DOCTOR OF PHILOSOPHY

Representations of Crossed Modules and Cat^1-Groups

Forrester-Barker, Magnus

Award date:
2003

Awarding institution:
University of Wales, Bangor

Representations of Crossed Modules and Cat^1-Groups

Magnus Forrester-Barker

September 2003

Primum cerebro, deinde opera; hoc modo aedificatur domus.

Ior [55]

Acknowledgements

I am indebted to my supervisor, Professor Tim Porter, for his patient and generous assistance throughout the course of my research. Thanks are also due to other mathematics staff and postgraduates at the University of Wales Bangor, especially Professor Ronnie Brown and Dr Chris Wensley, and to other mathematicians across the world, including John Baez, Tony Bak and Pilar Carrasco, for their advice and encouragement. I am also grateful to EPSRC for funding the first three years of my research.

I would also like to thank my family and friends who have put up with me and have helped me to keep going when I least felt like it. In particular, I would like to thank Siân Lewis for her assistance in proofreading the early sections of this work.

Summary

Cat^1-groups and crossed modules are equivalent formulations of 2-groups (a two-dimensional generalisation of the concept of group). A linear representation of the cat^1-group $\mathfrak{C}$ is defined to be a 2-functor $\phi : \mathfrak{C} \to \mathbf{Ch}_K^{(1)}$ into the 2-category of length 1 chain complexes of vector spaces over a fixed field. This definition, and to a large extent the theory of cat^1-group representations, is based on analogy with the classical theory of group representations. There exist cat^1 analogues of many definitions and results from group representation theory. These include the notion of a faithful representation and the existence of regular representations given by Cayley's theorem. However, there are also divergences between the theories. For example, the regular representation for cat^1-groups is not necessarily faithful. Every cat^1-group, $\mathfrak{C}$, has an associated cat^1-group algebra $\overline{K(\mathfrak{C})}$, which is obtained by first applying the group algebra functor to $\mathfrak{C}$ and then factoring the top level by a given ideal in order to introduce relations necessary to make the kernel conditions work in the algebra. Representations of a cat^1-group are equivalent to modules over its cat^1-group algebra. Since representations are 2-functors, there is a 2-functor 2-category $\mathbf{Rep}_{\mathfrak{C}}$ of representations, morphisms between them (natural transformations), and homotopies between the morphisms (modifications). Many of the results on the structure of group representations, for example Maschke's theorem, will generalise to the next dimension, although we have only just begun to scratch the surface of this theory. Since it is possible to pass freely between cat^1-groups and crossed modules, it is also possible to describe representation theory for 2-groups in the language of crossed modules.

Contents

Introduction

Aims and Background

In group theory, the idea of a representation is to find a group of permutations or linear transformations with the same structure as a given, abstract, group (see, for example, [48]). Formally, a representation of a group G is a homomorphism $G \to \Gamma$ where Γ is a 'concrete' group such as S_n (leading to a *permutation representation*) or $GL_n(\mathbb{C})$ (leading to a *linear representation*). Note that while some authors (usually those concentrating on matrix representations) use the term "linear representation" in a more restricted sense, meaning representations $G \to K$ where K is a field, we shall use the term more generally for any situation where the target is a group of linear transformations.

Permutation representations of groups are intimately related to actions of groups [49]. If $\phi : G \to S_X$ is a representation, then G acts on X by ${}^g x := \phi(g)(x)$; conversely, an action of G on a set X defines a representation by associating to each $g \in G$ the permutation $\phi(g)$ that takes $x \in X$ to ${}^g x$ (the fact that this is a permutation is an immediate consequence of the definition of group actions). Linear representations may be defined as homomorphisms of G into $GL_n(K)$ for a field K, giving an invertible square matrix for each element of G. They may also be defined [67] as homomorphisms into $GL(V)$ where V is a K-vector space, giving a linear isomorphism of V for each element of G. These two versions are, of course, essentially the same since there is a direct correspondence between linear isomorphisms and invertible matrices. More abstractly, linear representations may be defined in terms of modules over the group algebra of G. This extra level of abstraction gives linear representations a major advantage over permutation representations, allowing easier generalisation to infinite groups and a much richer development of the abstract theory. Also, any permutation representation may be canonically converted to a linear representation, so nothing is lost by concentrating on the latter.

The theory of representations is an important and well-developed branch of group theory (see, for example, the books [24] and [25] by Curtis and Reiner). It is computationally valuable, as calculations are generally much easier with well understood operations on objects such as matrices and permutations than with abstract operations on abstract objects. Also, representation theory helps to elucidate the structure of abstract groups, as well as providing important links between group theory and other areas of algebra.

The work of R. Brown on higher-dimensional analogues of the Van Kampen theorem led to a more general programme of "higher-dimensional group theory" [9, 12]. Many classical results have analogues in higher dimensions; the Van Kampen theorem, which largely motivated Brown's research, is of course one example. While many areas of group theory have been succesfully generalised to higher dimensions, there has previously been no systematic development of representation theory beyond dimension one. Since representation theory is so useful and interesting for groups, it is reasonable to suppose that a higher-dimensional version might also be useful and interesting. The principal goal of this thesis is to explore this supposition. This is not merely an attempt to fill in a few gaps in somebody else's programme; a successful representation theory of 2-groups should have the same benefits for higher-dimensional group theory that the theory of group representations has for group theory, both for facilitating computation and for understanding the abstract structures better. Some reasons why we might want to study higher-dimensional groups in the first place will be touched on below.

Brown's programme includes more general higher-dimensional groupoids, but for simplicity we shall concentrate on the case of groups in dimension two. It should be fairly straightforward to generalise our results to groupoids with many objects. Likewise we shall concentrate on the generalisation from dimension one to dimension two, which Brown [9] describes as "a significant one". Again, generalisation to higher dimensions should be possible; indeed, conceptually it may well be easier than the initial leap from dimension one to two, although the notation is necessarily even more complicated and visualisation just about impossible.

There are several candidates for the title of "2-dimensional group" [18], and while these are equivalent, they are not trivially so. Of these, we shall concentrate on crossed modules of groups and cat^1-groups . The former were historically the first on the scene, appearing in the late 1940s in the context of homotopy theory; they are now well-known to be equivalent to internal categories in **Cat** [29]. The latter, dating back to about 1982, are relatively easy to picture as 2-categories with a single object, in which all 1- and 2-cells are invertible. In dealing with higher dimensional algebra, it is common to take a category theoretic approach, and this is the way we shall proceed in this thesis. Therefore it will be helpful to remind ourselves of the categorical interpretation of groups, and to see what representations look like in this context.

Viewing a group G as a category with a single object and invertible morphisms, we find that a functor $G \to \mathbf{Set}$ corresponds to a permutation representation of G (each element of G is mapped to a bijection on a fixed set, and functoriality preserves the structure of G). Similarly, linear representations (representing the elements of G as matrices/linear transformations) are given by functors $G \to \mathbf{Vect}_K$ for K a fixed field. This immediately suggests that representations of 2-groups should be 2-functors into a suitable 2-category. For permutation representations this may well be the category of groupoids, and for linear representations we investigate the 2-category $\mathbf{Ch}_K^{(1)}$ of length one chain complexes of vector spaces. Linear representations of a 2-group $\mathfrak{C}$ will be defined as 2-functors $\mathfrak{C} \to \mathbf{Ch}_K^{(1)}$. In fact, the functorial image of $\mathfrak{C}$ given by a representation naturally lies within a 2-subgroupoid of $\mathbf{Ch}_K^{(1)}$ obtained by taking only the invertible chain maps over a single length one chain complex together with homotopies between them. This substructure is denoted by $\mathbf{Aut}(\delta)$ and has the structure of a cat^1-group. Therefore a representation of an abstract cat^1-group $\mathfrak{C}$ realises it as a cat^1-subgroup of a cat^1-group with linear structure, in the same way that a representation of an abstract group G realises that group as a subgroup of a general linear group.

Taking the above as a working definition of 2-group representations, the first major task is to prove such things exist. As Heath [35] pointed out in the commentary to his translation of Euclid, "a definition asserts nothing as to the existence or non-existence of the thing defined". To ensure that we are not studying non-existent things, we simply need to come up with some examples of representations. For group representations, we can find *ad hoc* examples for particular groups, and there is an existence theorem (Cayley's theorem) that constructs a representation (the regular representation) for any group. Following this lead, we shall examine both specific examples and a generalised version of Cayley's theorem for 2-groups.

A representation theory that merely defined representations and showed that they exist would not be very exciting, but the group case suggests that a lot more should be possible (a mere glance at the size of Curtis and Reiner's tome [25] indicates that this is a big subject). We arrived at the definition of cat^1-group representations by analogy with group representations, and analogies suggest themselves for many other facets of group representation theory. Analogy is a powerful mathematical tool [60], but it must be used with caution. While analogies will suggest potential definitions and results, they do not prove that these definitions are sensible or the results true (though they may suggest ways of attempting the proof).

Faithful representations of groups are those which are injective as homomorphisms; these are particularly important, since they preserve the structure of the group more completely than any other representations. They correspond to faithful functors. Perhaps the hardest thing about generalising them to dimension two is finding a definition of faithful 2-functor. Another important strand of elementary representation theory is the idea of reducibility. This involves breaking representations down into smaller, simpler parts which cannot be further broken down, and is somewhat akin to prime decomposition of integers. A central result in this area is Maschke's theorem, which roughly states that, under fairly general conditions, representations are well-behaved and break up nicely. Again, such theory immediately suggests itself for generalisation to the next dimension, and while the extra level adds a certain amount of complication the problem is not insurmountable.

Representations are functors, so natural transformations provide morphisms between them. This naturally leads to a definition of the category of representations as a functor category. For 2-groups, modifications provide "homotopies" between the morphisms, and the category of representations becomes a 2-functor 2-category. As with any category we may ask what properties it has, for example whether it is abelian, or monoidal, and so on.

As stated earlier, while we have concentrated on the case of 2-dimensional groupoids with a single object, the results should generalise in a fairly straightforward way both to higher dimensions and to n-groupoids with many objects. Both of these situations would require more complex, cumbersome notation, but the concepts involved are scarcely more difficult. There are several algebraic models for homotopy $(n+1)$-types available, including cat^n-groups [51], crossed n-cubes [28] and hypercrossed modules [21] (for 3-types, these are equivalent to 2-crossed modules [22]). The links between these models are explored in [57]. Together with longer chain complexes of vector spaces and the corresponding higher dimensional versions of $\mathbf{Aut}(\delta)$ these would take care of the higher dimensional group versions. Moerdijk and Svensson [56] treat 2-groupoids as a wider case of algebraic 2-types, and these too could be generalised to higher dimensions.

The field K crops up many times within these pages, and we shall be spending a lot of time working with K-algebras and vector spaces. However, except where noted (for example, the construction of section 2.3), the definitions and results given work equally well when K is allowed to be a more general commutative unitary ring (and the term "vector space" replaced by "K-module"). Occasionally it will be necessary to allow K to be an integral domain ($\mathbb{Z}$ being the classic example). For the most part, however, the reader may take K to be a field and will not lose very much generality by assuming it to be $\mathbb{R}$ or $\mathbb{C}$ throughout. The exposition is mostly given in terms of fields for expositional convenience, so that the more familiar language of vector spaces can be used and the existence of bases assumed.

Structure and Principal Results

At the start of each chapter is a brief summary of what that chapter contains. The idea for this was borrowed from Milne [54]. For convenience here is an overview of the contents and arrangement of this thesis.

Chapter 1 contains accounts of some necessary preliminaries for our studies – a review of the elements of linear representations of groups; the definitions of crossed modules and cat^1-groups; the properties of a 2-category, $\mathbf{Ch}_K^{(1)}$, which generalises the category $\mathbf{Vect}_K$ of K-vector spaces; and the construction of the group algebra $K(G)$ of a group G and its use in representation theory.

The 2-category $\mathbf{Ch}_K^{(1)}$ defined in chapter 1 is an important step, but not quite sufficient for a full exploration of representation theory. In chapter 2, the cat^1-group, $\mathbf{Aut}(\delta)$ of automorphisms of a single linear transformation, which is a subcategory of $\mathbf{Ch}_K^{(1)}$, is investigated. The matrix formulation is considered in some depth, since this is particularly suitable for calculations, and several examples and special cases are explored. At the end of this chapter, we state the definition of a cat^1-group representation, although we must postpone actually working with them until we have explored the notion of a cat^1-group algebra.

Chapter 3 develops the module theoretic aspects of cat^1-group representation theory. After reminding the reader of the definition of a cat^1-algebra, we define the cat^1-group algebra of a cat^1-group. While the idea of this is exactly what we might expect, based on the notion of a group algebra, the details turn out to have a subtle twist. After this, modules over a cat^1-group algebra are defined and explored.

In chapter 4 we return to the definition of a cat^1-group representation, and prove that interesting examples actually do exist. The apex of this chapter is a cat^1-group version of Cayley's Theorem, giving a constructive proof of the existence of regular representations. We then look at some illustrative examples, and define the notion of faithfulness for cat^1-group representations. The chapter closes by briefly considering a direct description of crossed module representations.

Since we are taking a category theoretic approach, it is natural to ask about the category of representations of a given cat^1-group, and we do so in chapter 5. After setting the scene with the category of representations for a group, we look at that for a cat^1-group, which (unsurprisingly) is a 2-category. We then consider some special properties of this category. The chapter concludes with a consideration of how the degree of a group representation might be generalised to dimension two.

One of the most important ideas in group representation theory, and the subject of chapter 6, is the important concept of reducibility and irreducibility. The major landmark aimed for is an analogue of Maschke's theorem, which states that for group representations any representation is semisimple provided that the order of the group does not divide the characteristic of the field underlying the representation space. This means that, under these quite general conditions, every representation splits up nicely into well-behaved chunks. While a full two-dimensional version of Maschke's theorem will not be achieved, this chapter will make progress towards that goal by analysing the classical, one-dimensional result and considering ways in which it could be generalised.

Unfortunately, constraints of time and space prevent the development of a complete representation theory within this thesis. Chapter 7 considers the progress made so far and looks at some aspects of the theory that I have not yet had time to develop, but which would be interesting subjects for further research. These include the questions of Hopf algebra and Lie algebra structures on the cat^1-group algebra, motivated by (i) the Hopf algebra structure of the group algebra $K(G)$ (prompting the question: "is there some kind of Hopf 2-algebra structure on $\overline{K(\mathfrak{C})}$?) and (ii) some diagrams in Quillen [65] that relate group algebras to Lie algebras.

Appendix A contains some examples of matrices corresponding to $\mathbf{Ch}_K^{(1)}$ for the case $K = \mathbb{R}$. This is a worked example for section 1.3.6, but is removed to the appendix to facilitate the flow of the narrative. Appendix B is a dictionary showing the correspondence between analogous structures in group representations and cat^1-group representations. The thesis closes with an indexed glossary of notation, a bibliography and an index.

