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Computing 3-Dimensional Groups :
Crossed Squares and Cat^2-Groups

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Abstract

The category **XSq** of crossed squares is equivalent to the category **Cat2** of cat^2-groups. Functions for computing with these structures have been developed in the package XMod written using the GAP computational discrete algebra programming language. This paper contains a table listing the numbers ι^2_G of isomorphism classes of cat^2-groups on groups G of order at most 30 – a total of 1007 cat^2-groups. Secondly, it contains general formulae for ι^2_G in a number of special cases.

Key Words: cat^2-group, crossed square, GAP, XMod
Classification: 18D35, 18G50.

1 Introduction

Crossed modules of groups were first defined by Whitehead in [26]. Connected (weak homotopy) 2-types are modelled algebraically by (quasi-isomorphism classes of) crossed modules (see [17]). However these algebraic structures are the essential data for 2-groups, which are exactly the same as internal categories in the category **Gp** of groups (see [24]). The Brown-Spencer theorem (see [10]) constructs the associated 2-group of a crossed module, which is now regarded as a "2-dimensional group". The 2-group viewpoint provides a useful way of interpreting the structure of a crossed module, and gives some applications (see [24]).

Turning to 3-types, there are several different algebraic models: crossed squares of groups; cat^2-groups; 2-crossed modules [14]; quadratic modules [6]; braided, regular crossed modules and (2-truncated) simplicial groups [11]. Some links between these structures are discussed in [5]. We consider here the equivalent categories **XSq** and **Cat2** of crossed squares and cat^2-groups (see [21]). These two algebraic structures represent 3-types and provide an interpretation for 3-groups (see [24]). Connected 3-types are modelled by quasi-isomorphism classes of crossed squares.

The inclusion crossed square is the simplest algebraic example of a crossed square. Given a pair of normal subgroups M, N of a group G, we can form a square

$$\begin{array}{ccc} M \cap N & \longrightarrow & N \\ \downarrow & & \downarrow \\ M & \longrightarrow & G \end{array}$$

in which each homomorphism is an inclusion crossed module, and there is an h-map

$$\begin{array}{rccccl} h: & M \times N & \longrightarrow & M \cap N & & \\ & (m,n) & \longmapsto & [m,n] & = & m^{-1}n^{-1}mn. \end{array}$$

The principal topological example of a crossed square is the fundamental crossed square. Given a pointed triad of spaces $A \subseteq X$, $B \subseteq X$ with $a \in A \cap B$, second relative homotopy groups $\pi_2(A, A \cap B, a)$, $\pi_2(B, A \cap B, a)$, and the first homotopy group $\pi_1(A \cap B, a)$, we obtain a square

$$\begin{array}{ccc} \pi_3(X; A, B, a) & \longrightarrow & \pi_2(B, A \cap B, a) \\ \downarrow & & \downarrow \\ \pi_2(A, A \cap B, a) & \longrightarrow & \pi_1(A \cap B, a). \end{array}$$

In this case the h-map

$$h : \pi_2(A, A \cap B, a) \times \pi_2(B, A \cap B, a) \longrightarrow \pi_3(X; A, B, a)$$

is the triad Whitehead product (see [15, 26]).

This paper is concerned with the latest developments in the general programme of "computational higher-dimensional group theory" which forms part of the "higher-dimensional group theory" programme described, for example, by Brown in [8].

The 2-dimensional part of these programmes is concerned with group objects in the categories of groups or groupoids. These objects and their morphisms form the equivalent categories **XMod** of crossed modules or **Cat1** of cat^1-groups. The initial computational part of this programme was described in Alp and Wensley [2]. The output from this work was the package XMod [1] for GAP [19] which, at the time, contained functions for constructing crossed modules and cat^1-groups of groups, and their morphisms, and conversions from one to another.

The next development of XMod used the package groupoids [22] to compute crossed modules of groupoids. Later still, a GAP package XModAlg [3] was written to compute cat^1-algebras and crossed modules of algebras, as described in [4].

We are concerned here with the 3-dimensional part of the programme which deals with objects in **XSq** and **Cat2**. The mathematical basis of all these structures is described in §2, and some computational details are included in §3. General formulae for some simple abelian groups are contained in §4. In §5 we enumerate the $1,007$ isomorphism classes of cat^2-groups on the 92 groups of order at most 30.

The contents of this paper are purely algebraic. Readers wishing to understand the applications of the theory are encouraged to study references such as [5, 9, 24, 17]. The XMod package also follows a purely algebraic approach, and does not compute any specifically topological results. The interested reader may wish to investigate the GAP package HAP [16] which also computes with cat^1-groups.

2 Crossed Squares and Cat^2-Groups

The notion of a *crossed module* $\mathcal{X} = (\partial : S \to R)$ was introduced by Whitehead [26]. It consists of a group homomorphism $\partial : S \to R$, together with a left action R on S (written $(r, s) \to {}^r s$ for $r \in R$ and $s \in S$) satisfying the following conditions:

$$\partial({}^r s) \;=\; r(\partial s)r^{-1} \;\; \forall\, s \in S,\; r \in R, \quad {}^{(\partial s_2)}s_1 \;=\; s_2 s_1 s_2^{-1} \;\; \forall\, s_1, s_2 \in S,$$

(the *pre-crossed module property* and the *Peiffer identity*).

A *morphism of crossed modules* $(\sigma, \rho) : \mathcal{X}_1 \to \mathcal{X}_2$ consists of two group homomorphisms $\sigma : S_1 \to S_2$ and $\rho : R_1 \to R_2$ such that $\partial_2 \circ \sigma = \rho \circ \partial_1$ and $\sigma({}^r s) = {}^{(\rho r)}(\sigma s)$ for all $s \in S_1,\; r \in R_1$.

Standard constructions for a crossed module morphism include the inclusion of a normal subgroup, where the action is conjugation; the inner automorphism map $S \to \mathrm{Inn}(S)$; the zero map when S is an R-module; maps with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1}r$; and direct products $(\partial_1 \times \partial_2 : S_1 \times S_2 \to R_1 \times R_2)$ with direct product action.

Loday [21] reformulated the notion of a crossed module as a cat^1-group. A *projection* on G is an endomorphism $p : G \to G$ satisfying $p \circ p = p$. A cat^1-group $\mathcal{C} = (G \rightrightarrows R)$ is a triple $(G; t, h)$ consisting of a group G with two homomorphisms: the *tail map* t and the *head map* h, having a common image R and satisfying the following axioms.

$$t \circ h = h, \quad h \circ t = t, \quad \text{and} \quad [\ker t, \ker h] = 1. \tag{1}$$

When only the first two of these axioms are satisfied, the structure is a *pre-cat^1-group*. It follows immediately (by expanding $t \circ h \circ t$ and $h \circ t \circ h$) that t and h are both projections. A cat^1-group is *symmetric* if $t = h$ and, from (1), a sufficient condition for this is that $t \circ h = h \circ t$.

A *morphism of cat^1-groups* $\mathcal{C}_1 = (G_1; t_1, h_1) \to \mathcal{C}_2 = (G_2; t_2, h_2)$ is a homomorphism of groups $f : G_1 \to G_2$ such that $f \circ t_1 = t_2 \circ f$ and $f \circ h_1 = h_2 \circ f$.

