $e \in S$ an idempotent. Then the subset $e S e=\{$ ese: $s \in S\}$ is a subsemigroup. Observe that $e$ is an identity in $e S e$. For this reason $e S e$ is called, a little illogically, a local submonoid of $S$; 'illogical' because we are not assuming that $S$ itself is a monoid. A semigroup $S$ is said to have a property locally if each local submonoid has that property.

Let $S$ be a semigroup. Then $P(S)$, the set of all subsets of $S$, is also a semigroup when we define the product of $A$ and $B$ to be

$$
A B=\{a b: a \in A \text { and } b \in B\} .
$$

We call this the set product. Observe that $T$ is a subsemigroup of $S$ precisely when $T^{2} \subseteq T$.

Let $S$ be a semigroup and $e, f \in E(S)$. Define the relation $\leq$ on $E(S)$ by $e \leq f$ if and only if $e f=f e=e$. It is easy to check that this relation is a partial order on $E(S)$. It is called the natural partial order on the set of idempotents.

If $S$ is a semigroup and $\leq$ a partial order on $S$, then we say that the order is compatible with the multiplication if $a \leq b$ and $c \leq d$ implies $a c \leq b d$ for all $a, b, c, d \in S$. If $S$ is a band, then it is called a normal band if the natural partial order is compatible with the multiplication (this is equivalent to the usual definition: see [11], page 141, Exercise 18). As a result of McAlister's work [20] the normal bands can also be characterised as those bands in which every local submonoid is a semilattice.

Let $(S, \circ)$ and $(T, \star)$ be semigroups. Then a function $\theta: S \rightarrow T$ is called a homomorphism if for all $a, b \in S$,

$$
\theta(a \circ b)=\theta(a) \star \theta(b)
$$

If $S$ and $T$ are monoids with respective identities $1_{S}$ and $1_{T}$ then a monoid homomorphism is required to map $1_{S}$ to $1_{T}$. An injective homomorphism is called an embedding and a bijective homomorphism is called an isomorphism. If $\theta: S \rightarrow T$ is a surjective homomorphism we often say that $S$ is a cover of $T$. If $T$ is a homo-
morphic image of a subsemigroup of $S$ then we say that $T$ divides $S$.

If $X$ is a set then $T(X)$ denotes the set of all functions from $X$ to itself. It is a monoid with respect to the operation of functional composition o. We call $(T(X), \circ)$ the full transformation monoid on $X$. Observe that we compose functions 'from right-to-left' in this thesis and so $(\alpha \circ \beta)(x)$ means $\alpha(\beta(x))$. The following is the semigroup-theoretic analogue of Cayley's theorem in group theory. The proof can be found in [11] (Theorem 1.1.2). We include it for the sake of completeness.

Theorem 1 Every semigroup can be embedded in a full transformation monoid.

### 1.1.2 The first isomorphism theorem

In this section, we outline the important properties of homomorphisms.

Recall that a relation $\rho$ on a set $X$ is said to be an equivalence relation if it is reflexive, symmetric and transitive.

Let $\rho$ be an equivalence relation on a set X . Then

$$
\rho(x)=\{y \in X:(y, x) \in \rho\}
$$

is called the $\rho$-equivalence class containing $x$. The set of $\rho$-classes forms a partition of $X$, in the sense that they are pairwise disjoint and their union is the whole of $S$. The set of $\rho$-classes is denoted by $S / \rho$.

Let $\theta: X \rightarrow Y$ be a function. Then the kernel of $\theta$, denoted by $\operatorname{ker}(\theta)$, is defined by

$$
\operatorname{ker}(\theta)=\{(x, y) \in X \times X: \theta(x)=\theta(y)\} .
$$

It is easy to check that the kernel is an equivalence relation on $X$.
Let $S$ be a semigroup. Then an equivalence relation $\rho$ on $S$ is called a congruence if it is a subsemigroup of $S \times S$. That is if

$$
(a, b),(c, d) \in \rho \Rightarrow(a c, b d) \in \rho .
$$

Sometimes it is convenient to split up the definition of a congruence into two parts: we say that an equivalence relation $\rho$ is a left congruence if $(a, b) \in \rho$ implies $(c a, c b) \in \rho$ for all $c \in S$; and we say that it is a right congruence if $(a, b) \in \rho$ implies $(a c, b c) \in \rho$ for all $c \in S$. It is easy to check that an equivalence is a congruence if and only if it is both a left and a right congruence.

Let $\rho$ be a congruence on the semigroup S. Define a binary operation * on $S / \rho$ by

$$
\rho(a) \star \rho(b)=\rho(a b) .
$$

Then it is easy to check that $(S / \rho, \star)$ is a semigroup. Observe that the set product $\rho(a) \rho(b) \subseteq \rho(a b)$ holds but that in general equality does not. When it does the congruence is said to be perfect. Congruences on groups are perfect. However, we shall not have need of this notion in this thesis. We shall denote the product in $S / \rho$ by concatenation, but the reader should always take care to remember how the product is defined to avoid confusion.

It is easy to check that if $\theta: S \rightarrow T$ is a homomorphism between semigroups then $\operatorname{ker}(\theta)$ is a congruence on $S$. The function $\rho^{\natural}: S \rightarrow S / \rho$ given by $s \mapsto \rho(s)$ is a surjective homomorphism called the natural homomorphism. The following result, called the first isomorphism theorem, describes the exact relationship between homomorphisms and congruences.

Theorem 2 Let $\theta: S \rightarrow T$ be a homomorphism between semigroups. Denote the natural homomorphism from $S$ to $S / \operatorname{ker}(\theta)$ by $\phi$. Then there is a unique injective homomorphism $\psi: S / \operatorname{ker}(\theta) \rightarrow$ $T$ such that $\theta=\psi \phi$. If $\theta$ is surjective, then $\psi$ is an isomorphism.

The above result is proved in [11] (Theorem 1.5.2). It follows that homomorphic images of $S$ can be constructed, up to isomorphism, from congruences on $S$.

It is easy to check that the intersection of any family of congruences on a semigroup $S$ is also a congruence on $S$. Thus for every
relation $\rho$ on $S$ there is a smallest congruence containing $\rho$ called the congruence generated by $\rho$. We denote this congruence by $\rho^{\sharp}$.

We shall now give a more explicit description of $\rho^{\sharp}$. Let $c, d \in S$ and $x, y \in S^{1}$ such that $c=x a y$ and $d=x b y$ where either $(a, b) \in \rho$ or $(b, a) \in \rho$. We say that $c$ is connected to $d$ by means of an elementary $\rho$-transition. We usually write ' $c \rightarrow d$ '. The following result is proved as Proposition 1.5.9 of [11].

Proposition 3 Let $\rho$ be a relation on a semigroup $S$, and let $a, b \in$ $S$. Then $(a, b) \in \rho^{\sharp}$ if and only if either $a=b$ or there is a sequence

$$
a=z_{1} \rightarrow z_{2} \rightarrow \ldots \rightarrow z_{n}=b
$$

of elementary $\rho$-transitions connecting $a$ to $b$.

### 1.1.3 Ideals

Ideals play an important role in ring theory. In semigroup theory they are not as important, but they are still very useful.

Let S be a semigroup. A nonempty subset $I$ of $S$ is called a left ideal if for all $a \in I$ and $s \in S$ we have sa $I$; a right ideal if as $\in I$; and an ideal if it is both a left and right ideal. It is clear that $S$ is an ideal of $S$ and, if $S$ has a zero, then $\{0\}$ is an ideal of $S$.

In ring theory, ideals are important because they can be used to construct all congruences. This is no longer true in the case of semigroups. However, it is true that ideals can be used to construct some congruences. If $I$ is an ideal in the semigroup $S$ then the semigroup $S / I$, called a Rees quotient of $S$ by $I$, has as underlying set $(S \backslash I)^{0}$ and multiplication defined as follows: if $a, b \in S \backslash I$ and $a b \in S \backslash I$ then the product is defined to be $a b$, whereas in all other cases the product is zero.

If $a \in S$, then the smallest left (respectively, right) ideal containing $a$ is $S^{1} a$ (resp. $a S^{1}$ ) called the principal left (resp. right) ideal generated by $a$. The principal ideal of $S$ generated by $a$ is $S^{1} a S^{1}$.

The principal ideals in a semigroup are used to define the important Green's relations, denoted $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$, as follows. Let $S$ be a semigroup. The equivalence relation $\mathcal{L}$ (resp. $\mathcal{R}$ ) is defined by $a \mathcal{L} b$ (resp. $a \mathcal{R} b$ ) if and only if $S^{1} a=S^{1} b$ (resp. $a S^{1}=b S^{1}$ ). The relation $\mathcal{H}$ is the intersection of $\mathcal{L}$ and $\mathcal{R}$, and the relation $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{L}$ and $\mathcal{R}$ (this is equivalent to saying: $\mathcal{D}$ is the intersection of all equivalence relations on $S$ containing both $\mathcal{L}$ and $\mathcal{R})$. The equivalence relation $\mathcal{J}$ is defined by $a \mathcal{J} b$ if and only if $S^{1} a S^{1}=S^{1} b S^{1}$. If $\mathcal{K}$ is one of Green's relations then the equivalence class containing $a$ is denoted by $K_{a}$. It is easy to check that $\mathcal{L}$ is a right congruence and that $\mathcal{R}$ is a left congruence. The $\mathcal{L}$ - and $\mathcal{R}$-classes can be ordered as follows: we write $R_{a} \leq R_{b}$ if $a S^{1} \subseteq b S^{1}$, and $L_{a} \leq L_{b}$ if $S^{1} a \subseteq S^{1} b$. The following lemma is frequently useful. We provide a proof for the sake of completeness.

Lemma 4 Let $e$ and $f$ be idempotents.
(i) $L_{e} \leq L_{f}$ implies $f e$ is an idempotent, $f e \leq f$ and $f e \mathcal{L} e$.
(ii) $R_{e} \leq R_{f}$ implies ef is an idempotent, ef $\leq f$ and ef $\mathcal{R} e$.

Proof We prove (i); the proof of (ii) is similar. By definition, $L_{e} \leq L_{f}$ means $S^{1} e \subseteq S^{1} f$. Thus $e \in S^{1} f$. It follows that $e f=e$. Observe that $(f e)^{2}=f e f e=f e$; that $f(f e)=f e$ and that $(f e) f=f e f=f e$ and so $f e \leq f$; and finally, it is immediate that fe $\mathcal{L} e$.

The $\mathcal{H}$-relation is particularly important in locating groups in semigroups. By a subgroup in a semigroup we mean a subsemigroup which is a group with respect to the induced multiplication.

Proposition 5 An $\mathcal{H}$-class in a semigroup contains an idempotent if and only if it is a group. Every subgroup in a semigroup is contained in a unique $\mathcal{H}$-class with an idempotent.

For a proof see Corollary 2.2.6 of [11].
A semigroup is said to be bisimple if it consists of exactly one $\mathcal{D}$-class; a semigroup with zero is said to be 0 -bisimple if it consists of exactly two $\mathcal{D}$-classes. A semigroup $S$ is simple if it has no proper ideals. A semigroup $S$ with zero is 0 -simple if $\{0\}$ is the only proper two-sided ideal of $S$ and $S^{2} \neq\{0\}$; the second condition is only to exclude the null semigroup of order 2 .

A non-zero idempotent $e$ of a semigroup $S$ is said to be primitive if $f \leq e$ implies $f=e$ or $f=0$. A semigroup $S$ is said to be completely simple if it is simple and contains a primitive idempotent; a semigroup is said to be completely 0 -simple if it is 0 -simple semigroup and contains a primitive idempotent.

### 1.1.4 Rees matrix semigroups

In this section, we describe an important technique for constructing semigroups. Let $S$ be a semigroup, let $I$ and $\Lambda$ be sets, and let $P$ be a $\Lambda \times I$-matrix whose entries are from $S$ and are denoted $p_{\lambda i}$ where $(\lambda, i) \in \Lambda \times I$. The semigroup $M=M(S ; I, \Lambda ; P)$ consists of all triples $I \times S \times \Lambda$ equipped with the product

$$
(i, a, \lambda)(j, b, \mu)=\left(i, a p_{\lambda j} b, \mu\right)
$$

It is easy to check that this really is an associative binary operation. We say that $M$ is a Rees matrix semigroup over $S$ with sandwich matrix $P$. If $S$ is a semigroup with zero then the elements

$$
I=\{(i, 0, \lambda):(i, \lambda) \in I \times \Lambda\}
$$

form an ideal in $M$. In this case, the Rees quotient $M / I$ is denoted $M^{0}(S ; I, \Lambda ; P)$ and is called a Rees matrix semigroup over a semigroup with zero.

Let $S$ be a monoid. Then the sandwich matrix $P$ is said to be regular if each row and each column contains an invertible element.

In 1940, David Rees found a way of constructing all completely 0 -simple and completely simple semigroups using Rees matrix semigroups. It is easy to check that a Rees matrix semigroup over a
group is completely simple, and that a Rees matrix semigroup over a 0 -group having a regular sandwich matrix is completely 0 -simple. The following result, known as the Rees Theorem, shows that the converses also hold. Observe that for a Rees matrix semigroup over a 0 -group, 'regular' means that each row and each column of the sandwich matrix contains a non-zero element.

Theorem 6 Every completely 0-simple semigroup is isomorphic to a regular Rees matrix semigroup over a 0-group; and every completely simple semigroup is isomorphic to a Rees matrix semigroup over a group.

The above theorem (proved as Theorem 3.2.3 in [11]) is the basis of all investigations into the structure of completely simple and completely 0 -simple semigroups. It is also the starting point for the main research in this thesis.

### 1.2 Regular semigroups

At the end of the last section, we described completely simple and completely 0 -simple semigroups. These are the first examples of an important class of semigroups: the regular semigroups. Full transformation monoids are also regular. Many of the deeper results of semigroup theory are either known only for regular semigroups or were first proved for regular semigroups. In this thesis, we shall show how some results originally proved for regular semigroups can be extended to some non-regular semigroups.

### 1.2.1 Basic definitions

Let $S$ be a semigroup. An element $s \in S$ is said to be regular if there exists an element $t \in S$ such that sts $=s$. The semigroup $S$ is called regular if all of its elements are regular. More generally, the regular elements in an arbitrary semigroup $S$ are denoted by $\operatorname{Reg}(S)$. An element $s^{\prime}$ is said to be an inverse of an element $s$ if
$s=s s^{\prime} s$ and $s^{\prime}=s^{\prime} s s^{\prime}$. It is clear that an element with an inverse is regular; however, the converse is also true. For if $s=s t s$, then it is easy to check that $t s t$ is an inverse of $s$. Thus regular elements are precisely the elements which have inverses. We denote the set of inverses of an element $s$ by $V(s)$.

The following contains important properties of inverses; see Theorem 2.3.4 [11] for a proof.

Proposition 1 Let $S$ be a semigroup.
(i) Let $a^{\prime} \in V(a)$. Then $a a^{\prime} \in R_{a} \cap L_{a^{\prime}}$, and $a^{\prime} a \in L_{a} \cap R_{a^{\prime}}$.
(ii) If $b$ is such that there exists $e \in R_{a} \cap L_{b} \cap E(S)$ and $f \in$ $L_{a} \cap R_{b} \cap E(S)$, then there exists $a^{\prime} \in H_{b} \cap V(a)$ such that $a a^{\prime}=e$ and $a^{\prime} a=f$.

The following results are of great importance (see Proposition 2.3.5 and Proposition 2.4.1 [11]).

Proposition 2 Let $S$ be a semigroup. Then
(i) If $a$ and $b$ are regular, then $a \mathcal{L} b$ if and only if there exists $a^{\prime} \in V(a)$ and $b^{\prime} \in V(b)$ such that $a^{\prime} a=b^{\prime} b$.
(ii) If $a$ and $b$ are regular, then $a \mathcal{R} b$ if and only if there exists $b^{\prime} \in V(b)$ such that $a a^{\prime}=b b^{\prime}$.
(iii) Let $e$ and $f$ be idempotents. Then $e \mathcal{D} f$ if and only if there exists $a \in S$ and $a^{\prime} \in V(a)$ such that $a^{\prime} a=e$ and $a a^{\prime}=f$.

It follows from Proposition 2, that in a regular semigroup each $\mathcal{L}$-class and each $\mathcal{R}$-class contains an idempotent.

In an arbitrary regular semigroup, an element will have many inverses. We say that a semigroup is inverse if each element has a unique inverse. In inverse semigroups, the unique inverse of $s$ is usually denoted by $s^{-1}$. Another way of characterising inverse semigroups is as follows; for a proof see [11] (Theorem 5.1.1).

Theorem 3 A regular semigroup is inverse if and only if its idempotents form a commutative subsemigroup.

The following result is well-known. We provide a proof for completeness.

Proposition 4 (i) The regular semigroups with a unique idempotent are precisely the groups.
(ii) The regular semigroups with zero having exactly one non-zero idempotent are the 0-groups.

Proof (i) Clearly groups are regular semigroups with unique idempotents. Let $S$ be a regular semigroup with unique idempotent $e$. Let $a \in S$. Since $E(S)$ commutes, each element $a \in S$ has a unique inverse $a^{\prime} \in S$, and $a^{\prime} a=e=a a^{\prime}$. It is evident that $e$ is an identity of $S$. Thus $S$ is a group.
(ii) It is clear that 0 -groups are regular and have exactly one non-zero idempotent. Let $S$ be a regular semigroup with zero having one non-zero idempotent. Since $E(S)=\{0, e\}$ commutes, each non-zero element $a$ has a unique inverse $a^{\prime}$, and $a^{\prime} a=e=a a^{\prime}$. Thus every element of $S \backslash\{0\}$ is $\mathcal{H}$-related to $e$. By Proposition 1.1.5, this means that $S \backslash\{0\}$ is a group. Thus $S$ is a 0-group.

The idempotents of regular semigroups enjoy some nice properties which make them easy to work with. Let $S$ be any semigroup. If $e$ and $f$ are idempotents in $S$ then the sandwich set $S(e, f)$ is defined by $S(e, f)=f V(e f) e$. Thus the sandwich set is non-empty precisely when ef is regular. It is easy to check (or see Nambooripad [25]), that if $S(e, f)$ is non-empty then

$$
h \in S(e, f) \Leftrightarrow h^{2}=h, f h e=h, \text { and } e h f=e f .
$$

The proof of the following may be found as Theorem 2.5.4 in [11].

Proposition 5 Let $s, t \in S$, where $S$ is a regular semigroup. Let $s^{\prime} \in V(s)$ and $t^{\prime} \in V(t)$ and $h \in S\left(s^{\prime} s, t t^{\prime}\right)$. Then $t^{\prime} h s^{\prime} \in V(s t)$.

### 1.2.2 Local submonoids

Local submonoids have turned out to be of great interest in the theory of regular semigroups. The following results are well-known; we prove them for the sake of completeness.

Lemma 6 Let $S$ be a semigroup.
(i) If $S$ is regular then every local submonoid of $S$ is regular.
(ii) If e $\mathcal{D} f$, where e and $f$ are idempotents, then eSe is isomorphic to $f S f$.
(iii) If $S=S e S$ then every local submonoid of $S$ is isomorphic to a local submonoid of eSe.

Proof (i) Let $e S e$ be a local submonoid. Let $a \in e S e$ and let $a^{\prime} \in V(a)$ in $S$. Then $a^{\prime \prime}=e a^{\prime} e \in e S e$ and

$$
a a^{\prime \prime} a=a e a^{\prime} e a=a a^{\prime} a=a
$$

and

$$
a^{\prime \prime} a a^{\prime \prime}=\left(e a^{\prime} e\right) a\left(e a^{\prime} e\right)=e a^{\prime} a a^{\prime} e=e a^{\prime} e=a^{\prime \prime} .
$$

Thus eSe is regular.
(ii) By Proposition 1(iii), there exists $a \in S$ and $a^{\prime} \in V(a)$ such that $a^{\prime} a=e$ and $a a^{\prime}=f$. Define $\theta: e S e \rightarrow f S f$ by $\theta(x)=a x a^{\prime}$. It is easy to check that this defines an isomorphism.
(iii) Let $f \in S$ be any idempotent. Then $f=a e b=(a e)(e b)$ for some $a, b \in S$. Put $x=a e$ and $y=e b$. Then $f=x y$. It is easy to check that $y f \in V(f x)$. Put $i=y f x$, an idempotent. Then $i \in e S e$, and $f x y f=f$. Hence $f \mathcal{D} i \in e S e$. By (ii), $f S f$ is isomorphic to $i S i \subseteq e S e$, and $i S i=i(e S e) i$.

Local submonoids provide a different way of characterising completely simple and completely 0 -simple semigroups.

Proposition 7 Let $S$ be a regular semigroup.
(i) $S$ is completely simple if and only if it is locally a group.
(ii) $S$ is completely 0-simple if and only if it is 0-simple and locally a 0-group.

Proof (i) By Theorem 3.3.3(4) of [11], $S$ completely simple is equivalent to being a regular semigroup in which every idempotent is primitive. But an idempotent $e$ is primitive in a regular semigroup if and only if $e S e$ is a group: this is because $e$ primitive means precisely that $e$ is the only idempotent in $e S e$; we have already proved that $e S e$ is regular; and the regular semigroups with a unique idempotent are the groups (Proposition 4(i)).
(ii) A semigroup is completely 0 -simple if it is regular and has a primitive idempotent. By the Rees theorem, it is easy to check that every idempotent is primitive. But in a semigroup with zero $S$, the idempotent $e$ is primitive if $E(e S e)=\{0, e\}$. But local submonoids of regular semigroups are regular. The regular semigroups with zero having exactly one non-zero idempotent are just the groups with zero (Proposition 4(ii)).

Conversely, let $S$ be a 0 -simple regular semigroup in which all local submonoids are 0-groups. Then clearly, every non-zero idempotent is primitive. Thus $S$ is completely 0 -simple.

Homomorphisms between regular semigroups are just semigroup homomorphisms. A notion intermediate between a homomorphism and an isomorphism is the following: a homomorphism $\theta: S \rightarrow T$ between regular semigroups is said to be a local isomorphism if the restriction of $\theta$ to each local submonoid of $S$ is injective. The following important result was proved in [20] as Lemma 1.3.

Lemma 8 If $\theta: S \rightarrow T$ is a local isomorphism between regular semigroups, then $\theta$ is injective when restricted to every subset of $S$ of the form $a S b$ where $a, b \in S$.