A note on notation

There is a profusion of different notation and terminology used by different authors when dealing with any mathematical subject, and those treated here are no exception. I have tried to make clear the notation I am employing, which has often been synthesised and adapted from a variety of sources. It has often been necessary to make fairly arbitrary choices. A case in point is the decision between left and right actions; both have their supporters, and I have adopted the former. Clearly, new concepts do not have existing notation or terminology, so these have been invented from scratch, often by analogy with existing cases. In general, I shall economise on notation and omit ornamentation (such as subscripts) that is rendered superfluous by the context. In particular, summations are always taken over all the indices unless otherwise stated (e.g., $\sum r_{c,p}$ is taken to be the sum over $c \in C$ and $p \in P$, while $\sum_{c \in C} r_{c,p}$ is the sum over C for a fixed value of p). Finally, I have used the standard symbol $\square$ (which, somehow, has come to stand for *quod erat demonstrandum*) to indicate the end of a proof. This symbol is used even where no actual proof is given, since it helps to make the result stand out from the surrounding text.

Working with vectors, we move freely between writing a vector $\mathbf{x}$ in the form $\sum_i \alpha_i \mathbf{v}_i$ (where $\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle$ is a basis) and writing it as an n-tuple $(\alpha_1, \ldots, \alpha_n)$. In either case the coefficients depend on the choice of basis; when this is not made explicit for an n-tuple over $\mathbb{C}$ or $\mathbb{R}$ the standard basis, $\mathbf{e}_1 = (1, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots,$ may be assumed. We shall tacitly assume that any given basis is ordered in ascending numerical order of subscripts, since this shall facilitate writing n-tuples (we shall not use n-tuples when dealing with bases indexed by multiple subscripts). In general it will be most convenient to use row vectors. However, when working with matrices we shall instead use column vectors $(\alpha_1, \ldots, \alpha_n)^{\mathsf{T}}$.

This thesis was typeset using LaTeX.

Chapter 1

Preliminaries

In which some foundations are laid for our study: group representations, various types of 2-groups, a 2-dimensional category of vector spaces, and the extremely useful group algebra functor.

Before commencing the study of cat^1-group representations we must assemble an arsenal of basic tools to use. This chapter contains no new material (although section 1.3 is a somewhat disguised special case of well-known material that appears elsewhere), but is intended to keep this thesis reasonably self-contained and to fix the notation that we shall employ later on. Section 1.1 consists of a brief overview of some of the most basic definitions in the theory of linear representations of groups. Section 1.2 will review the definitions of crossed modules and cat^1-groups. In section 1.3 we shall examine a 2-groupoid built from linear transformations of vector spaces; the aim of this is to provide a suitable target for the functors to be constructed when we define cat^1-group representations. Section 1.4 details the group algebra functor.

1.1 Linear Representations of Groups

Let K be a field and V a K-vector space. The collection of linear isomorphisms $V \to V$ forms a group under composition, denoted $GL(V)$. If V is finite dimensional, with dimension n, then $V \cong K^n$ and linear isomorphisms correspond to invertible $n \times n$ matrices with coefficients in K, so $GL(V)$ is equivalent to $GL_n(K)$, the *general linear* *group*[1].

Definition 1.1.1. Let G be a group and V a K-vector space. A *K-linear representation* of G with *representation space* V is a homomorphism

$$\phi : G \to GL(V).$$

The dimension of V is called the *degree* of ϕ.

The representation ϕ assigns to each element $g \in G$ a linear isomorphism $\phi(g) : V \to V$. The image $\phi(G)$ is a subgroup of $GL(V)$ and, by the fundamental homomorphism theorem, $G/\ker\phi \cong \phi(G)$. Particularly important is the case when ϕ is a monomorphism, for then $\ker\phi = \{1_G\}$ and G itself is isomorphic to a subgroup of $GL(V)$.

Definition 1.1.2. A representation $\phi : G \to GL(V)$ is *faithful* when ϕ is a monomorphism.

Example 1.1.3. The following are simple examples of representations over $\mathbb{C}$ (they also work over $\mathbb{R}$):

(i) Let G be any group and $n \in \mathbb{N}^+$ and define $u_n : G \to GL(\mathbb{C}^n)$ by $u_n(g) = \text{id}$ for every $g \in G$. This is the *trivial representation* of G in degree n, and is clearly not faithful unless G is the trivial group. The case $n = 1$ gives $u = u_1 : G \to \mathbb{C}^\times$ with $u(g) = 1$ for every $g \in G$; this is the *unity representation*[2].

(ii) For permutation groups, a more interesting degree 1 representation than the unity representation is defined as follows. Define $\zeta : S_n \to \mathbb{C}^\times$ by $\zeta(x) = 1$ when x is an even permutation and $\zeta(x) = -1$ when x is an odd permutation. The fact that ζ is a homomorphism amounts to the fact that a product of even or odd permutations is even, while a product of an even and an odd permutation is odd [49].

(iii) More generally, any degree 1 representation of a finite group G over $\mathbb{C}$ maps each element of G to a root of unity in $\mathbb{C}^\times$.

[1] $GL_n(K)$ is one of Weyl's *classical groups* [33], at least when $K = \mathbb{C}$. Throughout most of this thesis, K may be thought of as $\mathbb{C}$ or $\mathbb{R}$ if desired.

[2] The term "unity representation" is a translation of Serre's *représentation unité* [67].

(iv) Suppose $\sigma : G \to S_n$ is a permutation representation of degree n, where S_n is considered to be the group of permutations of the integers $1, \ldots, n$. Let $\{\mathrm{e}_1, \ldots \mathrm{e}_n\}$ be a basis for $\mathbb{C}^n$; then $\phi : G \to GL(\mathbb{C}^n)$, defined by $\phi_g(\mathrm{e}_k) := \mathrm{e}_{\sigma_g(k)}$, is a degree n linear representation.

Since linear isomorphisms of finite-dimensional vector spaces are equivalent to non-singular matrices, the definition of a representation with finite degree can easily be reformulated in terms of matrices. This approach is less amenable to theoretical development, or generalisation to representations of infinite degree, but it does facilitate calculations.

Definition 1.1.4. Let G be a group. A *matrix representation* of G over K is a homomorphism

$$\Phi : G \to GL_n(K);$$

$n \in \mathbb{N}^+$ is called the *degree* of Φ. The representation Φ is *faithful* if it is a monomorphism.

Example 1.1.5. Here are some matrix representations over $\mathbb{C}$:

(i) The trivial representations of example 1.1.3(i) correspond to matrix representations $U_n : G \to GL_n(\mathbb{C})$ with $U_n(g) = I_n$ for every $g \in G$.

(ii) Let $C_n = \langle x : x^n \rangle$ be the cyclic group of order n, and let $\omega_j\,(1 \leqslant j \leqslant n)$ be the nth roots of unity (which exist and are distinct since $\mathbb{C}$ is algebraically closed; see [69]). Define the map $\Omega_n : C_n \to GL_n(\mathbb{C})$ by $\Omega_n(x) := \mathrm{diag}(\omega_1, \ldots, \omega_n)$. Then for $1 \leqslant k \leqslant n$, $(\Omega_n(x))^k = \mathrm{diag}(\omega_1^k, \ldots, \omega_n^k) = \Omega_n(x^k)$. In particular, since $\{w_j\}$ includes primitive nth roots of unity, $\Omega_n(x^k) \neq I_n$ when $k < n$. Thus the Ω_n $(n \in \mathbb{N}^+)$ provide faithful matrix representations of any finite cyclic group.

(iii) The smallest non-abelian group is $S_3 = \langle \alpha, \beta : \alpha^3 = \beta^2 = 1, \alpha\beta = \beta\alpha^2 \rangle$, of order 6. Let A, B be the matrices:

$$A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where ω is a primitive cube root of unity. Then $\phi : S_n \to GL_2(\mathbb{C})$, $\alpha \mapsto A, \beta \mapsto B$ is a faithful representation of degree 2.

(iv) Suppose $\sigma : G \to S_n$ is a permutation representation of G, and let $\{\mathrm{e}_1, \dots \mathrm{e}_n\}$ be the standard basis for C_n, with the e_k column vectors. Then the matrix representation corresponding to the linear representation of example 1.1.3(iv) can be written down by taking the ith column of the matrix Φ_g to be $\mathrm{e}_{\sigma_g(i)}$.

It is always true that if V has dimension n, then $GL(V) \cong GL_n(K)$, but the exact isomorphism depends on the choice of basis for V. Therefore the matrix representation obtained by taking matrices corresponding to elements $\phi(g)$ of a linear representation (the representation *afforded* by ϕ) depends on the choice of basis.

Lemma 1.1.6. *Let $A, B \in GL_n(K)$. Then A, B are afforded by the same linear isomorphism $\phi : K^n \to K^n$, for (possibly) different bases, if and only if there exists a matrix $S \in GL_n(K)$ such that $B = SAS^{-1}$.*

Proof:
Standard linear algebra (see, for example, [6]). □

Definition 1.1.7. Two matrix representations $\Phi, \Psi : G \to GL_n(K)$ are *equivalent* if there is a matrix $S \in GL_n(K)$ such that

$$\Psi(g) = S\Phi(g)S^{-1}$$

for every $g \in G$.

Corollary 1.1.8. *Two matrix representations are equivalent if and only if they are afforded by the same linear representation (with possibly different bases).*

Proof:
Immediate from Lemma 1.1.6. □

The matrix formulation lies at the pragmatic, concrete end of the spectrum of representation theory. At the theoretical, abstract end of this spectrum is found the module theoretic approach pioneered by Noether. The key result that enables this approach is the bijection betweeen representations of a group and modules over its group algebra, which we shall examine in section 1.4.

1.1.0.1 Flavours of Representation Theory

We have defined representations over any field K. In fact they can be defined somewhat more generally over commutative rings with unity. Classically, representations were studied over fields of characteristic zero (such as $\mathbb{R}, \mathbb{C}$); the theory of such representations is called *ordinary representation theory* in [26]. The study of representations over fields of characteristic $p \neq 0$ (e.g. the fields $\mathbb{Z}_p$ for p a prime) was developed largely by Richard Brauer. This theory, called *modular representation theory*, diverges most significantly from ordinary representation theory when p divides the order of the group (for instance, Maschke's theorem then fails). Representations over integral domains (such as $\mathbb{Z}$) are called *integral representations*.

The same terminology can be borrowed for the 2-group representations we shall be studying. We shall usually formulate the results for general fields, although in many cases the reader can substitute $\mathbb{R}$ or $\mathbb{C}$ if so desired. In fact, most of the results will work for more general commutative rings, although we have chosen to state and, where applicable, prove them only for fields for the sake of exposition. Occasionally, it will be necessary to work in the greater generality of integral domains.

1.2 2-Group Analogues

Groups can be generalised to higher dimensions in several different ways, so we must make a choice as to how to proceed ([18, 50]). We shall restrict our attention to the two-dimensional case in which the two-dimensional elements are lozenge-shaped. Even here, there are at least five equivalent structures; a discussion can be found in [18]. We shall concentrate on crossed modules and cat^1-groups. In this section all crossed modules will be crossed modules of groups.

1.2.1 Crossed Modules

Crossed modules were first introduced by J. H. C. Whitehead [72] as a tool for homotopy theory. They also occur naturally in many other situations (see examples 1.2.3 below).

Definition 1.2.1. A *crossed module* $\mathfrak{X} = (C, P, \partial, \alpha)$ consists of groups C, P together with a homomorphism $\partial : C \to P$ and a left action $\alpha : P \times C \to C$ of P on C, written ${}^{p}c := \alpha(p, c)$, satisfying the conditions

CM1 $\partial({}^{p}c) = p\partial(c)p^{-1}$

CM2 ${}^{\partial(c)}c' = cc'c^{-1}$.

We shall sometimes assume that C, P are finite, with $|C| = \mathsf{c}$ and $|P| = \mathsf{p}$. When the action is unambiguous, we may write $\mathfrak{X}$ as the triple[3] (C, P, ∂). Crossed modules may also be drawn as $C \xrightarrow{\partial} P$, or $P \times C \xrightarrow{\alpha} C \xrightarrow{\partial} P$ if the action must be stated explicitly.

Remark: The terminology associated with crossed modules is not entirely standardised, but the following useful terms are sometimes encountered. The homomorphism $\partial : C \to P$ is called the *boundary*, while the groups C and P are referred to as, respectively, the *top group* and the *base* of the crossed module. The two crossed module axioms also have names, which are inconsistently applied. **CM1** is sometimes known as *equivariance*; **CM2** is called the *Peiffer identity* (see [13] for an explanation of this name). A structure with the same data as a crossed module and satisfying the equivariance condition but not the Peiffer identity is called a *precrossed module*.

The crossed module axioms impose some restrictions on the kernel and image of ∂. The following result is well known. A proof is included for convenience.

Proposition 1.2.2. *Let $\partial : C \to P$ be a crossed module. Then*

(i) $\partial(C) \lhd P$, and

(ii) $\ker \partial$ is a $P/\partial(C)$-module.

Proof:
Standard group theory ensures that $\ker \partial \lhd C$ and $\partial(C) \leqslant P$.

(i) $\partial(C) = \{\partial(c) : c \in C\} \leqslant P$. Suppose $x \in \partial(C)$ and $p \in P$; then $x = \partial(c)$ for some $c \in C$, and $p\partial(c)p^{-1} = \partial({}^{p}c)$ by equivariance. Now, ${}^{p}c \in C$, hence $pxp^{-1} \in \partial(C)$ as required.

(ii) $\ker \partial = \{c \in C : \partial(c) = 1_P\} \lhd C$. Let $c \in C,\ k \in \ker \partial$. The Peiffer identity ensures that ${}^{\partial k}c = kck^{-1}$, but $\partial k = 1_P$ so ${}^{\partial k}c = c$, whence $ck = kc$ and $\ker \partial \leqslant Z(C)$, the centre of C.

The action of P on C induces an action of P on $\ker \partial$; it is sufficient to check that ${}^{p}k \in \ker \partial$ whenever $k \in \ker \partial$. This is true because $\partial({}^{p}k) = p\partial(k)p^{-1} = p1_Pp^{-1} = 1_P$. Therefore $\ker \partial$ is a P-module.

[3]Most authors leave the action implicit as a matter of course. Lavendhomme and Roisin [47] explicitly include it, defined as a homomorphism $P \to Aut(C)$ (using different letters for the relevant groups).

From *(i)*, $\partial(C) \lhd P$, so $P/\partial(C)$ is defined. The action of P on $\ker\partial$ induces an action of $P/\partial(C)$ on $\ker\partial$ with ${}^{\nu(p)}k := {}^{p}k$ for $p \in P, k \in \ker\partial$ and $\nu : P \to P/\partial(C)$ the natural map. This is well-defined, since if $q \in P$ with $\nu(q) = \nu(p)$, there is a $c \in C$ with $q = p\partial c$ and so ${}^{q}k = {}^{p\partial c}k = {}^{p}ckc^{-1} = {}^{p}k$ since $\ker\partial$ is central. □

Remark: Since part *(i)* of proposition 1.2.2 depends only on equivariance, it is true for any precrossed module. However, part *(ii)* uses the Peiffer identity, so it need not be true of a general precrossed module. The proof of *(ii)* shows that the kernel of any crossed module is central in C and hence abelian.

Example 1.2.3. Certain generic situations give rise to crossed modules. Some are detailed here, others may be found in [1, 32, 58]. Our first two examples may be thought of as converses to proposition 1.2.2.