We use the following equivalence between **XMod** and **Cat1**. It was shown in [21, Lemma 2.2] that

$$(G \overset{t,h}{\rightrightarrows} R) \quad \text{determines} \quad \mathcal{X} = (\partial : S \to R) \quad \text{where} \quad S = \ker t,\ \partial = h|_S, \tag{2}$$

and the action is conjugation. Conversely, if $(\partial : S \to R)$ is a crossed module, then setting $G = S \rtimes R$ and defining t, h by $t(s,r) = (1,r)$ and $h(s,r) = (1,(\partial s)r)$ for $s \in S,\ r \in R$, produces a cat^1-group $(G; t, h)$.

The notion of a crossed square is due to Guin-Walery and Loday [20]. An *oriented crossed square of groups* $\mathbb{X}$ is a commutative square of groups $[L, M, N, P]$, together with left actions of P on L, M, N, and a *crossed pairing* map $\boxtimes : M \times N \to L$. Then M acts on N and L via P and N acts on M and L via P. This structure is illustrated in the following left-hand diagram.

$$\mathbb{X} \;=\; \begin{array}{ccc} L & \xrightarrow{\ \kappa\ } & M \\ {\scriptstyle\lambda}\downarrow & \searrow^{\pi} & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow[\ \nu\ ]{} & P \end{array} \qquad\qquad \tilde{\mathbb{X}} \;=\; \begin{array}{ccc} L & \xrightarrow{\ \lambda\ } & N \\ {\scriptstyle\kappa}\downarrow & \searrow^{\pi} & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow[\ \mu\ ]{} & P \end{array} \tag{3}$$

The following axioms must be satisfied for all $l \in L,\ m, m' \in M,\ n, n' \in N$ and $p \in P$.

1. With the given actions, the homomorphisms $\kappa, \lambda, \mu, \nu$ and $\pi = \mu \circ \kappa = \nu \circ \lambda$ are crossed modules, and both κ, λ are P-equivariant,
2. $(mm' \boxtimes n) \;=\; ({}^m m' \boxtimes {}^m n)\,(m \boxtimes n)$ and $(m \boxtimes nn') \;=\; (m \boxtimes n)\,({}^n m \boxtimes {}^n n')$,
3. $\kappa(m \boxtimes n) \;=\; m({}^n m^{-1})$ and $\lambda(m \boxtimes n) \;=\; ({}^m n)n^{-1}$,
4. $(\kappa l \boxtimes n) \;=\; l({}^n l^{-1})$ and $(m \boxtimes \lambda l) \;=\; ({}^m l)l^{-1}$,
5. ${}^p(m \boxtimes n) \;=\; ({}^p m \boxtimes {}^p n)$.

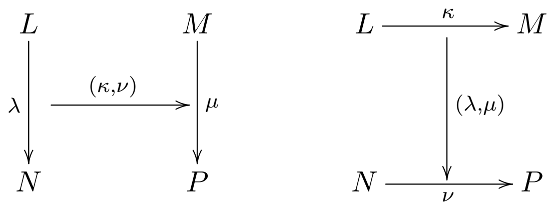
Note that axiom 1. implies that $(\mathrm{id}, \mu), (\mathrm{id}, \nu), (\kappa, \mathrm{id})$ and (λ, id) are morphisms of crossed modules.

The *transpose* $\tilde{\mathbb{X}}$ of $\mathbb{X}$, obtained by interchanging M and N, is shown in the right-hand diagram in (3). Since crossed pairing identities are similar to those for commutators, the crossed pairing for $\tilde{\mathbb{X}}$ is $\tilde{\boxtimes}$ where $(n \mathbin{\tilde{\boxtimes}} m) = (m \boxtimes n)^{-1}$. Transposition gives an equivalence relation on the set of oriented crossed squares, and a *crossed square* is an equivalence class. We shall follow the usual convention of omitting the adjective "oriented" and refer to $\mathbb{X}$ as a crossed square. It is important to remember that, when giving enumeration results, we have counted equivalence classes.

Standard constructions for crossed squares include the following sets of groups $[L, M, N, P]$:

- $[M \cap N, M, N, P]$, where M, N are normal subgroups of P;
- $[L, \mathrm{Inn}\ L, \mathrm{Inn}\ L, \mathrm{Aut}\ L]$ where $\kappa = \lambda$ maps $l \in L$ to the inner automorphism $l' \mapsto ll'l^{-1}$;
- $[M \otimes N, M, N, P]$ where where $M \otimes N$ is a nonabelian tensor product of groups [13];
- the *direct product* of crossed squares has groups $[L_1 \times L_2, M_1 \times M_2, N_1 \times N_2, P_1 \times P_2]$ with direct product actions and crossed pairing $\boxtimes((m_1, m_2), (n_1, n_2)) = (\boxtimes_1(m_1, n_1), \boxtimes_2(m_2, n_2))$.

The crossed square $\mathbb{X}$ in (3) can be thought of as a horizontal or vertical crossed module of crossed modules:



where (κ, ν) is the boundary of the crossed module with domain $(\lambda : L \to N)$ and codomain $(\mu : M \to P)$, (see also section 9.2 of [25]).

There is an evident notion of morphism of crossed squares which preserves all the structure, so that we obtain a category **XSq**, the category of crossed squares.

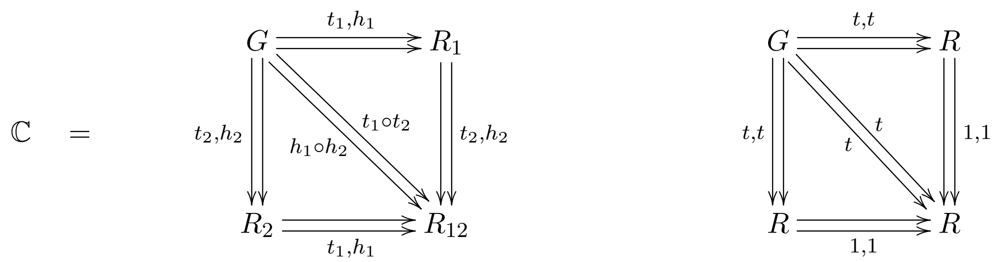
Although, when first introduced by Loday and Walery [20], the notion of crossed square of groups was not linked to that of cat^2-groups, it was in this form that Loday gave their generalisation to an n-fold structure, *cat^n-groups* (see [21]). When $n = 1$ this is the notion of cat^1-group given earlier.

When $n = 2$ we obtain a cat^2-group. Again we have a group G, but this time with two *independent* cat^1-group structures on it. An *oriented pre-cat^2-group* is a 5-tuple, $\mathbb{C} = (G; t_1, h_1; t_2, h_2) = [\mathcal{C}_1, \mathcal{C}_2]$, where $\mathcal{C}_1 = (G; t_1, h_1)$ and $\mathcal{C}_2 = (G; t_2, h_2)$ are pre-cat^1-groups, and

$$t_1 \circ t_2 = t_2 \circ t_1, \quad h_1 \circ h_2 = h_2 \circ h_1, \quad t_1 \circ h_2 = h_2 \circ t_1, \quad t_2 \circ h_1 = h_1 \circ t_2. \tag{4}$$

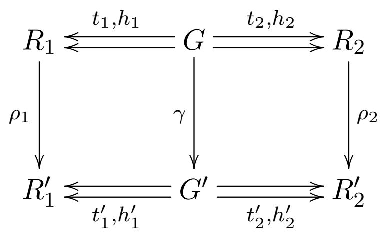
This is an *oriented cat^2-group* when $\mathcal{C}_1, \mathcal{C}_2$ are both cat^1-groups. We say $\mathbb{C}$ is *symmetric* if $\mathcal{C}_1 = \mathcal{C}_2$. By (4) this can only happen when $t_1 \circ h_1 = h_1 \circ t_1$, so $\mathcal{C}_1$ is symmetric. Thus symmetric cat^2-groups and in one-one correspondence with symmetric cat^1-groups. The transpose $\tilde{\mathbb{C}}$ of $\mathbb{C}$ is obtained by interchanging $[t_1, h_1]$ with $[t_2, h_2]$. Again, transposition is an equivalence relation, and a *cat^2-group* is an equivalence class. We shall omit the qualifier "oriented" whenever possible.