Let $S$ be an orthodox regular semigroup. Define the relation $\gamma$ on $S$ by

$$
(a, b) \in \gamma \Leftrightarrow V(a) \cap V(b) \neq \emptyset
$$

For a proof of the following see Theorem 6.2 .5 of [11] and Proposition 1.4 of [20]. Observe that locally inverse, orthodox, regular semigroups are precisely the regular semigroups with a normal band of idempotents because the locally inverse bands are precisely the normal bands.

Theorem 9 Let $S$ be an orthodox regular semigroup. Then $\gamma$ is the smallest congruence on $S$ with the property that $S / \gamma$ is inverse. The natural homomorphism $\gamma^{\natural}$ is a local isomorphism if and only if the idempotents of $S$ form a normal band.

### 1.2.3 The natural partial order

On every regular semigroup, a partial order can be defined in terms of the algebraic properties of the semigroup. Let $S$ be a regular semigroup. Define $a \leq b$ if and only if $R_{a} \leq R_{b}$ and $a=e b$ for some $e \in E\left(R_{a}\right)$. It can be proved that this really is a partial order, and it is easy to check that it coincides with the order already defined on the idempotents of $S$. It is called the natural partial order on a regular semigroup. It was introduced independently by Hartwig [8] and Nambooripad [26]. This order provides alternative characterisations of completely simple and completely 0 -simple semigroups.

Proposition 10 A regular semigroup without zero is completely simple if and only if the natural partial order is equality. A 0 simple regular semigroup is completely 0 -simple if and only if the natural partial is equality when restricted to the non-zero elements.

For a proof see Theorem 1.4 of [26]. In both these cases, the order is trivially compatible with the multiplication. The order
is also compatible with the multiplication on inverse semigroups. Nambooripad completely characterised those regular semigroups with a compatible natural partial order in the following way. We have already observed in Lemma 6(i) that every local submonoid in a regular semigroup is regular. We say that a regular semigroup is locally inverse if each local submonoid is inverse; by Theorem 3 this is equivalent to requiring the the idempotents in each local submonoid commute. The proof of the following may be found in [11] (Theorem 6.1.3).

Theorem 11 A regular semigroup has a compatible natural partial order if and only if it is locally inverse.

### 1.3 The structure of locally inverse regular semigroups

In this section, we describe the work which forms the inspiration for this thesis; it is based around the theory of regular locally inverse semigroups.

The Rees theorem describes the structure of a very special class of locally inverse regular semigroups. From this perspective, it is natural to ask to what extent the Rees theorem can be generalised to arbitrary locally inverse regular semigroups.

By Theorem 1.1.6 and Proposition 1.2.7, completely simple semigroups are precisely the regular semigroups which are locally groups, and they can be described by means of Rees matrix semigroups over groups. Thus a natural starting point for studying regular semigroups which are locally inverse is to study Rees matrix semigroups over inverse semigroups.

The first problem is that Rees matrix semigroups over regular semigroups are not, in general, regular. However, this potential drawback was overcome by McAlister. Denote by

$$
R M(S ; I, \Lambda ; P)
$$

the set of regular elements of the Rees matrix semigroup $M(S ; I, \Lambda ; P)$. Then we have the following result (Lemma 2.1(ii) [19], and Lemma 2.6 [21]).

Theorem 1 If $S$ is regular then $R M=R M(S ; I, \Lambda ; P)$ is a semigroup, it is regular, and every local submonoid of $R M$ is isomorphic to a local submonoid of $S$.

The semigroups $R M(S ; I, \Lambda ; P)$ are called regular Rees matrix semigroups.

Since local submonoids of inverse semigroups are inverse (they are regular and it is immediate that their idempotents commute), it follows that regular Rees matrix semigroups over inverse semigroups are regular locally inverse semigroups. It is now natural to wonder how general this construction is. It would be nice if every locally inverse regular semigroup was isomorphic to such a Rees matrix semigroup; this is not the case, but it turns out that the final answer is not too remote from this desirable situation.

The first step in answering this question was obtained by McAlister [19] and had its origins in an earlier paper of Allen [1]. McAlister proved what he called his 'local structure theorem'.

Theorem 2 Let $S$ be a regular semigroup, and suppose that $S=$ SeS for some idempotent $e \in S$. Then $S$ is a locally isomorphic image of a regular Rees matrix semigroup over eSe.

We have already proved in Lemma 1.2.6(iii), that every local submonoid of $S$ is isomorphic with a local submonoid of $e S e$. Thus if $e S e$ is inverse then $S$ is locally inverse. It follows that for locally inverse regular semigroups of this type we can describe them in terms of regular Rees matrix semigroups over inverse semigroups. For example, if $S$ is a bisimple locally inverse regular semigroup, then $S=S e S$ for every idempotent in $S$, and so $S$ can be described in the above way. In particular, every completely simple semigroup is bisimple and regular and, in this case, all local submonoids are
groups. However, in this case regularity is automatic because the Rees matrix semigroup is over a group. Thus every completely simple semigroup is a locally isomorphic image of a Rees matrix semigroup over a group. Thus the local structure theorem immediately implies that completely simple semigroups are locally isomorphic images of regular Rees matrix semigroups over groups. Thus the local structure theorem alone almost enables us to prove the Rees theorem: the Rees theorem converts that local isomorphism into an isomorphism. In a subsequent paper, McAlister went one better [20].

Theorem 3 Let $S$ be a regular semigroup. Then $S$ is locally inverse if and only if $S$ is a locally isomorphic image of a regular Rees matrix semigroup over an inverse semigroup.

The above theorem is the best generalisation we have of the Rees theorem to locally inverse regular semigroups.

In two subsequent papers, [21] and [22], McAlister brought the work for regular locally inverse semigroups full circle. We have seen that if $T$ is regular, $T=T e T$ and $e T e$ is inverse then $T$ is locally inverse. Clearly, every regular subsemigroup of $T$ will also be locally inverse. It is natural to wonder whether every locally inverse regular semigroup arises in this way. If $S$ is a regular subsemigroup of the regular semigroup $T$, it is said to be a quasi-ideal if $S T S=S$.

Theorem 4 A regular semigroup $S$ is locally inverse if and only if it can be embedded as a quasi-ideal in a regular semigroup $T$ such that $T=T e T$ and $e T e$ is inverse.

Theorem 3 and Theorem 4 are the starting points for this thesis. Indeed, Chapter 2 is devoted to generalising Theorem 3 and Chapter 3 is devoted to generalising Theorem 4.

### 1.4 Category theory

We need only a few definitions from category theory.

A partial binary operation on a set $C$ is a partial function from $C \times C$ to $C$, denoted by $(x, y) \mapsto x \cdot y$. We shall write $\exists x \cdot y$ to mean that the product $x \cdot y$ is defined. An element $e \in C$ is said to be an identity if $\exists e \cdot x$ implies $e \cdot x=x$ and $\exists x \cdot e$ implies $x \cdot e=x$. The set of identities of $C$ is denoted by $C_{o}$. The pair ( $C, \cdot$ ) is said to be a category if the following three axioms hold:
(C1) $x \cdot(y \cdot z)$ exists if and only if $(x \cdot y) \cdot z$ exists, in which case they are equal.
(C2) $x \cdot(y \cdot z)$ exists if and only if $x \cdot y$ and $y \cdot z$ exist.
(C3) For each $x \in C$ there exist identities $e, f \in C_{o}$ such that $\exists x \cdot e$ and $\exists f \cdot x$.

From axiom (C3), it follows that the identities $e$ and $f$ are uniquely determined by $x$. We write $e=\mathbf{d}(x)$ and $f=\mathbf{r}(x)$, where $\mathbf{d}(x)$ is the domain identity and $\mathbf{r}(x)$ is the range identity. Observe that $\exists x \cdot y$ if and only if $\mathbf{d}(x)=\mathbf{r}(y)$.

The elements of a category are called arrows. If $e, f \in C_{o}$ then

$$
\operatorname{hom}(e, f)=\{x \in C: \mathbf{d}(x)=e \text { and } \mathbf{r}(x)=f\},
$$

the set of arrows from e to $f$.
A subset $C^{\prime}$ of of a category $C$ is said to be a subcategory if $x \in C^{\prime}$ implies $\mathbf{d}(x), \mathbf{r}(x) \in C^{\prime}$ and $C^{\prime}$ is closed under the partial product. A subcategory $C^{\prime}$ of $C$ is said to be full if $e, f \in C_{o}^{\prime}$ and $x \in \operatorname{hom}(e, f)$ in $C$ implies $x \in C^{\prime}$. Thus full subcategories are determined by the identities they contain.

## Chapter 2

## Rees matrix covers for semigroups with locally commuting idempotents

McAlister Jroved that every regular locally inverse semigroup can be covered by a regular Rees matrix semigroup over an inverse semigroup by means of a homomorphism which is locally an isomorphism [20]. In this chapter, we generalise this result to the class of semigroups with local units whose local submonoids have commuting idempotents and possessing what we term a 'McAlister sandwich function'.

### 2.1 Introduction

McAlister's work was solely concerned with regular semigroups. However, the problems he solved are of interest for non-regular semigroups as well. We shall now introduce the generalisation of locally inverse regular semigroups with which this thesis will be concerned.

In a regular semigroup $S$ every element $a \in S$ has a left and right identity: for example, if $a^{\prime}$ is an inverse of $a$ then $a a^{\prime}$ is a left identity and $a^{\prime} a$ is a right identity. More generally, we say that
a semigroup has local units if every element has a left and right identity which is an idempotent. This is a much more general class than the regular semigroups: all monoids have local units, and if $S$ is an arbitrary semigroup then $E(S) S E(S)$ has local units.

A locally inverse regular semigroup is one in which the local submonoids are inverse. However, local submonoids of regular semigroups are always regular, so an equivalent formulation is: the idempotents in every local submonoid commute. A semigroup in which the idempotents in each local submonoid commute will be said to have locally commuting idempotents.

Our aim in this, and the next, chapter will be to generalise Theorem 1.3.3 and Theorem 1.3.4 to semigroups with local units which have locally commuting idempotents.

In the non-regular case, we have to be careful by what we mean by a 'local isomorphism'. In Lemma 1.2.8, we recalled that local isomorphisms are in fact injective on every subset of the form $a S b$ where $a, b \in S$. For semigroups which are not necessarily segular, the concept of a strict local isomorphism was devised in [18]; this is a surjective homomorphism $\theta: S \rightarrow T$ which is injective on every subset of the form $a S b$ where $a$ and $b$ are any elements such that $a \in S a$ and $b \in b S$. Evidently, local isomorphisms between regular semigroups are equivalent to strict local isomorphisms. It is easy to check that a strict local isomorphism between semigroups with local units is the same thing as a homomorphism which is injective on all subsets of the form $e S f$ where $e$ and $f$ are idempotents: this is the form of the definition of 'strict local isomorphism' we shall use in this thesis.

We now need to explain how we came to see semigroups with local units as being a suitable class for which generalisations of results for regular semigroups could be found.

The first step in generalising McAlister's work to non-regular semigroups was obtained by Márki and Steinfeld [18]. They generalised the local structure theorem (Theorem 1.3.2) in the following way.

Theorem 1 Let $S$ be an arbitrary semigroup such that $S=S e S$. Then $S$ is a homomorphic image of a Rees matrix semigroup over eSe which is a strict local isomorphism.

The local structure theorem was also generalised in the regular case by Lawson [12] with his introduction of the notion of an 'enlargement'. Let $S$ be a regular subsemigroup of a regular semigroup $T$. Then $T$ is said to be an enlargement of $S$ if $S=S T S$ and $T=T S T$. Observe that if $T=T e T$ then $T$ is an enlargement of $e T e$.

Theorem 2 Let $T$ be a regular semigroup, and an enlargement of the regular subsemigroup $S$. Then $T$ is a locally isomorphic image of a regular Rees matrix semigroup over the subsemigroup $S$.

Subsequently, Lawson and Márki obtained the following joint generalisation of Theorems 1 and 2. We need the following definition. Let $\theta: A \rightarrow B$ be a surjective homomorphism between semigroups. We say that idempotents (resp. regular elements) lift along $\theta$ if for every idempotent $e \in B$ (resp. for every regular element $b \in B$ ) there exists an idempotent $e^{\prime} \in A$ (resp. a regular element $a \in A$ ) such that $\theta\left(e^{\prime}\right)=e($ resp. $\theta(a)=b)$.

Theorem 3 Let $T$ be an enlargement of $S$ and suppose in addition that $S^{2}=S$. Then $T$ is a strict local isomorphic image of a Rees matrix semigroup over $S$; furthermore, idempotents and regular elements can be lifted along this local isomorphism.

Semigroups in which $S^{2}=S$ are said to be factorisable. Clearly, semigroups with local units are factorisable. The above theorem was the starting point for our work. It suggested that McAlister's work could be generalised to classes of non-regular semigroups. We would have liked to generalise McAlister's work to factorisable semigroups but we do not know how to do this. Instead, we decided to concentrate on the more tractable problem of semigroups with local units.

Semigroups with local units and Rees matrix semigroups are also the subject of the Morita theory developed by Talwar in [27]; and this theory is extended to factorisable semigroups in [28]. We believe that the results we obtain in this thesis will ultimately turn out to be contributions to the Morita theory of semigroups. We have not pursued this connection here.

### 2.2 Properties of regular elements

Let $S$ be an arbitrary semigroup. The set $\operatorname{Reg}(S)$ of regular elements of $S$ will play an important role in our work, although it is important to stress that it need not be a subsemigroup.

In Section 1.2.3, we recalled how Nambooripad [26] defined a natural partial order on any regular semigroup; and that independently Hartwig [8] showed that the regular elements of any semigroup could be naturally ordered. We use Nambooripad's form of the definition, but follow Hartwig in applying it to the regular elements of any semigroup. Specifically, let $S$ be an arbitrary semigroup. A relation $\leq$ is defined on the set $\operatorname{Reg}(S)$ as follows. Let $s, t \in S$. Then $s \leq t$ if and only if $R_{s} \leq R_{t}$ and $s=f t$ for some $f \in E\left(R_{s}\right)$. We include the proof of the following result for completeness.

Proposition 1 Let $S$ be an arbitrary semigroup. Then the relation $\leq$ is a partial order on the set of regular elements of $S$.

Proof Let $s \in \operatorname{Reg}(S)$. Then by assumption, there exists $s^{\prime} \in V(s)$. Thus $s=\left(s s^{\prime}\right) s$. Hence $R_{s}=R_{s}, s=\left(s s^{\prime}\right) s$ and $s s^{\prime} \in E\left(R_{s}\right)$. It follows that $s \leq s$, and so $\leq$ is reflexive.

Suppose that $s \leq t$ and $t \leq s$. Then $R_{s}=R_{t}$ and there exist idempotents $e, f \in E\left(R_{s}\right)=E\left(R_{t}\right)$ such that $s=f t$ and $t=e s$. But $f \in R_{s}=R_{t}$ implies that $f t=t$. Thus $s=t$, and $\leq$ is antisymmetric.

Finally, suppose that $s \leq t$ and $t \leq v$. Then $R_{s} \leq R_{t}$ and $R_{t} \leq R_{v}$ and there are idempotents $f \in E\left(R_{s}\right)$ and $e \in E\left(R_{t}\right)$ such
that $s=f t$ and $t=e v$. Clearly, $R_{s} \leq R_{v}$ and $s=(f e) v$. But by Lemma 1.1.4(ii), $R_{f} \leq R_{e}$ and so $f e \in E\left(R_{s}\right)$. Hence $s \leq v$, and $\leq$ is transitive.

The relation $\leq$ is called the Hartwig-Nambooripad order [8], [25] or the natural partial order defined on the regular elements. Observe that if $e$ and $f$ are idempotents then $e \leq f$ precisely when $e=e f=f e$, which is the usual order on the idempotents of a semigroup. There are a number of alternative ways of characterising this order; the proofs of the following can all be deduced from [26]. Again, we include proofs for the sake of completeness.

Proposition 2 Let $S$ be a semigroup and let $s, t \in \operatorname{Reg}(S)$. Then the following are equivalent:
(i) $s \leq t$.
(ii) For each $f \in E\left(R_{t}\right)$ there exists $e \in E\left(R_{s}\right)$ such that $e \leq f$ and $s=e t$.
(iii) For each $f^{\prime} \in E\left(L_{t}\right)$ there exists $e^{\prime} \in E\left(L_{s}\right)$ such that $e^{\prime} \leq f^{\prime}$ and $s=t e^{\prime}$.
(iv) There exist idempotents $e$ and $f$ such that $s=e t=t f$.

Proof (i) $\Rightarrow$ (ii). Let $s \leq t$. Then by definition, $R_{s} \leq R_{t}$ and $s=i t$ for some $i \in E\left(R_{s}\right)$. Let $f \in E\left(R_{t}\right)$. Then $R_{i}=R_{s} \leq R_{t}=R_{f}$, and so $R_{i} \leq R_{f}$. In particular, $f i=i$. Put $e=i f$. Then by Lemma 1.1.4(ii), we have that $e^{2}=e, e \leq f$ and $i \mathcal{R} e$. It follows that $e \in E\left(R_{s}\right)$. Finally, et $=i f t=i t=s$.
(ii) $\Rightarrow$ (iii). Let $f^{\prime} \in E\left(L_{t}\right)$. By Theorem 2.3.4(2) of [11], choose $t^{\prime} \in V(t) \cap R_{f^{\prime}}$. Then $t^{\prime} t=f^{\prime}$ and $t t^{\prime} \in E\left(R_{t}\right)$. Thus by (ii), there exists $e \in E\left(R_{s}\right)$ such that $e \leq t t^{\prime}$ and $s=e t$. Put $e^{\prime}=t^{\prime} e t$. Then $s=e t=t\left(t^{\prime} e t\right)=t e^{\prime}$. It is easy to check that $e^{\prime} \leq f^{\prime}$. Also, $t e^{\prime}=t t^{\prime} e t=e t=s$ and $t^{\prime} s=t^{\prime} e t=e^{\prime}$, so that $s \mathcal{L} e^{\prime}$. Hence result.
(iii) $\Rightarrow$ (iv). Let $f^{\prime} \in E\left(L_{t}\right), e^{\prime} \in E\left(L_{s}\right)$, where $e^{\prime} \leq f^{\prime}$ and $s=t e^{\prime}$. Since $f^{\prime} \in E\left(L_{t}\right)$ there exists $t^{\prime} \in V(t)$ such that $f^{\prime}=t^{\prime} t$.

Thus $s=t e^{\prime}=t e^{\prime} f^{\prime}=\left(t e^{\prime} t^{\prime}\right) t$. But

$$
\left(t e^{\prime} t^{\prime}\right)^{2}=t e^{\prime}\left(t^{\prime} t\right) e^{\prime} t^{\prime}=t e^{\prime} f^{\prime} e^{\prime} t^{\prime}=t e^{\prime} t^{\prime},
$$

and so $t e^{\prime} t^{\prime}$ is an idempotent.
(iv) $\Rightarrow$ (i). Let $s=e t=t f$ where $e$ and $f$ are idempotents. From $s=t f$ we have that $R_{s} \leq R_{t}$. Let $s^{\prime} \in V(s)$. From $s=e t$ we obtain $e s=s$ and so $e s s^{\prime}=s s^{\prime}$. Put $i=s s^{\prime} e$. Then by Lemma 1.1.4(ii), we have that $i^{2}=i, s=i t$ and $i \mathcal{R} s$.

We shall now derive some properties of the natural partial order on semigroups with locally commuting idempotents.

Proposition 3 Let $S$ be a semigroup with locally commuting idempotents.
(i) $|S(e, f)| \leq 1$ for all $e, f \in E(S)$.
(ii) If $x, y, u, v \in \operatorname{Reg}(S)$ and $x \leq u, y \leq v$ and $x y$, $u v \in \operatorname{Reg}(S)$, then $x y \leq u v$.
(iii) If $x, y \in \operatorname{Reg}(S)$ and $e$ is an idempotent such that $x e=x$ and $e y=y$ then $x y$ is regular.

Proof (i) Let $h, k \in S(e, f)$. We show that $h=k$. We have that

$$
f h e=h, e h f=e f \text { and } f k e=k, e k f=e f .
$$

It is easy to check that eh, ek, $h f$, and $k f$ are all idempotents. Furthermore,

$$
e h, e k \in E(e S e) \text { and } h f, k f \in E(f S f) .
$$

Thus ehek $=e k e h$ and $h f k f=k f h f$ since the idempotents in every local submonoid commute. Hence ehk $=e k h$ and $h k f=k h f$. But

$$
e h k=e h f k e=e f k e=e k .
$$

By the same token, $e k h=e h$. Thus $e k=e h$. Similarly, $h f=k f$. Now

$$
k=f k e=f k e k=f k e h=k h
$$

and

$$
h=f h e=h f h e=k f h e=k h .
$$

Thus $k=h$.
(ii) Let $u^{\prime} \in V(u)$ and $v^{\prime} \in V(v)$. By Proposition 2(ii) and (iii), there exist idempotents $e$ and $f$ such that

$$
e \leq u^{\prime} u, e \mathcal{L} x, x=u e \text { and } f \leq v v^{\prime}, f \mathcal{R} y, y=f v
$$

Thus $x y=u e f v$. By assumption, $x y$ is regular. But ey $\mathcal{L} x y$ and so ey is regular, and ey $\mathcal{R}$ ef and so ef is regular. Hence $S(e, f)$ is non-empty. Let $h \in S(e, f)$. Then $f h e=h$ and ehf $=e f$, and so $x y=u e f v=u e h f v=u(e h)(h f) v$. Observe that $h e=h$ and $h u^{\prime} u=h$ and so

$$
u^{\prime} u h \mathcal{L} h, u^{\prime} u h \leq u^{\prime} u \text { and } \text { eh } \mathcal{L} h, e h \leq e \leq u^{\prime} u \text {. }
$$

Thus $u^{\prime} u h \mathcal{L} e h$ and $u^{\prime} u h, e h \leq u^{\prime} u$. But $E\left(u^{\prime} u S u^{\prime} u\right)$ is a commutative semigroup, and so $u^{\prime} u h=e h$. Similarly, $h v v^{\prime}=h f$. Hence

$$
x y=u(e h)(h f) v=u\left(u^{\prime} u h\right)\left(h v v^{\prime}\right) v=u h v .
$$

Now

$$
h v=\left(h v v^{\prime}\right)\left(v v^{\prime}\right) v=\left(v v^{\prime}\right)\left(h v v^{\prime}\right) v=v\left(v^{\prime} h v\right) .
$$

Thus

$$
x y=u h v=u v\left(v^{\prime} h v\right)
$$

where $v^{\prime} h v$ is an idempotent. Similarly,

$$
x y=u h v=\left(u h u^{\prime}\right) u v
$$

where $u h u^{\prime}$ is an idempotent. Hence $x y \leq u v$ by Proposition 2(iv).
(iii) Let $x^{\prime} \in V(x)$. Then $x^{\prime} x e=x^{\prime} x$. Thus by Lemma 1.1.4(i), $e x^{\prime} x$ is an idempotent and $x^{\prime} x \mathcal{L} e x^{\prime} x \leq e$. Hence $x \mathcal{L} e x^{\prime} x$. By Proposition 1.2.1, there is $x^{\prime \prime} \in V(x)$ such that $x^{\prime \prime} x=e x^{\prime} x$. Thus
we have proved that if $x e=x$ then there is $x^{\prime} \in V(x)$ such that $x^{\prime} x \leq e$. Similarly, if $e y=y$ then there exists $y^{\prime} \in V(y)$ such that $y y^{\prime} \leq e$. With these choices of inverses we calculate

$$
x y\left(y^{\prime} x^{\prime}\right) x y=x\left(y y^{\prime}\right)\left(x^{\prime} x\right) y=x\left(x^{\prime} x\right)\left(y y^{\prime}\right) y=x y
$$

since $x^{\prime} x, y y^{\prime} \leq e$ and so they commute. Thus $x y$ is regular.