(i) Suppose $N \lhd G$ is a normal subgroup. Then G acts on N by conjugation; this action and the inclusion $\iota : N \hookrightarrow G$ form a *conjugation crossed module*, (N, G, ι).

(ii) If M is a G-module, there is a well-defined G-action on M. This, together with the zero homomorphism $0 : M \to G$ (sending everything in M to the identity in G) yields a *G-module crossed module*, $(M, G, 0)$.

(iii) A *central extension* of the group P is an epimorphism $\pi : E \twoheadrightarrow P$ where $K = \ker\pi \in Z(E)$, the centre of E. A map $s : P \to E$ such that $\pi s_p = p$ for each $p \in P$ picks out a transversal of $P \cong E/K$. Although s may not be a homomorphism,

$$s_{pp'} = s_p s_{p'} k$$

for some $k \in K$. This fact and the centrality of K ensure that the action ${}^{p}e := s_p e s_p^{-1}$ is well-defined and, together with π, yields a *central extension crossed module*, (E, P, π).

(iv) Let G be any group and Aut(G) its group of automorphisms. There is an obvious action of Aut(G) on G, and a homomorphism $\phi : G \to \text{Aut}(G)$ sending each $g \in G$ to the inner automorphism of conjugation by g. These together form an *automorphism crossed module*, $(G, \text{Aut}(G), \phi)$.

(v) Any group G may be thought of as a crossed module in two ways. Since G always has the two normal subgroups $\{1\}$ and G, we can form the conjugation crossed modules $\{1\} \hookrightarrow G$ and $\mathrm{id} : G \to G$. Note that the homomorphism $G \to \{1\}$ with the trivial action forms a crossed module whenever G is abelian, otherwise the Peiffer identity fails and the result is a precrossed module.

These examples provide strong motivation for studying crossed modules. As Norrie [58] wrote in her thesis: "crossed modules simultaneously generalize normal subgroups, modules over a group and central extensions of groups. Furthermore any group together with its automorphism group gives rise to a crossed module, and any group may itself be regarded as a crossed module."

Example 1.2.4. In addition to the generic examples of crossed modules considered above, it will be useful to have at hand some small, easily checked, examples of individual crossed modules. These will provide test cases for later constructions.

(i) Let $C_2 = \langle x : x^2 \rangle$ and $I = \{1\}$. Then $C_2 \to I$ is a crossed module (there is no choice for either the homomorphism or the action).

(ii) Let $C_3 = \langle x : x^3 \rangle$ and $C_2 = \langle y : y^2 \rangle$. The zero homomorphism and the trivial action, 1, (with ${}^{y}x = x$) make $(C_3, C_2, 0, 1)$ a crossed module.

(iii) With the same groups and boundary as (ii), define the *twisting* action, τ, of C_2 on C_3 as ${}^{y}x := x^2$. This also yields a crossed module $(C_3, C_2, 0, \tau)$.

(iv) Let $C_4 = \langle x : x^4 \rangle$ and $C_2 = \langle y : y^2 \rangle$ (where $y = x^2$). Action by conjugation fixes each element of C_4, and together with the boundary ∂ defined by $x \mapsto y$, gives a crossed module, (C_4, C_2, ∂). Since C_4 is abelian, action by conjugation fixes every element of C_4, and we suppress explicit mention of the action.

Definition 1.2.5. If $\mathfrak{X}_1 = (C_1, P_1, \partial_1, \alpha_1)$ and $\mathfrak{X}_2 = (C_2, P_2, \partial_2, \alpha_2)$ are crossed modules, then a *crossed module morphism* $\phi : \mathfrak{X}_1 \to \mathfrak{X}_2$ consists of a pair of group homomorphisms $\phi_C : C_1 \to C_2$ and $\phi_P : P_1 \to P_2$ that commute with ∂_i and preserve the action. That is, $\partial_2\phi_C = \phi_P\partial_1$ and $\phi_C({}^{p}c) = {}^{\phi_P(p)}\phi_C(c)$. These conditions are encapsulated in the commutativity of the following diagrams:

$$\begin{tikzcd} C_1 \arrow[r, "\phi_C"] \arrow[d, "\partial_1"'] & C_2 \arrow[d, "\partial_2"] \\ P_1 \arrow[r, "\phi_P"'] & P_2 \end{tikzcd} \quad , \qquad \begin{tikzcd} P_1 \times C_1 \arrow[r, "\alpha_1"] \arrow[d, "(\phi_P, \phi_C)"'] & C_1 \arrow[d, "\phi_C"] \\ P_2 \times C_2 \arrow[r, "\alpha_2"'] & C_2 \end{tikzcd}$$

Example 1.2.6. Let $\mathfrak{X} = (C_3, C_2, 0, 1)$ and $\mathfrak{Y} = (C_3, C_2, 0, \tau)$ be the crossed modules of example 1.2.4 (ii) and (iii) respectively. Then $\phi : \mathfrak{X} \to \mathfrak{Y}$ with $\phi_{C_3} : x \mapsto x^2$ and $\phi_{C_2} : y \mapsto 1$ is a crossed module map; in fact it is the only non-trivial map between these crossed modules. There is no non-trivial map $\mathfrak{Y} \to \mathfrak{X}$.

There is an obvious composition of crossed module morphisms that leads to the category **XMod** of crossed modules (of groups) and their morphisms.

An important construction on crossed modules, which relates to their homotopy theoretic origins, is the *classifying space* [11, 31, 51]. This is a functor B from **XMod** to the category of CW-complexes such that for any $\mathfrak{X} \in$ **XMod**, $B\mathfrak{X}$ is a connected CW-complex with $\pi_1 B\mathfrak{X} \cong \operatorname{coker} \delta = P/\partial(C)$, $\pi_2 B\mathfrak{X} \cong \ker \delta$ and $\pi_i B\mathfrak{X} = 0$ for every $i \neq 1, 2$. Also, for any connected CW-complex with $\pi_i = 0$ for $i \neq 1, 2$, there is a crossed module with that complex as its classifying space. In this way, crossed modules of groups are algebraic models for 2-types. We may slightly abuse notation and write $\pi_i \mathfrak{X}$ for $\pi_i B\mathfrak{X}$. The classifying space functor is a generalisation of Segal's classifying space functor for categories (and hence groups), described in [45, 46].

1.2.2 Cat^1-groups

Cat^1-groups (originally called 1-cat groups) are the first in a series of models for homotopy n-types introduced by Loday [51]. They are sometimes referred to simply as cat-groups [31] if the higher cat^n-groups are not also being considered[4]; the term *categorical group* [3] is used for similar structures in which inverses for the group operations are only defined up to isomorphism [37].

Definition 1.2.7. A *cat^1-group* $\mathfrak{C} = (G, P, i, s, t)$ consists of groups G and P, an embedding $i : P \rightarrowtail G$ and epimorphisms $s, t : G \twoheadrightarrow P$ satisfying:

CG1 $si = ti = \mathsf{id}_P$

CG2 $[\ker s, \ker t] = \{1_G\}$

We shall often assume that both groups are finite, and write g for $|G|$. A morphism $\gamma : \mathfrak{C}_1 \to \mathfrak{C}_2$ of cat^1-groups consists of a pair $\gamma_G : G_1 \to G_2$ and $\gamma_P : P_1 \to P_2$ that commute with the homomorphisms of $\mathfrak{C}_1$ and $\mathfrak{C}_2$. With the obvious composition, there is a category **Cat1** of cat^1-groups and their morphisms.

[4]Although we shall not be considering cat^n-groups for $n \geqslant 2$, we shall continue to use the term cat^1-group in the interests of greater generality.

Remark: As with crossed modules, the groups G and P of a cat^1-group may be referred to as the *top group*[5] and *base* respectively. The morphisms s, t (standing for *source* and *target* respectively) are called *structural (homo)morphisms* and i is the *inclusion* (a justifiable term, since P can always be thought of as a subgroup of G). **CM1** may be termed the *identity condition* and **CM2** is the *kernel condition*. A structure with the same data as a cat^1-group and satisfying the identity condition but not the kernel condition is called a *precat^1-group*.

Example 1.2.8. We shall not need to define any specific examples of cat^1-groups directly since, as we shall see below, we can obtain cat^1-groups from any crossed modules. However, in considering trivial cat^1-groups, note that the top group may not be trivial unless the base is also trivial, for the identity condition would be violated. The simplest cat^1-groups are those for which the base and top group are the same, and all morphisms the identity. Any group G forms a precat^1-group with trivial base.

There is a useful structural decomposition of the top group of a cat^1-group as a semidirect product involving the base and one of the structural homomorphisms. First we recall the definition of a semidirect product.

Definition 1.2.9. If G and H are groups, with a left action of H on G, the *semidirect product* of G by H is the group $G \rtimes H = \{(g, h) : g \in G, h \in H\}$ with multiplication $(g, h)(g', h') := (g\,{}^h g', hh')$. The inverse of (g, h) is $({}^{h^{-1}}g^{-1}, h^{-1})$.

Of course, the semidirect product $G \rtimes H$ has the same underlying set as the direct product $G \times H$, so $|G \rtimes H| = |G||H|$. Since, for any cat^1-group, $\ker s \lhd G$ and $iP \leqslant G$, there is an action of iP on $\ker s$ by conjugation. Hence, the semidirect product $\ker s \rtimes P$ is defined.

Lemma 1.2.10. *For a cat^1-group (G, P, i, s, t),*

$$G \cong \ker s \rtimes P.$$

Proof:
Let $\phi : G \to \ker s \rtimes P$ with $\phi(g) := (gis(g^{-1}), s(g))$ and $\psi : \ker s \rtimes P \to G$ with $\psi(c, p) := ci(p)$. It is straightforward to check that ϕ, ψ are homomorphisms and $\psi = \phi^{-1}$. □

[5] Brown and Loday call this the *big group* in their definition of cat^n-groups [14].

From a crossed module $\mathfrak{X} = (C, P, \partial)$ we can construct a cat^1-group, $\mathfrak{C}(\mathfrak{X})$:

$$\begin{tikzcd} C \rtimes P \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & P \arrow[l, bend left, "i"] \end{tikzcd}$$

Here $s, t : C \rtimes P \to P$, $i : P \to C \rtimes P$ are defined as $s(c, p) = p$, $t(c, p) = \partial(c)p$ and $i(p) = (1_C, p)$. Then $s\mid_P = t\mid_P = \mathrm{id}_P$ and $[\ker s, \ker t] = 1_{C\rtimes P}$.

Note that $(c, p) \in \ker s \Leftrightarrow p = 1_P$, i.e. $\ker s = \{(c, 1_P)\} \cong C$; hence $t(c, 1_P) = \partial(c)1_P = \partial(c)$, so $\partial = t\mid_{\ker s}$ and we can recover $\mathfrak{X}$ from $\mathfrak{C}(\mathfrak{X})$. The same trick enables us to construct a crossed module $\mathfrak{X}(\mathfrak{C})$ from any given cat^1-group $\mathfrak{C}$, if we first use lemma 1.2.10 to decompose the top group. These constructions lead to the well-known equivalence between crossed modules and cat^1-groups [51].

Example 1.2.11. Each of the crossed modules of example 1.2.4 yields a cat^1-group according to the given recipe:

(i) From $C_2 \to I$ we get the cat^1-group $(C_2, I, i, 0, 0)$ where i is the inclusion $(1 \mapsto 1)$ and s, t both the zero map.

(ii) The semidirect product corresponding to $(C_3, C_2, 0, 1)$ is $C_3 \rtimes C_2 \cong C_6$ and so $\mathfrak{C}(\mathfrak{X}) = (C_6, C_2, i, s, s)$, where the structural homomorphisms are identical and send odd powers of the generator of C_6 to the generator of C_2 and even powers to the identity.

(iii) $(C_3, C_2, 0, \tau)$ gives semidirect product $C_3 \rtimes C_2 \cong S_3$, so the corresponding cat^1-group is $\mathfrak{C}(\mathfrak{Y}) = (S_3, C_2, i, s, s)$. The inclusion i maps y to a transposition in S_3, while s maps every odd permutation to the generator of C_2 and every even permutation to the identity.

(iv) (C_4, C_2, δ) with action by conjugation yields $C_4 \rtimes C_2 \cong C_4 \times C_2$ and leads to the cat^1-group $(C_4 \times C_2, C_2, i, s, t)$ with s the projection onto C_2 and t the homomorphism sending both $(x, 1)$ and $(1, y)$ (that is, the generators of $C_4 \times C_2$) to y.

All of these cat^1-groups are included in the first few rows of Alp and Wensley's table [1].

Since cat^1-groups are equivalent to crossed modules, it follows that they too must be algebraic models for 2-types. It is also possible to construct the classifying space of a cat^1-group, and establish this fact directly. The construction, detailed in [19] for cat^n-groups, starts by forming the nerve of $\mathfrak{C}$, a simplicial group. The classifying space, $B\mathfrak{C}$, of $\mathfrak{C}$ is defined to be the classifying space of its nerve. As with the crossed module case, the homotopy groups of $B\mathfrak{C}$ can be found, and we shall write $\pi_i\mathfrak{C}$ for $\pi_i B\mathfrak{C}$. Garzon and Miranda [31] state that $\pi_1\mathfrak{C} = \text{coker}\,(s,t)$ (the coequaliser of s,t, using the notation of [7]) $= P/t(\ker s)$ and $\pi_2\mathfrak{C} = \ker s \cap \ker t$, while all other π_i are trivial. These results can also be obtained from general formulae given in [57] (using the fact that a cat^1-group is, equivalently, a simplicial group with Moore complex of length 1). Some authors, including Garzon and Miranda, shift the suffixes and have π_0 and π_1 respectively for the groups we are denoting as π_1 and π_2. The choice of suffix convention employed depends on whether you principally interpret a cat^1-group as a 1- or a 2-dimensional structure (see page 21).

1.2.3 Composition

We have seen that crossed modules and cat^1-groups are equivalent. That is, they may be considered as different ways of looking at the same thing. They are also equivalent to 2-categories [7, 44, 70] with certain restrictions. This categorical view is particularly useful for two principal reasons. Firstly, it provides a more visual understanding of the structures [2], and while a visual intuition breaks down in higher dimensions – and cannot entirely be relied upon even in the lowest dimensions – it can provide useful insights. Also, the full weight of categorical machinery may be brought to bear on the problem in hand; this usually opens the way to more elegant methods than a brute force approach.

A cat^1-group is more-or-less immediately an internal category in the category of groups [21, 51]. This fact is uncovered in section 12.8 of [52], *en route* to showing the equivalence of internal categories and crossed modules of groups, although MacLane does not explicitly use the term 'cat^1-group' in his account. The essence of the argument is that the kernel condition of the cat^1-group is equivalent to a categorical composition which is a group homomorphism (i.e. there is an interchange law between this composition and the multiplication in the top group G).