To emphasise the relationship with crossed squares we give the following left-hand diagram for $\mathbb{C}$, where R_{12} is the image of $t_1 \circ t_2 = t_2 \circ t_1$. On the right we show the symmetric case.



(5)

A morphism of pre-cat^2-groups from $\mathbb{C}$ to $\mathbb{C}'$ is a triple (γ, ρ_1, ρ_2), as shown in the diagram



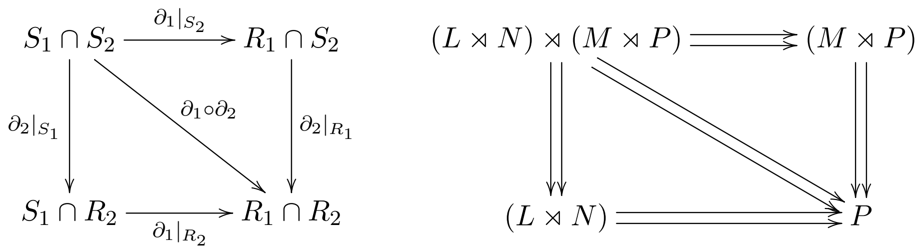
where $\gamma : G \to G'$, $\rho_1 = \gamma|_{R_1}$ and $\rho_2 = \gamma|_{R_2}$ are homomorphisms satisfying:

$$\rho_1 \circ t_1 = t_1' \circ \gamma, \qquad \rho_1 \circ h_1 = h_1' \circ \gamma, \qquad \rho_2 \circ t_2 = t_2' \circ \gamma, \qquad \rho_2 \circ h_2 = h_2' \circ \gamma.$$

We thus obtain categories **PreCat2** and **Cat2**, the categories of (pre-)cat^2-groups.

Notice that, unlike the situation with crossed squares where the diagonal is a crossed module, it is *not* required that the diagonal in (5) is a cat^1-group – it may just be a pre-cat^1-group. The simplest case of this situation is described in Example 2.1 below.

Recall that Loday, in [21], proved that there is an equivalence between the category **Cat2** and the category **XSq** (see also [23]). Applying the equivalence between **Cat1** and **XMod** in (2) to the cat^2-group $\mathbb{C}$ in (5), we obtain the left-hand diagram of group homomorphisms in (6) where each morphism is a crossed module for the natural action, conjugation in G. The required crossed pairing is given by the commutator in G since if $x \in R_1 \cap S_2$ and $y \in S_1 \cap R_2$ then $[x, y] \in S_1 \cap S_2$. Note that equation (4) implies $\partial_1 \circ \partial_2 = \partial_2 \circ \partial_1$. It is routine to check the remaining crossed square axioms.

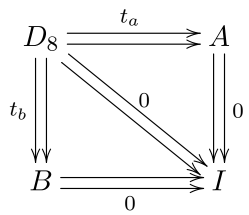


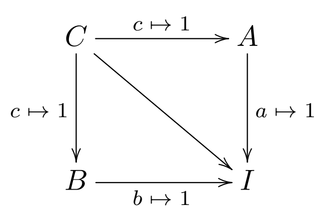
(6)

Conversely, we may consider the crossed square $\mathbb{X}$ in (3) as a morphism of crossed modules $(\kappa, \nu) : (\lambda : L \to N) \to (\mu : M \to P)$. Using the equivalence between crossed modules and cat^1-groups this gives a morphism $\partial : (L \rtimes N, t, h) \longrightarrow (M \rtimes P, t', h')$ of cat^1-groups. There is an action of $(m, p) \in M \rtimes P$ on $(l, n) \in L \rtimes N$ given by ${}^{(m,p)}(l, n) = ({}^m({}^pl)(m \boxtimes {}^pn),\ {}^pn)$. Using this action, we form its associated cat^2-group with source $(L \rtimes N) \rtimes (M \rtimes P)$, as shown in the right-hand diagram in (6).

Example 2.1. Let $D_8 = \langle a, b \mid a^2, b^2, (ab)^4 \rangle$ be the dihedral group of order 8, and let $c = [a, b] = (ab)^2$ so that $a^b = ac$ and $b^a = bc$. (The standard permutation representation is given by $a = (1, 2)(3, 4), b = (1, 3), ab = (1, 2, 3, 4), c = (1, 3)(2, 4)$.)

Define $t_a, t_b : D_8 \to D_8$ by $t_a : a, b \mapsto a, 1$ and $t_b : a, b \mapsto 1, b$. Then construct cat^1-groups $\mathcal{C}_a = (D_8; t_a, t_a)$ and $\mathcal{C}_b = (D_8; t_b, t_b)$. Diagram (5) and the left-hand diagram in (6) become





where $A = \langle a \rangle, B = \langle b \rangle, C = \langle c \rangle$ and I is the trivial group. The crossed pairing is given by $\boxtimes(a, b) = c$. The composite $t_a \circ t_b$ is the zero map, and $[D_8, D_8] = C$, so the diagonal is *not* a cat^1-group.

For dimensions $n \geqslant 3$, a *cat^n-group* consists of a group G with n independent cat^1-group structures $(G; t_i, h_i)$, $1 \leq i \leq n$, such that $t_i t_j = t_j t_i,\ h_i h_j = h_j h_i$ and $t_i h_j = h_j t_i$ for all $i \neq j$. Ellis and Steiner in [18] defined a generalisation of a crossed square to higher dimensions, called a *crossed n-cube*.

The following result is needed for §4. Since all the axioms are immediately satisfied, the proof is straightforward.

Lemma 2.2. *Let G be a direct product $H \times K$.*
(i) Let $\mathcal{C}$ be a cat^1-group on G such that $tH \leqslant H$ and $tK \leqslant K$. Then restricting the maps t, h to H gives a cat^1-group on H, and similarly for K.
(ii) Let $\mathbb{C}$ be a cat^2-group on G such that $t_1 H$ and $t_2 H$ are subgroups of H while $t_1 K$ and $t_2 K$ are subgroups of K. Then restricting the maps t_1, h_1, t_2, h_2 to H gives a cat^2-group on H, and similarly for K.

3 Computer Implementation

GAP [19] is an open-source system for discrete computational algebra. The system consists of a library of implementations of mathematical structures: groups, vector spaces, modules, algebras, graphs, codes, designs, etc.; plus databases of groups of small order, character tables, etc. The system has world-wide usage in the area of education and scientific research. GAP is free software and user contributions to the system are supported. These contributions are organized in a form of GAP packages and are distributed together with the system. Contributors can submit additional packages for inclusion after a reviewing process.

The Small Groups library by Besche, Eick and O'Brien in [7] provides access to descriptions of the groups of small order. The groups are listed up to isomorphism. The library contains all groups of order at most 2000 except 1024.