Property (iii) above will be used repeatedly in what follows to show that certain products of regular elements are again regular.

The following lemma, and its left-right dual, will be needed in Section 2.4.

Lemma 4 Let $x, y \in \operatorname{Reg}(S)$ such that $x \leq y$ and ey $=y$ for some idempotent $e$. Then there exists an idempotent $f \leq e$ such that $x=f y$.

Proof Let $y^{\prime} \in V(y)$. Then eyy ${ }^{\prime}=y y^{\prime}$. By Lemma 1.1.4(ii), we have that $y y^{\prime} e \in E(S), y y^{\prime} e \leq e$ and $y y^{\prime} e \mathcal{R} y$. Thus by Proposition 2(ii), there exists an idempotent $f \leq y y^{\prime} e$ such that $x=f y$. Clearly, $f \leq e$.

### 2.3 An associated semigroup

Let $S$ be a semigroup with local units having locally commuting idempotents. We may associate a category with $S$ as follows. Put

$$
C(S)=\{(e, x, f) \in E(S) \times S \times E(S): e x f=x\}
$$

with product given by $(e, x, f)(f, y, j)=(e, x y, j)$ and undefined in all other cases.

Our aim is to convert $C(S)$ into a semigroup with local units with a normal band of idempotents. To do this, we need to introduce a major assumption on the structure of the semigroup $S$.

A function $p: E(S) \times E(S) \rightarrow \operatorname{Reg}(S)$, where we write $p_{u, v}=$ $p(u, v)$, is called a McAlister sandwich function if it satisfies the following three conditions:
(M1) $p_{u, v} \in u S v$ and $p_{u, u}=u$.
(M2) $p_{u, v} \in V\left(p_{v, u}\right)$.
(M3) $p_{u, v} p_{v, f} \leq p_{u, f}$.
To see that condition (M3) makes sense, we have to show that the product $p_{u, v} p_{v, f}$ is always regular. To this end, observe that by condition (M2), both $p_{u, v}$ and $p_{v, f}$ are regular; by condition (M1), we have that $p_{u, v} v=p_{u, v}$ and $v p_{v, f}=p_{v, f}$; thus the regularity of $p_{u, v} p_{v, f}$ follows from Proposition 2.3(iii). Observe that the above argument implies that any product of the form

$$
p_{a, b} p_{b, c} p_{c, d} \ldots
$$

is regular.
All regular locally inverse semigroups have McAlister sandwich functions by Lemma 2.2 of [20]. In Sections 2.6 and 2.7, we shall discuss ways of constructing such sandwich functions on non-regular semigroups.

Proposition 1 Let $S$ be a semigroup with local units with locally commuting idempotents equipped with a McAlister sandwich function. Define a semigroup multiplication on $C(S)$ which extends the category product by

$$
(e, x, f) \cdot(i, y, j)=\left(e, x p_{f, i} y, j\right)
$$

Then $(C(S), \cdot)$ is a semigroup with local units whose idempotents form a normal band. If $S$ is regular then $(C(S), \cdot)$ is regular.

Proof It is evident that $(C(S), \cdot)$ is a semigroup with local units. We begin by locating the idempotents. Observe that $(e, x, f)^{2}=$ $(e, x, f)$ if and only if $x p_{f, e} x=x$. Thus, in particular, $x$ is regular.

Suppose that $(e, x, f)$ is an idempotent. By condition (M2), $p_{f, e} p_{e, f} p_{f, e}=p_{f, e}$. Thus $x=x p_{f, e} p_{e, f} p_{f, e} x$. Now $x p_{f, e}, p_{f, e} x \in$ $E(S)$, and $x p_{f, e} \leq e$ and $p_{f, e} x \leq f$ using condition (M1). The
product $x p_{f, e} p_{e, f}$ is regular by Proposition 2.3(iii) using condition (M1). Thus by Proposition 2.3(ii) we have that

$$
x p_{f, e} p_{e, f} \leq e p_{e, f}=p_{e, f} .
$$

By Proposition 2.3(iii) and condition (M1) the product $x p_{f, e} p_{e, f} p_{f, e} x$ is regular and so $x p_{f, e} p_{e, f} p_{f, e} x \leq p_{e, f} f=p_{e, f}$ by Proposition 2.3(ii). Hence $x \leq p_{e, f}$.

Conversely, suppose that $x$ is regular and $x \leq p_{e, f}$. Then $x=f^{\prime} p_{e, f}=p_{e, f} e^{\prime}$ for some idempotents $e^{\prime}, f^{\prime} \in S$ by Proposition 2.2(iv). Thus

$$
x p_{f, e} x=f^{\prime} p_{e, f} p_{f, e} p_{e, f} e^{\prime}=f^{\prime} p_{e, f} e^{\prime}=x .
$$

We have therefore proved that

$$
E(C(S), \cdot)=\left\{(e, x, f) \in C(S): x \in \operatorname{Reg}(S) \text { and } x \leq p_{e, f}\right\} .
$$

We now show that the idempotents form a band. Let $(e, x, f)$ and ( $k, y, l$ ) be idempotents. Then by the result above

$$
x \leq p_{e, f} \text { and } y \leq p_{k, l} .
$$

By definition $(e, x, f) \cdot(k, y, l)=\left(e, x p_{f, k} y, l\right)$. By Proposition 2.3(iii) and condition (M1), the product $x p_{f, k}$ is regular as is $p_{e, f} p_{f, k}$. Thus by Proposition 2.3(ii), $x p_{f, k} \leq p_{e, f} p_{f, k}$. Similarly, $x p_{f, k} y$ and $p_{e, f} p_{f, k} p_{k, l}$ are both regular and so by Proposition 2.3(ii) $x p_{f, k} y \leq$ $p_{e, f} p_{f, k} p_{k, l}$. But by two applications of condition (M3), we have that $p_{e, f} p_{f, k} p_{k, l} \leq p_{e, l}$. Hence $\left(e, x p_{f, k} y, l\right)$ is an idempotent.

Finally, to show that the band is normal, we check that the idempotents in the local submonoids commute (using the fact that a band is normal precisely when it is locally inverse; see [11], page 141, Exercise 18). Observe that if $(i, w, j) \leq(e, z, f)$ then $i=$ $e$ and $j=f$. Let $(e, z, f)$ be an idempotent and let $(e, x, f),(e, y, f) \leq$ $(e, z, f)$ be idempotents. We prove that

$$
(e, x, f) \cdot(e, y, f)=(e, y, f) \cdot(e, x, f) .
$$

By definition,

$$
(e, x, f) \cdot(e, y, f)=\left(e, x p_{f, e} y, f\right) \text { and }(e, y, f) \cdot(e, x, f)=\left(e, y p_{f, e} x, f\right) \text {. }
$$

Now $y=y p_{f, e} z$ and so

$$
x p_{f, e} y=x p_{f, e} y p_{f, e} z .
$$

But $x p_{f, e}$ and $y p_{f, e}$ are idempotents, and using condition (M1) we have that $x p_{f, e}, y p_{f, e} \leq e$. Thus $x p_{f, e} y=y p_{f, e} x p_{f, e} z$. But $x=$ $x p_{f, e} z$ thus $x p_{f, e} y=y p_{f, e} x$. Hence $(e, x, f)(e, y, f)=(e, y, f)(e, x, f)$.

The fact that $S$ regular implies $(C(S), \cdot)$ is regular is straightforward.

We shall denote the semigroup $(C(S), \cdot)$ by $C(S)^{p}$.

### 2.4 A semigroup with commuting idempotents

The semigroup $C(S)^{p}$ has a normal band of idempotents. In this section, we shall show that we can define a congruence $\delta$ on this semigroup in such a way that $C(S)^{p} / \delta$ has commuting idempotents; in addition, the natural homomorphism $\delta^{\natural}$ will be a strict local isomorphism. In the case of regular semigroups, this result follows from Theorem 1.2.9. Recall that on an orthodox regular semigroup $T$ the minimum inverse congruence $\gamma$ on $T$ can be defined by

$$
(a, b) \in \gamma \Leftrightarrow V(a) \cap V(b) \neq \emptyset
$$

As a first step, we characterise $\gamma$ on the semigroup $C(S)^{p}$ in the case where $S$ is regular (and therefore locally inverse).

Proposition 1 Let $S$ be a regular locally inverse semigroup, and let $(e, x, f),(i, y, j) \in C(S)^{p}$. Then $(e, x, f) \gamma(i, y, j)$ if and only if $x=p_{e, i} y p_{j, f}$ and $y=p_{i, e} x p_{f, j}$.

Proof We shall use Proposition 3(iii) repeatedly throughout this proof.

Suppose first that $(e, x, f) \gamma(i, y, j)$. Let $(a, b, c) \in V(e, x, f) \cap$ $V(i, y, j)$. Then

$$
x=x p_{f, a} b p_{c, e} x \text { and } y=y p_{j, a} b p_{c, i} y
$$

and

$$
b=b p_{c, e} x p_{f, a} b \text { and } b=b p_{c, i} y p_{j, a} b .
$$

Also, because an element multiplied by an inverse is an idempotent, and using the characterisation of idempotents in Proposition 2.3.1, we have that

$$
x p_{f, a} b \leq p_{e, c}, b p_{c, e} x \leq p_{a, f}, y p_{j, a} b \leq p_{i, c}, \text { and } b p_{c, i} y \leq p_{a, j} .
$$

By assumption,

$$
b p_{c, e} x p_{f, a} b=b p_{c, i} y p_{j, a} b
$$

Thus

$$
x p_{f, a}\left(b p_{c, e} x p_{f, a} b\right) p_{c, e} x=x p_{f, a}\left(b p_{c, i} y p_{j, a} b\right) p_{c, e} x .
$$

Now

$$
x p_{f, a}\left(b p_{c, e} x p_{f, a} b\right) p_{c, e} x=x p_{f, a} b p_{c, e} x=x .
$$

Thus

$$
x=\left(x p_{f, a} b\right) p_{c, i} y p_{j, a}\left(b p_{c, e} x\right) .
$$

It follows that

$$
x \leq p_{e, c} p_{c, i} y p_{j, a} p_{a, f} \leq p_{e, i} y p_{j, f} .
$$

Now

$$
p_{e, i} y p_{j, f}=\left(p_{e, i} y p_{j, a}\right) b\left(p_{c, i} y p_{j, f}\right)=\left(p_{e, i} y p_{j, a}\right) b p_{c, e} x p_{f, a} b\left(p_{c, i} y p_{j, f}\right)
$$

which is equal to

$$
p_{e, i}\left(y p_{j, a} b\right) p_{c, e} x p_{f, a}\left(b p_{c, i} y\right) p_{j, f} \leq\left(p_{e, i} p_{i, c} p_{c, e}\right) x\left(p_{f, a} p_{a, j} p_{j, f}\right)
$$

and

$$
\left(p_{e, i} p_{i, c} p_{c, e}\right) x\left(p_{f, a} p_{a, j} p_{j, f}\right) \leq p_{e, e} x p_{f, f}=x .
$$

Hence $x=p_{e, i} y p_{j, f}$. We may similarly show that $y=p_{i, e} x p_{f, j}$.
To prove the converse, suppose that $x=p_{e, i} y p_{j, f}$ and $y=$ $p_{i, e} x p_{f, j}$. We shall show that $V(e, x, f) \cap V(i, y, j) \neq \emptyset$. Observe that

$$
y=p_{i, e} x p_{f, j}=\left(p_{i, e} p_{e, i}\right) y\left(p_{j, f} p_{f, j}\right)
$$

and, similarly,

$$
x=\left(p_{e, i} p_{i, e}\right) x\left(p_{f, j} p_{j, f}\right) .
$$

Now $x \in e S f$ implies that there is $x^{\prime} \in V(x) \cap f S e$. Thus $\left(f, x^{\prime}, e\right) \in$ $C(S)$. Next observe that

$$
y\left(p_{j, f} x^{\prime} p_{e, i}\right) y=\left(p_{i, e} x p_{f, j}\right) p_{j, f} x^{\prime} p_{e, i}\left(p_{i, e} x p_{f, j}\right)=p_{i, e} x x^{\prime} x p_{f, j}=y
$$

and

$$
\left(p_{j, f} x^{\prime} p_{e, i}\right) y\left(p_{j, f} x^{\prime} p_{e, i}\right)=p_{j, f} x^{\prime} p_{e, i} p_{i, e} x p_{f, j} p_{j, f} x^{\prime} p_{e, i}=p_{j, f} x^{\prime} p_{e, i} .
$$

Thus $p_{j, f} x^{\prime} p_{e, i} \in V(y)$. It is now easy to check that

$$
\left(f, x^{\prime}, e\right) \in V(e, x, f) \cap V(i, y, j) .
$$

Thus $V(e, x, f) \cap V(i, y, j) \neq \emptyset$.

Now let $S$ be a semigroup with local units with locally commuting idempotents, equipped with a McAlister sandwich function. Motivated by Proposition 1, define the relation $\delta$ on the semigroup $C(S)^{p}$ by

$$
(e, x, f) \delta(i, y, j) \Leftrightarrow x=p_{e, i} y p_{j, f} \text { and } y=p_{i, e} x p_{f, j} .
$$

Theorem 2 The relation $\delta$ is an idempotent pure congruence on the semigroup $C(S)^{p}$, and the idempotents in the quotient semigroup $C(S)^{p} / \delta$ commute. Furthermore, $\delta^{\natural}$ is a strict local isomorphism.

Proof The proof is long and we have to be careful to manipulate regular elements correctly.

## 1. $\delta$ is an equivalence relation

Both reflexivity and symmetry are straightforward to check. We prove transitivity explicitly. Let

$$
(e, x, f) \delta(i, y, j) \text { and }(i, y, j) \delta(k, z, l)
$$

We prove that

$$
(e, x, f) \delta(k, z, l)
$$

By definition,

$$
x=p_{e, i} y p_{j, f} \text { and } y=p_{i, e} x p_{f, j}
$$

and

$$
y=p_{i, k} z p_{l, j} \text { and } z=p_{k, i} y p_{j, l}
$$

Now

$$
x=p_{e, i} y p_{j, f}=p_{e, i}\left(p_{i, k} z p_{l, j}\right) p_{j, f} .
$$

By condition (M3), we have that

$$
p_{e, i} p_{i, k} \leq p_{e, k} \text { and } p_{l, j} p_{j, f} \leq p_{l, f}
$$

Thus by Lemma 2.2 .4 and its dual, there are idempotents $\alpha \leq e$ and $\beta \leq f$ such that

$$
p_{e, i} p_{i, k}=\alpha p_{e, k} \text { and } p_{l, j} p_{j, f}=p_{l, f} \beta
$$

Hence
$x=\alpha p_{e, k} z p_{l, f} \beta=\alpha p_{e, k}\left(p_{k, i} y p_{j, l}\right) p_{l, f} \beta=\alpha p_{e, k} p_{k, i}\left(p_{i, e} x p_{f, j}\right) p_{j, l} p_{l, f} \beta$.
In particular, $\alpha x=x=x \beta$. Now

$$
p_{e, k} p_{k, i} p_{i, e} \leq p_{e, e}=e
$$

and $\alpha \leq e$. But $E(e S e)$ is a semilattice. Thus

$$
\alpha\left(p_{e, k} p_{k, i} p_{i, e}\right)=\left(p_{e, k} p_{k, i} p_{i, e}\right) \alpha
$$

Similarly,

$$
\beta\left(p_{f, j} p_{j, l} p_{l, f}\right)=\left(p_{f, j} p_{j, l} p_{l, f}\right) \beta
$$

Hence

$$
x=p_{e, k} p_{k, i} p_{i, e} \alpha x \beta p_{f, j} p_{j, l} p_{l, f}=p_{e, k} p_{k, i}\left(p_{i, e} x p_{f, j}\right) p_{j, l} p_{l, f}
$$

and

$$
p_{e, k} p_{k, i}\left(p_{i, e} x p_{f, j}\right) p_{j, l} p_{l, f}=p_{e, k}\left(p_{k, i} y p_{j, l}\right) p_{l, f}=p_{e, k} z p_{l, f} .
$$

We may show, in a similar way, that $z=p_{k, e} x p_{f, l}$. Hence

$$
(e, x, f) \delta(k, z, l),
$$

as required.
2. $\delta$ is a congruence

We prove that $\delta$ is left compatible with the multiplication. The proof that it is right compatible is similar. Let $(e, x, f) \delta(i, y, j)$ and let ( $a, b, c$ ) be arbitrary. Then

$$
(a, b, c)(e, x, f)=\left(a, b p_{c, e} x, f\right) \text { and }(a, b, c)(i, y, j)=\left(a, b p_{c, i} y, j\right) .
$$

We prove that

$$
\left(a, b p_{c, e} x, f\right) \delta\left(a, b p_{c, i} y, j\right) .
$$

To do this, we need to show that

$$
b p_{c, e} x=b p_{c, i} y p_{j, f} \text { and } b p_{c, i} y=b p_{c, e} x p_{f, j} .
$$

We shall prove the former equality explicitly; the latter equality is established in a similar way. By assumption,

$$
x=p_{e, i} y p_{j, f} \text { and } y=p_{i, e} x p_{f, j} .
$$

Now

$$
p_{c, e} x p_{f, j}=p_{c, e}\left(p_{e, i} y p_{j, f}\right) p_{f, j}=p_{c, e} p_{e, i} p_{i, e} x p_{f, j} p_{j, f} p_{f, j} .
$$

Now $p_{c, e} p_{e, i}=\gamma p_{c, i}$ for some idempotent $\gamma \leq c$, and $p_{c, i} p_{i, e}=\alpha p_{c, e}$ for some idempotent $\alpha \leq c$. Thus

$$
p_{c, e} x p_{f, j}=\gamma \alpha p_{c, e} x p_{f, j} p_{j, f} p_{f, j} .
$$

Since $\alpha, \gamma \in c S c$ we have that $\gamma \alpha=\alpha \gamma$. Hence $\alpha\left(p_{c, e} x p_{f, j}\right)=$ $p_{c, e} x p_{f, j}$. We now have that

$$
p_{c, i} y p_{j, f}=p_{c, i} p_{i, e} x p_{f, j} p_{j, f}=\alpha\left(p_{c, e} x p_{f, j}\right) p_{j, f}=p_{c, e} x p_{f, j} p_{j, f} .
$$

However

$$
x=p_{e, i} y p_{j, f}
$$

and so by condition (M2) we have that $x p_{f, j} p_{j, f}=x$. Thus

$$
p_{c, i} y p_{j, f}=p_{c, e} x .
$$

It follows that

$$
b p_{c, e} x=b p_{c, i} y p_{j, f}
$$

as required.

## 3. Idempotents in $C(S)^{p} / \delta$ commute

First of all, we characterise the idempotents in $C(S)^{p} / \delta$. Suppose that $\delta(e, x, f)$ is an idempotent in $C(S)^{p} / \delta$. Then

$$
\left(e, x p_{f, e} x, f\right) \delta(e, x, f)
$$

Thus

$$
x p_{f, e} x=p_{e, e} x p_{f, f}=x
$$

Hence $(e, x, f)$ is an idempotent in $C(S)^{p}$. Thus $\delta(e, x, f)$ is an idempotent in $C(S)^{p} / \delta$ if, and only if, $(e, x, f)$ is an idempotent in $C(S)^{p}$. In particular, $x$ is regular.

Let $\delta(e, x, f)$ and $\delta(i, y, j)$ be idempotents in $C(S)^{p} / \delta$. We shall prove that they commute. We therefore need to show that

$$
\delta\left(e, x p_{f, i} y, j\right)=\delta\left(i, y p_{j, e} x, f\right) ;
$$

that is, we need to prove that

$$
x p_{f, i} y=p_{e, i}\left(y p_{j, e} x\right) p_{f, j} \text { and } y p_{j, e} x=p_{i, e}\left(x p_{f, i} y\right) p_{j, f} .
$$

We shall prove the former equality; the proof of the latter equality is similar. Observe also that

$$
x \leq p_{e, f} \text { and } y \leq p_{i, j}
$$

from the proof of Proposition 2.3.1 and the fact that $(e, x, f)$ and $(i, y, j)$ are idempotents in $C(S)^{p}$.

We have that

$$
x p_{f, i} y=\left(x p_{f, i} y\right) p_{j, e}\left(x p_{f, i} y\right)
$$

since $\left(e, x p_{f, i} y, j\right)$ is an idempotent by Proposition 2.3.1. Thus

$$
x p_{f, i} y=\left(x p_{f, i}\right) y p_{j, e} x\left(p_{f, i} y\right) \leq\left(p_{e, f} p_{f, i}\right) y p_{j, e} x\left(p_{f, i} p_{i, j}\right)
$$

which gives

$$
x p_{f, i} y \leq p_{e, i}\left(y p_{j, e} x\right) p_{f, j}
$$

using Proposition 2.2.(ii),(iii) and the fact that all elements involved are regular. Similarly,

$$
y p_{j, e} x \leq p_{i, e}\left(x p_{f, i} y\right) p_{j, f} .
$$

Hence

$$
x p_{f, i} y \leq p_{e, i}\left(y p_{j, e} x\right) p_{f, j} \leq p_{e, i} p_{i, e}\left(x p_{f, i} y\right) p_{j, f} p_{f, j} \leq p_{e, e}\left(x p_{f, i} y\right) p_{j, j}
$$

and so

$$
x p_{f, i} y \leq e\left(x p_{f, i} y\right) j
$$

and this is equal to $x p_{f, i} y$. Thus

$$
x p_{f, i} y=p_{e, i}\left(y p_{j, e} x\right) p_{f, j},
$$

as required
4. $C(S)^{p} / \delta$ is a semigroup with local units

Let $\delta(e, x, f)$ be an element of $C(S)^{p} / \delta$. Observe that $(e, e, e)$ and $(f, f, f)$ are both idempotents of $C(S)^{p}$, by condition (M1). Thus $\delta(e, e, e)$ and $\delta(f, f, f)$ are both idempotents in $C(S)^{p} / \delta$, and clearly $\delta(e, x, f) \delta(f, f, f)=\delta(e, x, f)$ and $\delta(e, e, e) \delta(e, x, f)=\delta(e, x, f)$.
5. $\delta^{\natural}$ is a strict local isomorphism

Let $(a, b, c)$ and $(d, e, f)$ be idempotents in $C(S)^{p}$. Then elements
of $(a, b, c) \cdot C(S) \cdot(d, e, f)$ will certainly have the form $(a, z, f)$ for suitable $z$. Let

$$
(a, x, f),(a, y, f) \in(a, b, c) \cdot C(S) \cdot(d, e, f) .
$$

Then if $\delta(a, x, f)=\delta(a, y, f)$, it follows that $x=p_{a, a y p_{f, f}}=y$.