We now show that these are equivalent to 2-groups[6]. P is a group, so its elements can be taken as arrows (1-cells) with a unique vertex (denoted $\star$). The multiplication in P becomes composition in the categorical view. We may write this as $q\#_0 p := qp$ (meaning 'first p then q'); the subscript 0 indicates that the two elements are joined by a common face in dimension zero; in this case, there is only one possible such face ($\star$) so composition is always defined. We shall use the notations of composition ($\#_0$) and multiplication (juxtaposition) interchangeably. The idea behind this notation is that each composition for an n-cell is denoted as $\#_k$ with $k = 0, 1, \dots n-1$; k is the dimension of the common boundary by which the adjacent cells are joined. This notation is useful in 2-dimensional algebra and certainly invaluable in higher dimensions. It is quite common in recent papers on higher-dimensional algebra (e.g., [23]), though I am not sure who introduced it.

By lemma 1.2.10, G can be decomposed as $\ker s \rtimes P$, so a typical element is (c, p), with $c \in \ker s$, $p \in P$. Then $s(c,p) = p$ and $t(c,p) = \partial(c)p$ so we can view (c,p) as a 2-cell $p \Rightarrow \partial(c)p$, with source $s(c,p)$ and target $t(c,p)$ (hence the names):

$$\begin{tikzcd}
\star \arrow[r, bend left=50, "p"] \arrow[r, bend right=50, "\partial(c)p"'] \arrow[r, phantom, "\Downarrow (c,p)"] & \star
\end{tikzcd}$$

Of course, G is itself a group, so it is also a category with one object, $\star$ (which may be taken to be the same one as for P); its multiplication is interpreted categorically as composition. Suppose (c,p) and (d,q) are in G, then $(d,q)\#_0(c,p) = (d,q)(c,p) = (d^q c, qp)$. Now $\partial(d^q c) = \partial(d)q\partial(c)q^{-1}$, so $(d^q c, qp)$ is a 2-cell $qp \Rightarrow \partial(d)q\partial(c)p$. Pictorially,

$$\begin{tikzcd}
\star \arrow[r, bend left=50, "p"] \arrow[r, bend right=50, "\partial(c)p"'] \arrow[r, phantom, "\Downarrow (c,p)"] & \star \arrow[r, bend left=50, "q"] \arrow[r, bend right=50, "\partial(d)q"'] \arrow[r, phantom, "\Downarrow (d,q)"] & \star & = & \star \arrow[r, bend left=50, "q\#_0 p"] \arrow[r, bend right=50, "\partial(d^q c)q\#_0 p"'] \arrow[r, phantom, "\Downarrow (d^q c, q\#_0 p)"] & \star
\end{tikzcd}$$

The picture makes it obvious why this is called *horizontal composition*.

[6] It is worth noting that the term '2-group' is used by group theorists to mean a group with order a power of 2 [66]. This is quite different from the category theoretic meaning used here, i.e. a 2-category with invertible 1- and 2-cells and only one object [3].

There is an alternative composition that can also be defined. Suppose $(c', \partial(c)p)$ is another element in G, i.e. $c' \in \ker s$. Then we can compose (c, p) and $(c', \partial(c)p)$ by defining

$$(c', \partial(c)p)\#_1(c, p) := (c'c, \partial(c'c)p)$$

For this composition, the subscript 1 indicates that the two elements to be composed share a common face in dimension one (an element of P). Due to the natural way of drawing 2-cells, this composition is known as *vertical composition.*

$$\begin{array}{c}
\begin{tikzcd} \star \arrow[r, bend left=50, "p"] \arrow[r, bend right=50, "\partial(c)p"'] \arrow[r, phantom, "\Downarrow (c,p)"] & \star \end{tikzcd} \\
\begin{tikzcd} \star \arrow[r, bend left=50, "\partial(c)p"] \arrow[r, bend right=50, "\partial(c')\partial(c)p"'] \arrow[r, phantom, "\Downarrow (c',\partial(c)p)"] & \star \end{tikzcd}
\end{array}
=
\begin{tikzcd} \star \arrow[r, bend left=50, "p"] \arrow[r, bend right=50, "\partial(c'c)p"'] \arrow[r, phantom, "\Downarrow (c'\#_0 c,p)"] & \star \end{tikzcd}$$

The second condition on cat^1-groups establishes an interchange law between the horizontal and vertical compositions, hence we have a 2-category. Since both compositions are clearly invertible and there is only one object, it is in fact a 2-group.

Conversely, any 2-group is a cat^1-group. The source and target maps from 2-cells to 1-cells, together with the identity map from 1- to 2-cells, function as the structural morphisms and immediately satisfy the identity condition. The kernel condition is slightly more subtle, but comes fairly directly from the interchange law between the two compositions in the 2-group, which essentially states that vertical composition is a homeomorphism [64].

At this point we may also note in passing that a cat^1-group can also be interpreted as a strict monoidal groupoid. This follows from the observation in [50] that a 2-category with one object is a monoidal category. Because of this fact, a cat^1-group may be viewed either as a two-dimensional structure (a 2-groupoid with a single 0-cell) or as a one-dimensional structure (a monoidal category with inverses). We shall usually take the first view, and this will be reflected in the notation we employ. The reader should be aware that different sources may use different notational conventions.

1.3 A 2-Groupoid of Very Short Chain Complexes

From a categorical viewpoint, a linear representation of a group G becomes a functor from G to the category $\mathbf{Vect}_K$ of K-vector spaces. In section 1.2.1 we saw that a cat^1-group is essentially a 2-group, and in section 4.1 we shall be exploring cat^1-group representations as a 2-categorical generalisation of group representations. In order to do this, we need a 2-categorical analogue of $\mathbf{Vect}_K$, and it is to this problem that we now turn. The construction we shall describe in this section is essentially a special case of Gabriel and Zisman's 2-category of complexes [30].

1.3.1 First Steps

Let K be a field and let C_0, C_1 be vector spaces over K. If $\delta : C_1 \to C_0$ is a linear transformation, then $\mathcal{C} : C_1 \xrightarrow{\delta} C_0$ is a length 1 chain complex of vector spaces. $\mathcal{C}$ can be considered as $\ldots \to 0 \to C_1 \xrightarrow{\delta} C_0 \to 0 \to \ldots$ and so composition trivially gives the zero map and δ is a differential (a discussion of the terminology and basic theory of chain complexes can be found in [39]). Thus, every linear transformation can be considered as a chain complex. It will sometimes be convenient to blur the distinction between the linear transformation and its chain complex, and refer to δ itself as a chain complex.

Suppose that in addition to $\mathcal{C}$ we have a chain complex $\mathcal{D} : D_1 \xrightarrow{\delta^D} D_0$ (write δ^C for the differential in $\mathcal{C}$ to distinguish it). Then a morphism between $\mathcal{C}$ and $\mathcal{D}$ is defined as follows.

Definition 1.3.1. A *chain map* $f : \mathcal{C} \to \mathcal{D}$ consists of components $f_1 : C_1 \to D_1$ and $f_0 : C_0 \to D_0$ such that $\delta^D f_1 = f_0 \delta^C$; i.e. the following diagram commutes:

$$\begin{tikzcd} C_1 \arrow[r, "f_1"] \arrow[d, "\delta^C"'] & D_1 \arrow[d, "\delta^D"] \\ C_0 \arrow[r, "f_0"'] & D_0 \end{tikzcd}$$

Suppose $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{E}$ are chain maps. Then the composite $g\#_0 f : \mathcal{C} \to \mathcal{E}$ is defined by $(g\#_0 f)_i := g_i f_i$, where $i = 0, 1$ and the composition on the right hand side is the usual one for linear maps. The notation used for the composition on the left hand side is the same as that described in section 1.2.3, and is introduced in order to facilitate exploration of 2-cells later on. Each chain map has a clearly defined source and target, and maps can be composed when the target of one coincides with the source of the next; composition is clearly associative. For each chain complex $\mathcal{C}$, there is an identity, $\mathrm{id}_{\mathcal{C}}$, under this composition, where $(\mathrm{id}_{\mathcal{C}})_i$ is the identity on C_i. Hence the collection of all length 1 chain complexes of vector spaces over K, and all chain maps between them, forms a category.

Definition 1.3.2. Let K be a field. The category of length 1 chain complexes of K-vector spaces, and chain maps between them, is denoted $\mathbf{Ch}_K^{(1)}$.

This structure provides the foundation for a 2-groupoid. By restricting our attention to those chain maps that are invertible we obtain a subgroupoid of $\mathbf{Ch}_K^{(1)}$, which we shall write as $\mathbf{invCh}_K^{(1)}$. From definition 1.3.1 it is clear that a chain map $f : \mathcal{C} \to \mathcal{D}$ is invertible precisely when both its components are invertible.

Definition 1.3.3. A *chain isomorphism* is an invertible chain map $f : \mathcal{C} \to \mathcal{D}$.

Hence the morphisms of $\mathbf{invCh}_K^{(1)}$ are precisely the chain isomorphisms of $\mathbf{Ch}_K^{(1)}$.

As an aside, readers with a penchant for homological algebra will observe that for any $\mathcal{C} \in \mathbf{Ch}_K^{(1)}$, $H_1(C) = \ker \delta^C$ and $H_0(C) = \operatorname{coker} \delta^C = C_0/\delta^C(C_1)$, while $H_n(C) = 0$ for every other $n \in \mathbb{Z}$. We shall not explicitly consider homology any further in this thesis.

1.3.2 The Next Dimension

The next task is to find a groupoid enrichment for $\mathbf{Ch}_K^{(1)}$ (see [39] for an introduction to this concept). This will make it a 2-category in which the vertical composition is invertible; hence, $\mathbf{invCh}_K^{(1)}$ will be a 2-groupoid.

One of the standard motivating examples of a 2-category (given, for instance, in [52]) is **Top**, in which the 2-cells are given by homotopies between the continuous maps. This suggests that, for other suitable categories, homotopy might provide a 2-categorical structure. Chain complexes of vector spaces have a well-behaved homotopy theory (in fact, several equivalent theories). We shall consider homotopies of length 1 chain complexes over $\mathbf{Vect}_K$, with a view to demonstrating their suitability as a 2-cell structure for $\mathbf{Ch}_K^{(1)}$. See [39] for a more detailed treatment of the elementary homotopy theory of chain complexes in general.

For any chain complex $\mathcal{C}$, there is a chain complex $\mathcal{C} \otimes I$ defined as

$$(\mathcal{C} \otimes I)_n := C_n \oplus C_n \oplus C_{n-1}$$

with differential

$$\delta^{\mathcal{C}\otimes I}(x, y, z) := (\delta x - z, \delta y + z, -\delta z).$$

As usual, the superscript on the δ will be omitted wherever possible. For $\mathcal{C} \in \mathbf{Ch}_K^{(1)}$, most of the C_i are trivial and so $\mathcal{C} \otimes I$ reduces to a chain complex of length 2:

$$0 \to C_1 \xrightarrow{\delta_2} C_1 \oplus C_1 \oplus C_0 \xrightarrow{\delta_1} C_0 \oplus C_0 \to 0.$$

This construction yields a functor $- \otimes I : \mathbf{Ch}_K^{(1)} \to \mathbf{Ch}_K^{(2)}$, where $\mathbf{Ch}_K^{(2)}$ is the 3-category of length 2 chain complexes (to be discussed in section 1.3.5.2). There are natural transformations $e_0, e_1 : \mathrm{id}_{\mathbf{Ch}_K^{(1)}} \to - \otimes I$ and $\sigma : - \otimes I \to \mathrm{id}_{\mathbf{Ch}_K^{(1)}}$ with $e_0(C)(x) = (x, 0, 0)$, $e_1(C)(y) = (0, y, 0)$ and $\sigma(C)(x, y, z) = (x + y)$. This functor and these natural transformations provide a cylinder structure on $\mathbf{Ch}_K^{(1)}$, which allows homotopy to be defined.

Definition 1.3.4. Let $f, f' : \mathcal{C} \to \mathcal{D}$ be chain maps in $\mathbf{Ch}_K^{(1)}$. Then f is *homotopic* to f', written $f \simeq f'$, if there is a chain map $h : \mathcal{C} \otimes I \to \mathcal{D}$ with $he_0(C) = f$ and $he_1(C) = f'$.

In practice, homotopies, along with chain complexes, chain maps, and other graded stuff, are usually viewed as "black boxes" which take the given input and produce the required output without the user having to worry about the details. At times, however, it is useful to be able to see inside the box, so to speak, and examine what is happening at the level of the individual objects/maps. The following diagram illustrates the definition of homotopy for $\mathbf{Ch}_K^{(1)}$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d, "\delta_2^{\mathcal{C}\otimes I}"] & 0 \arrow[d] \\
C_1 \arrow[r, shift left, "(e_0(C))_1"] \arrow[r, shift right, "(e_1(C))_1"'] \arrow[d, "\delta^{\mathcal{C}}"'] & C_1 \oplus C_1 \oplus C_0 \arrow[r, "h_1"] \arrow[d, "\delta_1^{\mathcal{C}\otimes I}"] & D_1 \arrow[d, "\delta^{\mathcal{D}}"] \\
C_0 \arrow[r, shift left, "(e_0(C))_0"] \arrow[r, shift right, "(e_1(C))_0"'] \arrow[d] & C_0 \oplus C_0 \arrow[r, "h_0"] \arrow[d] & D_0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

The condition $f = he_0(C)$ implies that $h(x,0,0) = f(x)$ for any $x \in \mathcal{C}$; likewise $h(0,y,0) = f'(y)$. Define $h'(z) := h(0,0,z)$, then for any $(x,y,z) \in \mathcal{C} \otimes I$, $h(x,y,z) = f(x) + f'(y) + h'(z)$. It is straightforward to check that $\delta h' + h'\delta = f' - f$. Note that because $\mathcal{C}$ and $\mathcal{D}$ are both length 1 chain complexes, the homotopy is trivial in dimension two.

Definition 1.3.5. If $f, f' : \mathcal{C} \to \mathcal{D}$ in $\mathbf{Ch}_K^{(1)}$ then a map $h' : C_0 \to D_1$ with $h'\delta^C = f_1' - f_1$ and $\delta^D h' = f_0' - f_0$ is called a *chain homotopy* from f to f':

$$\begin{tikzcd} C_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_1'"'] \arrow[d, "\delta^C"'] & D_1 \arrow[d, "\delta^D"] \\ C_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_0'"'] \arrow[ur, dashed, "h'"] & D_0 \end{tikzcd}$$

The chain homotopy condition may be written more succinctly as $\delta h' + h'd = f' - f$, bearing in mind that these are graded maps (δ being of degree -1, h' of degree +1, and the chain maps f, f' of degree 0). This formula is also valid for longer chain complexes.

From the foregoing discussion, it is clear that every homotopy on $\mathbf{Ch}_K^{(1)}$ yields a chain homotopy. Conversely, suppose h' is a chain homotopy from f to f'. Then $h(x,y,z) := f(x) + f'(y) + h'(z)$ is a homotopy. Indeed, the chain homotopy condition may be rearranged and then substituted into this formula to yield:

$$h(x,y,z) = f(x) + (\delta h' + h'\delta + f)(y) + h'(z).$$

Therefore any homotopy is uniquely specified by its chain homotopy and its starting point, and we may write $h \leftrightarrow (h', f)$ interchangeably. Note that for chain maps we use primes (′) to distinguish between different maps having the same source and target, while for homotopies we use h' to denote the chain homotopy component of the homotopy h, i.e. $h'(z) = h(0,0,z)$. In practice it should always be clear from the context whether we are referring to chain maps or to homotopies, so no confusion should occur.