3.1 2-Dimensional Groups

The XMod package for GAP contains functions for computing with crossed modules, cat^1-groups and their morphisms, and was first described in [1]. An equivalent notion of cat^1-group is implemented in XMod, where the tail and head maps are no longer required to be endomorphisms on G. Instead it is required that t and h have a common image R, and an *embedding* $e : R \to G$ is added. The axioms in (1) then become:

$$t \circ e \circ h = h, \quad h \circ e \circ t = t, \quad \text{and} \quad [\ker t, \ker h] = 1, \tag{7}$$

and again it follows that $t \circ e \circ t = t$ and $h \circ e \circ h = h$. We denote such a cat^1-group by $\mathcal{C} = (e; t, h : G \to R)$. Note that (id, e) is an isomorphism from $\mathcal{C}$ to $\mathcal{C}' = (\mathrm{id}; t \circ e, h \circ e : G \to eR)$ where the maps *are* endomorphisms.

The package provides an operation **Cat1Select** which may be used to select a cat^1-group $\mathcal{C}$ from a data file. This file contains data on all isomorphism classes of cat^1-structures on groups of size up to 70 (ordered according to the GAP numbering of small groups). The GAP package HAP has more recently extended this information to groups of size up to 255. Cat^1-groups may be converted into crossed modules, and vice-versa, using the functions **XModOfCat1Group** and **Cat1GroupOfXMod**.

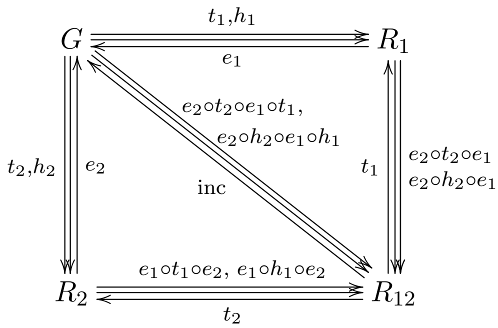
The operation **AllCat1Groups(G)** may be used to produce a list of all the cat^1-groups with source G. This function starts with a list L of the projections on G, selects pairs (t, h) from L, and tests whether these satisfy axioms (1). While this is acceptable for many small groups, such lists can make heavy use of memory. It is a fundamental principle in GAP to avoid the unnecessary storing of long lists by providing iterators. An *iterator* is a function which returns a record containing functions **NextIterator**, **IsDoneIterator** and **ShallowCopy**. The package provides iterators **All-**

Cat1GroupsIterator(G), and **AllCat1GroupsWithImageIterator(G,R)** which constructs cat^1-groups $(G \rightrightarrows R)$ for a given subgroup R. The equivalent function in HAP is **CatOneGroupsByGroup**.

3.2 3-dimensional Groups

We have developed new operations for XMod which construct (pre-)cat^2-groups and their morphisms. There are also functions for crossed squares and their morphisms, and functions to convert between cat^2-groups and crossed squares.

As with pre-cat^1-groups, we use an equivalent notion for pre-cat^2-groups. An *oriented pre-cat^2-group* has the form



where R_1, R_2 need not be subgroups of G, but R_{12} is taken to be the common image of $e_2 \circ t_2 \circ e_1 \circ t_1$ and $e_1 \circ t_1 \circ e_2 \circ t_2$, a subgroup of G. The other orientation is obtained by reflecting in the diagonal.

The following GAP session illustrates the use of the function **Cat2Group(C1,C2)** which constructs a cat^2-group from two cat^1-groups. Notice that the cat^2-group `C2ab` is the second example with a diagonal which is only a pre-cat^1-group.

```
gap> a := (1,2,3,4)(5,6,7,8);;
gap> b := (1,5)(2,6)(3,7)(4,8);;
gap> c := (2,6)(4,8);;
gap> G := Group( a, b, c );;
gap> SetName( G, "c4c2:c2" );
gap> t1a := GroupHomomorphismByImages( G, G, [a,b,c], [(),(),c] );;
gap> C1a := PreCat1Group( t1a, t1a );;
gap> t1b := GroupHomomorphismByImages( G, G, [a,b,c], [a,(),()] );;
gap> C1b := PreCat1Group( t1b, t1b );;
gap> C2ab := Cat2Group( C1a, C1b );
(pre-)cat2-group with generating (pre-)cat1-groups:
1 : [c4c2:c2 => Group( [ (), (), (2,6)(4,8) ] )]
2 : [c4c2:c2 => Group( [ (1,2,3,4)(5,6,7,8), (), () ] )]
gap> IsCat2Group( C2ab );
true
gap> Size( C2ab );
[ 16, 2, 4, 1 ]
gap> IsCat1Group( Diagonal2DimensionalGroup( C2ab ) );
false
```

The basic algorithm for the function **AllCat2Groups(G)** is very simple. It returns a list L of cat^2-groups, and is shown in Algorithm 1.

Algorithm 1: AllCat2Groups

```
Input: G, a group
Output: L, a list
begin
    L ← [ ]
    for [R1,R2] in unordered pairs of subgroups of G do
        for A in AllCat1GroupsWithImage(G,R1) do
            for B in AllCat1GroupsWithImage(G,R2) do
                C ← Cat2Group(A,B)
                if C ≠ fail then
                    Add(L,C)
```

Note that this algorithm is used to provide an iterator **AllCat2GroupsIterator(G)**; that the pair of subgroups $[R_1, R_2]$ is provided by the standard GAP functions **AllSubgroupsIterator** and **UnorderedPairsIterator**; and that A, B are constructed using the iterator for cat^1-groups with a given image described above. Note also that an oriented cat^2-group and its transpose are only counted once.

The package also includes an iterator **AllCat2GroupsWithImagesIterator(G,R1,R2)** which returns cat^2-groups with chosen subgroups R_1, R_2. The utility of this becomes clear when, for example, G has the form C_p^3, the cube of a cyclic group of prime order, with generators $\{a, b, c\}$. As we shall see in the following section there are, up to isomorphism, very few cases to consider. So it is only necessary to call this function with pairs R_1, R_2 chosen from $[G, \langle a, b\rangle, \langle a, c\rangle, \langle a\rangle, \langle b\rangle, I]$, and then apply appropriate multiplicities.

4 Formulae for special cases

Our aim in the next two sections is to list, for each group G of order at most 30, the following values. First, the number ϵ_G of projections on G. Then the number γ_G^1 of cat^1-groups on G, followed by the number ι_G^1 of their isomorphism classes. (The numbers ϵ_G, γ_G^1 and ι_G^1 can be found in [2].) Then the number σ_G of *symmetric* cat^1-groups, followed by the number τ_G of their isomorphism classes. (We have already observed that σ_G, τ_G are also the numbers of symmetric cat^2-groups and of their isomorphism classes.) Finally, we list the number γ_G^2 of cat^2-groups on G and the number ι_G^2 of their isomorphism classes.

We define the *compatibility matrix* M_G^2 of G to be the symmetric matrix with rows and columns indexed by the cat^1-groups $\mathcal{C}_i$ on G, where $(M_G^2)_{ij} = 1$ if $\mathcal{C}_i$ and $\mathcal{C}_j$ combine to form a cat^2-group. We denote by μ_G^2 the number of ones in M_G^2. Off-diagonal ones in symmetric positions correspond to the two orientations of a cat^2-group, so γ_G^2 is the number of ones in the upper-triangular part of M_G^2. Thus $\gamma_G^2 = (\mu_G^2 + \sigma_G)/2$ and $\mu_G^2 = 2\gamma_G^2 - \sigma_G$.