### 2.5 The covering theorem

Let $S$ be a semigroup with local units having locally commuting idempotents and equipped with a McAlister sandwich function. We may therefore construct the semigroup with local units $C(S)^{p} / \delta=$ $U(S)$ which has commuting idempotents. The map $\delta^{\natural}: C(S)^{p} \rightarrow$ $U(S)$ is a strict local isomorphism. Define $q: E(S) \times E(S) \rightarrow U(S)$ by $q(v, u)=q_{v, u}=\delta(v, v u, u)$. We may therefore form the Rees matrix semigroup $\mathcal{M}=\mathcal{M}(U(S) ; E(S), E(S) ; Q)$.

Put $E^{\prime}=\{(e, \delta(e, e, e), e): e \in E(S)\}$. Then $E^{\prime}$ is a set of idempotents of $\mathcal{M}$. The semigroup $E^{\prime} \mathcal{M} E^{\prime}$ is a subsemigroup of $\mathcal{M}$ and a semigroup with local units.

Lemma $1 E^{\prime} \mathcal{M} E^{\prime}=\{(u, \delta(u, x, v), v) \in \mathcal{M}\}$.
Proof Observe that

$$
(u, \delta(u, x, v), v)=(u, \delta(u, u, u), u)(u, \delta(u, x, v), v)(v, \delta(v, v, v), v) .
$$

Thus $\{(u, \delta(u, x, v), v) \in \mathcal{M}\}$ is contained in $E^{\prime} \mathcal{M} E^{\prime}$. On the other hand,

$$
(e, \delta(e, e, e), e)(u, \delta(i, x, j), v)(f, \delta(f, f, f), f)
$$

which is equal to $\left(e, \delta\left(e, \operatorname{eup}_{u, i} x p_{j, v} v f, f\right), f\right)$, which is of the required form.

Put $\mathcal{E} \mathcal{M}=E^{\prime} \mathcal{M} E^{\prime}$. Observe that in the regular case, $\mathcal{E} \mathcal{M}=$ $\mathcal{R M}$ the set of regular elements of $\mathcal{M}$ (see the proof of Theorem 2.1 at the foot of page 731 of [20]).

Define $\theta: \mathcal{E} \mathcal{M} \rightarrow S$ by $\theta(u, \delta(u, x, v), v)=x$; this is well-defined because if $\delta(u, x, v)=\delta(u, y, v)$ then $x=y$ by the last part of the proof of Theorem 2.4.2.

Proposition 2 The function $\theta$ is a surjective strict local isomorphism along which idempotents can be lifted.

Proof Let $s \in S$. Then because $S$ has local units, we can find idempotents $e, f \in S$ such that $e s=s=s f$. But then $\theta(e, \delta(e, s, f), f)=$ $s$, and so $\theta$ is surjective.

To show that $\theta$ is a homomorphism, let

$$
(u, \delta(u, x, v), v),(g, \delta(g, y, k), k) \in \mathcal{E} \mathcal{M}
$$

Then

$$
(u, \delta(u, x, v), v)(g, \delta(g, y, k), k)=(u, \delta(u, x y, k), k)
$$

The result is now clear.
To show that $\theta$ is a strict local isomorphism, it is sufficient to check that if

$$
\theta(u, \delta(u, x, v), v)=\theta(u, \delta(u, y, v), v)
$$

then

$$
(u, \delta(u, x, v), v)=(u, \delta(u, y, v), v)
$$

but this is immediate from the definition.
Suppose now that $e \in E(S)$. Then $\theta(e, \delta(e, e, e), e)=$,$e and$ $(e, \delta(e, e, e), e,) \in E(\mathcal{E} \mathcal{M})$. Thus idempotents lift along $\theta$.

We have proved the following covering theorem.
Theorem 3 Let $S$ be a semigroup with local units having locally commuting idempotents. If $S$ has a McAlister sandwich function, then there exists a semigroup $U$ with local units whose idempotents commute, a square Rees matrix semigroup $\mathcal{M}=\mathcal{M}(U ; I, I ; Q)$ over $U$, and a subsemigroup $T$ of $\mathcal{M}$ which has local units, and a surjective homomorphism $\theta: T \rightarrow S$ which is a strict local isomorphism along which idempotents can be lifted.

### 2.6 McAlister sandwich functions I

It is natural to ask when a semigroup $S$ has a McAlister sandwich function. In this section, we adapt results from [13] to prove that if $S$ possesses an idempotent $e$ such that every element of $e E(S)$ is regular, then $S$ has a McAlister sandwich function constructed in the same way as the one in McAlister's original paper [20]. In particular, if the regular elements of $S$ form a subsemigroup then $S$ has a McAlister sandwich function.

Let $S$ be a semigroup with local units with locally commuting idempotents. Suppose that $S$ has an idempotent $e$ such that every element of $e E(S)$ is regular. By Proposition 2.2.3, for every $u \in$ $E(S)$ the set $S(e, u)$ contains exactly one element. Denote this element by $u^{\circ}$. It is straightforward to check that the element $u^{\circ}$ has the following properties:

- $u^{\circ}$ is an idempotent.
- $u u^{\circ}=u^{\circ}$.
- $u^{\circ} e=u^{\circ}$.
- $e u^{\circ} u=e u$.

Define $q: E(S) \times E(S) \rightarrow S$ by $q(u, v)=u$ if $u=v$ and $u^{\circ} v$ otherwise. Put $q(u, v)=q_{u, v}$. The following generalises Lemma 2.2 of [20].

Proposition 1 With the above definitions we have:
(i) $q_{u, v} \in u S v$.
(ii) $q_{v, u} \in V\left(q_{u, v}\right)$.
(iii) The product $q_{u, v} q_{v, f}$ is regular, and $q_{u, v} q_{v, f} \leq q_{u, f}$.

Proof (i) If $u=v$ then $q_{u, v}=u$ and so $q_{u, v} \in u S v$. If $u \neq v$ then $q_{u, v}=u^{\circ} v$. But $u q_{u, v}=u u^{\circ} v=u^{\circ} v=q_{u, v}$ and $q_{u, v} v=q_{u, v}$.
(ii) If $u=v$ then the result is clear. Suppose that $u \neq v$. Then

$$
q_{u, v} q_{v, u} q_{u, v}=\left(u^{\circ} v\right)\left(v^{\circ} u\right)\left(u^{\circ} v\right)=u^{\circ} v^{\circ} u^{\circ} v=u^{\circ}\left(e v^{\circ} e\right)\left(e u^{\circ} e\right) v
$$

But $e v^{\circ}, e u^{\circ} \leq e$ and so, by assumption, they commute. Thus

$$
u^{\circ}\left(e v^{\circ} e\right)\left(e u^{\circ} e\right) v=u^{\circ}\left(e u^{\circ} e\right)\left(e v^{\circ} e\right) v=u^{\circ} v^{\circ} v
$$

but

$$
u^{\circ} v^{\circ} v=u^{\circ}\left(e v^{\circ} v\right)=u^{\circ} e v=u^{\circ} v=q_{u, v}
$$

Hence $q_{u, v} q_{v, u} q_{u, v}=q_{u, v}$. By symmetry $q_{v, u} q_{u, v} q_{v, u}=q_{v, u}$.
(iii) By (ii), both $q_{u, v}$ and $q_{v, f}$ are regular, and by (i), we have that $q_{u, v} v=q_{u, v}$ and $v q_{v, f}=q_{v, f}$. Thus $q_{u, v} q_{v, f}$ is regular by Proposition 2.2.3(iii). We now prove that $q_{u, v} q_{v, f} \leq q_{u, f}$. If either $u=v$ or $v=f$ then the result is clear. Thus we may assume that $u \neq v$ and $v \neq f$. By definition

$$
q_{u, v} q_{v, f}=u^{\circ} v v^{\circ} f=u^{\circ} v^{\circ} f
$$

Now

$$
u^{\circ} v^{\circ} f=u^{\circ} u^{\circ} v^{\circ} f=u^{\circ} e u^{\circ} e v^{\circ} f
$$

and $e u^{\circ}, e v^{\circ} \leq e$. Thus $e u^{\circ} e v^{\circ}=e v^{\circ} e u^{\circ}$. Hence

$$
q_{u, v} q_{v, f}=u^{\circ} v^{\circ} u^{\circ} f=\left(u^{\circ} v^{\circ} u\right) u^{\circ} f .
$$

Suppose that $u=f$. Then

$$
q_{u, v} q_{v, f}=q_{u, v} q_{v, u} \leq u=q_{u, f}
$$

using (ii). If, on the other hand, $u \neq f$, then $q_{u, v} q_{v, f}=\left(u^{\circ} v^{\circ} u\right) q_{u, f}$. But

$$
u^{\circ} v^{\circ} u=\left(u^{\circ} v\right)\left(v^{\circ} u\right)=q_{u, v} q_{v, u}
$$

is an idempotent less than $u$. It follows by Proposition 2.2.3(ii) that

$$
q_{u, v} q_{v, f}=\left(q_{u, v} q_{v, u}\right) q_{u, f} \leq u q_{u, f}=q_{u, f}
$$

As a result we have the following theorem.

Theorem 2 Let $S$ be a semigroup with local units having locally commuting idempotents in which the regular elements form a subsemigroup. Then $S$ divides a Rees matrix semigroup over a semigroup with commuting idempotents.

### 2.7 McAlister sandwich functions II

In this section, we show that there is another way of constructing examples of semigroups with McAlister sandwich functions. This approach will form the basis of the work in the next chapter. Let $S$ be a subsemigroup of the semigroup $T$. We say that $T$ is an enlargement of $S$ if $S=S T S$ and $T=T S T$. We begin with the motivating example of an enlargement; it is taken from [14].

Lemma 1 Let $T=T e T$, where $e$ is an idempotent. Then $T$ is an enlargement of eTe.

Proof Firstly

$$
(e T e) T(e T e)=e(T e T) e T e=e(T e T) e=e T e
$$

and

$$
T(e T e) T=T e(T e T)=T e T=T
$$

as required.

We now consider when such a $T$ has locally commuting idempotents.

Lemma 2 Let $T=T e T$, where $e$ is an idempotent. Then $e T e$ has commuting idempotents if and only if $T$ has locally commuting idempotents.

Proof Suppose that eTe has commuting idempotents. Every local submonoid of $T$ is isomorphic to a local submonoid of $e T e$ by Lemma 1.2.6(iii). Thus $T$ has locally commuting idempotents. The converse is immediate.

Lemma 3 Let $T=T e T$, where $e$ is an idempotent. Suppose that $S$ is a subsemigroup of $T$ such that $S=S T S$. Then there is a subsemigroup $T^{\prime}$ of $T$ and a subsemigroup $U=U^{2}$ of eTe such that $T^{\prime}$ is an enlargement of both $S$ and $U$.

Proof Let $U=T S T \cap e T e$ and $T^{\prime}=T S T$. We prove first that $U=e T S T e$. Observe that

$$
e(T S T) e=(e T) S(T e) \subseteq T S T
$$

and

$$
e(T S T) e \subseteq e T e
$$

Thus

$$
e(T S T) e \subseteq T S T \cap e T e
$$

To prove the reverse inclusion, let $x \in T S T \cap e T e$. Then $x=t s t^{\prime}$ where $t, t^{\prime} \in T$ and $s \in S$, and $x=$ exe where $x \in T$. Thus

$$
x=e t s t^{\prime} e=(e t) s\left(t^{\prime} e\right) \in e T S T e .
$$

Hence $e(T S T) e=T S T \cap e T e$.
To prove that $U^{2}=U$, observe that

$$
U^{2}=(e T S T e)(e T S T e)=e T S(T e T) S T e=e T S T S T e
$$

since $T=T e T$. But then

$$
e T S T S T e=e T S T e=U
$$

since $S T S=S$.
We now show that $T^{\prime}$ is an enlargement of $U$. We have that

$$
U T^{\prime} U=U T S T U=(e T S T e) T S T(e T S T e)
$$

using the fact that $T e T=T$ twice this is equal to

$$
e T S T S T S T e=e T S T S T e=e T S T e=U
$$

using the fact that $S T S=S$ twice. Hence

$$
U T^{\prime} U=U
$$

Next we have that

$$
T^{\prime} U T^{\prime}=(T S T) U(T S T)=T S T(e T S T e) T S T
$$

and from the fact that $T=T e T$ and $S=S T S$ we get

$$
T^{\prime} U T^{\prime}=T S T=T^{\prime}
$$

We have therefore shown that $T^{\prime}$ is an enlargement of $U$.
We now show that $T^{\prime}$ is an enlargement of $S$. First of all we have to show that $S$ is actually contained in $T^{\prime}$. Observe that

$$
S=S T S=(S T S) T S=(S T) S(T S) \subseteq T S T=T^{\prime}
$$

Now we check the defining properties of enlargements:

$$
S T^{\prime} S=S(T S T) S=S T S=S
$$

and

$$
T^{\prime} S T^{\prime}=(T S T) S(T S T)=T S T S T S T=T S T S T=T S T=T^{\prime}
$$

The following is immediate.
Theorem 4 Let $T$ be a semigroup with local units such that $T=$ TeT, where $e$ is an idempotent, and eTe has commuting idempotents. Let $S$ be a subsemigroup of $T$ with local units such that $S=S T S$. Then there is a semigroup with local units $T^{\prime}$ with locally commuting idempotents which is an enlargement of the subsemigroups $S$ and $U$, where $U^{2}=U$ has commuting idempotents.

We now come to the main result of this section. The inspiration for it came from [21].

Theorem 5 Let $S$ be a semigroup with local units and let $U$ be $a$ semigroup with commuting idempotents such that $U^{2}=U$. If $T$ is a semigroup which is an enlargement of both $S$ and $U$, then $S$ has a McAlister sandwich function.

Proof By assumption, $T$ is an enlargement of $U$ and $U^{2}=U$. Thus by the proof of Proposition 2.1 of [14], every idempotent of $T$ is $\mathcal{D}$-related to an idempotent of $U$. Thus, in particular, every idempotent $e$ of $S$ is $\mathcal{D}$-related to some idempotent in $U$. By Proposition 1.2.1(iii), this means that there exists $t_{e} \in T$ and $t_{e}^{\prime} \in V\left(t_{e}\right)$ such that

$$
t_{e} t_{e}^{\prime}=e \text { and } t_{e}^{\prime} t_{e} \in E(U)
$$

Observe, in particular, that $t_{e}$ is regular.
Define a function $q: E(S) \times E(S) \rightarrow S$ by

$$
q(f, e)=q_{f, e}=t_{f} t_{e}^{\prime}
$$

To show it is well-defined, we have to prove that $q_{f, e} \in S$. However,

$$
f q_{f, e} e=f\left(t_{f} t_{e}^{\prime}\right) e=t_{f} t_{e}^{\prime}=q_{f, e}
$$

Thus

$$
q_{f, e} \in f T e \subseteq S T S=S
$$

as required. We now prove that $q$ is a McAlister sandwich function for $S$.
(M1) holds: observe that $q_{e, e}=t_{e} t_{e}^{\prime}=e$ and $q_{f, e}=t_{f} t_{e}^{\prime}$ where

$$
f\left(t_{f} t_{e}^{\prime}\right)=\left(t_{f} t_{f}^{\prime}\right) t_{f} t_{e}^{\prime}=t_{f} t_{e}^{\prime}
$$

and

$$
\left(t_{f} t_{e}^{\prime}\right) e=t_{f} t_{e}^{\prime} t_{e} t_{e}^{\prime}=t_{f} t_{e}^{\prime}
$$

(M2) holds: we calculate

$$
q_{e, f} q_{f, e} q_{e, f}=\left(t_{e} t_{f}^{\prime}\right)\left(t_{f} t_{e}^{\prime}\right)\left(t_{e} t_{f}^{\prime}\right)
$$

which is $t_{e}\left(t_{f}^{\prime} t_{f}\right)\left(t_{e}^{\prime} t_{e}\right) t_{f}^{\prime}$. Now the bracketed elements are both idempotents in $U$ and so commute. Thus we can rewrite the product as $t_{e}\left(t_{e}^{\prime} t_{e}\right)\left(t_{f}^{\prime} t_{f}\right) t_{f}^{\prime}$ which is just $q_{e, f}$. Interchanging $e$ and $f$ we arrive at $q_{f, e} \in V\left(q_{e, f}\right)$.
(M3) holds: by definition

$$
q_{e, f} q_{f, i}=t_{e} t_{f}^{\prime} t_{f} t_{i}^{\prime}
$$

Now this is equal to

$$
\left(t_{e} t_{f}^{\prime} t_{f} t_{e}^{\prime}\right) t_{e} t_{i}^{\prime}
$$

using the fact that $t_{f}^{\prime} t_{f}$ and $t_{e}^{\prime} t_{e}$ are idempotents in $U$ and so commute. However, it is easy to check that

$$
\alpha=t_{e} t_{f}^{\prime} t_{f} t_{e}^{\prime} \in S
$$

is an idempotent. Thus

$$
q_{e, f} q_{f, i}=\alpha q_{e, i}
$$

Similarly

$$
q_{e, f} q_{f, i}=q_{e, i}\left(t_{i} t_{f}^{\prime} t_{f} t_{i}^{\prime}\right)=q_{e, i} \beta
$$

where $\beta \in S$ is an idempotent. Hence by Proposition 2.2.2(iv),

$$
q_{e, f} q_{f, i} \leq q_{e, i}
$$

The proof of the following is immediate from Theorems 4 and 5. It provides us with another way of constructing semigroups which satisfy the main conditions of this chapter.

Corollary 6 Let $T$ be a semigroup with local units such that $T=$ TeT, where $e$ is an idempotent, and eTe has commuting idempotents. If $S$ is a subsemigroup of $T$ with local units such that $S=S T S$, then $S$ has a McAlister sandwich function.

## Chapter 3

## An embedding theorem for semigroups with locally commuting idempotents

McAlister proved that a regular semigroup $S$ was locally inverse if and only if it could be embedded in a regular semigroup $T$ such that $T=T e T$, for some idempotent $e$, and $e T e$ is inverse, and satisfying $S=S T S$ [20], [22]. In this chapter, we show how this result can be generalised to the class of semigroups with local units whose local submonoids have commuting idempotents. Our characterisation makes essential use of McAlister sandwich functions.

The constructions in Sections 3.1 and 3.2 were motivated by constructions, originally for regular semigroups, to be found on pages 168 to 181 of [22]. Our contribution has been to remove the assumption of regularity and replace it by the existence of local units, and to clarify the presentation of the main arguments by phrasing them in terms of categories.

### 3.1 Basic constructions

Let $C$ be a category. We say that $C$ is strongly connected if for any pair of identities $(e, f)$ the set $\operatorname{hom}(e, f)$ is non-empty.

Let $C$ be a strongly connected category. A consolidation $q$ for $C$ is a function $q: C_{o} \times C_{o} \rightarrow C$ such that $q(e, f) \in \operatorname{hom}(f, e)$, and $q(e, e)=e$ for every $e \in C_{o}$. We will write $q_{e, f}$ rather than $q(e, f)$. The pair ( $C, q$ ), which we usually write as $C^{q}$, can be used to define a semigroup structure on $C$ as follows: define $\circ$ on $C$ by

$$
x \circ y=x q_{e, f} y
$$

where $\mathbf{d}(x)=e$ and $\mathbf{r}(y)=f$. We will always denote products with respect to consolidations by means of o . The following is easy to prove.

Lemma 1 Let $C$ be a category equipped with a consolidation $q$. Then $C^{q}$ is a semigroup with local units.

We shall now define a special class of categories which we shall later convert into semigroups with local units by means of consolidations. The definition is not standard. Let $C$ be a category. We shall say that it is bipartite if the following conditions hold:
(B1) There are full disjoint subcategories $A$ and $B$ of $C$, such that $C_{o}=A_{o} \cup B_{o}$.
(B2) There is a set of isomorphisms $\mathcal{A} \subseteq C$ such that $\mathcal{A}^{-1} \mathcal{A}=A_{o}$, $\mathcal{A A}^{-1}=B_{0}$, and $B=\mathcal{A} A \mathcal{A}^{-1}$.
(B3) $C=A \cup B \cup \mathcal{A} A \cup \mathcal{A}^{-1} B$.

Thus $\mathcal{A} A$ consists of all the arrows which start in $A$ and finish in $B$, whereas $\mathcal{A}^{-1} B$ consists of all the arrows which start in $B$ and finish in $A$. We shall denote bipartite categories with these ingredients by $C(A, B, \mathcal{A})$.

Lemma 2 Let $C=C(A, B, \mathcal{A})$ be a strongly connected bipartite category and let $q$ be a consolidation on $C$. Then $A^{q}$ and $B^{q}$ are subsemigroups of $C^{q}$, and $C^{q}$ is an enlargement of both of them.

Proof It is clear that $A^{q}$ and $B^{q}$ are subsemigroups of $C^{q}$. We shall prove that $C^{q}$ is an enlargement of $A^{q}$; the proof that $C^{q}$ is an enlargement of $B^{q}$ follows by symmetry.

To prove that $A \circ C \circ A \subseteq A$, let $a \in A$ where $a \in \operatorname{hom}(e, f)$; and $c \in C$ where $c \in \operatorname{hom}(i, j)$. Then $a \circ c \circ a=a q_{e, j} c q_{i, f} a$. But $a q_{e, j} c q_{i, f} a$ begins and ends in $A$ and $A$ is a full subcategory of $C$ and so $a q_{e, j} c q_{i, f} a \in A$. Thus $A \circ C \circ A \subseteq A$. The reverse inclusion is immediate.