It would be cumbersome to always write out the full commutative diagrams for chain maps homotopies, but the "black box" idea enables us to avoid this most of the time. A homotopy $h = (h', f) : f \simeq f'$ runs between two arrows (1-cells) of $\mathbf{Ch}_K^{(1)}$, and so can be viewed as a 2-cell:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow h"] & \mathcal{D}. \end{tikzcd}$$

The 2-cell may be written more compactly as $h : f \Rightarrow f'$, although it will sometimes be more convenient to draw the full picture, particularly when horizontal and vertical composition are considered. In practice, we shall flit freely between all of these notations. Of course, it will sometimes also be useful to open up the black box and work on a component level.

Note that while the endpoints are implicit in any homotopy h, the chain homotopy h' is not unique to $f \simeq f'$. It is possible to have $f \neq g$ and $f' \neq g'$ such that both $\delta h' + h'\delta = f' - f$ and $\delta h' + h'\delta = g' - g$, i.e. $f \simeq f'$ and $g \simeq g'$. Therefore a 2-cell is not completely determined by the chain homotopy. We shall return to this observation later.

Calculations with homotopies may be done using either the chain homotopy or the cylindrical homotopy formulation, since these are equivalent. We shall usually use chain homotopies. In order to get a feel for calculations using chain homotopies, we shall check that homotopy is an equivalence relation. Note that homotopy is automatically reflexive (this is a basic consequence of the cylinder structure), but not neccessarily symmetric or transitive. These are, however, desirable properties for a homotopy theory and fortunately the homotopy of chain complexes, like the standard homotopy of topology, has them.

Proposition 1.3.6. *Homotopy on* $\mathbf{Ch}_K^{(1)}$ *is an equivalence relation.*

Proof:
It is sufficient to check the three properties directly.

Reflexivity: For any $f \in \mathbf{Ch}_K^{(1)}$, we have $0\delta^C = 0 = f_1 - f_1$ and $\delta^D 0 = 0 = f_0 - f_0$; thus the zero map is a chain homotopy for $f \simeq f$. Denote the corresponding homotopy by $1_f : f \Rightarrow f$ (for consistency, we may write $1'_f = 0$).

Symmetry: Suppose $h : f \simeq f'$. Then $f_1 - f'_1 = -h'\delta^C$ and $f_0 - f'_0 = -\delta^D h' = \delta^D(-h')$, so $-h'$ is a chain homotopy $f' \simeq f$ and there is a corresponding $-h : f' \Rightarrow f$, as required.

Transitivity: Suppose $h : f \simeq f'$ and $\hat{h} : f' \simeq f''$. Then, in addition to the equations for f_i, f_i' and h' already established, $\hat{h}'\delta^C = f_1'' - f_1''$, $\delta^D\hat{h} = f_0'' - f_0'$. Then $(h'+\hat{h}')\delta^C = h'\delta^C + \hat{h}\delta^C = f_1' - f_1 + f_1'' - f_1' = f_1'' - f_1$ and $\delta^D(h'+\hat{h}') = \delta^D h' + \delta^D \hat{h}' = f_0' - f_0 + f_0'' - f_0' = f_0'' - f_0$ hence $(h' + \hat{h}')$ is a chain homotopy. Thus homotopy is transitive and hence an equivalence relation. □

Transitivity allows us to define a composition of homotopies.

Definition 1.3.7. Let $h : f \simeq f'$ and $\hat{h} : f' \simeq f''$ $(f, f', f'' : \mathcal{C} \to \mathcal{D})$. Then the *vertical composite* $(\hat{h}\#_1 h) : f \simeq f''$ is the homotopy with chain homotopy component:

$$(\hat{h}\#_1 h)' := \hat{h}' + h'. \tag{1.1}$$

Using 2-cells, this composition is pictured as follows:

$$\begin{array}{c} \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow h"] & \mathcal{D} \end{tikzcd} \\ \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f'"] \arrow[r, bend right, "f''"'] \arrow[r, phantom, "\Downarrow \hat{h}"] & \mathcal{D} \end{tikzcd} \end{array} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f"] \arrow[r, bend right, "f''"'] \arrow[r, phantom, "\Downarrow \hat{h}\#_1 h"] & \mathcal{D} \end{tikzcd}.$$

The name *vertical composition* refers to the way 2-cells are usually drawn[7], and distinguishes it from the horizontal composition to be defined below. The collection of all homotopies between chain maps in $\mathbf{Ch}_K^{(1)}$ will now be considered as a collection of 2-cells in $\mathbf{Ch}_K^{(1)}$. Each homotopy has clearly defined source and target chain maps and (by composition) source and target chain complexes. The composition is associative. Reflexivity of homotopy gives us an identity 1_f for each chain map f. Symmetry ensures that $(-h\#_1 h)' = -h' + h' = 0$, so every 2-cell is invertible under vertical composition. Thus $\mathbf{Ch}_K^{(1)}$ has a vertical groupoid structure on its 2-cells (with composition written as $\#_1$), and a horizontal category structure on its 1-cells (with composition $\#_0$). Note that, whereas to get a groupoid structure horizontally (as in $\mathbf{invCh}_K^{(1)}$) we must explicitly restrict our attention to invertible chain maps, all 2-cells are intrinsically invertible under

[7]The notation for vertical composition ($\#_1$), and that for horizontal composition ($\#_0$), is the notation introduced in section 1.2.3 in the context of cat[1]-groups.

vertical composition, due to the properties of homotopy for chain complexes. It remains to extend the horizontal composition to the 2-cells and establish a relationship between the two types of 2-cell composition.

1.3.3 Whiskers

As an intermediate step to defining horizontal composition for 2-cells, we can define a composition between a 1-cell and a 2-cell. Let $h : f \simeq f' : \mathcal{C} \to \mathcal{D}$ be a homotopy and $g : \mathcal{D} \to \mathcal{E}$ a chain map. With the chain homotopy h' we get the following picture:

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_1'"'] \arrow[d, "\delta^C"'] & D_1 \arrow[r, "g_1"] \arrow[d, "\delta^D"] & E_1 \arrow[d, "\delta^E"] \\
C_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_0'"'] \arrow[ur, dashed, "h'"] & D_1 \arrow[r, "g_0"'] & E_0
\end{tikzcd} .$$

Since h' and g_1 are compatible linear transformations, they may be composed; the composite g_1h' satisfies:

$$\delta^E g_1 h' = g_0 \delta^D h' = g_0(f_0' - f_0) = g_0 f_0' - g_0 f_0$$

and

$$g_1 h' \delta^C = g_1(f_1' - f_1) = g_1 f_1' - g_1 f_1.$$

In other words, g_1h' is a chain homotopy from $g\#_0 f$ to $g\#_0 f'$.

Definition 1.3.8. For $h : f \simeq f' : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{E}$, the *whiskering* $g\#_0 h : g\#_0 f \simeq g\#_0 f'$ is the homotopy with chain homotopy component

$$(g\#_0 h)' := g_1 h'. \tag{1.2}$$

This choice of terminology is explained by the picture:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow h"] & \mathcal{D} \arrow[r, "g"] & \mathcal{E}
\end{tikzcd} = \begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "g\#_0 f"] \arrow[r, bend right, "g\#_0 f'"'] \arrow[r, phantom, "\Downarrow g\#_0 h"] & \mathcal{E}
\end{tikzcd}$$

Since the natural "whisker" is the 1-cell g, which appears on the left in the notation $g\#_0 h$ but on the right in the diagram, we refer to it as a *postwhisker* (and to the operation as *postwhiskering*) rather than a whisker on the left or on the right.

As might be expected, whiskering can also be defined for a 1-cell followed by a 2-cell, i.e. *prewhiskering*. In this case (for $f : \mathcal{C} \to \mathcal{D}$ and $k : g \simeq g' : \mathcal{D} \to \mathcal{E}$) the homotopy $k\#_0 f$ is defined by

$$(k\#_0 f)' := k' f_0. \tag{1.3}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] & \mathcal{D} \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow k"] & \mathcal{E} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "g\#_0 f"] \arrow[r, bend right, "g'\#_0 f"'] \arrow[r, phantom, "\Downarrow k\#_0 f"] & \mathcal{E} \end{tikzcd}$$

1.3.4 Horizontal Composition

Now suppose we have 2-cells $h : f \simeq f' : \mathcal{C} \to \mathcal{D}$ and $k : g \simeq g' : \mathcal{D} \to \mathcal{E}$. The following picture suggests a possible definition of horizontal composition.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow h"] & \mathcal{D} \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow k"] & \mathcal{E} \end{tikzcd} := \begin{tikzcd} \mathcal{C} \arrow[r, bend right, "f'"'] & \mathcal{D} \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow k"] & \mathcal{E} \end{tikzcd} \#_1 \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow h"] & \mathcal{D} \arrow[r, bend left, "g"] & \mathcal{E} \end{tikzcd}.$$

This is clearly well-defined in the sense that the right hand side is the vertical composite of two whiskerings, which are both defined and have a common 1-face. In order to check that it is a sensible definition, we need to ensure that the sum is indeed a homotopy. As usual, we do this by considering the chain homotopy component.

Lemma 1.3.9. *If $h : f \simeq f'$ and $k : g \simeq g'$ are homotopies, then $(k\#_0 f')\#_1(g\#_0 h)$ is a homotopy.*

Proof:
Put $\phi = (k\#_0 f')\#_1(g\#_0 h)$. Then

$$\phi' = g_1 h' + k' f_0'$$

and it is sufficient to prove that ϕ' is a chain homotopy. Now,

$$\delta^E \phi' = \delta^E(g_1 h' + k' f_0') = \delta^E g_1 h' + \delta^E k' f_0' = g_0 \delta^D h' + \delta^E k' f_0'$$

$$= g_0 f_0' - g_0 f_0 + g_0' f_0' - g_0 f_0' = g_0' f_0' - g_0 f_0.$$

Similarly,

$$\phi'\delta^C = g_1h'\delta^C + k'f_0'\delta^C = g_1h'\delta^C + k'\delta^D f_1'$$

$$= g_1f_1' - g_1f_1 + g_1'f_1' - g_1f_1' = g_1'f_1' - g_1f_1.$$

Hence ϕ' is a chain homotopy and $(k\#_0h) := \phi : g\#_0f \simeq g'\#_0f'$ as required. □

Armed with this lemma, we may now formally define horizontal composition.

Definition 1.3.10. Let $h : f \simeq f' : \mathcal{C} \to \mathcal{D}$ and $k : g \simeq g' : \mathcal{D} \to \mathcal{E}$ be homotopies. Then the *horizontal composite* $k\#_0h := (k\#_0f')\#_1(g\#_0h) : g\#_0f \to g'\#_0f'$ is the homotopy with chain homotopy component

$$(k\#_0h)' = g_1h' + k'f_0'. \tag{1.4}$$

In defining $(k\#_0h)$, it was necessary to arbitrarily choose how to split the 2-cell composition as a sum of whiskerings. We chose to use $k\#_0h := (k\#_0f')\#_1(g\#_0h)$, but could equally well have taken $k\#_0h := (g'\#_0h)\#_1(k\#_0f)$ instead. It is easy to see that this would also yield a homotopy, but it is less immediately obvious whether this is the same as the one we have taken for our definition.

Lemma 1.3.11. $(g'\#_0h)\#_1(k\#_0f) = (k\#_0f')\#_1(g\#_0h)$.

Proof:
Consider the chain homotopy

$$[(g'\#_0h)\#_1(k\#_0f)]' = g_1'h' + k'f_0. \tag{1.5}$$

Rearranging the defining equations of chain homotopy we get $g_1' = g_1 + k'\delta^D$ and $f_0 = f_0' - \delta^D h'$. Substituting these into (1.5) gives

$$\begin{aligned} g_1'h' + k'f_0 &= (g_1 + k'\delta^D)h' + k'(f_0' - \delta^D h') \\ &= g_1h' + k'f_0' + k'\delta^D h' - k'\delta^D h' \\ &= g_1h' + k'f_0' \end{aligned}$$

which is, of course, $(k\#_0h)'$ as given by (1.4). □

Thus it makes no difference which of the two formulae is chosen; more important is to be consistent once the initial choice is made — we shall use the original choice given in definition 1.3.10.

We are now ready to establish the interchange law for the vertical and horizontal compositions as defined.

Suppose that $h : f \simeq f'$, $\hat{h} : f' \simeq f''$, $k : g \simeq g'$ and $\hat{k} : g' \simeq g''$ are homotopies ($f, f', f'' : \mathcal{C} \to \mathcal{D}$ and $g, g', g'' : \mathcal{D} \to \mathcal{E}$). We can form horizontal composites $k\#_0 h$ and $\hat{k}\#_0\hat{h}$ and then compose these vertically:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow h"] \arrow[r, phantom, shift right=3ex, "\Downarrow \hat{h}"] & \mathcal{D} \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow k"] \arrow[r, phantom, shift right=3ex, "\Downarrow \hat{k}"] & \mathcal{E} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left=40, "g\#_0 f"] \arrow[r] \arrow[r, bend right=40, "g''\#_0 f''"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow k\#_0 h"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow \hat{k}\#_0\hat{h}"] & \mathcal{E} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "g\#_0 f"] \arrow[r, bend right=50, "g''\#_0 f''"'] \arrow[r, phantom, "\Downarrow \phi"] & \mathcal{E} \end{tikzcd}$$

where $\phi := (\hat{k}\#_0\hat{h})\#_1(k\#_0 h)$.

Alternatively, we can start by forming the vertical composites and then compose these horizontally:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow h"] \arrow[r, phantom, shift right=3ex, "\Downarrow \hat{h}"] & \mathcal{D} \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow k"] \arrow[r, phantom, shift right=3ex, "\Downarrow \hat{k}"] & \mathcal{E} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f''"'] \arrow[r, phantom, "\Downarrow \hat{h}\#_1 h"] & \mathcal{D} \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g''"'] \arrow[r, phantom, "\Downarrow \hat{k}\#_1 k"] & \mathcal{E} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "g\#_0 f"] \arrow[r, bend right=50, "g''\#_0 f''"'] \arrow[r, phantom, "\Downarrow \psi"] & \mathcal{E} \end{tikzcd}$$

where $\psi := (\hat{k}\#_1 k)\#_0(\hat{h}\#_1 h)$.