There is a similar $0-1$ matrix M_G^1 containing μ_G^1 ones. Its rows and columns are indexed by the projections t_i on G with $(M_G^1)_{ij} = 1$ when t_i and t_j combine to form an oriented cat^1-group. Again $\gamma_G^1 = (\mu_G^1 + \sigma_G)/2$ and $\mu_G^1 = 2\gamma_G^1 - \sigma_G$. Note that M_G^1 has the form of a block diagonal matrix with one block for each subgroup of G.

4.1 The case $G = A \times B$ with $|A|$ coprime to $|B|$

Since $|A|$ is coprime to $|B|$, an endomorphism of G must consist of an endomorphism of A together with one for B. Hence $\epsilon_G = \epsilon_A \epsilon_B$ and $\gamma^1_G = \gamma^1_A \gamma^1_B$ and $\sigma_G = \sigma_A \sigma_B$.

Proposition 4.1. *When $G = A \times B$ with $|A|$ coprime to $|B|$ then $\gamma^2_G = \gamma^2_A \gamma^2_B + (\gamma^2_A - \sigma_A)(\gamma^2_B - \sigma_B)$.*

Proof: The matrix M^2_G is the Kronecker product of M^2_A with M^2_B so $\mu^2_G = \mu^2_A \mu^2_B$. The formula for γ^2_G follows by applying $\gamma^2_G = (\mu^2_G + \sigma_G)/2$. □

For example, using values in the table of §5 for C_2^2 and for C_3^2 we may calculate for $G = C_6^2$ that $\gamma^2_G = 36 \times 93 + (36 - 8)(93 - 14) = 5,560$.

4.2 The case when G is cyclic

Proposition 4.2. *When $G = C_{p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}}$ is cyclic, and its order is the product of m distinct primes p_i, having multiplicities k_i, then*

$$\epsilon_G = \gamma^1_G = \iota^1_G = \sigma_G = \tau_G = 2^m \quad \textit{and} \quad \gamma^2_G = \iota^2_G = 2^{m-1}(2^m + 1).$$

Proof: When $G = C_{p^k}$ is cyclic, with p prime, the only projections are the identity and zero maps. So there are just two cat^1-groups, both symmetric and all isomorphism classes are singletons. All pairs are compatible, so

$$M^2_G = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and $\mu^2_G = 4$. Similarly, in the general case, there are 2^m subgroups, 2^m projections and 2^m cat^1-groups. All of these are symmetric and no two are isomorphic. M^2_G is the m-th Kronecker power of the matrix above, so $\mu^2_G = 4^m$ and $\gamma^2_G = (4^m + 2^m)/2$. □

4.3 The case $G = C_p^n$ with p prime

When G is an elementary p-group the numbers γ^1_G and γ^2_G can get very large, and computations may run out of memory. Indeed the largest numbers of cat^2-groups in the table below are $325,363$ for C_2^4 and $24,222$ for C_3^3. However, some general formulae may be obtained.

Proposition 4.3. *For p a prime and $G = C_p^n$, the n-th power of the cyclic group C_p,*

$$\sigma_G = \epsilon_G = \sum_{k=0}^{n} p^{k(n-k)} \prod_{j=1}^{k} \frac{(p^{n-j+1} - 1)}{(p^j - 1)} \qquad \textit{and} \qquad \gamma^1_G = \sum_{k=0}^{n} p^{2k(n-k)} \prod_{j=1}^{k} \frac{(p^{n-j+1} - 1)}{(p^j - 1)}.$$

Proof: Let $\theta : G \to G$ be a projection with image $R \cong C_p^k$. The structure of the subgroup lattice of G is well known, and the common product term in the two formulae gives the number of subgroups of G isomorphic to R. We may choose a generating set $\{g_1, g_2, \ldots, g_n\}$ for G such that $\{g_1, g_2, \ldots, g_k\}$ is a generating set for R. Then $\theta g_i = g_i$ for $1 \leqslant i \leqslant k$, and each of $(n - k)$ generators $\{g_{k+1}, \ldots, g_n\}$ may be mapped to any of the p^k elements in R, so the number of projections with image R is $p^{k(n-k)}$.

If $t, h : G \to G$ have a common image R then $t \circ h = h$ since t is the identity on R. Similarly $h \circ t = t$. Since G is abelian, $\ker t$ and $\ker h$ commute. Hence there is a cat^1-group with t, h as the tail and head maps. It follows that the number of cat^1-groups with image R is the square of the number of projections with image R. □

Note, in particular, that γ^1_G is equal to $\frac{1}{2}p(p^2+1)+(p^2+2)$ when $n = 2$, and $2+p^2(p^2+1)(p^2+p+1)$ when $n = 3$.

4.3.1 Cat^2-groups on $G = C_p^2$

G has $p+1$ subgroups isomorphic to C_p and Proposition 4.3 states that there are $2+p(p+1)$ projections and $2+p^2(p+1)$ cat^1-groups. In the case $p=2$ there are 8 projections which combine to form 14 cat^1-groups $\mathcal{C}_i$, and 8 of these are symmetric. These form 4 isomorphism classes with numbers $1, 8, 9, 14$ as representatives, of the form $(G \rightrightarrows I)$; $(G \rightrightarrows \langle b \rangle)$, both non-symmetric and symmetric; and $(G \rightrightarrows G)$.

The matrix M_G^2 shown below has $\mu_G^2 = 64$, so $\gamma_G^2 = 36$.

This matrix illustrates the behaviour for every prime p: $\mathcal{C}_1 = (G \rightrightarrows I)$ forms a cat^2-group with every $\mathcal{C}_i$, and every $\mathcal{C}_i$ forms one with $\mathcal{C}_{14} = (G \rightrightarrows G)$. There is an additional symmetric cat^2-group (ones on the diagonal) only when $\mathcal{C}_i$ is symmetric (tail and head maps are equal). Finally, each symmetric $\mathcal{C}_i$ forms a cat^2-group with precisely one of the other symmetric $\mathcal{C}_j$'s. We conclude that the cat^2-groups on $C_p \times C_p$ comprise 9 isomorphism classes with the following structure:

$$
\begin{array}{ccc} G & \rightrightarrows & I \\ \downdownarrows & & \downdownarrows \\ I & \rightrightarrows & I \end{array}
\quad
\begin{array}{ccc} G & \rightrightarrows & I \\ \downdownarrows & & \downdownarrows \\ C_i & \rightrightarrows & I \end{array}
\quad
\begin{array}{ccc} G & \rightrightarrows & I \\ \downdownarrows & & \downdownarrows \\ G & \rightrightarrows & I \end{array}
\quad
\begin{array}{ccc} G & \rightrightarrows & C_i \\ \downdownarrows & & \downdownarrows \\ C_i & \rightrightarrows & C_i \end{array}
\quad
\begin{array}{ccc} G & \rightrightarrows & C_i \\ \downdownarrows & & \downdownarrows \\ G & \rightrightarrows & C_i \end{array}
\quad
\begin{array}{ccc} G & \rightrightarrows & C_i \\ \downdownarrows & & \downdownarrows \\ C_j & \rightrightarrows & I \end{array}
\quad
\begin{array}{ccc} G & \rightrightarrows & G \\ \downdownarrows & & \downdownarrows \\ G & \rightrightarrows & G \end{array}
$$

where the second and fifth cases illustrate both symmetric and non-symmetric classes.