To prove that and $C \subseteq C \circ A \circ C$, let $c \in C$ where $c \in \operatorname{hom}(i, j)$. Then $c \in A \cup B \cup \mathcal{A} A \cup \mathcal{A}^{-1} B$. If $c \in A$ then clearly $c \in C \circ A \circ C$. If $c \in B$ then $c=y a x^{-1}$ for some $x, y \in \mathcal{A}$ and $a \in A$; thus $c \in C \circ A \circ C$. If $c \in \mathcal{A} A$ then clearly $c \in C \circ A \circ C$. Finally, if $c \in \mathcal{A}^{-1} B=A \mathcal{A}^{-1}$ then clearly $c \in C \circ A \circ C$. Thus in all cases the inclusion holds. The proof of the reverse inclusion is immediate.

Let $C=C(A, B, \mathcal{A})$ be a strongly connected bipartite category equipped with a consolidation $r$. The two results which follow, which are fundamental to our work, concern the behaviour of certain congruences on $C^{r}$.

Lemma 3 Let $C=C(A, B, \mathcal{A})$ be a bipartite category, and let $r$ be a consolidation on $C$. Let $p$ be the restriction of $r$ to $A$, and let $q$ be the restriction of $r$ to $B$. Let $\pi_{1}$ be a congruence on $A^{p}$ and $\pi_{2}$ be a congruence on $B^{q}$. Let $\pi$ be the congruence generated by $\pi_{1} \cup \pi_{2}$ on $C^{r}$. Then

$$
\pi \cap(A \times A)=\pi_{1}
$$

if and only if the following conditions hold:
(1) $\left(a, a^{\prime}\right) \in \pi_{1}$ and $x \in \mathcal{A}^{-1}$ implies $\left(x \circ a, x \circ a^{\prime}\right) \in \pi_{1}$.
(2) $\left(a, a^{\prime}\right) \in \pi_{1}$ and $y \in \mathcal{A}$ implies $\left(a \circ y, a^{\prime} \circ y\right) \in \pi_{1}$.
(3) $\left(b, b^{\prime}\right) \in \pi_{2}$ and $x \in A$ and $y \in \mathcal{A}$ implies $\left(x \circ b \circ y, x \circ b^{\prime} \circ y\right) \in \pi_{1}$.
(4) $\left(b, b^{\prime}\right) \in \pi_{2}$ and $x \in \mathcal{A}^{-1}$ and $y \in A$ implies $\left(x \circ b \circ y, x \circ b^{\prime} \circ y\right) \in$ $\pi_{1}$.
(5) $\left(b, b^{\prime}\right) \in \pi_{2}$ and $x \in \mathcal{A}^{-1}$ and $y \in \mathcal{A}$ implies $\left(x \circ b \circ y, x \circ b^{\prime} \circ y\right) \in$ $\pi_{1}$.
(6) $\left(b, b^{\prime}\right) \in \pi_{2}$ and $x \in A$ and $y \in A$ implies $\left(x \circ b \circ y, x \circ b^{\prime} \circ y\right) \in \pi_{1}$.

Proof If $\pi \cap(A \times A)=\pi_{1}$, then it is easy to check that all the conditions hold. Conversely, assume that all the conditions hold. We prove that

$$
\pi \cap(A \times A)=\pi_{1} .
$$

Let $\left(a_{1}, a_{2}\right) \in \pi$ where $a_{1}, a_{2} \in A$. We shall prove that $\left(a_{1}, a_{2}\right) \in \pi_{1}$. From Proposition 1.1.3, there is a sequence of elementary $\pi_{1} \cup \pi_{2}$ transitions

$$
a_{1}=z_{1} \rightarrow z_{2} \rightarrow \ldots \rightarrow z_{n}=a_{2},
$$

where $z_{i}=x_{i} \circ u_{i} \circ y_{i}, z_{i+1}=x_{i} \circ v_{i} \circ y_{i}$, and $\left(u_{i}, v_{i}\right) \in \pi_{1} \cup \pi_{2}$.
The crucial observation on which the proof of this part of the lemma rests is the following. We can define a congruence $\rho$ on $C^{r}$ whose set of congruence classes is

$$
\left\{A, \mathcal{A} A, A \mathcal{A}^{-1}, B\right\} .
$$

Clearly, $\pi_{1}, \pi_{2} \subseteq \rho$ and so $\pi \subseteq \rho$. Since $z_{1}=a_{1} \in A$ then all the $z_{i} \in A$. Thus $x_{i} \circ u_{i} \circ y_{i}, x_{i} \circ v_{i} \circ y_{i} \in A$ and $\left(u_{i}, v_{i}\right) \in \pi_{1} \cup \pi_{2}$.

We now work out the consequences of this observation. Let us suppose first that $\left(u_{i}, v_{i}\right) \in \pi_{1}$. Then $u_{i}, v_{i} \in A$. We now have to find all possible ways of choosing $x_{i}$ and $y_{i}$ so that $z_{i}$ and $z_{i+1}$ belong to $A$. It follows that $x_{i}$ must come from either $A$ or $A \mathcal{A}^{-1}$ and $y_{i}$ must come from $A$ or $\mathcal{A} A$. The cases involving $A$ are immediate since $\pi_{1}$ is a congruence on $A$. Furthermore, the other two cases can be dealt with independently since $u_{i}$ and $v_{i}$ both come from $A$. It is now easy to see that conditions (1) and (2) are all we need to deal with these cases.

Now we turn to the case where $\left(u_{i}, v_{i}\right) \in \pi_{2}$. Then $u_{i}, v_{i} \in B$. We now have to find all possible ways of choosing $x_{i}$ and $y_{i}$ so that $z_{i}$ and $z_{i+1}$ belong to $A$. It follows that $x_{i}$ must come from $A$ or $\mathcal{A}^{-1} B$, and $y_{i}$ must come from $A$ or $B \mathcal{A}$. It is now easy to see that conditions (3), (4), (5) and (6) are all we need to deal with these cases.

We shall now strengthen Lemma 3 in the case where we have more information about the consolidation. Let $C=C(A, B, \mathcal{A})$ be a strongly connected bipartite category. Let $p$ be a consolidation on $A$ and let $q$ be a consolidation on $B$. Then we can construct a consolidation $r$ on $C$ in the following way. Choose an isomorphism $\alpha \in \mathcal{A}$ from $i$ to $j$. Define the consolidation $r$ on $C$ as follows:

$$
r_{e, f}= \begin{cases}p_{e, f} & \text { if } e, f \in A_{o} \\ q_{e, j} \alpha p_{i, f} & \text { if } e \in B_{o} \text { and } f \in A_{o} \\ p_{e, i} \alpha^{-1} q_{j, f} & \text { if } e \in A_{o} \text { and } f \in B_{o} \\ q_{e, f} & \text { if } e, f \in B_{o}\end{cases}
$$

Thus $r$ agrees with $p$ and $q$ on $A$ and $B$ respectively and then does the simplest thing otherwise. The consolidation $r$ is determined by $p, q$ and the choice of $\alpha$. Put $r=r(p, q, \alpha)$.

Lemma 4 Let $C=(A, B, \mathcal{A})$ be a bipartite category. Let $p$ be a consolidation on $A$, and let $q$ be a consolidation on $B$. Let $\alpha \in \mathcal{A}$ and let $r=r(p, q, \alpha)$ be the consolidation on $C$ defined above. Let $\pi_{1}$ be a congruence on $A^{p}$ and $\pi_{2}$ be a congruence on $B^{q}$. Let $\pi$ be the congruence generated by $\pi_{1} \cup \pi_{2}$ on $C^{r}$. Then we have the following:
(i) $\pi \cap(A \times A)=\pi_{1}$ if and only if the following conditions hold:
(1) $\prime\left(a, a^{\prime}\right) \in \pi_{1}$ implies $\left(\alpha^{-1} \circ a, \alpha^{-1} \circ a^{\prime}\right) \in \pi_{1}$.
(2) $\prime\left(a, a^{\prime}\right) \in \pi_{1}$ implies $\left(a \circ \alpha, a^{\prime} \circ \alpha\right) \in \pi_{1}$.
(3) $\prime\left(b, b^{\prime}\right) \in \pi_{2}$ implies $\left(\beta^{-1} \circ b \circ \gamma, \beta^{-1} \circ b^{\prime} \circ \gamma\right) \in \pi_{1}$ for all $\beta, \gamma \in \mathcal{A}$.
(ii) $\pi \cap(B \times B)=\pi_{2}$ if and only if the following conditions hold:
$(1) \prime \prime\left(b, b^{\prime}\right) \in \pi_{2}$ implies $\left(\alpha \circ b, \alpha \circ b^{\prime}\right) \in \pi_{2}$.
(2) $॥\left(b, b^{\prime}\right) \in \pi_{2}$ implies $\left(b \circ \alpha^{-1}, b^{\prime} \circ \alpha^{-1}\right) \in \pi_{2}$.
(3) $\prime \prime\left(a, a^{\prime}\right) \in \pi_{1}$ implies $\left(\beta \circ a \circ \gamma^{-1}, \beta \circ a^{\prime} \circ \gamma^{-1}\right) \in \pi_{2}$ for all $\beta, \gamma \in \mathcal{A}$.

Proof (i) It is evident that conditions (1) $\prime$, (2) $\prime$ and (3) $\prime$ above are special cases of conditions (1)-(6) in Lemma 3. It therefore is enough to show that we can deduce conditions (1)-(6) from conditions (1) $\iota$, (2) ノ and (3) ノ.
(1) holds: let $\left(a, a^{\prime}\right) \in \pi_{1}$ and let $x \in \mathcal{A}^{-1}$. We prove that $\left(x \circ a, x \circ a^{\prime}\right) \in \pi_{1}$. Let $x \in \operatorname{hom}(n, m), a \in \operatorname{hom}(f, e)$ and $a^{\prime} \in$ $\operatorname{hom}\left(f^{\prime}, e^{\prime}\right)$. By (1) $\prime$, we have that

$$
\left(\alpha^{-1} \circ a, \alpha^{-1} \circ a^{\prime}\right) \in \pi_{1}
$$

But

$$
\alpha^{-1} \circ a=\alpha^{-1} r_{j, e} a=\alpha^{-1} q_{j, j} \alpha p_{i, e} a
$$

and by the definition of a consolidation $q_{j, j}=j$. Thus

$$
\alpha^{-1} \circ a=p_{i, e} a
$$

Similarly, $\alpha^{-1} \circ a^{\prime}=p_{i, e^{\prime}} a^{\prime}$. Thus

$$
\left(p_{i, e} a, p_{i, e^{\prime}} a\right) \in \pi_{1}
$$

Now the element $x \circ q_{n, j} \circ \alpha=x q_{n, j} \alpha$ starts and ends in $A$ and so belongs to $A$ by fullness. By assumption, $\pi_{1}$ is a congruence on $A$. Thus

$$
\left(\left(x q_{n, j} \alpha\right) \circ\left(p_{i, e} a\right),\left(x q_{n, j} \alpha\right) \circ\left(p_{i, e^{\prime}} a\right)\right) \in \pi_{1}
$$

Using the fact that $p_{i, i}=i$ we obtain

$$
\left(x q_{n, j} \alpha p_{i, e} a, x q_{n, j} \alpha p_{i, e^{\prime}} a\right) \in \pi_{1}
$$

But

$$
x \circ a=x q_{n, j} \alpha p_{i, e} a
$$

and

$$
x \circ a^{\prime}=x q_{n, j} \alpha p_{i, e^{\prime}} a^{\prime} .
$$

Hence

$$
\left(x \circ a, x \circ a^{\prime}\right) \in \pi_{1},
$$

as required.
(2) holds: let $\left(a, a^{\prime}\right) \in \pi_{1}$ and let $y \in \mathcal{A}$. We prove that ( $a \circ y, a^{\prime} \circ$ $y) \in \pi_{1}$. Let $a \in \operatorname{hom}(f, e), a^{\prime} \in \operatorname{hom}\left(f^{\prime}, e^{\prime}\right)$ and $y \in \operatorname{hom}(m, n)$. By condition (2) $\prime$, we have that $\left(a \circ \alpha, a^{\prime} \circ \alpha\right) \in \pi_{1}$. But

$$
a \circ \alpha=a r_{f, j} \alpha=a p_{f, i} \alpha^{-1} q_{j, j} \alpha
$$

and so $a \circ \alpha=a p_{f, i}$. Similarly, $a^{\prime} \circ \alpha=a p_{f^{\prime}, i}$. Thus

$$
\left(a p_{f, i}, a p_{f^{\prime}, i}\right) \in \pi_{1} .
$$

Observe that $\alpha^{-1} \circ y \in A$ because it begins and ends in $A$, and $A$ is a full subcategory of $C$. Since $\pi_{1}$ is a congruence on $A$ we have that

$$
\left(\left(a p_{f, i}\right) \circ\left(\alpha^{-1} \circ y\right),\left(a p_{f^{\prime}, i}\right) \circ\left(\alpha^{-1} \circ y\right)\right) \in \pi_{1} .
$$

But

$$
a \circ y=a r_{f, n} y=a p_{f, i} \alpha^{-1} q_{j, n} y=\left(a p_{f, i}\right) \circ\left(\alpha^{-1} \circ y\right)
$$

and, similarly,

$$
a^{\prime} \circ y=a p_{f^{\prime}, i} \alpha^{-1} q_{j, n} y=\left(a p_{f^{\prime}, i}\right) \circ\left(\alpha^{-1} \circ y\right) .
$$

Hence

$$
\left(a \circ y, a^{\prime} \circ y\right) \in \pi_{1} .
$$

Thus (2) holds as required.
(3) holds: let $\left(b, b^{\prime}\right) \in \pi_{2}, x \in A$ and $y \in \mathcal{A}$. We prove that $\left(x \circ b \circ y, x \circ b^{\prime} \circ y\right) \in \pi_{1}$. Let $b \in \operatorname{hom}(f, e), b^{\prime} \in \operatorname{hom}\left(f^{\prime}, e^{\prime}\right)$, $x \in \operatorname{hom}(n, m)$ and $y \in \operatorname{hom}(k, l)$. By condition (3) $)$, we have that

$$
\left(\alpha^{-1} \circ b \circ y, \alpha^{-1} \circ b^{\prime} \circ y\right) \in \pi_{1} .
$$

By assumption, $x \in A$ and so since $\pi_{1}$ is a congruence on $A$ we have that

$$
\left(x \circ \alpha^{-1} \circ b \circ y, x \circ \alpha^{-1} \circ b^{\prime} \circ y\right) \in \pi_{1} .
$$

However, it is easy to check that

$$
x \circ b \circ y=x r_{n, e} b r_{f, l} y=x p_{n, i} \alpha^{-1} q_{j, e} b q_{f, l} y=x \circ \alpha^{-1} \circ b \circ y
$$

and

$$
x \circ b^{\prime} \circ y=x p_{n, i} \alpha^{-1} q_{j, e^{\prime}} b^{\prime} q_{f^{\prime}, f} y=x \circ \alpha^{-1} \circ b^{\prime} \circ y .
$$

Thus (3) holds as required.
(4) holds: let $\left(b, b^{\prime}\right) \in \pi_{2}, x \in \mathcal{A}^{-1}$ and $y \in A$. We prove that $\left(x \circ b \circ y, x \circ b^{\prime} \circ y\right) \in \pi_{1}$. Let $b \in \operatorname{hom}(f, e), b^{\prime} \in \operatorname{hom}\left(f^{\prime}, e^{\prime}\right)$, $y \in \operatorname{hom}(n, m)$ and $x \in \operatorname{hom}(k, l)$. By condition (3) $)$, we have that

$$
\left(x \circ b \circ \alpha, x \circ b^{\prime} \circ \alpha\right) \in \pi_{1} .
$$

Now $y \in A$ and $\pi_{1}$ is a congruence on $A$ and so

$$
\left(x \circ b \circ \alpha \circ y, x \circ b^{\prime} \circ \alpha \circ y\right) \in \pi_{1} .
$$

But

$$
x \circ b \circ y=x r_{k, e} b r_{f, m} y=x q_{k, e} b q_{f, j} \alpha p_{i, m} y=x \circ b \circ \alpha \circ y .
$$

Similarly,

$$
x \circ b^{\prime} \circ y=x q_{k, e^{\prime}} b^{\prime} q_{f^{\prime}, j} \alpha p_{i, m} y=x \circ b^{\prime} \circ \alpha \circ y .
$$

Thus (4) holds.
(5) holds: this is just condition (3)/.
(6) holds: let $\left(b, b^{\prime}\right) \in \pi_{2}$, and $x, y \in A$. We prove that ( $x \circ b \circ$ $\left.y, x \circ b^{\prime} \circ y\right) \in \pi_{1}$. Let $b \in \operatorname{hom}(f, e), b^{\prime} \in \operatorname{hom}\left(f^{\prime}, e^{\prime}\right), y \in \operatorname{hom}(n, m)$ and $x \in \operatorname{hom}(k, l)$. By condition (3) , we have that

$$
\left(\alpha^{-1} \circ b \circ \alpha, \alpha^{-1} \circ b^{\prime} \circ \alpha\right) \in \pi_{1} .
$$

Now $x, y \in A$ and so since $\pi_{1}$ is a congruence on $A$ we have that

$$
\left(x \circ \alpha^{-1} \circ b \circ \alpha \circ y, x \circ \alpha^{-1} \circ b^{\prime} \circ \alpha \circ y\right) \in \pi_{1} .
$$

But
$x \circ \alpha^{-1} \circ b \circ \alpha \circ y=x r_{k, i} \alpha^{-1} r_{j, e} b r_{f, j} \alpha r_{i, n} y=x p_{k, i} \alpha^{-1} q_{j, e} b q_{f, j} \alpha p_{i, n} y$.
Whereas

$$
x \circ b \circ y=x r_{k, e} b r_{f, n} y=x p_{k, i} \alpha^{-1} q_{j, e} b q_{f, j} \alpha p_{i, n} y .
$$

Thus

$$
x \circ \alpha^{-1} \circ b \circ \alpha \circ y=x \circ b \circ y .
$$

Similarly

$$
x \circ \alpha^{-1} \circ b^{\prime} \circ \alpha \circ y=x \circ b^{\prime} \circ y .
$$

Thus condition (6) also holds.
(ii) Observe first that the conclusions of Lemma 3 can be applied to the congruence $\pi_{2}$ with obvious modifications. We therefore have the following six necessary and sufficient conditions for $\pi \cap(B \times B)=$ $\pi_{2}$ :
(1) $\left(b, b^{\prime}\right) \in \pi_{2}$ and $x \in \mathcal{A}$ implies $\left(x \circ b, x \circ b^{\prime}\right) \in \pi_{2}$.
(2) $\left(b, b^{\prime}\right) \in \pi_{2}$ and $y \in \mathcal{A}^{-1}$ implies $\left(b \circ y, b^{\prime} \circ y\right) \in \pi_{2}$.
(3) $\left(a, a^{\prime}\right) \in \pi_{1}$ and $x \in B$ and $y \in \mathcal{A}^{-1}$ implies $\left(x \circ a \circ y, x \circ a^{\prime} \circ y\right) \in$ $\pi_{2}$.
(4) $\left(a, a^{\prime}\right) \in \pi_{1}$ and $x \in \mathcal{A}$ and $y \in B$ implies $\left(x \circ a \circ y, x \circ a^{\prime} \circ y\right) \in \pi_{2}$.
(5) $\left(a, a^{\prime}\right) \in \pi_{1}$ and $x \in \mathcal{A}$ and $y \in \mathcal{A}^{-1}$ implies $\left(x \circ a \circ y, x \circ a^{\prime} \circ y\right) \in$ $\pi_{2}$.
(6) $\left(a, a^{\prime}\right) \in \pi_{1}$ and $x \in B$ and $y \in B$ implies $\left(x \circ a \circ y, x \circ a^{\prime} \circ y\right) \in \pi_{2}$.

It is evident that conditions $(1)\|,(2)\|$ and (3) $\|$ are special cases of the above conditions so, as in (i), it remains to prove that (1) $॥$, (2) $/ \prime$ and (3) $/$ imply the above six conditions. The arguments used are similar to the arguments employed in (i), and will be omitted.

### 3.2 The main theorems

Lemma 3.1.4 will now be applied to a particular class of bipartite categories.

Let $S$ be any semigroup with local units and let $C(S)$ be the category of Section 2.3. Then $C(S)$ is strongly connected because for any ordered pair of identities $(e, e, e)$ and ( $f, f, f$ ), the hom-set hom $((e, e, e),(f, f, f))$ contains the element $(f, f e, e)$. By definition, a consolidation for $C(S)$ is a function

$$
\xi: C(S)_{o} \times C(S)_{o} \rightarrow C(S)
$$

such that

$$
\xi_{(e, e, e),(f, f, f)} \in \operatorname{hom}((f, f, f),(e, e, e)) \text { and } \xi_{(e, e, e),(e, e, e)}=(e, e, e) .
$$

Thus $\xi_{(e, e, e),(f, f, f)}$ is of the form $\left(e, \xi_{e, f}^{\prime}, f\right)$ where $\xi_{e, f}^{\prime} \in e S f$. It follows that consolidations such as $\xi$ are completely determined by functions $\xi^{\prime}: E(S) \times E(S) \rightarrow S$ which satisfy $\xi_{e, e}^{\prime}=e$ and $\xi_{e, f}^{\prime} \in e S f$.

We now make some notational alterations which will enable us more easily to construct a bipartite category from $S$.

Put $E=E(S)$ and let $\bar{E}=\{\bar{e}: e \in E\}$. Put

$$
W=\{(\bar{e}, s, \bar{f}) \in \bar{E} \times S \times \bar{E}: s \in e S f\}
$$

regarded as a category in the above way. Consequently, $W$ is nothing more than $C(S)$ with identities relabelled. Let $q: \bar{E} \times \bar{E} \rightarrow S$ be any function which satisfies $q_{\bar{e}, \bar{e}}=e$ and $q_{\bar{e}, \bar{f}} \in e S f$. As above this gives rise to a consolidation on the category $W$ with elements $\left(\bar{e}, q_{\bar{e}, \bar{f}}, \bar{f}\right)$. We denote by $W^{q}$ the semigroup with local units which results from this consolidation.

Let $\delta$ be a congruence on $W^{q}$ and put $T=W^{q} / \delta$. We shall assume that $\delta^{\natural}: W^{q} \rightarrow W^{q} / \delta$ is a strict local isomorphism. Denote the $\delta$-equivalence class of ( $\bar{e}, s, \bar{f}$ ) by $[\bar{e}, s, \bar{f}]$.