Looking at the corresponding chain homotopies, and using (1.1) and (1.4), we get:

$$\phi' = ((\hat{k}\#_0\hat{h})\#_1(k\#_0 h))' = (\hat{k}\#_0\hat{h})' + (k\#_0 h)' = g_1'\hat{h}' + \hat{k}'f_0'' + g_1 h' + k' f_0' \quad (1.6)$$

and

$$\phi' = ((\hat{k}\#_1 k)\#_0(\hat{h}\#_1 h))' = g_1(\hat{h}\#_1 h)' + (\hat{k}\#_1 k)' f_0'' = g_1\hat{h}' + g_1 h' + \hat{k}' f_0'' + k' f_0''. \quad (1.7)$$

Then (1.6) - (1.7) gives

$$g_1'\hat{h}' + k' f_0' - g_1\hat{h}' - k' f_0'' = (g_1' - g_1)\hat{h}' + k'(f_0' - f_0'')$$

$$= k'\delta^D\hat{h}' - k'\delta^D\hat{h}' = 0,$$

whence $\phi' = \phi'$. Since these chain homotopies are identical, the corresponding homotopies are also identical; thus we have:

$$(\hat{k}\#_0\hat{h})\#_1(k\#_0 h) = (\hat{k}\#_1 k)\#_0(\hat{h}\#_1 h).$$

Since the interchange law is satisfied, $\mathbf{Ch}_K^{(1)}$ is a 2-category. The interchange law shows that horizontal composition is a homomorphism with respect to the (vertical) groupoid structure on the Hom-sets of $\mathbf{Ch}_K^{(1)}$. The foregoing discussion proves the following:

Theorem 1.3.12. $\mathbf{Ch}_K^{(1)}$ *is a **Grpd**-enriched category.*

□

The reader is referred to [43] for a treatment of enriched category theory. By restricting the chain maps to those which are invertible, $\mathbf{invCh}_K^{(1)}$ is a 2-groupoid, or groupoid-enriched groupoid.

1.3.5 The Long and Short of It

From our starting point of length 1 chain complexes, let us make two brief excursions in opposite directions.

1.3.5.1 Even Shorter Chain Complexes

The following elementary observation may be useful to help strengthen the claim that $\mathbf{Ch}_K^{(1)}$ is a 2-categorical generalisation of $\mathbf{Vect}_K$. A length 0 chain complex of vector spaces is just a vector space: $C = \ldots \to 0 \to C \to 0 \to \ldots$. Any linear transformation of vector spaces is a chain map in this case (the commutativity condition is trivial) and all homotopies, being degree 1 maps, are trivial. Therefore the category of length 0 chain complexes, and chain maps between them, which would be denoted as $\mathbf{Ch}_K^{(0)}$, is just $\mathbf{Vect}_K$. It has no non-trivial 2-category structure (given by homotopy, at least).

$\mathbf{Ch}_K^{(0)}$, of course, appears as a full subcategory of $\mathbf{Ch}_K^{(1)}$, consisting of those chain complexes with $C_1 = 0$.

1.3.5.2 Longer Chain Complexes

Length 0 chain complexes give us the category $\mathbf{Ch}_K^{(0)} = \mathbf{Vect}_K$. Length 1 chain complexes give us $\mathbf{Ch}_K^{(1)}$, which as we have seen, is a groupoid-enriched category, i.e. a 2-category with minimal extra structure. There is no intrinsic reason why things should stop at this level. Given any natural number n, it is possible to define the $(n+1)$-category $\mathbf{Ch}_K^{(n)}$ of length n chain complexes. There is also an ∞-category, $\mathbf{Ch}_K^{(\infty)}$, which includes all finite cases ([50] discusses ∞-categories in general).

In the sequel, we shall mostly be working with subgroupoids of $\mathbf{Ch}_K^{(1)}$ and their equivalents, but occasionally it will be necessary to consider longer chain complexes of vector spaces (or of modules over a more general ring). These would also be necessary in order to generalise our results to higher dimensions.

The definition of a chain map extends very obviously to chain complexes of arbitrary length; the definition of homotopy is equally obvious. The levelwise definition of chain homotopy requires just a little more work. If f, g are chain maps $\mathcal{C} \to \mathcal{D}$ (so that $\mathcal{C} : \ldots \to C_2 \to C_1 \to C_0 \to C_{-1} \to \ldots$ etc.) and $f \simeq g$ then a chain homotopy consists of maps $h'_n : C_n \to D_{n+1}$ satisfying $g_n - f_n = \partial^D_{n+1} h'_n + h'_{n-1} \partial^C_n$ for each $n \in \mathbb{Z}$:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "h'_n"'] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "h'_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d_{n+1}"'] & D_n \arrow[r, "d_n"'] & D_{n-1} \arrow[r, "d_{n-1}"'] & \cdots
\end{tikzcd}$$

The chain homotopy conditions for length 1 chain complexes are just a special case in which most of the maps are trivial.

In this thesis, and conceivably in any generalisation to higher dimensions, it is sufficient to consider non-negative chain complexes. A non-negative chain complex is one in which the subscripts are natural numbers[8]. The category of all such chain complexes, in the slightly more general setting of modules over a ring, is called $\mathbf{Ch}$ and is discussed in some detail by Kamps and Porter [40]. They show that $\mathbf{Ch}$ is in fact a 2-groupoid enriched Gray category. Our $\mathbf{Ch}_K^{(1)}$ is clearly a subcategory of $\mathbf{Ch}$. Since it has nothing at level 2 or beyond, the Gray-category structure is clearly trivial.

1.3.6 A Matrix Formulation For Calculations

Perhaps one of the most beautiful results of elementary linear algebra is that, up to isomorphism, there is only one K-vector space for each (finite) dimension n. The upshot

[8]We take $\mathbb{N}$ to include 0.

of this is that any element in an abstract vector space V of dimension n can be considered as an n-tuple in K^n. Linear transformations between vector spaces are equivalent to matrices over K; assuming standard bases, a linear transformation $\phi : K^n \to K^m$ uniquely determines and is determined by an $m \times n$ matrix Φ (or[9] M_ϕ or $M(\phi)$) with coefficients in K. In particular, a linear isomorphism $K^n \to K^n$ is equivalent to an element of $GL_n(K)$. Matrices have the advantage over abstract linear transformations that, at least for low dimensions, calculations can be performed easily by hand or by computer.

In $\mathbf{Ch}_K^{(1)}$, the objects are chain complexes of length 1. As we have seen, these are essentially the same as linear transformations. Hence a chain complex $\mathcal{C}$ with differential $\delta^C : C_1 \to C_0$ can be represented by an $n_0 \times n_1$ matrix Δ^C, where n_i is the dimension of C_i.

Suppose $\mathcal{D}$ is another chain complex, with differential $d^D : D_1 \to D_0$, where the dimension of D_i is m_i. A chain map $f : \mathcal{C} \to \mathcal{D}$ is given by a pair of matrices F_1 ($m_1 \times n_1$) and F_0 ($m_0 \times n_0$). The commutativity of the chain map with the differentials is then expressed as

$$F_0 \Delta^C = \Delta^D F_1, \tag{1.8}$$

which is an $m_0 \times n_1$ matrix as required. Any chain map $f : \mathcal{C} \to \mathcal{D}$ in $\mathbf{invCh}_K^{(1)}$ is invertible, so in this case D_i also has dimension n_i and the corresponding square matrices are non-singular, i.e. $F_1 \in GL_{n_1}(K)$ and $F_0 \in GL_{n_0}(K)$. Equation (1.8) can then be rewritten as

$$\Delta^C = F_0^{-1} \Delta^D F_1.$$

Using matrices, the abstract composition of chain maps is, of course, replaced by straightforward matrix multiplication. We will look at this in more detail later (see section 2.1.2).

Moving up to dimension two, a homotopy is determined by its starting point (a chain map f) and its chain homotopy. A chain homotopy is a linear transformation, so corresponds to a matrix. Suppose, in addition to the maps in the previous paragraph, we

[9] In general, we shall use a lowercase Greek letter for an abstract linear transformation, and the corresponding uppercase letter for its matrix. Occasionally, however, it will be useful to resort to one of the other obvious notations.

have another chain map $f' : \mathcal{C} \to \mathcal{D}$ and a homotopy $h : f \simeq f'$. Then there is a chain homotopy $h' : C_0 \to D_1$ with a corresponding $n_1 \times n_0$ matrix[10] H such that

$$H\Delta^C = F_1' - F_1 \text{ and } \Delta^D H = F_0' - F_0. \tag{1.9}$$

If $h : f \to f'$ and $\hat{h} : f' \to f''$, then the vertical composite $\hat{h}\#_1 h$, which is given by the chain homotopy (1.1), corresponds to the matrix sum $\hat{H} + H$. Since both the chain homotopies in question have the same source and target 0-cells, the matrices are always compatible for addition.

Whiskering and horizontal composition are likewise modelled on the formulae for chain homotopies given in equations (1.2), (1.3), and (1.4). Suppose we have homotopies $h : f \simeq f' : \mathcal{C} \to \mathcal{D}$ and $k : g \simeq g' : \mathcal{D} \to \mathcal{E}$. Then the chain homotopy components of $g\#_0 h$, $k\#_0 f'$, and $k\#_0 h$ are represented by the matrices $G_1 H$, KF_0', and $G_1 H + KF_0'$ respectively.

In this section, we have assumed that the standard basis is used for each vector space. In fact, it is possible to use any basis, although the matrices obtained will vary with different bases. In chapter 2 we shall consider, in the special case of automorphisms over a specific linear transformation, the effect of a change of basis on the matrix formulation.

Example 1.3.13. The foregoing discussion may be somewhat illuminated by considering some actual examples of matrices for $\mathbf{Ch}_K^{(1)}$. Since these necessarily take up a fair amount of space they have been placed in appendix A, so as not to disrupt the flow of the narrative.

1.4 Group Algebras

The group algebra construction is particularly useful in representation theory. Not only does it provide a way of getting an algebra from any given group, but also it allows representations to be studied by way of modules. For convenience, the exposition in this section will be given for the case where K is a field. In fact, the definitions and results hold with little extra complication, when K is a more general commutative ring; later on we shall sometimes need the case $K = \mathbb{Z}$.

[10]Since only the chain homotopy is to be rendered as a matrix, the use of primes to distinguish chain homotopies from homotopies is no longer necessary. If desired, the homotopy h can be thought of as the triple of matrices (F_0, H, F_1) satisfying equations (1.8) and (1.9).

1.4.1 Definition and Adjunction

Let G be any group and X_G the underlying set[11] of G. Let $\mathbf{e}_g$ denote the element in X_G corresponding to $g \in G$. Then form the vector space $K(G)$ (over a fixed field K) with basis $X_G = \{\mathbf{e}_g : g \in G\}$. This is a vector space of dimension $\#(X_G)$. A typical element of $K(G)$ is of the form $\sum_{g\in G} r_g\mathbf{e}_g$, with $r_g \in K$ and only finitely many $r_g \neq 0$ [59]. Suppose $\sum s_g\mathbf{e}_g$ is another element in $K(G)$, then

$$\sum r_g\mathbf{e}_g + \sum s_g\mathbf{e}_g := \sum (r_g + s_g)\mathbf{e}_g;$$

this addition is clearly commutative, since K is an abelian group. If $s \in K$, then the scalar multiplication is defined as

$$s\sum r_g\mathbf{e}_g := \sum sr_g\mathbf{e}_g.$$

The group operation in G induces a multiplication in $K(G)$. If $\sum r_g\mathbf{e}_g$ and $\sum s_h\mathbf{e}_h$ are in $K(G)$, then

$$(\sum_g r_g\mathbf{e}_g)(\sum_h s_h\mathbf{e}_h) := \sum_{g,h} r_g s_h\mathbf{e}_{gh}.$$

Together with the addition and scalar multiplication this makes $K(G)$ an algebra.

Definition 1.4.1. Let G be a group and K a field. Then $K(G)$, as defined above, is called the *group algebra* of G.

We shall also use the term *group algebra* when K is merely a commutative ring, although the term *group ring* is often used when $K = \mathbb{Z}$ (technically, we should ignore the scalar multiplication if this name is used). $K(G)$ may be written as $\bigoplus_{g\in G} K\mathbf{e}_g$, where $K\mathbf{e}_g$ is a 1-dimensional K-vector space. The definition of multiplication for $K(G)$ ensures that $K\mathbf{e}_g K\mathbf{e}_h = K\mathbf{e}_{gh}$. Also $1 \in K\mathbf{e}_1$ is a 2-sided identity. This means that $K(G)$ is a special case of the definition of π-algebra used by Turaev [71].

Let $f : G \to H$ be a homomorphism of groups. Define $K(f) : K(G) \to K(H)$ by $K(f)(\mathbf{e}_g) := \mathbf{e}_{f(g)}$; $K(f)$ is a homomorphism of group algebras. It has the properties that $K(\mathrm{id}_G) = \mathrm{id}_{K(G)}$ and, if $f' : H \to J$ is a group homomorphism, $K(f'f) = K(f')K(f)$. These facts are summarised in the following proposition:

[11] The notation X_G is chosen for the underlying set to avoid confusion with $|G|$ used for the order of the group G. When speaking about sets we will refer to the cardinality of the set X, written as $\#(X)$. Of course, for any group, $|G| = \#(X_G)$.

Proposition 1.4.2. *$K(.)$: **Gr** $\to$ **Alg**$_K$ is a functor.*

$\square$

In fact, a group algebra can be endowed with a comultiplication and an antipode to make it into a Hopf algebra (see [53]). We shall not require this extra structure; indeed, for the present we shall mostly be concerned with the vector space underlying $K(G)$ (i.e. we ignore the multiplication). To avoid cumbersome notation we shall write $K(G)$ for this underlying vector space as well. If G is finite, let $\mathrm{g} = \#(X_G)$; then $K(G) \cong K^{\mathrm{g}}$ and we may write K^{g} for $K(G)$ to emphasise the dimension of the vector space. To abbreviate notation, we will usually write ϕ for $K(f)$, using a Roman letter for the group homomorphism and the corresponding Greek letter for the linear transformation.

The group algebra functor provides a canonical construction for a K-algebra from any given group. Conversely, there are at least two canonical ways of extracting a group from a given K-algebra. One is to forget the multiplications and take the additive (abelian) group of the algebra; this gives the forgetful functor $\mathbf{Alg}_K \to \mathbf{Ab}$. Alternatively, the subset of the algebra consisting of elements which are invertible under multiplication forms a subgroup (with the operation of multiplication, of course) called the *group of units* of the algebra; this gives a functor $U(.) : \mathbf{Alg}_K \to \mathbf{Gr}$. In general, the group of units of a non-commutative algebra need not be abelian.

The following result is well-known, but a proof is given since the ideas contained within it are useful later.

Proposition 1.4.3. *The group algebra functor $K(.)$: **Gr** $\to$ **Alg**$_K$ is left adjoint to the unit group functor $U(.)$: **Alg**$_K$ $\to$ **Gr**.*

Proof:
Let G be a group and A a K-algebra, and suppose $f : G \to U(A)$. Define a map $\theta_{G,A} : \mathbf{Gr}(G, U(A)) \to \mathbf{Alg}_K(K(G), A)$ by

$$\theta_{G,A}(f)(\mathbf{e}_g) := \mathbf{e}_{f(g)}$$

(this defines $\theta_{G,A}$ completely, since $\{\mathbf{e}_g : g \in G\}$ is a basis for $K(G)$ and, for every $g \in G$, $\theta_{G,A}(f)(\mathbf{e}_g) \in KU(A) \subseteq A$).