$M_G^2 =$

	(ta,tb)	(ha,hb)	symm?	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	$(1,1)$	$(1,1)$	Y	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	$(a,1)$	$(a,1)$	Y	1	1				1								1
3	$(a,1)$	(a,a)		1													1
4	(a,a)	$(a,1)$		1													1
5	(a,a)	(a,a)	Y	1				1								1	1
6	$(1,b)$	$(1,b)$	Y	1	1				1								1
7	$(1,b)$	(b,b)		1													1
8	(b,b)	$(1,b)$		1													1
9	(b,b)	(b,b)	Y	1								1	1				1
10	$(ab,1)$	$(ab,1)$	Y	1								1	1				1
11	$(ab,1)$	$(1,ab)$		1													1
12	$(1,ab)$	$(ab,1)$		1													1
13	$(1,ab)$	$(1,ab)$	Y	1				1								1	1
14	(a,b)	(a,b)	Y	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Proposition 4.4. *When $G = C_p \times C_p$ with p prime, $\gamma_G^2 \; = \; 4\gamma_G^1 - \frac{1}{2}\epsilon_G - 4$.*

Proof: We consider the symmetric matrix M_G^2 for this general case. To avoid counting a cat^2-group twice, we take the number of ones in the matrix, add the number on the diagonal, and divide by 2. The number of ones around the perimeter is $4\mu_G^1 - 4$ and the number on the diagonal is ϵ_G. If, as claimed, each symmetric $\mathcal{C}_i$ forms a cat^2-group with precisely one other symmetric $\mathcal{C}_j$, then the total number is $(4\mu_G^1 - 4) + 2(\epsilon_G - 2)$ and so $\gamma_G^2 \; = \; 4\gamma_G^1 - \frac{1}{2}\epsilon_G - 4 \; = \; 3 + \frac{1}{2}p(p+1)(4p+3)$.

To verify the claim, let $\phi_i, 0 \leqslant i \leqslant p-1$, be the projections $G \to A$ mapping $[a,b]$ to $[a,a^i]$. Then $\phi_j \circ \phi_i$ has images $[a,a^i]$ while those of $\phi_i \circ \phi_j$ are $[a,a^j]$. So ϕ_i, ϕ_j are compatible only when $i=j$. Now let $\psi_j : [a,b] \mapsto [b^j, b]$. Then $\psi_j \circ \phi_i$ and $\phi_j \circ \psi_j$ have images $[b^j, b^{ij}]$ and $[a^{ij}, a^i]$ respectively. For compatibilty $i = j = 0$ and the only compatible pair with images A and B is (ϕ_0, ψ_0). (In the case $p=2$ this is the pair $(2,6)$ in the matrix above.) By symmetry in G, each symmetric cat^1-group whose range is a C_p is compatible with just one symmetric cat^1-group whose range is a different C_p. Hence we obtain the given formula for γ_G^2. □

Proposition 4.5. *When $G = C_p^2$ the number of isomorphism classes of cat^n-groups on G is $\iota_G^n = (n+1)^2$.*

Proof: We start with the case $n = 2$ and continue to use the notation of the previous proof. The four isomorphism classes $[\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}]$ of cat^1-groups on G have representatives $\mathcal{C}_1 = (0_G, 0_G), \mathcal{C}_2 = (\phi_0, \phi_0), \mathcal{C}_3 = (\phi_0, \phi_1), \mathcal{C}_4 = (1_G, 1_G)$. (In the case $p = 2$ these are in rows 1,2,3 and 14 of the matrix above.) When forming a cat^2-group on G we have to pick a pair of compatible cat^1-groups. It is easy to see that $\mathcal{C}_1$ and $\mathcal{C}_4$ are both compatible with all the cat^1-groups, giving $(2\gamma_G^1 - 1)$ cat^2-groups in 7 isomorphism classes. We have seen that the remaining $\epsilon_G - 2$ symmetric cat^1-groups form $\epsilon_G - 2$ symmetric cat^2-groups, making an eighth class, and pair off to form $\frac{1}{2}(\epsilon_G - 2)$ non-symmetric ones, a ninth class. Thus the $\gamma_G^2 = (2\gamma_G^1 - 1) + \frac{3}{2}(\epsilon_G - 2)$ cat^2-groups form 9 isomorphism classes with representatives

$$(\mathcal{C}_1, \mathcal{C}_1), (\mathcal{C}_1, \mathcal{C}_2), (\mathcal{C}_1, \mathcal{C}_3), (\mathcal{C}_1, \mathcal{C}_4), (\mathcal{C}_2, \mathcal{C}_4), (\mathcal{C}_3, \mathcal{C}_4), (\mathcal{C}_4, \mathcal{C}_4), (\mathcal{C}_2, \mathcal{C}_2), (\mathcal{C}_2, \mathcal{C}_2').$$

where $\mathcal{C}_2' = (\psi_0, \psi_0)$. We say that these classes have types $[\mathbf{11}, \mathbf{12}, \mathbf{13}, \mathbf{14}, \mathbf{24}, \mathbf{34}, \mathbf{44}, \mathbf{22}, \mathbf{22'}]$.

For the general case we note that the isomorphism classes may be partitioned into 4 sets. Let S_1^n be the set of classes having a $\mathcal{C}_1$ in one direction, but no $\mathcal{C}_4$; let S_4^n be the set with a $\mathcal{C}_4$ but no $\mathcal{C}_1$; and let S_{14}^n, S_0^n be the classes having both a $\mathcal{C}_1$ and a $\mathcal{C}_4$, or neither of these. Let $\xi_n = |S_1^n| = |S_4^n|, \eta_n = |S_{14}^n|$ and $\zeta_n = |S_0^n|$, so that the total number of classes is $\tau_n = 2\xi_n + \eta_n + \zeta_n$, which is $6 + 1 + 2 = 9$ when $n = 2$. We claim that $\xi_n = 2n - 1, \eta_n = (n-1)^2$ and $\zeta_n = 2$ for all n, except that $\zeta_1 = 1$. The proof is by induction on n. The initial cases are $\xi_1 = 1, \eta_1 = 0, \zeta_1 = 1, \xi_2 = 3, \eta_2 = 1, \zeta_2 = 2$. Classes in S_1^n are formed by adding a $\mathcal{C}_1$ to those in S_1^{n-1} or in S_0^{n-1}, so $\xi_n = \xi_{n-1} + \zeta_{n-1} = 2(2n-3) + 2 = 2n - 1$. There is a similar argument for S_4^n. Classes in S_{14}^n are formed by adding a $\mathcal{C}_4$ to those in S_1^{n-1}; a $\mathcal{C}_1$ to those in S_4^{n-1}; or both a $\mathcal{C}_1$ and a $\mathcal{C}_4$ to those in S_{14}^{n-2}. Those that are duplicated are the ones formed by adding both a $\mathcal{C}_1$ and a $\mathcal{C}_4$ to those in S_0^{n-2}. Hence $\eta_n = 2\xi_{n-1} + \eta_{n-2} - \zeta_{n-2} = 2(2n-3) + (n-3)^2 - 2 = (n-1)^2$. Representatives for classes contributing to $\zeta_n = 2$ are $\mathbf{2} \ldots \mathbf{22}$ and $\mathbf{2} \ldots \mathbf{22'}$. It remains to calculate $\tau_n = 2\xi_n + \eta_n + \zeta_n = (4n-2) + (n-1)^2 + 2 = (n+1)^2$. □

4.3.2 Cat^2-groups on $G = C_p \times C_p \times C_p$

We see here how the numbers of cat^2-groups increases rapidly with p for these uncomplicated groups, while the number of isomorphism classes remains constant. For p in $[2, 3, 5, 7]$ the following formula for γ_G^2 gives [1,711, 24,222, 870,328, 10,253,106].