We shall now construct a bipartite category $M$ from the semigroup $T$. Let

$$
M=\{(\alpha,[\bar{e}, s, \bar{f}], \beta): \alpha=e \text { or } \bar{e} \text { and } \beta=f \text { or } \bar{f}\},
$$

where triples in $M$ are multiplied in the obvious way. Put

$$
A=\{(e,[\bar{e}, s, \bar{f}], f) \in E \times T \times E: s \in e S f\}
$$

and

$$
B=\{(\bar{e},[\bar{e}, s, \bar{f}], \bar{f}): s \in e S f\}
$$

Let

$$
\mathcal{A}=\{(e,[\bar{e}, e, \bar{e}], \bar{e}): e \in E\}
$$

For notational convenience, put $\mathbf{e}=(e,[\bar{e}, e, \bar{e}], e)$ and $\overline{\mathbf{e}}=$ $(\bar{e},[\bar{e}, e, \bar{e}], \bar{e})$.

Proposition 1 With the above definitions, $M=C(A, B, \mathcal{A})$ is a bipartite category.

Proof If $(\alpha, \mathbf{x}, \beta)$ and $(\gamma, \mathbf{y}, \epsilon)$ are elements of $M$ then their product is only defined when $\beta=\gamma$ in which case it is $(\alpha, \mathbf{x y}, \epsilon)$. It is immediate from the form of the product in $T$ that this multiplication is well-defined. It is easy to verify that $M$ is a category with respect to this partial product with identities

$$
\{\mathbf{e}: e \in E\} \cup\{\overline{\mathbf{e}}: e \in E\}
$$

It is clear that $A$ and $B$ are full disjoint subcategories; that $\mathcal{A}$ is a set of isomorphisms; and that conditions (B1), (B2) and (B3) are satisfied.

We shall now define a consolidation on $M$ of the type described before Lemma 3.1.4.

- The consolidation $\mathbf{p}$ in $A$ consists of elements of the form

$$
\mathbf{p}_{\mathbf{e}, \mathbf{f}}=(e,[\bar{e}, e f, \bar{f}], f)
$$

- The consolidation $\mathbf{q}$ in $B$ consists of elements of the form

$$
\mathbf{q}_{\bar{e}, \overline{\mathbf{f}}}=\left(\bar{e},\left[\bar{e}, q_{\bar{e}, \bar{f}}, \bar{f}\right], \bar{f}\right) .
$$

- Choose any element

$$
\alpha=(\bar{u},[\bar{u}, u, \bar{u}], u)
$$

of $\mathcal{A}$.
Define the consolidation on $M$ defined by $\mathbf{r}=\mathbf{r}(\mathbf{p}, \mathbf{q}, \alpha)$. We now calculate the form of $\mathbf{r}$ explicitly:

$$
\mathbf{r}_{\mathbf{e}, \overline{\mathbf{f}}}=\mathbf{p}_{\mathbf{e}, \mathbf{u}} \alpha^{-1} \mathbf{q}_{\overline{\mathbf{u}}, \overline{\mathbf{f}}}=\left(e,\left[\bar{e}, e q_{\bar{u}, \bar{f}}, \bar{f}\right], \bar{f}\right) ;
$$

and

$$
\mathbf{r}_{\bar{e}, \mathbf{f}}=\mathbf{q}_{\bar{e}, \overline{\mathbf{u}}} \alpha \mathbf{p}_{\mathbf{u}, \mathrm{f}}=\left(\bar{e},\left[\bar{e}, q_{\bar{e}, \bar{u}} f, \bar{f}\right], f\right) .
$$

A more succinct description of $\mathbf{r}$ can be obtained as follows. It is

$$
\mathbf{r}_{\alpha, \beta}=\left(\alpha,\left[\bar{e}, r_{\alpha, \beta}, \bar{f}\right], \beta\right)
$$

where

$$
r_{\alpha, \beta}= \begin{cases}q_{\bar{e}, \bar{f}} & \text { if } \alpha=\bar{e} \text { and } \beta=\bar{f} \\ e f & \text { if } \alpha=e \text { and } \beta=f \\ e q_{\bar{u}, \bar{f}} & \text { if } \alpha=e \text { and } \beta=\bar{f} \\ q_{\bar{e}, \bar{u}} f & \text { if } \alpha=\bar{e} \text { and } \beta=f\end{cases}
$$

Lemma 2 With the notation above we have the following.
(i) Define $\theta: A^{\mathbf{r}} \rightarrow S$ by $\theta(e,[\bar{e}, s, \bar{f}], f)=s$. Then $\theta$ is a welldefined strict local isomorphism onto $S$.
(ii) Define $\phi: B^{\mathbf{r}} \rightarrow T$ by $\phi(\bar{e},[\bar{e}, s, \bar{f}], \bar{f})=[\bar{e}, s, \bar{f}]$. Then $\phi$ is a well-defined strict local isomorphism onto $T$.

Proof (i) The restriction of the consolidation $\mathbf{r}$ to $A$ takes the form ( $e,[\bar{e}, e f, \bar{f}], f$ ). Thus the product in $A$ is given by

$$
(e,[\bar{e}, s, \bar{f}], f) \circ(i,[\bar{i}, t, \bar{j}], j)
$$

which is equal to

$$
(e,[\bar{e}, s, \bar{f}], f)(f,[\bar{f}, f i, \bar{i}], i)(i,[\bar{i}, t, \bar{j}], j)
$$

which simplifies to

$$
(e,[\bar{e}, s t, \bar{j}], j) .
$$

Next we have to check that $\theta$ is well-defined. Suppose that

$$
(e,[\bar{e}, x, \bar{f}], f)=(e,[\bar{e}, y, \bar{f}], f) .
$$

Then $x, y \in e S f$ and $\delta(\bar{e}, x, \bar{f})=\delta(\bar{e}, y, \bar{f})$. By assumption, $\delta$ is a strict local isomorphism on $W$, and clearly

$$
(\bar{e}, x, \bar{f}),(\bar{e}, y, \bar{f}) \in(\bar{e}, e, \bar{e}) \circ W^{q} \circ(\bar{f}, f, \bar{f}) .
$$

Thus $x=y$. It follows that $\theta$ is a well-defined homomorphism. It is surjective because $S$ has local units. It is easy to check that it is a strict local isomorphism.
(ii) The restriction of the consolidation $\mathbf{r}$ to $B$ takes the form $\left(\bar{e},\left[\bar{e}, q_{\bar{e}, \bar{f}}, \bar{f}\right], \bar{f}\right)$. Thus the product in $B$ is given by

$$
(\bar{e},[\bar{e}, s, \bar{f}], \bar{f}) \circ(\bar{i},[\bar{i}, t, \bar{j}], \bar{j})
$$

which is equal to

$$
(\bar{e},[\bar{e}, s, \bar{f}], \bar{f})\left(\bar{f},\left[\bar{f}, q_{\bar{f}, \bar{i}}, \bar{i}\right], \bar{i}\right)(\bar{i},[\bar{i}, t, \bar{j}], \bar{j})
$$

which simplifies to

$$
(\bar{e},[\bar{e}, s, \bar{f}][\bar{i}, t, \bar{j}], \bar{j}) .
$$

It is now evident that $\phi$ is a well-defined homomorphism onto $T$, and it is easy to check that it is a strict local isomorphism.

Let $\operatorname{ker}(\theta)=\pi_{1}$ and $\operatorname{ker}(\phi)=\pi_{2}$. We can now prove the main result of this section, and the one on which our subsequent theory depends.

Theorem 3 With the definitions above, let $\pi$ be the congruence generated by $\pi_{1} \cup \pi_{2}$ on $M$. Then $\pi \cap(A \times A)=\pi_{1}$ and $\pi \cap(B \times B)=$ $\pi_{2}$.

Proof To prove that $\pi \cap(A \times A)=\pi_{1}$, we verify that the conditions (1) $\prime$, (2) $\prime$ and (3) $\prime$ of Lemma 3.1.4(i) hold.
(1) holds: let

$$
(e,[\bar{e}, s, \bar{f}], f) \pi_{1}(i,[\bar{i}, s, \bar{j}], j) .
$$

Direct calculation shows that

$$
\alpha^{-1} \circ(e,[\bar{e}, s, \bar{f}], f)=(u,[\bar{u}, u s, \bar{f}], f)
$$

and

$$
\alpha^{-1} \circ(i,[\bar{i}, s, \bar{j}], j)=(u,[\bar{u}, u s, \bar{j}], j)
$$

Thus

$$
\alpha^{-1} \circ(e,[\bar{e}, s, \bar{f}], f) \pi_{1} \alpha^{-1} \circ(i,[\bar{i}, s, \bar{j}], j)
$$

as required.
(2) holds: let

$$
(e,[\bar{e}, s, \bar{f}], f) \pi_{1}(i,[\bar{i}, s, \bar{j}], j)
$$

Direct calculation shows that

$$
(e,[\bar{e}, s, \bar{f}], f) \circ \alpha=(e,[\bar{e}, s u, \bar{u}], u)
$$

and

$$
(i,[\bar{i}, s, \bar{j}], j) \circ \alpha=(i,[\bar{i}, s u, \bar{u}], u) .
$$

Thus

$$
(e,[\bar{e}, s, \bar{f}], f) \circ \alpha=(i,[\bar{i}, s, \bar{j}], j) \circ \alpha
$$

as required.
(3)/ holds: let

$$
(\bar{e},[\bar{e}, s, \bar{f}], \bar{f}) \pi_{2}(\bar{i},[\bar{i}, t, \bar{j}], \bar{j}) .
$$

Thus by definition

$$
[\bar{e}, s, \bar{f}]=[\bar{i}, t, \bar{j}] .
$$

Let

$$
\beta=(\bar{a},[\bar{a}, a, \bar{a}], a)
$$

and

$$
\gamma=(\bar{b},[\bar{b}, b, \bar{b}], b)
$$

both be elements of $\mathcal{A}$. Then

$$
\beta^{-1} \circ(\bar{e},[\bar{e}, s, \bar{f}], \bar{f}) \circ \gamma=\left(a,\left[\bar{a}, q_{\bar{a}, \bar{e}} s q_{\bar{f}, \bar{b}}, \bar{b}\right], b\right),
$$

and

$$
\beta^{-1} \circ(\bar{i},[\bar{i}, t, \bar{j}], \bar{j}) \circ \gamma=\left(a,\left[\bar{a}, q_{\bar{a}, \bar{i}} q_{\bar{j}, \bar{b}}, \bar{b}\right], b\right) .
$$

But

$$
(\bar{e}, s, \bar{f}) \delta(\bar{i}, t, \bar{j})
$$

by assumption. Thus

$$
\left(\bar{a}, q_{\bar{a}, \bar{e}} s_{\bar{f}, \bar{b}}, \bar{b}\right) \delta\left(\bar{a}, q_{\bar{a}, \bar{i}} t q_{\bar{j}, \bar{b},}, \bar{b}\right) .
$$

But by assumption, $\delta$ is a local isomorphism on $W^{q}$ and so

$$
q_{\bar{a}, \bar{e}} s q_{\bar{f}, \bar{b}}=q_{\bar{a}, \bar{i}} q_{\bar{j}, \bar{b}},
$$

as required.
To prove that $\pi \cap(B \times B)=\pi_{2}$, we verify that the conditions (1) $॥$, (2) $॥$ and (3) $॥$ of Lemma 3.1.4(ii) hold.
(1)/I holds: let

$$
(\bar{e},[\bar{e}, s, \bar{f}], \bar{f}) \pi_{2}(\bar{i},[\bar{i}, t, \bar{j}], \bar{j}) .
$$

Thus $(\bar{e}, s, \bar{f}) \delta(\bar{i}, t, \bar{j})$. Then

$$
\alpha \circ(\bar{e},[\bar{e}, s, \bar{f}], \bar{f})=\left(\bar{u},\left[\bar{u}, q_{\bar{u}, \bar{e}} s, \bar{f}\right], \bar{f}\right)
$$

and

$$
\alpha \circ(\bar{i},[\bar{i}, t, \bar{j}], \bar{j})=\left(\bar{u},\left[\bar{u}, q_{\bar{u}, i}, \bar{j}\right], \bar{j}\right) .
$$

$\operatorname{But}(\bar{e}, s, \bar{f}) \delta(\bar{i}, t, \bar{j})$ implies that $(\bar{u}, u, \bar{u})(\bar{e}, s, \bar{f}) \delta(\bar{u}, u, \bar{u})(\bar{i}, t, \bar{j})$. Thus $\left[\bar{u}, q_{\bar{u}, \bar{e}}, \bar{f}\right]=\left[\bar{u}, q_{\bar{u}, \bar{i}} t, \bar{j}\right]$. Hence

$$
\alpha \circ(\bar{e},[\bar{e}, s, \bar{f}], \bar{f}) \pi_{2} \alpha \circ(\bar{i},[\bar{i}, t, \bar{j}], \bar{j})
$$

as required.
(2)/I holds: let

$$
(\bar{e},[\bar{e}, s, \bar{f}], \bar{f}) \pi_{2}(\bar{i},[\bar{i}, t, \bar{j}], \bar{j}) .
$$

Thus $(\bar{e}, s, \bar{f}) \delta(\bar{i}, t, \bar{j})$. Then

$$
(\bar{e},[\bar{e}, s, \bar{f}], \bar{f}) \circ \alpha^{-1}=\left(\bar{e},\left[\bar{e}, s q_{\bar{f}, \bar{u}}, \bar{u}\right], \bar{u}\right)
$$

and

$$
(\bar{i},[\bar{i}, t, \bar{j}], \bar{j}) \circ \alpha^{-1}=\left(\bar{i},\left[\bar{i}, t q_{\bar{j}, \bar{u}} \bar{u}\right], \bar{u}\right) .
$$

But

$$
(\bar{e}, s, \bar{f}) \delta(\bar{i}, t, \bar{j})
$$

implies that

$$
(\bar{e}, s, \bar{f})(\bar{u}, u, \bar{u}) \delta(\bar{i}, t, \bar{j})(u, u, \bar{u}) .
$$

Thus $\left[\bar{e}, s q_{\bar{f}, \bar{u}}, \bar{u}\right]=\left[\bar{i}, t q_{\bar{j}, \bar{u}} \bar{u}\right]$. Hence

$$
(\bar{e},[\bar{e}, s, \bar{f}], \bar{f}) \circ \alpha^{-1} \pi_{2}(\bar{i},[\bar{i}, t, \bar{j}], \bar{j}) \circ \alpha^{-1}
$$

as required.
(3)॥ holds: let

$$
(e,[\bar{e}, s, \bar{f}], f) \pi_{1}(i,[\bar{i}, t, \bar{j}], j) .
$$

Thus $s=t$. Let $\beta=(\bar{a},[\bar{a}, a, \bar{a}], a)$ and $\gamma=(\bar{b},[\bar{b}, b, \bar{b}], b)$ be elements of $\mathcal{A}$. Then

$$
\beta \circ(e,[\bar{e}, s, \bar{f}], f) \circ \gamma^{-1}=(\bar{a},[\bar{a}, a s b, \bar{b}], \bar{b})
$$

and

$$
\beta \circ(i,[\bar{i}, t, \bar{j}], j) \circ \gamma^{-1}=(\bar{a},[\bar{a}, a t b, \bar{b}], \bar{b}) .
$$

But $s=t$. Thus $(\bar{a},[\bar{a}, a s b, \bar{b}], \bar{b})=(\bar{a},[\bar{a}, a t b, \bar{b}], \bar{b})$ which trivially implies that these two elements are $\pi_{2}$-related.

The following lemma will enable us to state our main theorem.
Lemma 4 Let $\theta: V \rightarrow W$ be a surjective homomorphism.
(i) If $V$ has local units so does $W$.
(ii) If $U \subseteq V$ is a subsemigroup such that $U=U V U$, then $\theta(U)=$ $\theta(U) W \theta(U)$.
(iii) If $V$ is an enlargement of the subsemigroup $U$, then $W$ is an enlargement of $\theta(U)$.
(iv) If $e$ is an idempotent in $V$ such that $V=V e V$, then $W=$ $W \theta(e) W$.

Proof (i) Straightforward.
(ii) To prove $\theta(U) W \theta(U)=\theta(U)$, it is enough to prove that $\theta(U) W \theta(U) \subseteq \theta(U)$. Let $\theta(u), \theta\left(u^{\prime}\right) \in \theta(U)$ and $w \in W$. Since $\theta$ is onto $W$ there exists $v \in V$ such that $\theta(v)=w$. Now $u v u^{\prime} \in$ $U W U=U$. Thus $\theta\left(u v u^{\prime}\right) \in \theta(U)$. It follows that the inclusion holds.
(iii) Given (ii), it is enough to prove that $W \theta(U) W=W$; in fact, it is enough to prove that $W \subseteq W \theta(U) W$. Let $w \in W$. Then since $\theta$ is surjective there exists $v \in V$ such that $\theta(v)=w$. But $V=V U V$. Thus $v=v^{\prime} u v^{\prime \prime}$ for some $v^{\prime}, v^{\prime \prime} \in V$ and $u \in U$. Thus $w=\theta(v)=\theta\left(v^{\prime}\right) \theta(u) \theta\left(v^{\prime \prime}\right) \in W \theta(U) W$.
(iv) Immediate.

The key result which we deduce from Section 3.1 and the results of this section is as follows.

Theorem 5 Let $S$ be a semigroup with local units. Suppose that the category $C(S)$ is equipped with a consolidation $q$, such that $C(S)^{q}$ admits a strict local isomorphism onto a semigroup $T$. Then both $S$ and $T$ can be embedded in a semigroup with local units $N$ in such a way that $N$ is an enlargement of both $S$ and $T$.

Proof By Proposition 1, we can construct a bipartite category $M$ equipped with a consolidation r containing subsemigroups $A^{\mathrm{r}}$ and $B^{\mathbf{r}}$. By Lemma 3.1.2, $M^{\mathbf{r}}$ is an enlargement of both $A^{\mathbf{r}}$ and $B^{\mathrm{r}}$. By Lemma 2, there is a surjective strict local isomorphism $\theta: A^{\mathbf{r}} \rightarrow S$ and a surjective strict local isomorphism $\phi: B^{\mathrm{r}} \rightarrow T$.

Put $\pi_{1}=\operatorname{ker}(\theta)$ and $\pi_{2}=\operatorname{ker}(\phi)$. By Theorem 3, the congruence $\pi$ generated by $\pi_{1} \cup \pi_{2}$ is such that $\pi \cap(A \times A)=\pi_{1}$ and $\pi \cap(B \times B)=\pi_{2}$. Put $N=M^{\mathrm{r}} / \pi$. Then $S$ is isomorphic to $S^{\prime}=A^{\mathrm{r}} / \pi$ and $T$ is isomorphic to $T^{\prime}=B^{\mathrm{r}} / \pi$. But by Lemma 4, $N$ is a semigroup with local units which is an enlargement of both $S^{\prime}$ and $T^{\prime}$.

We can now provide our first characterisation theorem.
Theorem 6 Let $S$ be a semigroup with local units in which each local submonoid has commuting idempotents. Then the following are equivalent:
(i) $S$ has a McAlister sandwich function.
(ii) There is a semigroup $N$ with local units which is an enlargement of both $S$ and a semigroup with local units $T$ having commuting idempotents.

Proof (i) $\Rightarrow$ (ii). Let $S$ a semigroup with local units in which each local submonoid has commuting idempotents, which is equipped with a McAlister sandwich function. By Theorem 2.4.2, the category $C(S)$ can be equipped with a consolidation $q$ in such a way that $C(S)^{q}$ admits a strict local isomorphism onto a semigroup $T$ with local units and commuting idempotents. Theorem 5 now delivers the desired conclusion.
(ii) $\Rightarrow$ (i). Let $S$ be a semigroup with local units in which each local submonoid has commuting idempotents. Suppose that $S$ can be embedded into a semigroup with local units $N$ satisfying the stated conditions. Then $S$ has a McAlister sandwich function by Theorem 2.7.5.

The following is immediate by the preceding theorem and Section 2.6.

Corollary 7 Let $S$ be a semigroup with local units whose regular elements form a subsemigroup. Suppose that the local submonoids
of $S$ have commuting idempotents Then $S$ can be embedded in a semigroup with local units $N$ in such a way that $N$ is an enlargement of $S$ and $N$ is an enlargement of a subsemigroup with local units $T$ having commuting idempotents.

We conclude by refining our characterisation theorem using the ideas described by McAlister in [22].

Theorem 8 Let $S$ be a semigroup with local units having locally commuting idempotents. Then the following are equivalent:
(i) S has a McAlister sandwich function.
(ii) There is a semigroup with local units $P$ equipped with an idempotent $e$ such that $P=P e P$ and ePe has commuting idempotents into which $S$ can be embedded so that $S=S P S$.

Proof The implication (ii) $\Rightarrow$ (i) follows from Corollary 2.7.6. We may therefore concentrate on the implication (i) $\Rightarrow$ (ii). Thus $S$ is a semigroup with local units having locally commuting idempotents and equipped with a McAlister sandwich function. By Theorem 6(ii), we can embed $S$ into a semigroup with local units $N$ which is an enlargement of $S$, and $N$ is an enlargement of a subsemigroup $T$ which has local units and commuting idempotents. If $T$ has an identity then there is nothing to prove. So we assume that $T$ does not have an identity. By Theorem 5, we have that $N=M^{\mathrm{r}} / \pi$. We may now follow McAlister's arguments in [22] because they at no point use regularity. Consider the semigroup $M^{\mathrm{r}}$. Exactly as in Lemma 5.3 of [22] we may define a pair $(\lambda, \rho)$ of linked left and right idempotent translations of $M$. Thus an idempotent $\omega$ may be adjoined to $M$ to obtain a semigroup $\bar{M}=M^{\omega}$ which contains $M$ as an ideal. Evidently $\bar{M}$ has local units. The semigroup $\bar{M}$ contains $A$ as a subsemigroup; observe that

$$
A \bar{M} A=A M A \cup A \omega A=A
$$

since $A=A M A$ and $A \omega A \subseteq A$ from the definition of $\omega$ in [22] and the form of the consolidation multiplication in $M$. Also $\omega \bar{M} \omega=$
$B \cup \omega$, which is essentially $B$ with an identity adjoined, and $\bar{M}=$ $\bar{M} \omega \bar{M}$ exactly as in [22]. By Lemma 5.4 of [22], $\rho=\pi \cup\{(\omega, \omega)\}$ is a congruence on $\bar{M}$. Let $P=\bar{M} / \rho$, and let $e=\rho(\omega)$. By Lemma 4, both $S=S P S$ and $P=P e P$. It is clear that $e P e$ is isomorphic to $T^{1}$ and so has commuting idempotents.