Suppose $\phi : K(G) \to A$. Then ϕ is completely determined by $\{\phi(\mathbf{e}_g) : g \in G\}$, and for each $g \in G$

$$1_A = \phi(\mathbf{e}_g \mathbf{e}_{g^{-1}}) = \phi(\mathbf{e}_g)\phi(\mathbf{e}_{g^{-1}}),$$

so $\phi(\mathrm{e}_g) \in U(A)$. Define the map $\Theta_{G,A} : \mathbf{Alg}_K(K(G), A) \to \mathbf{Gr}(G, U(A))$ by

$$\Theta_{G,A}(\phi)(g) := \phi(\mathrm{e}_g).$$

Now $[\Theta_{G,A}\theta_{G,A}(f)](g) = f(g)$ and $[\theta_{G,A}\Theta_{G,A}(\phi)](g) = \phi(g)$, so $\theta_{G,A}$ is a bijection and

$$\mathbf{Gr}(G, U(A)) \cong \mathbf{Alg}_K(K(G), A)$$

as required.

It remains to show that θ and Θ are natural in G and A. For θ to be natural in G requires, for a homomorphism $\mu : G' \to G$,

$$\theta_{G',A}(f \circ \mu) = \theta_{G,A}(f) \circ K\mu.$$

It is clear that both sides of this equation are K-algebra morphisms $KG' \to A$, so let $\mathrm{e}_{g'}$ be a basis element of KG'. Then

$$\theta_{G',A}(f \circ \mu)(\mathrm{e}_{g'}) = \mathrm{e}_{f \circ \mu(\mathrm{e}_{g'})},$$

while

$$\theta_{G,A}(f) \circ K\mu(\mathrm{e}_{g'}) = \theta_{G,A}(f)(\mathrm{e}_{\mu g'}) = \mathrm{e}_{f(\mu g')}.$$

Now, $f(\mu g') = f \circ \mu g'$, whence equality. The remaining cases are proved similarly. □

Corollary 1.4.4. *The group algebra functor preserves colimits.*

Proof:
By proposition 1.4.3, the group algebra functor is a left adjoint. The result then follows from standard category theory (see, for example, [52]). □

Note that $K(.)$ need not preserve limits.

1.4.2 $K(G)$-Modules and Representation Theory

The principal reason for the importance of the group algebra functor in representation theory is the fact that there is a bijective correspondence between K-linear representations of a group G and $K(G)$-modules. This allows the more powerful abstract machinery of module theory to be used to study representations. Since this fact is so important, it will be worth our while to study it briefly here.

Lemma 1.4.5. *Let V be a K-vector space. Then the collection, $End_K(V)$, of all K-endomorphisms on V is a K-algebra, with $GL(V)$ as its group of units.*

□

Definition 1.4.6. Let A be a K-algebra. A *left A-module* is a K-vector space, V, having a K-linear morphism $A \times V \to V, (a, \mathbf{v}) \mapsto a\mathbf{v}$ such that the following axioms are satisfied for every $a, a' \in A, \mathbf{v}, \mathbf{v}' \in V, \alpha \in K$:

I. $a(\mathbf{v} + \mathbf{v})' = a\mathbf{v} + a\mathbf{v}'$,

II. $(a + a')\mathbf{v} = a\mathbf{v} + a'\mathbf{v}$,

III. $(aa')\mathbf{v} = a(a'\mathbf{v})$,

IV. $1_A\mathbf{v} = \mathbf{v}$,

V. $(\alpha a)\mathbf{v} = \alpha(a\mathbf{v}) = a(\alpha\mathbf{v})$.

An alternative way of viewing an A-module is to regard it as a left action of the K-algebra A on the K-vector space V. The scalar multiplication $a\mathbf{v}$ is rewritten as ${}^{a}\mathbf{v}$ and the conditions go through *mutatis mutandis*. In particular, since $K(G)$ is a K-algebra, this definition gives us the notion of a (left)[12] $K(G)$-module, or equivalently a (left) $K(G)$-action on a vector space.

Theorem 1.4.7. *Let G be a group and V a K-vector space. There is a bijective correspondence between representations $\phi : G \to GL(V)$ and $K(G)$-module structures on V.*

Proof:

Suppose $\phi : G \to GL(V)$ is a representation, i.e. a homomorphism of groups. Then ϕ extends uniquely to a K-algebra morphism $\phi^* : K(G) \to \mathrm{End}_K(V)$, with:

$$\phi^*(\sum \alpha_g \mathbf{e}_g) = \sum \alpha_g \mathbf{e}_{\phi g}.$$

[12] Since K is commutative there is no distinction between left and right K-modules. However, the K-algebra A is not necessarily commutative so left and right A-modules are distinct. We shall only be using left modules.

Define a scalar multiplication

$$\begin{aligned} K(G) \times V &\to V \\ (a, \mathbf{v}) &\mapsto a\mathbf{v} := \phi^*(a)(\mathbf{v}), \end{aligned}$$

where $a \in K(G)$ and $\mathbf{v} \in V$. With this multiplication, V is a $K(G)$-module: condition I of definition 1.4.6 holds because $\phi^*(a)$ is K-linear; conditions II, III and IV hold because ϕ^* is K-linear; condition V follows from the K-linearity of both the map ϕ^* and the images $\phi^*(a)$ of each $a \in K(G)$.

Conversely, suppose V is a $K(G)$-module. Then, define $\Phi : K(G) \to \mathrm{End}_K(V)$ by sending each $a \in K(G)$ to $\Phi(a) : V \to V$ with $\Phi(a)(\mathbf{v}) := a\mathbf{v}$. Conditions I and V ensure that each $\Phi(a)$ is K-linear and hence Φ is well-defined. Conditions II - V ensure that Φ itself is a K-linear morphism. For any $g \in G$,

$$\Phi(\mathbf{e}_g)\Phi(\mathbf{e}_{g^{-1}})(\mathbf{v}) = \Phi(\mathbf{e}_g\mathbf{e}_{g^{-1}})(\mathbf{v}) = \Phi(1)(\mathbf{v}) = \mathbf{v},$$

so $\Phi(\mathbf{e}_g)$ is a unit in $\mathrm{End}_K(V)$. Hence Φ restricts to a homomorphism $\phi : G \to GL(V)$ which is a representation, as required. □

Chapter 2

Automorphisms of a Linear Transformation

In which the automorphism cat^1-group of a linear transformation is explored, together with its matrix formulation. Finally, we shall meet the definition of a linear representation.

In section 1.3 we considered the groupoid-enriched category, $\mathbf{Ch}_K^{(1)}$, of length 1 chain complexes over $\mathbf{Vect}_K$, whose objects are linear transformations of K-vector spaces, with 1- and 2-cells given respectively by chain maps and homotopies. In order to develop the representation theory of crossed modules and cat^1-groups, we shall need the extra algebraic structure obtained by concentrating on the chain isomorphisms and homotopies defined on a single linear transformation. After the definition is given and examined, several examples are considered. In these we concentrate on the matrix formulation, which is particularly useful for calculations. The examples culminate in the general case for a linear transformation of vector spaces. The definition of a linear representation of a cat^1-group is given in section 2.4, although a detailed consideration of this definition, with examples, will be postponed until chapter 4.

2.1 The Automorphism Cat^1-Group of a Linear Transformation

Let $\delta : C_1 \to C_0$ be a linear transformation of vector spaces; this can and will be considered as an object in $\mathbf{Ch}_K^{(1)}$ in the way explained earlier. The collection of all chain isomorphisms $\delta \to \delta$ and homotopies between them forms a 2-subcategory of $\mathbf{Ch}_K^{(1)}$. In fact, it is clear that this will be a 2-group. From the discussion in section 1.2.3, we know that this is also a cat^1-group. As isomorphisms from an object to itself are commonly known as *automorphisms*[1], we may call this structure an automorphism cat^1-group.

2.1.1 The Definition of Aut(δ)

Definition 2.1.1. Let $\delta : C_1 \to C_0$ be a linear transformation of K-vector spaces. The *automorphism cat^1-group* of δ, $\mathbf{Aut}(\delta)$, consists of:

- the group $\mathbf{Aut}(\delta)_1$ of all chain automorphisms $\delta \to \delta$,
- the group $\mathbf{Aut}(\delta)_2$ of all homotopies on $\mathbf{Aut}(\delta)_1$,
- morphisms $s, t : \mathbf{Aut}(\delta)_2 \to \mathbf{Aut}(\delta)_1$, selecting the source and target of each homotopy,
- the morphism $i : \mathbf{Aut}(\delta)_1 \to \mathbf{Aut}(\delta)_2$, which provides the identity homotopy on each chain automorphism.

It may be instructive at this point to look more closely at the cells of $\mathbf{Aut}(\delta)$ from the 2-category perspective. There is but one 0-cell, δ, and so $\mathbf{Aut}(\delta)_0$ is a singleton; most of the time it remains quietly behind the scenes. $\mathbf{Aut}(\delta)_1$ consists of 1-cells:

$$\delta \xrightarrow{f} \delta \,,$$

while $\mathbf{Aut}(\delta)_2$ contains 2-cells:

$$\begin{tikzcd} \delta \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow h"] & \delta \end{tikzcd} .$$

[1]The meaning of the term "chain automorphism" should be obvious.

This is the "black box" interpretation of the cells, which is often the best way to view them. When required, however, a chain automorphism $f : \delta \to \delta$ may be unpacked as a pair (f_1, f_0) of linear isomorphisms such that the following diagram commutes:

$$\begin{tikzcd} C_1 \arrow[r, "f_1"] \arrow[d, "\delta"'] & C_1 \arrow[d, "\delta"] \\ C_0 \arrow[r, "f_0"'] & C_0 \end{tikzcd} .$$

A homotopy $h : f \simeq f'$ decomposes as a pair (h', f), with h' a chain homotopy

$$\begin{tikzcd} C_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_1'"'] \arrow[d, "\delta"'] & C_1 \arrow[d, "\delta"] \\ C_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_0'"'] \arrow[ur, dashed, "h'"] & C_0 \end{tikzcd}$$

and f (a chain automorphism) the source of h. Together these, along with the target f', satisfy the chain homotopy conditions $f_0' - f_0 = \delta h'$ and $f_1' - f_1 = h'\delta$. For convenience, we shall often abbreviate these as the single condition $f' - f = \delta h' + h'\delta$. This causes no problem as long as it is remembered that f, f' and h' are graded maps (both sides are degree 0 maps).

The structural homomorphisms in $\mathbf{Aut}(\delta)$ are straightforward. The maps s, t give respectively the source, f, and target, $f' = f + \delta h' + h'\delta$, of the homotopy (h', f), while i maps each chain map f to the identity homotopy $1_f : f \Rightarrow f$. The chain homotopy h' and the source chain map f together capture all the information of the homotopy h. Note that h' on its own may function as a chain homotopy for several different pairs of chain maps.

The group operation in $\mathbf{Aut}(\delta)_1$ is composition of chain automorphisms, for which we shall use the notation $g\#_0 f$ introduced in section 1.2.3. The identity is id_δ, the chain map consisting of the identity linear transformation at both levels. Since every $f \in \mathbf{Aut}(\delta)_1$ is a chain automorphism, it has an inverse f^{-1}, which is also a chain automorphism on δ and hence an element of $\mathbf{Aut}(\delta)_1$.

Horizontal composition provides the group operation for $\mathbf{Aut}(\delta)_2$; the related operation of whiskering is a degenerate case of this. The notation employed is again taken from the example of $\mathbf{Ch}_K^{(1)}$; in this case, if $h = (h', f)$ and $k = (k', g)$ are homotopies, the composite $k\#_0 h$ is the homotopy specified by the source chain map $g\#_0 f$ and the chain homotopy $(g_1 h' + k' f_0')$, where $f' = f + \delta h' + h'\delta$. The identity for this composition is the homotopy $(0, \mathrm{id}_\delta) : \mathrm{id}_\delta \Rightarrow \mathrm{id}_\delta$. The inverse of (h', f) is the element $(-f_1^{-1} h'(f_0')^{-1}, f^{-1})$.

The elements of $\mathbf{Aut}(\delta)_2$ can also be joined by vertical composition, $\#_1$, which is defined for pairs of 2-cells for which the target 1-cell of the first is the source of the second. That is, if $h = (h', f)$ and $\hat{h} = (\hat{h}', f + \delta h' + h'\delta)$ are in $\mathbf{Aut}(\delta)_2$, the vertical composite is $\hat{h}\#_1 h = (\hat{h}' + h', f)$. This is a groupoid operation, with each 1-cell f having an identity $1_f = (0, f)$ for vertical composition and every 2-cell (h', f) having the inverse $(-h', f + \delta h' + h'\delta)$. The horizontal and vertical compositions are linked by an interchange law, which leads to the kernel condition being satisfied in $\mathbf{Aut}(\delta)$ (so that it really is a cat^1-group).

Because $\mathbf{Aut}(\delta)$ is a cat^1-group it has a classifying space as discussed in section 1.2.2, hence the homotopy groups $\pi_i\mathbf{Aut}(\delta)$ can be found. As always, these are trivial for $i \neq 1, 2$, while formulae are given on page 19 which allow us to find π_1 and π_2 without first calculating $B\mathbf{Aut}(\delta)$ explicitly.

The notation $\pi_1\mathbf{Aut}(\delta)$ suggests that this group should be the fundamental group of $\mathbf{Aut}(\delta)$, i.e. the group of homotopy classes of the elements of $\mathbf{Aut}(\delta)_1$. This is indeed the case, and in fact the result is also true for a more general cat^1-group $\mathfrak{C}$. Recall that $\pi_1\mathfrak{C} = P/t(\ker s)$. Now, $t(\ker s) = \{\delta(c) | c \in C\}$ (consider the 2-cell structure of $C \rtimes P$ as described in section 1.2.3) and two elements $p, p' \in P$ are defined to be homotopic precisely when there is a 2-cell (c, p) with $\delta(c)p = p'$, which occurs whenever p and p' are in the same coset of $t(\ker s)$. Hence $p \simeq p'$ in $\mathfrak{C}$ precisely when $\bar{p} = \bar{p}'$ in $\pi_1\mathfrak{C}$, as required.

We shall not require a rigorous topological justification of the definition of $\pi_2\mathfrak{C}$ as $\ker s \cap \ker t$. Informally, a 2-cell in $\mathfrak{C}$ yields a disc in $B\mathfrak{C}$. Elements of $\pi_2\mathfrak{C}$ are discs whose boundaries are identified at a point, and these are the discs coming from 2-cells with trivial source and target, i.e. those which are in both $\ker s$ and $\ker t$. In the case of $\mathbf{Aut}(\delta)$ the chain homotopy condition imposes some severe restrictions on elements of π_2. Suppose $(h', f) \in \pi_2\mathbf{Aut}(\delta)$, then both the source and target of (h', f) are the identity; hence, $\delta h' + h'\delta = f' - f = 0$. If $\ker\delta = 0$, then of course $\delta h' + h'\delta = 0 \Rightarrow h' = 0$ and in this case homotopy collapses to equality (i.e. $f \simeq g \Rightarrow f = g$). However, if $\ker\delta$ is non-zero then $\delta h' + h'\delta = 0$ need not imply that $h' = 0$.

2.1.2 Matrices for $\mathbf{Aut}(\delta)$

We can now specialise the remarks of section 1.3.6 to obtain a description of $\mathbf{Aut}(\delta)$ as a 2-subgroup of $\mathbf{Ch}_K^{(1)}$ in matrix terms. The principal benefit of this approach is that it allows the potential use of powerful computational algebra packages such as Maple and GAP for direct calculation with $\mathbf{Aut}(\delta)$ (and hence of cat^1-group representations). It is, then, worth outlining in some detail the basic equational formulation required by this approach. It is largely a straightforward extension of standard linear algebra techniques, but it will also have the advantage of providing us with some more concrete, generic examples of cat^1-groups and hence crossed modules, including some which seem to have previously escaped notice.