Example 4.6. *When $G = C_p^3$ with p prime, $\gamma_G^2 = p^8 + 4p^7 + 9p^6 + 7p^5 + 7p^4 + 2p^3 + 3p^2 + 3$.*

We give a brief sketch of the argument. As before, we count μ_G^2, the number of ones in the matrix M_G^2, which has $\mu_G^1 = 2 + 2p^4(p^2 + p + 1)$ rows and columns. For each subgroup of G isomorphic to C_p or C_p^2 there are p^4 cat^1-groups with that subgroup as range. The subgroups of G may be partitioned into: the identity subgroup I; the subgroup A; a set U of $p^2 + p$ subgroups isomorphic to C_p; a set V of $p + 1$ subgroups generated by A and H for some $H \in U$; the set W of p^2 subgroups isomorphic to C_p^2 and not containing A; and G itself. The matrix M_G^2 has ones throughout the first and last rows and columns. The remaining entries partition into $4(p^2 + p + 1)$ square blocks, each with p^4 entries, labelled by the pairs of subgroups (H, K) as source and range.

Of the $p^4(p^2 + p + 1)$ cat^1-groups with source in $\{A\} \cup U$, just $p^2(p^2 + p + 1)$ are symmetric. By symmetry, the corresponding rows in M_G^2 all contain the same number of ones, and similarly for the non-symmetric cat^1-groups, so it is only necessary to consider one example of each. Let t_1, t_2 be the projections on G mapping $[a, b, c]$ to $[a, 1, 1]$ and $[a, a, 1]$ respectively, and let $\mathcal{C}_1 = \mathcal{C}(t_1, t_1)$ and $\mathcal{C}_2 = \mathcal{C}(t_1, t_2)$. The row corresponding to $\mathcal{C}_1$ has a 1 in the first column; a single 1 in block (A, A), on the diagonal; p^2 ones in each of the $p + 1$ blocks (A, H) with $H = \langle b^i c^j \rangle$; none in the remaining (A, H)

blocks when $H \in U$; p^2 ones in the $p+1$ blocks (A, K) with $K \in V$; none in the remaining (A, K) blocks with $K \in W$; and 1 in the final column. This gives the total as $2p^3 + 2p^2 + 4$.

The row corresponding to $\mathcal{C}_2$ has a 1 in the first column; p^2 ones in block (A, C); none in (A, A) and the remaining (A, H) with $H \in U$; a single 1 in the p blocks $(A, \langle A, H \rangle)$, with range one of the symmetric cat^1-groups where $[a, b, c] \mapsto [a, bc^j, 1]$ for $0 \leqslant j < p$; and 1 in the final column. This gives the total as $p^2 + p + 2$. Hence the total contribution from the blocks $(A, -)$ is $p^2(2p^3 + 2p^2 + 4) + (p^4 - p^2)(p^2 + p + 2)$. Multiplying this by $(p^2 + p + 1)$ and adding in the contribution from the first row, gives $p^8 + 4p^7 + 9p^6 + 7p^5 + 6p^4 + p^3 + 2p^2 + 2$ which is the total for the top half of M^2_G. This is $\frac{1}{2}\mu^2_G$ since, by symmetry, the bottom half is a reflection of the top half. Then calculate $\gamma^2_G = \frac{1}{2}(\mu^2_G + \epsilon_G)$.

Example 4.7. When $G = C_p^3$ with generating set $[a, b, c]$, there are $\iota^2_G = 23$ isomorphism classes of cat^2-groups. As we have seen earlier, the number of isomorphism classes is the same for all p. Here we construct 23 representative cat^2-groups using the following 12 projections, where the images of $[a, b, c]$ are listed.

projection	0_G	α_1	α_2	β_1	β_2	β_3	γ_1	γ_2	γ_3	δ_1	δ_2	1_G
has images	111	$11c$	$1b1$	$1cc$	$c1c$	$bb1$	$ab1$	$a1c$	$1bc$	abb	bbc	abc

From these we construct twelve cat^1-groups, the first six being representatives for the six isomorphism classes, and $\mathcal{C}_2 \cong \mathcal{C}'_2,\ \mathcal{C}_3 \cong \mathcal{C}'_3 \cong \mathcal{C}''_3,\ \mathcal{C}_4 \cong \mathcal{C}'_4 \cong \mathcal{C}''_4,\ \mathcal{C}_5 \cong \mathcal{C}'_5$.

cat^1-group	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$
tail, head	$[0_G, 0_G]$	$[\alpha_1, \alpha_1]$	$[\alpha_1, \beta_1]$	$[\gamma_1, \gamma_1]$	$[\gamma_1, \delta_1]$	$[1_G, 1_G]$
cat^1-group	$\mathcal{C}'_2$	$\mathcal{C}'_3$	$\mathcal{C}''_3$	$\mathcal{C}'_4$	$\mathcal{C}''_4$	$\mathcal{C}'_5$
tail, head	$[\alpha_2, \alpha_2]$	$[\alpha_2, \beta_3]$	$[\alpha_1, \beta_2]$	$[\gamma_3, \gamma_3]$	$[\delta_2, \delta_2]$	$[\gamma_3, \delta_2]$

Computations show that the following pairs of cat^1-groups generate the required set of 23 representative cat^2-groups:
$(\mathcal{C}_1, \mathcal{C}_1)$, $(\mathcal{C}_1, \mathcal{C}_2)$, $(\mathcal{C}_1, \mathcal{C}_3)$, $(\mathcal{C}_1, \mathcal{C}_4)$, $(\mathcal{C}_1, \mathcal{C}_5)$, $(\mathcal{C}_1, \mathcal{C}_6)$, $(\mathcal{C}_2, \mathcal{C}_6)$, $(\mathcal{C}_3, \mathcal{C}_6)$, $(\mathcal{C}_4, \mathcal{C}_6)$, $(\mathcal{C}_5, \mathcal{C}_5)$, $(\mathcal{C}_6, \mathcal{C}_6)$,
$(\mathcal{C}_2, \mathcal{C}_2)$, $(\mathcal{C}_2, \mathcal{C}'_2)$, $(\mathcal{C}_2, \mathcal{C}'_3)$, $(\mathcal{C}_2, \mathcal{C}_4)$, $(\mathcal{C}_2, \mathcal{C}'_4)$, $(\mathcal{C}_2, \mathcal{C}''_5)$, $(\mathcal{C}'_3, \mathcal{C}''_3)$, $(\mathcal{C}_3, \mathcal{C}'_4)$, $(\mathcal{C}_4, \mathcal{C}_4)$, $(\mathcal{C}_4, \mathcal{C}'_4)$, $(\mathcal{C}'_4, \mathcal{C}_5)$, $(\mathcal{C}_5, \mathcal{C}'_5)$.

4.4 Two nonabelian examples

It is not so straightforward to find formulae for nonabelian groups. However, for dihedral groups $G = D_{2p} = \langle a, b \mid a^p = b^2 = (ab)^2 = 1 \rangle$, with p prime, we can check that

$$\epsilon_G = p + 2,\ \ \gamma^1_G = \sigma^1_G = p + 1,\ \ \iota^1_G = \tau^1_G = 2,\ \ \gamma^2_G = 2p + 1,\ \ \iota^2_G = 3.$$

The idempotent endomorphisms are the identity map id; maps $\epsilon_i : a \mapsto 1, b \mapsto a^i b,\ (1 \leqslant i \leqslant p)$; and the zero map 0. The cat^1-groups are $\mathcal{C}_0 = (D_{2p}; \mathrm{id}, \mathrm{id})$ and $\mathcal{C}_i = (D_{2p}; \epsilon_i, \epsilon_i)$, while $(D_{2p}; 0, 0)$ is only a pre-cat^1-group. The $\mathcal{C}_i$ are all isomorphic. The cat^2-groups have the form $[\mathcal{C}_0, \mathcal{C}_0]$; $[\mathcal{C}_0, \mathcal{C}_i]$ and $[\mathcal{C}_i, \mathcal{C}_i]$, forming three isomorphism classes.