## Chapter 4

## Variants of regular semigroups

This chapter is independent of the preceding chapters, and is concerned once more with regular semigroups. Let $S$ be a regular semigroup, and let $a \in S$. Then a variant of $S$ with respect to $a$ is a semigroup with underlying set $S$ and multiplication o defined by $x \circ y=x a y$. In this chapter, we characterise those elements $a$ such that $(S, \circ)$ is also regular, and show that the set of such elements can function as a replacement for the unit group when $S$ does not have an identity. We also investigate the structure of arbitrary variants of regular semigroups concentrating on how the local structure of $S$ affects the structure of its variants. We raise a number of questions concerning the properties of regularity-preserving elements.

### 4.1 Introduction

Let $S$ be a semigroup and $a \in S$. A new product o may be defined on $S$ by putting $x \circ y=x a y$. It is clear that $(S, \circ)$ is a semigroup; it is called a variant of $S$. We usually write $(S, a)$ rather than $(S, \circ)$ to make the element $a$ explicit.

Variants of abstract semigroups were first studied by John Hickey
[9], although variants of concrete semigroups of relations had earlier been considered by Magill [16]. It is natural to wonder whether variants have applications to semigroup theory in general; the answer is in the affirmative, and here are three examples. First of all, variants arise naturally in connection with Rees matrix semigroups. Let $M=M(S ; I, \Lambda ; P)$ be a Rees matrix semigroup, let $i \in I$ and $\lambda \in \Lambda$, and put

$$
M_{i, \lambda}=\{(i, s, \lambda): s \in S\}
$$

Then it is easy to check that $M_{i, \lambda}$ is a subsemigroup of $M$ isomorphic to $\left(S, p_{\lambda i}\right)$. Indeed, $M$ is a disjoint union of such semigroups. Secondly, Hickey showed (Theorem 5.1 of [9]) that variants could be used to provide a natural interpretation of the HartwigNambooripad order (see Section 1.2.3). Thirdly, and the application which forms of the substance of this chapter, variants can be used to generalise the unit group of a semigroup. We now explain what this means.

Although every semigroup $S$ can be converted into a monoid $S^{1}$ by adjoining an identity, this is not always a useful process. For example, if $S$ has the property that every local submonoid belongs to some class of semigroups $\mathcal{C}$ then the same will only be true of $S^{1}$ if $S$ itself belongs to the class $\mathcal{C}$. Another example applies when $S$ is a Rees matrix semigroup: $S^{1}$ will itself not in general be a Rees matrix semigroup. Hickey suggested the following generalisation of the unit group in the case of regular semigroups $S$. An element $a \in S$ is said to be regularity preserving if $(S, a)$ is regular. Denote the set of all regularity-preserving elements of $S$ by $R P(S)$. When the set $R P(S)$ is non-empty it forms a completely simple subsemigroup of $S$ (Theorem 4.4 of [9]): this is the sought-for generalisation of the unit group. We develop this suggestion further in Section 4.2. We relate this work to the paper of Loganathan and Chandrasekaran [15] and discuss applications to the local structure of regular semigroups.

The remainder of this chapter, Section 4.3, is given over to
determining what we can say about the structure of variants of regular semigroups in general. Our point of departure here was Hickey's result (Lemma 3.4 of [9]) that every variant of a completely simple semigroup is isomorphic to a rectangular group; we show that in a number of cases local structure of a semigroup becomes global in variants.

Let $\mathcal{C}$ be a class of semigroups. Then $S$ is said to be locally $\mathcal{C}$ if each local submonoid of $S$ belongs to $\mathcal{C}$.

### 4.2 Regularity-preserving elements

In Section 4.1, we defined the set of regularity-preserving elements $R P(S)$ of a regular semigroup $S$, and noted that this set is supposed to generalise the unit group in monoids. Our first two results provide evidence for this idea.

Proposition 1 Let $S$ be a regular semigroup.
(i) If $S$ is a monoid, then $a \in S$ is regularity-preserving if, and only if, $a$ is invertible.
(ii) If $S$ is inverse, then $a \in S$ is regularity-preserving if, and only if, $S$ is a monoid and $a$ is invertible.

Proof (i) This result follows from Theorem 4.7 of [9]. We give a direct proof for the sake of completeness. Let $S$ be a monoid. Suppose first that $a$ is regularity-preserving. Then 1 is regular in $(S, a)$ and so there exists $x \in(S, a)$ such that $1=1 \circ x \circ 1$. That is $1=1 a x a 1$. Thus $1=a x a$. Now $a(x a)=1$ and $1 a=a$, so that $a \mathcal{R} 1 ;$ and $(a x) a=1$ and $a 1=a$, so that $a \mathcal{L} 1$. Hence $a \mathcal{H} 1$. But $H_{1}$ is just the group of units of $S$, and so $a$ is invertible.

Conversely, suppose that $a$ is invertible. We show that $a$ is regularity-preserving. Let $b \in S$. Then $b=b x b$ for some $x \in S$, and so

$$
b=b 1 x 1 b=b a a^{-1} x a^{-1} a b=b a\left(a^{-1} x a^{-1}\right) a b
$$

that is $b=b \circ\left(a^{-1} x a^{-1}\right) \circ b$. Thus $b$ is regular in $(S, a)$.
(ii) Suppose that the inverse semigroup $S$ has a regularitypreserving element $a$. Let $b \in S$ be an arbitrary element. By assumption, $b$ is regular in $(S, a)$. Thus there exists an element $x \in(S, a)$ such that $b=b \circ x \circ b$ and $x=x \circ b \circ x$. That is $b=b a x a b$ and $x=$ aabax. Multiplying the first equation by $a$ on the left and right we obtain $a b a=(a b a) x(a b a)$ and $x=x(a b a) x$. Thus, since $S$ is inverse, we have that $x=(a b a)^{-1}$. Hence $b=b a(a b a)^{-1} a b$. Thus

$$
b=b a\left(a^{-1} b^{-1} a^{-1}\right) a b=b\left(a a^{-1}\right)\left[b^{-1}\left(a^{-1} a\right) b\right]
$$

and so using the fact that idempotents commute in an inverse semigroup, we obtain

$$
b=b\left[b^{-1}\left(a^{-1} a\right) b\right]\left(a a^{-1}\right)=\left(b b^{-1}\right)\left(a^{-1} a\right) b\left(a a^{-1}\right)=\left(a^{-1} a\right) b\left(a a^{-1}\right) .
$$

Thus $b=a^{-1} a b a a^{-1}$. A special case of this equation is obtained by putting $a=b$. Hence $a=\left(a^{-1} a\right) a\left(a a^{-1}\right)$. Thus $\left(a^{-1} a\right) a=a$, and so $\left(a^{-1} a\right)\left(a a^{-1}\right)=a a^{-1}$; and $a\left(a a^{-1}\right)=a$, and so $\left(a^{-1} a\right)\left(a a^{-1}\right)=$ $a^{-1} a$. It follows that $a a^{-1}=a^{-1} a=e$, say. But we proved that $b=e b e$ for all $b \in S$, so it follows that $S=e S e$. Thus $S$ is a monoid with identity $e$, and so $a$ is invertible.

The converse is immediate by result (i).
Lemma 2 Let $S$ be a regular semigroup. If a is regularity-preserving then $S b S \subseteq S a S$ for every $b \in S$. In particular, $S=$ SaS.

Proof Let $a$ be regularity-preserving. Then any element $b \in S$ is regular in $(S, a)$. Thus there exists $x \in S$ such that $b=b \circ x \circ b$. Thus $b=b a x a b$. It follows that $b \in S a S$ and so $S b S \subseteq S a S$, as required. The fact that $S=S a S$ is now immediate.

Lemma 2 implies that all regularity-preserving elements are $\mathcal{J}$-related, and belong to a maximum $\mathcal{J}$-class. Thus the set of regularity-preserving elements of $S$ is at the 'top' of the semigroup. Furthermore, Proposition 1 shows that when the semigroup $S$ is a monoid, the set $R P(S)$ is the unit group. This constitutes our
motivation for regarding $R P(S)$ as a reasonable generalisation of the unit group to semigroups.

If the completely simple semigroup $R P(S)$ is to be of any use, we need to be able to locate its elements reasonably easily. The following three results provide some solutions to this problem.

Proposition 3 Let $S$ be a regular semigroup. Then $a \in S$ is regularity-preserving if and only if ba $\mathcal{R} b \mathcal{L}$ ab for every $b \in S$.

Proof Suppose that $b a \mathcal{R} b \mathcal{L} a b$ for every $b \in S$. We show that $a$ is regularity-preserving. By assumption, there exist $x, y \in S$ such that $b a x=b$ and $y a b=b$. Let $b^{\prime} \in V(b)$. Then

$$
b=b b^{\prime} b=(b a x) b^{\prime}(y a b)=b \circ\left(x b^{\prime} y\right) \circ b .
$$

Thus $b$ is regular in $(S, a)$. The converse is proved in Lemma 4.2 of [9].

The following result telis us that in fact we need only locate the regularity-preserving idempotents.

Lemma 4 Let $S$ be a regular semigroup. Then $a \in S$ is regularitypreserving if, and only if, a $\mathcal{H} e$ where $e$ is a regularity-preserving idempotent.

Proof Let $a$ be a regularity-preserving element in $S$. We have by Lemma 4.3(i) of [9], that the $\mathcal{H}$-class of $a$ is a group and so $a \mathcal{H} e$ for some idempotent $e$ in $S$. Now, $a \in e S \cap S e$ and so by Lemma 4.3(ii) of [9] the idempotent $e$ is regularity-preserving.

Conversely, suppose that $a \mathcal{H} e$ where $e$ is a regularity-preserving idempotent. Then $S a=S e$ and $a S=e S$ so that $e \in e S \cap S e=$ $a S \cap S a$. Thus $a$ is regularity-preserving by Lemma 4.3(ii) of [9].

Our main result on characterising regularity-preserving idempotents is the following.

Theorem 5 Let $S$ be a regular semigroup, and let $e \in E(S)$. Then the following are equivalent:
(i) $e$ is regularity-preserving.
(ii) fe $\mathcal{R} f \mathcal{L}$ ef for every $f \in E(S)$.
(iii) $V(f) \cap e S e \neq \emptyset$ for every $f \in E(S)$.
(iv) $V(a) \cap e S e \neq \emptyset$ for every $a \in S$.

Proof (i) $\Rightarrow$ (ii). This is immediate from Proposition 3.
(ii) $\Rightarrow$ (iii). Let $f \in E(S)$. Then by assumption, there exist elements $x, y \in S$ such that $f e x=f$ and yef $=f$. Observe that

$$
f(e x f y e) f=(f e x) f(y e f)=f f=f
$$

and

$$
(e x f y e) f(e x f y e)=e x f(y e f) e x f y e=e x(f e x) f y e=e x f y e
$$

Thus exfye $\in V(f) \cap e S e$.
(iii) $\Rightarrow$ (iv). Let $a \in S$ and $a^{\prime} \in V(a)$. By (iii), there are elements $u \in V\left(a a^{\prime}\right) \cap e S e$ and $v \in V\left(a^{\prime} a\right) \cap e S e$. Consider the element $v a^{\prime} u$. Observe that

$$
a\left(v a^{\prime} u\right) a=a\left[\left(a^{\prime} a\right) v\left(a^{\prime} a\right)\right] a^{\prime} u a=a a^{\prime} u a=\left(a a^{\prime}\right) u\left(a a^{\prime}\right) a=a
$$

and
$\left(v a^{\prime} u\right) a\left(v a^{\prime} u\right)=v a^{\prime}\left[\left(a a^{\prime}\right) u\left(a a^{\prime}\right)\right] a\left(v a^{\prime} u\right)=v\left[\left(a^{\prime} a\right) v\left(a^{\prime} a\right)\right] a^{\prime} u=v a^{\prime} u$.
Thus $v a^{\prime} u \in V(a)$. But $u e=u$ and $e v=v$. Hence $V(a) \cap e S e \neq \emptyset$.
(iv) $\Rightarrow$ (i). Let $a \in S$. Then by assumption, there is $a^{\prime} \in$ $V(a) \cap e S e$. Thus $a=a a^{\prime} a=a\left(e a^{\prime} e\right) a=a \circ a^{\prime} \circ a$. Thus $a$ is regular in ( $S, e$ ). Hence $e$ is regularity-preserving.

Property (ii) above provides a link between this paper and the one of Loganathan and Chandrasekaran [15]. If $e$ is a regularitypreserving idempotent of the regular semigroup $S$, then the function $\theta_{e}: S \rightarrow e S e$ defined by $\theta_{e}(s)=e s e$ is an example of a 'split
map'. Furthermore, split maps of this type play an important role in the general theory (Theorem 1.8 of [15]). The following result is just a reinterpretation of Example 1.4 and Lemma 1.5 of [15] in the light of our Theorem 5.

Proposition 6 Let $S$ be a regular semigroup and $e$ an arbitrary idempotent. Put

$$
S_{e}=\{a \in S: V(a) \cap e S e \neq \emptyset\}
$$

Then $S_{e}$ is the largest regular subsemigroup $U$ of $S$ containing $e$ such that $(U, e)$ is regular.

Proof The set $S_{e}$ is non-empty because it always contains $e$. Let $a, b \in S_{e}$. Then there exists $a^{\prime} \in V(a) \cap e S e$ and $b^{\prime} \in V(b) \cap$ $e S e$. Let $g \in S\left(a^{\prime} a, b b^{\prime}\right)$, the sandwich set of $a^{\prime} a$ and $b b^{\prime}$. Then by Proposition 1.2.5, we have that $b^{\prime} g a^{\prime} \in V(a b)$. Thus $b^{\prime} g a^{\prime} \in$ $V(a b) \cap e S e$. It follows that $S_{e}$ is closed under multiplication. Now let $a \in S_{e}$. Then there exists $a^{\prime} \in V(a) \cap e S e$. Observe that $e a e \in e S e$ and $a^{\prime}(e a e) a^{\prime}=a^{\prime} a a^{\prime}=a^{\prime}$ and $(e a e) a^{\prime}(e a e)=e\left(a a^{\prime} a\right) e=$ eae. Thus eae $\in V\left(a^{\prime}\right) \cap e S e$. It follows that $a^{\prime} \in S_{e}$. Hence $S_{e}$ is a regular subsemigroup of $S$ containing $e$. To show that $\left(S_{e}, e\right)$ is regular, let $a \in S_{e}$ and choose $a^{\prime} \in V(a) \cap e S e$. Then $a=a a^{\prime} a=a e a^{\prime} e a=a \circ a^{\prime} \circ a$ in $\left(S_{e}, e\right)$. Thus $\left(S_{e}, e\right)$ is regular.

Now let $T$ be any regular subsemigroup of $S$ containing $e$ such that $(T, e)$ is regular. Let $a \in T$. Then by assumption $a$ is regular in $(T, e)$. Thus there exists $x \in T$ such that $a=a \circ x \circ a$ and $x=x \circ a \circ x$. Thus $a=$ aexea and $x=$ xeaex. Multiplying the second equation on the left and right by $e$ we obtain exe $=($ exe $) a(e x e)$. Thus exe $\in V(a) \cap e S e$. Hence $a \in S_{e}$, and so $T \subseteq S_{e}$.

It is immediate that the idempotent $e$ is regularity-preserving if and only if $S_{e}=S$.

A number of different classes of regularity-preserving idempotents have been considered in the literature. Before examining
them, we need some definitions. Let $S$ be a regular semigroup. An idempotent $e$ is said to be medial if aea $=a$ for each $a \in\langle E(S)\rangle$; it is said to be normal medial if it is medial and $e\langle E(S)\rangle e$ is a semilattice; it is said to be a mididentity if $a e b=a b$ for all $a, b \in S$. The following result is trivial but makes the connection with regularitypreserving elements.

Lemma 7 Let $S$ be a regular semigroup. Then both medial idempotents and mididentities are regularity-preserving.

Proof Let $e$ be a medial idempotent. Let $f \in E(S)$. Then $f e f=$ $f$. It follows that fe $\mathcal{R} f \mathcal{L} e f$. Thus $e$ is regularity preserving by Theorem 5(ii).

The result concerning mididentities follows from Theorem 4.7 of [9]. We give our proof for the sake of completeness. Let $e$ be a mididentity. Then $f e f=f$ once again. Thus $e$ is regularitypreserving.

Blyth and McFadden [3] show how to construct all idempotentgenerated regular semigroups containing a medial idempotent, and all regular semigroups containing a normal medial idempotent. Our result Lemma 7 implies that these results can be interpreted as determining the structure of classes of regular semigroups equipped with various kinds of regularity-preserving idempotents. If $S$ contains a mididentity then $R P(S)$ is an orthodox completely simple semigroup by Corollary 4.8 of [9]; this result suggests studying the relationship between the existence of special kinds of regularitypreserving elements of $S$ and the structure of $R P(S)$, although we shall not pursue this question here.

We noted in the introduction that semigroups with local properties are a natural class in which to investigate regularity-preserving elements. We consider one example of this now. Let $S$ be a regular semigroup and let $T$ be an inverse subsemigroup of $S$. Following [23], we say that $T$ is an inverse transversal of $S$ if for each $s \in S$ the set $V(s) \cap T$ contains exactly one element.

Theorem 8 Let $S$ be a locally inverse regular semigroup. Then the idempotent $e \in S$ is regularity-preserving if and only if eSe is an inverse transversal of $S$.

Proof Let $e$ be a regularity-preserving idempotent. Since $S$ is locally inverse $e S e$ is an inverse semigroup. By Theorem 5(iv), the set $V(s) \cap e S e$ contains at least one element for each $s \in S$. We shall prove that it contains exactly one element. Let $u, v \in V(s) \cap e S e$. Then

$$
s=s u s, u=u s u \text { and } s=s v s, v=v s v .
$$

Observe that esu,esv $\in E(e S e)$. Since $e S e$ is inverse, $(e s u)(e s v)=$ $(e s v)(e s u)$, and so $e(s u s) v=e(s v s) u$. But $s=s u s$ and $s=s v s$, and so $e s v=e s u$. Multiplying on the left by $v$ and using $v s v=v$ we obtain $v=v s u$. Next observe that use, vse $\in E(e S e)$, and so $(u s e)(v s e)=(v s e)(u s e)$. It follows that $u(s v s) e=v(s u s) e$, and so $u s e=v s e$. Multiplying on the right by $u$, we obtain $u s u=v s u$, and so $u=v s u$. Hence $u=v$, as required.

The converse is immediate by the definition of an inverse transversal and Theorem 5(iv).

If $S$ is a regular subsemigroup of a regular semigroup $T$, then we say that it is a quasi-ideal if $S=S T S$. Observe that the locally inverse semigroups above have the further property that $e S e$ is also a quasi-ideal of $S$. Theorem 2.2 of [23] can now be stated as follows: a regular semigroup $S$ contains a quasi-ideal inverse transversal if and only if $S$ can be embedded as an ideal in a locally inverse semigroup with a regularity-preserving idempotent; furthermore, Theorem 3.2 of [23] can be stated as: the regularity-preserving idempotent in question is a normal medial idempotent if and only if $S$ in fact contains a multiplicative inverse transversal.

We put the above results in a more general context. From Lemma 1.2.6 and Theorem 1.3.4, a regular semigroup $S$ is locally inverse if and only if $S$ is a quasi-ideal of a locally inverse semigroup $T$ such that $T=T e T$. If we strengthen 'quasi-ideal' to 'ideal',
require that the idempotent $e$ be regularity-preserving, and require $S$ to contain a quasi-ideal inverse transversal, then we have the first of McAlister's two results above. Thus regularity-preserving elements can lead to special cases of the main results of this thesis.

Given that regularity-preserving elements are useful, it is natural to enquire into the structure of regular semigroups possessing them: we would expect that they should be constructed from regular monoids. One approach to this question may be deduced from [15]: particularly, Theorem 3.1 and Corollary 3.2. Another approach, which we set up here, is based on Rees matrix semigroups.

Let $S$ be a regular semigroup and $e$ a regularity-preserving idempotent. Then by Lemma $2, S=S e S$. By the Local Structure Theorem (Theorem 1.3.2), the semigroup $S$ is a locally isomorphic image of a regular Rees matrix semigroup over $e S e$; thus there is a surjective local isomorphism $\theta: \mathcal{R} \mathcal{M}(e S e ; I, \Lambda ; P) \rightarrow S$. The relationship between the regularity-preserving elements of $\mathcal{R M}=$ $\mathcal{F} \mathcal{M}(e S e ; I, \Lambda ; P)$ and those of $S$ is described by the following result.

Lemma 9 Let $\theta: T \rightarrow S$ be a surjective local isomorphism between regular semigroups. Then $t \in T$ is regularity-preserving in $T$ if, and only if, $\theta(t)$ is regularity-preserving in $S$.

Proof Let $t \in T$ be regularity-preserving in $T$. Let $s \in S$ be arbitrary. Since $\theta$ is surjective there exists $u \in T$ such that $\theta(u)=s$. By assumption, $u$ is regular in $(T, t)$. Thus there exists $a \in T$ such that $u=u \circ a \circ u=u t a t u$. Hence $s=s \theta(t) \theta(a) \theta(t) s$. That is, $s=s \circ \theta(a) \circ s$ in $(S, \theta(t))$. It follows that $s$ is regular in $(S, \theta(t))$ and so $\theta(t)$ is regularity-preserving in $S$.

Conversely, let $s \in S$ be a regularity-preserving element, and let $t \in T$ be any element of $T$ such that $\theta(t)=s$. We shall prove that $t$ is regularity-preserving in $T$. Let $a \in T$ be any element. By assumption, $\theta(a)$ is regular in $(S, s)$. Thus there exists $x \in S$. such that $\theta(a)=\theta(a) \circ x \circ \theta(a)$. That is $\theta(a)=\theta(a) \operatorname{sxs} \theta(a)$. Let $y \in T$ be such that $\theta(y)=x$. Then $\theta(a)=\theta(a) \theta(t) \theta(y) \theta(t) \theta(a)$

Thus $\theta(a)=\theta($ atyta $)$. Let $a^{\prime} \in V(a)$. Then $a$, atyta $\in a a^{\prime} T a^{\prime} a$. But $\theta$ is a local isomorphism between regular semigroups. Thus by definition, we have that $a=$ atyta. Thus $a=a \circ y \circ a$ in $(T, t)$. It follows that $t$ is regularity-preserving.