Even more straightforward is the case of a nonabelian, simple group G. The only idempotent endomorphisms are the identity and zero maps. The only cat^1-group is $\mathcal{C}_0 = (G; \mathrm{id}, \mathrm{id})$ and the only cat^2-group is $(\mathcal{C}_0, \mathcal{C}_0)$.

5 Tables of computed results

In the following tables the groups of size at most 30 are ordered by their GAP number. For each group G we list the numbers of projections; cat^1-groups, and their classes; symmetric cat^1-groups, and their classes; and cat^2-groups, and their classes: $\epsilon_G,\ \gamma^1_G,\ \iota^1_G,\ \sigma_G,\ \tau_G,\ \gamma^2_G$ and ι^2_G.

We may reduce the size of the table by noting the results for cyclic groups. We have seen in Proposition 4.2 that when $G = C_{p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}}$ then $\epsilon_G = \gamma_G^1 = \iota_G^1 = \sigma_G = \tau_G = 2^m$ and $\gamma_G^2 = \iota_G^2 = 2^{m-1}(2^m+1)$. When $m = 2$ we may list the 10 isomorphism classes as

$$[G,I,I,I],[G,I,G,I],[G,G,G,G],2[G,I,C_{p^k},I],2[G,C_{p^k},G,C_{p^k}],2[G,C_{p^k},C_{p^k},C_{p^k}],[G,C_{p_1^{k_1}},C_{p_2^{k_2}},I]$$

where, for example, $2[G,I,C_{p^k},I]$ denotes $\{[G,I,C_{p_1^{k_1}},I],[G,I,C_{p_2^{k_2}},I]\}$.

When $m = 1$ there are 16 cyclic groups of order at most 30; when $m = 2$ there are 12 such groups; and when $m = 3$ there is just the group $30/4 = C_{30}$.

The following table contains the results for those G which are not cyclic. The values $\{\epsilon_G, \gamma_G^1, \iota_G^1\}$ agree with those in [2] except for the group 16/14.

GAP #	G	ϵ_G	γ_G^1	ι_G^1	σ_G	τ_G	γ_G^2	ι_G^2
1/1	I	1	1	1	1	1	1	1
4/2	$K_4 = C_2 \times C_2$	8	14	4	8	3	36	9
6/1	S_3	5	4	2	4	2	7	3
8/2	$C_4 \times C_2$	10	18	6	10	4	47	14
8/3	D_8	10	9	3	9	3	21	6
8/4	Q_8	2	1	1	1	1	1	1
8/5	$(C_2)^3$	58	226	6	58	4	1,711	23
9/2	$C_3 \times C_3$	14	38	4	14	3	93	9
10/1	D_{10}	7	6	2	6	2	11	3
12/1	$C_3 \ltimes C_4$	5	4	2	4	2	7	3
12/3	A_4	6	5	2	5	2	9	3
12/4	D_{12}	21	12	4	12	4	41	10
12/5	$C_3 \times K_4$	16	28	8	16	6	136	32
14/1	D_{14}	9	8	2	8	2	15	3
16/2	$C_4 \times C_4$	26	98	5	26	3	231	11
16/3	$(C_4 \times C_2) \ltimes C_2$	18	25	4	17	3	57	7
16/4	$C_4 \ltimes C_4$	10	17	3	9	2	25	4
16/5	$C_8 \times C_2$	10	18	6	10	4	47	14
16/6	$C_8 \ltimes C_2$	6	5	2	5	2	9	3
16/7	D_{16}	18	9	2	9	2	17	3
16/8	QD_{16}	10	5	2	5	2	9	3
16/9	Q_{16}	2	1	1	1	1	1	1
16/10	$C_4 \times K_4$	82	322	12	82	6	2,875	53
16/11	$C_2 \times D_8$	82	97	9	57	6	649	29
16/12	$C_2 \times Q_8$	18	17	3	9	2	25	4
16/13	$(C_4 \times C_2) \ltimes C_2$	26	13	2	13	2	37	4
16/14	$(C_2)^4$	802	10,882	9	802	5	325,363	53
18/1	D_{18}	11	10	2	10	2	19	3
18/3	$C_3 \times S_3$	12	8	4	8	4	24	10
18/4	$(C_3 \times C_3) \ltimes C_2$	47	118	4	46	3	541	9
18/5	$C_6 \times C_3$	28	76	8	28	6	358	32
20/1	Q_{20}	7	6	2	6	2	11	3
20/3	$C_4 \ltimes C_5$	7	6	2	6	2	11	3
20/4	D_{20}	31	18	4	18	4	65	10
20/5	$C_5 \times K_4$	16	28	8	16	6	136	32

21/1	$C_3 \ltimes C_7$	9	8	2	8	2	15	3
22/1	D_{22}	13	12	2	12	2	23	3
24/1	$C_3 \ltimes C_8$	5	4	2	4	2	7	3
24/3	$SL(2,3)$	6	1	1	1	1	1	1
24/4	Q_{24}	5	4	2	4	2	7	3
24/5	$S_3 \times C_4$	27	12	4	12	4	41	10
24/6	D_{24}	33	20	4	20	4	75	10
24/7	$Q_{12} \times C_2$	25	36	6	20	4	115	14
24/8	$D_8 \ltimes C_3$	23	12	4	12	4	41	10
24/9	$C_{12} \times C_2$	20	36	12	20	8	178	52
24/10	$D_8 \times C_3$	20	18	6	18	6	75	20
24/11	$Q_8 \times C_3$	4	2	2	2	2	3	3
24/12	S_4	12	5	2	5	2	9	3
24/13	$A_4 \times C_2$	15	10	4	10	4	31	10
24/14	$S_3 \times K_4$	157	116	8	68	6	999	32
24/15	$C_6 \times K_4$	116	452	12	116	8	6,786	84
25/2	$C_5 \times C_5$	32	152	4	32	3	348	9
26/1	D_{26}	15	14	2	14	2	27	3
27/2	$C_9 \times C_3$	20	56	6	20	4	138	14
27/3	$(C_3 \times C_3) \ltimes C_3$	38	37	2	37	2	127	4
27/4	$C_9 \ltimes C_3$	11	10	2	10	2	19	3
27/5	$(C_3)^3$	236	2,108	6	236	4	24,222	23
28/1	Q_{28}	9	8	2	8	2	15	3
28/3	D_{28}	41	24	4	24	4	89	10
28/4	$C_7 \times K_4$	16	28	8	16	6	136	32
30/1	$S_3 \times C_5$	10	8	4	8	4	24	10
30/2	$D_{10} \times C_3$	14	12	4	12	4	38	10
30/3	D_{30}	25	24	4	24	4	92	10

There are just 6 of these groups which produce cat^2-groups whose diagonal is *not* a cat^1-group. For each of these we give a triple consisting of the group number, the number of such cat^2-groups, and the number of their isomorphism classes:

$$[8/3, 4, 1],\ [16/3, 16, 1],\ [16/11, 176, 5],\ [16/13, 12, 1],\ [24/10, 16, 3],\ [27/3, 54, 1].$$

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