From the calculation preceding Lemma 9, and from Lemma 9 itself, we have that $R P(\mathcal{R} \mathcal{M})=\theta^{-1}(R P(S))$. This result suggests that we investigate the regularity-preserving elements of regular Rees matrix semigroups. To do this, we shall need the following definitions. Let $S$ be a monoid. We say that $a \in S$ is left invertible if there is an element $a^{*}$ such that $a^{*} a=1$; we say that $a$ is right invertible if there is an element $a^{\dagger}$ such that $a a^{\dagger}=1$.

Theorem 10 Let $S$ be a regular monoid and let

$$
\mathcal{R} \mathcal{M}=\mathcal{R} \mathcal{M}(S ; I, \Lambda ; P)
$$

be a regular Rees matrix semigroup over $S$.
(i) Let $(i, a, \lambda) \in \mathcal{R} \mathcal{M}$ be such that $a$ is invertible, every element in the ith-column of $P$ is right-invertible, and every element in the $\lambda$ th-row of $P$ is left-invertible. Then $(i, a, \lambda)$ is a regularitypreserving element of $\mathcal{R} \mathcal{M}$.
(ii) If each row and each column of $P$ contains an invertible element, then every element of $\mathcal{M}=\mathcal{M}(S ; I, \Lambda ; P)$ is regular and the regularity-preserving elements of $\mathcal{M}$ are precisely those of the form described in (i).

Proof (i) Let $(j, b, \mu)$ be an arbitrary element of $\mathcal{R} \mathcal{M}$. We show that $(j, b, \mu)$ is regular in the variant $(\mathcal{R} \mathcal{M},(i, a, \lambda))$. Let $b^{\prime} \in V(b)$. By assumption, $p_{\lambda i}$ is invertible with inverse $p_{\lambda i}^{-1}$, the element $p_{\mu i}$ has a right inverse $p_{\mu i}^{\dagger}$, and $p_{\lambda j}$ has a left inverse $p_{\lambda j}^{*}$. Consider the element $(i, x, \lambda)$ where

$$
x=p_{\lambda i}^{-1} a^{-1} p_{\mu i}^{\dagger} b^{\prime} p_{\lambda j}^{*} a^{-1} p_{\lambda i}^{-1}
$$

Firstly, $(i, x, \lambda) \in \mathcal{R} \mathcal{M}$ because

$$
(i, x, \lambda)\left(i, a p_{\lambda j} b p_{\mu i} a, \lambda\right)(i, x, \lambda)=\left(i, x p_{\lambda i} a p_{\lambda j} b p_{\mu i} a p_{\lambda i} x, \lambda\right)
$$

and

$$
x p_{\lambda i} a p_{\lambda j} b p_{\mu i} a p_{\lambda i} x=x
$$

as direct computation shows. Secondly, in $(\mathcal{R M},(i, a, \lambda))$

$$
(j, b, \mu) \circ(i, x, \lambda) \circ(j, b, \mu)=(j, b, \mu)(i, a, \lambda)(i, x, \lambda)(i, a, \lambda)(j, b, \mu)
$$

which is equal to $\left(j, b p_{\mu i} a p_{\lambda i} x p_{\lambda i} a p_{\lambda j} b, \mu\right)$. But

$$
b p_{\mu i} a p_{\lambda i} x p_{\lambda i} a p_{\lambda j} b=b b^{\prime} b=b .
$$

Thus ( $i, a, \lambda$ ) is regularity-preserving.
(ii) Suppose now that each row and each column of $P$ contains an invertible element. By Theorem 4 of [17], the semigroup $\mathcal{M}(S ; I, \Lambda ; P)$ is automatically regular. Let $(i, a, \lambda)$ be a regularitypreserving element of $\mathcal{M}(S ; I, \Lambda ; P)$. We prove that $(i, a, \lambda)$ satisfies the conditions of (i) above. Firstly, we show that $a$ is invertible. By assumption, there is $j \in I$ such that $p_{\lambda j}$ is invertible. Also by assumption, $(j, 1, \lambda)$ is regular in $(\mathcal{M},(i, a, \lambda))$. Thus there exists $(k, b, \nu) \in \mathcal{M}$ such that

$$
(j, 1, \lambda)=(j, 1, \lambda) \circ(k, b, \nu) \circ(j, 1, \lambda)
$$

which is equal to

$$
(j, 1, \lambda)(i, a, \lambda)(k, b, \nu)(i, a, \lambda)(j, 1, \lambda)
$$

which gives $1=p_{\lambda i} a p_{\lambda k} b p_{\nu i} a p_{\lambda j}$ and so $1=\left(p_{\lambda j} p_{\lambda i} a p_{\lambda k} b p_{\nu i}\right) a$ using the fact that $p_{\lambda j}$ is invertible. Thus $a$ has a left inverse. Similarly, there is $\omega \in \Lambda$ such that $p_{\omega i}$ is invertible and $(i, 1, \omega)$ is regular in $(\mathcal{M},(i, a, \lambda))$. A similar argument to the above leads to the conclusion that $a$ has a right inverse. It follows that $a$ is invertible.

By Lemma $4,(i, a, \lambda)$ is $\mathcal{H}$-related to a regularity-preserving idempotent. This will be of the form $(i, c, \lambda)$ where $c \mathcal{H} a$. Thus $c$ is invertible. It follows that $c=p_{\lambda i}^{-1}$. In particular, $p_{\lambda i}$ is invertible.

We now prove that every element in the $\lambda$ th-row of $P$ is leftinvertible, and every element in the $i$ th-column of $P$ is right-invertible.

Let $k \in I$. We prove that $p_{\lambda k}$ has a left inverse. By assumption, $(k, 1, \lambda)$ is regular in $\left(\mathcal{M},\left(i, p_{\lambda i}^{-1}, \lambda\right)\right)$. Thus there exists $(m, d, \xi)$ such that $(k, 1, \lambda)=(k, 1, \lambda) \circ(m, d, \xi) \circ(k, 1, \lambda)$. That is

$$
(k, 1, \lambda)=(k, 1, \lambda)\left(i, p_{\lambda i}^{-1}, \lambda\right)(m, d, \xi)\left(i, p_{\lambda i}^{-1}, \lambda\right)(k, 1, \lambda) .
$$

It follows that $1=\left(p_{\lambda m} d p_{\xi i} p_{\lambda i}^{-1}\right) p_{\lambda k}$, and so $p_{\lambda k}$ has a left inverse.
Let $\rho \in \Lambda$. We prove that $p_{\rho i}$ has a right inverse. By assumption, $(i, 1, \rho)$ is regular in $\left(\mathcal{M},\left(i, p_{\lambda i}^{-1}, \lambda\right)\right)$. Thus there exists $(m, d, \xi)$ such that $(i, 1, \rho)=(i, 1, \rho) \circ(m, d, \xi) \circ(i, 1, \rho)$. That is

$$
(i, 1, \rho)=(i, 1, \rho)\left(i, p_{\lambda i}^{-1}, \lambda\right)(m, d, \xi)\left(i, p_{\lambda i}^{-1}, \lambda\right)(i, 1, \rho) .
$$

It follows that $1=p_{\rho i}\left(p_{\lambda i}^{-1} p_{\lambda m} d p_{\xi i}\right)$, and so $p_{\rho i}$ has a right inverse.

It is a simple consequence of the above result that the elements ( $i, a, \lambda$ ) in a completely 0 -simple semigroup which are regularitypreserving are precisely those for which the $\lambda$ th-row and $i$ th-column of $P$ contain no zeros. This also provides a simple visual proof of the fact that every element of a completely simple semigroup is regularity-preserving.

We can now provide some examples of naturally occurring regular semigroups possessing regularity-preserving elements.

## Examples

(i) The four-spiral semigroup [4] is the Rees matrix semigroup

$$
\mathcal{M}(T ;\{1,2\},\{1,2\} ; P)
$$

where $T$ is the bicyclic monoid generated by elements $a$ and $b$ subject to $a b=1$ and

$$
P=\left(\begin{array}{ll}
1 & b \\
1 & 1
\end{array}\right)
$$

By Theorem 10, the four-spiral semigroup has the following regularity-preserving elements: namely, $(1,1,1)$ and $(1,1,2)$.
(ii) The semigroup $\mathcal{M}^{0}(\{0,1\} ;\{1,2\},\{1,2\} ; P)$ where

$$
P=\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)
$$

is the smallest non-orthodox regular semigroup [3]. By Theorem 10 , it has exactly one regularity-preserving element: namely, $(1,1,1)$.
(iii) The semigroups $\mathcal{M}^{0}(\{0,1\} ; I, \Lambda ; P)$ and $\mathcal{M}^{0}(\{0,1\} ; I, \Lambda ; Q)$ where $I=\{1,2\}$ and $\Lambda=\{1,2,3\}$ and

$$
P=\left(\begin{array}{ll}
1 & 0 \\
1 & 1 \\
1 & 1
\end{array}\right) \text { and } Q=\left(\begin{array}{cc}
1 & 0 \\
1 & 1 \\
1 & 0
\end{array}\right)
$$

were used to construct counterexamples in the theory of amalgamations [7]. The former has two regularity-preserving elements, the latter has one.
(iv) The semigroup $\mathcal{M}^{0}(\{0,1\} ;\{1,2,3\},\{1,2,3\} ; P)$ where

$$
P=\left(\begin{array}{lll}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{array}\right)
$$

has exactly one regularity-preserving element; it generates the variety of combinatorial strict regular $*$-semigroups [2].

### 4.3 The structure of variants

In [9], Hickey proved (Lemma 3.4) that every variant of a completely simple semigroup is a rectangular group. Rectangular groups are precisely the orthodox completely simple semigroups (see, for example, page 139, Exercise 10(a) of [11]). Clearly, completely simple semigroups are locally orthodox. Thus Hickey's result shows
that in the case of completely simple semigroups local orthodoxy of the semigroup becomes orthodoxy in any variant of the semigroup. In this section, we shall generalise this result.

Lemma 1 Let $S$ be a semigroup, let $a \in S$, and let $i$ be an idempotent such that $a i=a$. Let $e \in E(S, a)$. Then $i e a \in E(i S i)$.

Proof Observe that $(i e a)^{2}=(i e a)(i e a)=i e a e a=i e a$, since $a i=a$ and $e a e=e$ Also $(i e a) i=i e a$ and $i(i e a)=i e a$.

A band $B$ is said to be normal if $x y z x=x z y x$ for all $x, y, z \in B$; a band is said to be rectangular if $x=x y x$ for all $x, y \in B$.

Proposition 2 Let $S$ be a semigroup, let $a \in S$, and let $i$ be an idempotent such that $a i=a$.
(i) If $E(i S i)$ is a band then $E(S, a)$ is a band.
(ii) If $E(i S i)$ is a normal band then $E(S, a)$ is a normal band.
(iii) If $E(i S i)$ is a rectangular band then $E(S, a)$ is a rectangular band.

Proof (i) Let $E(i S i)$ be a band, and let $e, f \in E(S, a)$. Then $e \circ e=$ $e$ and $f \circ f=f$. Thus eae $=e$ and $f a f=f$. We show that $e \circ f$ is an idempotent of $(S, a)$. Thus we shall show that eafaeaf $=e a f$. By Lemma 1, we have that iea, ifa $\in E(i S i)$. Since $i S i$ is orthodox it follows that (iea) $($ ifa $)=$ ieafa is an idempotent of $S$. Thus $($ ieafa $)($ ieafa $)=$ ieafa; that is ieafaeafa $=$ ieafa. Multiplying this equation on the left by ea, we obtain eaeafaeafa $=$ eaeafa. But eae $=e$ and so eafaeafa $=e a f a$. Multiplying this equation on the right by $f$ we obtain eafaeafaf $=$ eafaf. But $f a f=f$ and so eafaeaf $=e a f$, as required.
(ii) Let $E(i S i)$ be a normal band, and let $e, f, g \in E(S, a)$. We shall prove that $e \circ f \circ g \circ e=e \circ g \circ f \circ e$. By assumption, $e a e=e$, $f a f=f, g a g=g$. By Lemma 1, iea, ifa,iga $\in E(i S i)$. Since $E(i S i)$ is a normal band, we have that

$$
(i e a)(i f a)(i g a)(i e a)=(i e a)(i g a)(i f a)(i e a)
$$

Using $a i=a$ this equation simplifies to

$$
i(e a f a g a e) a=i(e a g a f a e) a .
$$

Multiplying on the right by $e$ and using $e a e=e$ we obtain

$$
i(e a f a g a e)=i(\text { eagafae }) .
$$

Multiplying the equation on the left by $a$ and using $a i=a$ we obtain

$$
a(e a f a g a e)=a(\text { eagafae }) .
$$

Multiplying on the left by $e$ and using eae $=e$, we obtain

$$
\text { eafagae }=\text { eagafae }
$$

as required.
(iii) Let $E(i S i)$ be a rectangular band, and let $e, f \in E(S, a)$. Then $e \circ e=e$ and $f \circ f=f$. Thus eae $=e$ and $f a f=f$. We show that $e \circ f \circ e=e$. Thus we shall show that eafae $=e$. By Lemma 1, we have that iea,ifa $\in E(i S i)$. Since $E(i S i)$ is a rectangular band, we have that $($ iea $)(i f a)(i e a)=i e a$. Using $a i=a$ we obtain ieafaea $=$ iea. Multiplying on the left by $e a$ we obtain eafaea $=e a$. Multiplying on the right by $e$ we obtain eafae $=e$, as required.

The reader might have been expecting a result which said that if $E(i S i)$ is commutative then $E(S, a)$ is commutative. The following example shows that this result is not true.

## Example

We use Example 2.2 from [19]. Let $S=\{1>a>b>0\}$ be the four-element chain regarded as a semilattice with respect to the operation of greatest lower bound. Let $I=\Lambda=\{1,2\}$ and let $P$ be the $2 \times 2$-matrix

$$
P=\left(\begin{array}{ll}
1 & a \\
b & 0
\end{array}\right) .
$$

Let $R M=R M(S ; I, \Lambda ; P)$ be the regular Rees matrix semigroup constructed from $S, I, \Lambda$ and $P$. This semigroup has 11 elements namely:

$$
\begin{aligned}
R M= & \{(1,1,1),(1, a, 1),(2, a, 1),(1, b, 1),(1, b, 2),(2, b, 1),(2, b, 2), \\
& (1,0,1),(1,0,2),(2,0,1),(2,0,2)\} .
\end{aligned}
$$

The semigroup $R M$ is a regular semigroup each local submonoid of which is inverse. We show that the idempotents of the variant ( $R M,(2, b, 2))$ do not commute. It is easy to check that

$$
(1,0,2),(1,0,1) \in E(R M,(2, b, 2))
$$

but that

$$
(1,0,2) \circ(1,0,1) \neq(1,0,1) \circ(1,0,2)
$$

Before proving our next result we need some preparation. In [10], Hickey defines the congruence $\delta^{a}$ on the variant $(S, a)$ by

$$
(x, y) \in \delta^{a} \Leftrightarrow a x a=a y a .
$$

A semigroup is locally orthodox if all local submonoids are orthodox.

Theorem 3 Let $S$ be a regular semigroup.
(i) All variants of $S$ are orthodox if, and only if, $S$ is locally orthodox.
(ii) All variants of $S$ have normal bands of idempotents if, and only if, every local submonoid of $S$ has a normal band of idempotents.
(iii) All variants of $S$ have rectangular bands of idempotents if, and only if, every local submonoid of $S$ has a rectangular band of idempotents.

Proof We prove (i); the proofs of (ii) and (iii) are similar. Suppose that $S$ is a regular locally orthodox semigroup. Let $a \in S$. Then
there is an idempotent $i$ such that $a i=a$ (for example, $i=a^{\prime} a$ where $a^{\prime}$ is any inverse of $a$ ). By assumption, $i S i$ is orthodox and so $(S, a)$ is orthodox by Proposition 2.

Conversely, suppose that all variants of $S$ are orthodox. Let $e \in E(S)$. By Lemma 3.2 of [10], the local submonoid $e S e$ is isomorphic to $(S, e) / \delta^{e}$. The function $\psi: S \rightarrow e S e$ given by $\psi(x)=e x e$ is a homomorphism from ( $S, e$ ) onto $e S e$ whose kernel is $\delta^{e}$. Let $i, j \in E(e S e)$. Then $i, j \in(S, e)$ and $\psi(i)=i$ and $\psi(j)=j$. Now in ( $S, e$ ), we have that $i \circ i=i e i=i$ and similarly $j \circ j=j$. Thus both $i$ and $j$ are idempotents in ( $S, e$ ). By assumption, $(S, e)$ is an orthodox semigroup. Thus $i \circ j$ is an idempotent in ( $S, e$ ). Since $\psi$ is a homomorphism $\psi(i \circ j)$ is an idempotent in $e S e$. But $i \circ j=i j$, and $\psi(i j)=i j$. Thus $i j$ is an idempotent in eSe. Hence $e S e$ is orthodox.

Finally, we address the following question. Variants of regular semigroups are not in general regular, but what can we say about the properties of such variants? We need some definitions before we an answer this question.

An element $b$ such that $b=b a b$ is said to be a weak inverse (or post-inverse) of $a$, whereas an element $b$ such that $a=a b a$ is said to be a preinverse of $a$. The set of weak inverses of $a$ will be denoted by $W(a)$, and the set of preinverses of $a$ will be denoted by $P(a)$. Observe that $V(a)=W(a) \cap P(a)$. A semigroup $S$ is said to be $E$-inversive if for each $a \in S$ there exists $x \in S$ such that $a x$ is an idempotent (see [24] for background references). Fountain et al [5] prove that a semigroup is $E$-inversive precisely when for each $a \in S$ there exists $b \in S$ such that $b=b a b$. Thus $E$-inversive semigroups are just those semigroups in which every element has a weak inverse. The class of $E$-inversive semigroups contains both regular semigroups and finite semigroups. The following simple result contains part of Lemma 3.1 of [9].

Proposition 4 Let $S$ be an $E$-inversive semigroup, and let $a \in S$.

Then $(S, a)$ is $E$-inversive, and $E(S, a)=W(a)$.
Proof Let $S$ be an $E$-inversive semigroup and let $a$ be an arbitrary element of $S$. Let $x$ be an element of the variant $(S, a)$. The product $a x a$ has a weak inverse $b$ in the semigroup $S$ and so $b=b(a x a) b$. But then $b=b \circ x \circ b$ and so $b$ is a weak inverse of $x$ in $(S, a)$. Thus $(S, a)$ is $E$-inversive.

The element $b \in(S, a)$ is an idempotent if, and only if, $b=b \circ b$, which means precisely that $b=b a b$. Hence $E(S, a)=W(a)$.

We now turn to the properties of variants of regular semigroups.
Proposition 5 Let $S$ be a regular semigroup, and let $a \in S$. Then $(S, a)$ is $E$-inversive, and the regular elements of $(S, a)$ form a regular subsemigroup.

Proof Let $S$ be a regular semigroup and let $a \in S$. We proved in Proposition 4 that $(S, a)$ is $E$-inversive. To show that the regular elements of a semigroup form a subsemigroup it is enough to prove that the product of any two idempotents is regular (Result 2 of [6]). Let $b, c \in E(S, a)$. Then $b a b=b$ and $c a c=c$. We prove that $b \circ c$ is a regular element in $(S, a)$. Consider the product abaca in $S$. Since $S$ is regular there is an element $x$ such that (abaca) $x(a b a c a)=a b a c a$. Multiplying this equation on the left by $b$ and on the right by $c$, we obtain $b(a b a c a) x(a b a c a) c=b(a b a c a) c$. But $b a b=b$ and $c a c=c$ and so $(b a c a) x(a b a c)=b a c$. Thus $(b \circ c) \circ x \circ(b \circ c)=b \circ c$, and so $b \circ c$ is regular in $(S, a)$.

In Lemma 3.1 of [9], the mididentities of $(S, a)$ are characterised as the preinverses of $a$, and the idempotent mididentities of $(S, a)$ are precisely the inverses of $a$. We now turn to the properties of variants of inverse semigroups.

Proposition 6 Let $S$ be an inverse semigroup and let $a \in S$. Then $E(S, a)$ is a commutative band, and the mididentities are the elements above $a^{-1}$ in $S$.

Proof We begin by locating the idempotents of $(S, a)$. Suppose that $b \in E(S, a)$. Then $b \circ b=b$ and so $b a b=b$. Thus $b^{-1}(b a b) b^{-1}=$ $b^{-1} b b^{-1}=b^{-1}$. Hence $b^{-1}=b^{-1} b a b b^{-1} \leq a$, and so $b \leq a^{-1}$. Conversely, suppose that $b \leq a^{-1}$. Then $b^{-1} \leq a$. Thus $b^{-1}=$ $b^{-1} b a b b^{-1}$. It follows that $b=b a b$ and so $b=b \circ b$. Thus $b$ is an idempotent of $(S, a)$ if, and only if, $b \leq a^{-1}$.

Now let $b, c \in E(S, a)$. We prove that $b \circ c=c \circ b$. Thus we need to prove that $b a c=c a b$. By the result above we have that $b, c \leq$ $a^{-1}$. Thus $b=b b^{-1} a^{-1}=a^{-1} b^{-1} b$ and $c=c c^{-1} a^{-1}=a^{-1} c^{-1} c$. Hence

$$
b a c=\left(a^{-1} b^{-1} b\right) a c=\left(a^{-1} b^{-1} b\right) a\left(a^{-1} c^{-1} c\right)
$$

which gives $b a c=a^{-1}\left(b^{-1} b\right)\left(c^{-1} c\right)$. On the other hand

$$
c a b=\left(a^{-1} c^{-1} c\right) a b=\left(a^{-1} c^{-1} c\right) a\left(a^{-1} b^{-1} b\right)=a^{-1}\left(c^{-1} c\right)\left(b^{-1} b\right) .
$$

Thus $b a c=c a b$ as required.
Finally, we locate the preinverses of $a$. Suppose that $b$ is a preinverse of $a$. Then $a=a b a$. Thus $a$ is a weak inverse of $b$, and so by the calculations above $a \leq b^{-1}$. Thus the mididentities of $(S, a)$ are the elements above $a^{-1}$.

We summarise our results on variants of inverse semigroups in the following theorem.

Theorem 7 Every variant of an inverse semigroup is an E-inversive semigroup with commuting idempotents. The idempotents of $(S, a)$ are the elements of $S$ beneath $a^{-1}$, the mididentities of $(S, a)$ are the elements of $S$ above $a^{-1}$, and the element $a^{-1}$ is the unique idempotent mididentity. The variant $(S, a)$ is regular if, and only if, $S$ is a monoid and a is invertible.

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