Once bases are chosen for C_1 and C_0, the linear transformation δ yields a unique matrix, Δ. If C_1 and C_0 have dimension n_1 and n_0 respectively, Δ is an $n_0 \times n_1$ matrix. For the moment, assume that the bases are fixed. When considering the matrices, we shall write $\mathbf{Aut}(\Delta)$ instead of $\mathbf{Aut}(\delta)$. We shall also adopt the notation $K^{m,n}$ (borrowed from [34]) to denote the set of all $m \times n$ matrices with coefficients in K. This is an mn-dimensional K-vector space under addition and scalar multiplication; $K^{m,m}$ is a K-algebra with matrix multiplication. Although the general linear group is not strictly a subobject of $K^{m,m}$ (since it is a group, rather than a linear space), we shall employ the usual notation that blurs this distinction. As always, $GL_m(K)$ denotes the collection of invertible $m \times m$ matrices, and the operations of addition and scalar multiplication as well as matrix multiplication.

An element of $\mathbf{Aut}(\delta)_1$ is a pair $F = (F_1, F_0)$ of matrices, $F_1 \in GL_{n_1}(K)$ and $F_0 \in GL_{n_0}(K)$ such that:

$$\Delta F_1 = F_0 \Delta. \tag{2.1}$$

Since F_1 and F_0 are both invertible this condition may be rewritten to give $\Delta = F_0 \Delta F_1^{-1}$ or $\Delta = F_0^{-1} \Delta F_1$. One might think that F_1 and F_0 are linked to the extent that once one is chosen the other is automatically fixed. In fact this is not so, and it is an easy exercise to find a counterexample (appendix A contains one for $\mathbf{invCh}_K^{(1)}$ which can easily be customised).

To get at the elements of $\mathbf{Aut}(\delta)_2$ it is easiest to use the decomposition of the homotopy $h : f \Rightarrow f'$ as the pair (h', f), chain homotopy and source chain map. Since h' is a linear transformation, it gives an $n_1 \times n_0$ matrix H. Again, the compatibility condition translates easily into matrix notation, with multiplication replacing the composition of linear transformations. This gives:

$$H\Delta = F_1' - F_1, \quad \Delta H = F_0' - F_0. \tag{2.2}$$

Note that $H \in K^{n_1,n_0}$, while $\Delta \in K^{n_0,n_1}$, so both $H\Delta$ and ΔH are defined. Thus an element of $\mathbf{Aut}(\delta)_2$ is a pair (H, F), with H and F satisfying conditions (2.2) and (2.1).

The maps s, t and i work as follows in the matrix formulation. Let $F = (F_1, F_0)$, $F' = (F_1', F_0') \in \mathbf{Aut}(\delta)_1$ and $(H, F) : F \Rightarrow F' \in \mathbf{Aut}(\delta)_2$; then $s(H, F) = F$, $t(H, F) = F + \Delta H + H\Delta = F'$ and $i(F) = (0, F)$. Note that the zero matrix is the chain homotopy component for any identity homotopy.

As with $\mathbf{Ch}_K^{(1)}$, the formulae for composition in the matrix approach are derived from the formulae in the more abstract approach. The composition of chain automorphisms simply becomes matrix multiplication at each level — this is now always defined since the matrices are all square. Vertical composition of homotopies is replaced by matrix addition. The slightly more complicated formula for horizontal composition only involves addition and composition of linear transformations, so this translates equally easily to a matrix formula involving addition and multiplication. The formulae for all these compositions can be found in the more general discussion starting on page 33.

Much of the time it is possible, and convenient, to keep the bases of the vector spaces fixed (for example, use the standard basis of $\mathbb{C}^n$). There are times, however, when it is necessary or desirable to change bases. This means, of course, that the same underlying linear transformation will give a different matrix. The mathematics of changing bases is standard linear algebra, which can be found in any standard text (such as [6]). Here we shall recall some of the basic results, mostly for notational purposes.

Firstly, suppose $V = \langle \mathbf{v}_1, \ldots \mathbf{v}_n \rangle$ and $W = \langle \mathbf{w}_1, \ldots, \mathbf{w}_n \rangle$ are bases for K^n. There is a unique non-singular matrix $P \in GL_n(K)$, which will be called the *change matrix* from V to W, such that if $\mathbf{x} \in K^n$ is a vector expressed in terms of coefficients with respect to the basis V, then $P\mathbf{x}$ is the same vector expressed with respect to the basis W. Where necessary we shall clarify which basis we are using by means of subscripts, writing $\mathbf{x}_V$ or $\mathbf{x}_W$ instead of $\mathbf{x}$. Of course, P^{-1} is the change matrix from W to V.

Now let $\phi : K^m \to K^n$ be a linear transformation and suppose V and W are bases for K^m, and V' and W' bases for K^n, with change matrices P and P' respectively. Denote by Φ_V the matrix obtained from ϕ using the bases V, V', and by Φ_W the matrix obtained using the bases W, W'. Then Φ_V and Φ_W are related by the formula:

$$\Phi_W = P'\Phi_V P^{-1}. \tag{2.3}$$

All of the elements of $\mathbf{Aut}(\delta)$ are defined in terms of linear transformations between the vector spaces C_1 and C_0 (the source and target of δ), so this formula can be applied, with the suitable change matrices, to find matrices for everything in $\mathbf{Aut}(\delta)$ using any basis desired.

We shall consider change of basis for $\mathbf{Aut}(\delta)$ as an all-or-nothing package. In other words, we insist on using the same pair of bases at any given time for δ and all its chain maps and homotopies. If a change of basis is made, it must be made to everything. If Δ_V is used for δ then we must also use $F_{i,V}$, H_V and so on.

2.2 Examples of $\mathbf{Aut}(\delta)$

In order to develop a better intuition of what $\mathbf{Aut}(\delta)$ actually looks like and how it works, let us consider some examples. We shall mostly concentrate on the matrix formulation of these examples, since this is the version most conducive to calculation. Most of the examples will be reasonably generic, but the first is both specific and fairly small. It is included to show the ease with which the foregoing methods can be applied to analyse individual cat^1-groups. In practice, the dimensions of C_1 and C_0 are usually sufficient to make hand calculation impractical but the same techniques can be used with standard computer linear algebra packages to handle larger examples.

2.2.1 A Small, Specific Example

Working over $\mathbb{C}$ and taking the standard basis for each vector space, define $\delta : \mathbb{C}^2 \to \mathbb{C}$ to be the linear transformation:

$$\begin{aligned}\delta : \mathbb{C}^2 &\to \mathbb{C}\\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto x.\end{aligned}$$

This corresponds to the matrix

$$\Delta = \begin{pmatrix} 1 & 0 \end{pmatrix} \in \mathbb{C}^{1,2}.$$

A chain automorphism in $\mathbf{Aut}(\Delta)_1$ will consist of a pair of non-singular matrices, one 2×2 and the other 1×1, satisfying equation (2.1). Suppose

$$F_1 = \begin{pmatrix} a & d \\ b & c \end{pmatrix}, \quad F_0 = \begin{pmatrix} e \end{pmatrix}$$

is a pair of suitably dimensioned matrices. Then

$$\Delta F_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a & d \\ b & c \end{pmatrix} = \begin{pmatrix} a & d \end{pmatrix},$$

while

$$F_0 \Delta = \begin{pmatrix} e \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} e & 0 \end{pmatrix}.$$

In order to satisfy (2.1), then, it is necessary to have $d = 0$ and $a = e$; in addition a and c must not be equal to zero (in order that F_1 be non-singular). Therefore, the elements of $\mathbf{Aut}(\Delta)_1$ will be precisely the matrix pairs $F = (F_1, F_0)$ of the form:

$$F_1 = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \quad F_0 = \begin{pmatrix} a \end{pmatrix}$$

where $a, b, c \in \mathbb{C},\ a, c \neq 0$.

The simplest possible has both matrices to be the identity in their respective dimensions. This gives $\mathsf{id}_\Delta = (\mathsf{id}_1, \mathsf{id}_0)$ with:

$$\mathsf{id}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{id}_0 = \begin{pmatrix} 1 \end{pmatrix}.$$

It is straightforward to check that $\Delta \mathsf{id}_1 = \mathsf{id}_0 \Delta (= \Delta)$. Clearly id_Δ is the identity chain map on Δ. However, it is not the only chain automorphism on Δ. For instance, $F' = (F_1', F_0')$ with:

$$F_1' = \begin{pmatrix} i & 0 \\ 1 & 1 \end{pmatrix}, \quad F_0' = \begin{pmatrix} i \end{pmatrix},$$

is another example (with $\Delta F_1' = F_0 \Delta = \begin{pmatrix} i & 0 \end{pmatrix}$). There are, of course, infinitely many.

$\mathbf{Aut}(\Delta)_1$ is, of course a group, so we need to specify not just the elements (which we have done already) but also the group operation. In this context, it is simply matrix multiplication at both levels. Suppose $F = (F_1, F_0)$ as above and $G = (G_1, G_0)$ is another chain automorphism, where

$$G_1 = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}, \quad G_0 = \begin{pmatrix} \alpha \end{pmatrix}.$$

Then $F\#_0G$ is the chain automorphism with $(F\#_0G)_0 = F_0G_0$ and

$$(F\#_0G)_1 = F_1G_1 = \begin{pmatrix} a\alpha & 0 \\ b\alpha + c\beta & c\gamma \end{pmatrix}.$$

We can find homotopies, the elements of $\mathbf{Aut}(\Delta)_2$, by a similar process. Suppose also that $F \simeq G$, i.e. there is a homotopy $(H, F) : F \Rightarrow G$. We already know F; the chain homotopy H (a 2×1 matrix) must satisfy $H\Delta = G_1 - F_1$ and $\Delta H = G_0 - F_0$. Let

$$H = \begin{pmatrix} \mu \\ \nu \end{pmatrix}.$$

Then

$$H\Delta = \begin{pmatrix} \mu & 0 \\ \nu & 0 \end{pmatrix}, \quad \Delta H = \begin{pmatrix} \mu \end{pmatrix},$$

while

$$G_1 - F_1 = \begin{pmatrix} \alpha - a & 0 \\ \beta - b & \gamma - c \end{pmatrix}, \quad G_0 - F_0 = \begin{pmatrix} \alpha - a \end{pmatrix}.$$

Therefore, to satisfy the homotopy conditions we must have $\alpha - a = \mu$, $\beta - b = \nu$ and $\gamma - c = 0$. Hence $F \simeq G$ if and only if $\gamma = c$, and in this case the unique chain homotopy such that $(H, F) : F \simeq G$ is

$$\begin{pmatrix} \alpha - a \\ \beta - b \end{pmatrix}. \tag{2.4}$$

For example, the chain maps id and F' defined above are homotopic, with chain homotopy:

$$H = \begin{pmatrix} i - 1 \\ 1 \end{pmatrix}.$$

As with the lower level, $\mathbf{Aut}(\Delta)_2$ is a group. Its elements are the homotopies (H, F). In this case, the group operation (horizontal composition) is slightly more complicated than merely multiplying the matrices. If $(H, F) : F \simeq G$ and $(H', F') : F' \simeq G'$ are homotopies, the horizontal composite $(H', F')\#_0(H, F)$ is defined to be the homotopy with source $F'\#_0 F \in \mathbf{Aut}(\Delta)_1$ and chain homotopy (given by the formula on page 35) $F'_1 H + H' G_0$.

Since $\mathbf{Aut}(\Delta)$ is a cat^1-group vertical composition is also defined, for suitable pairs of homotopies. If $(H, F) : F \simeq F'$ and $(H', F') : F' \simeq F''$ then their vertical composite is $(H', F')\#_1(H, F) : F \simeq F''$. Its source is the chain map F and its chain homotopy is the sum of the component chain homotopies $H' + H$.

In this example, the homotopy classes of chain maps are characterised by the coefficient c in the bottom right corner of the top group matrix. In other words, the fundamental group $\pi_1\mathbf{Aut}(\Delta) = \mathbb{C}$.

The elements of $\pi_2\mathbf{Aut}(\Delta) = \ker s \cap \ker t$ have both source and target equal to the identity. Therefore, (2.4) implies that $H = 0$ for every element of π_2. In other words, $(0, \mathrm{id})$ is the unique element of $\pi_2\mathbf{Aut}(\Delta)$, which is thus the trivial group.

This example is also a suitable vehicle to explore the effect of a change of basis on the matrices. Let S_1 and S_0 denote the standard bases on $\mathbb{C}^2$ and $\mathbb{C}^1$ respectively (with $\mathbf{e}_1^1 = (1, 0)$ etc.) and let $T_1 = \langle \mathbf{t}_1^1 = (i, 0), \mathbf{t}_2^1 = (1, -i)\rangle$ and $T_0 = \langle \mathbf{t}_1^0 = (i)\rangle$ be another pair of bases for the same spaces. The change matrix from S_1 to T_1 is:

$$P_1 = \begin{pmatrix} -i & -1 \\ 0 & i \end{pmatrix},$$

while that from S_0 to T_0 is:

$$P_0 = \begin{pmatrix} -i \end{pmatrix}.$$

The matrix Δ previously used to represent δ was found using the standard bases for $\mathbb{C}^2$ and $\mathbb{C}$. Writing it more explicitly as Δ_S, the formula (2.3) may be used to find Δ_T, the matrix representing δ with respect to the bases T_i. Thus, $\Delta_T = P_0 \Delta_S P_1^{-1}$, i.e.

$$\Delta_T = \begin{pmatrix} -i \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & -i \end{pmatrix}.$$

In the same way, the formula can be applied to give the chain maps and homotopies with respect to the new basis. This is left as an exercise for the reader.

2.2.2 Identity

For any finite vector space K^n, one linear transformation that is guaranteed to exist is the identity $1_n : K^n \to K^n$. Although $\mathbf{Aut}(1_n)$ may not be wildly exciting, its ubiquity makes it worthy of at least a passing examination.

Suppose $f : 1_n \to 1_n$ is a chain automorphism. Then the commutativity condition gives:

$$f_0 = f_0 1_n = 1_n f_1 = f_1.$$

Therefore any chain automorphism consists of the same linear automorphism repeated top and bottom; conversely any such pair gives a chain automorphism. Suppose f and g are homotopic chain automorphisms. Then there is a chain homotopy h' and we get the following diagram:

$$\begin{tikzcd}
K^n \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, equal] & K^n . \arrow[d, equal] \\
K^n \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[ur, "h'"] & K^n
\end{tikzcd}$$

The homotopy conditions both become $h = g - f$. Therefore any pair of chain automorphisms are homotopic to each other, with chain homotopy given by their difference. In particular, $\pi_1\mathbf{Aut}(1_n)$ and $\pi_2\mathbf{Aut}(1_n)$ are both trivial.

Translating to matrices, each chain automorphism $F = (F, F)$ consists of the same matrix $F \in GL_n(K)$ at both levels. Given chain automorphisms F and G, there is a homotopy (H, F) between them, where $H = G - F \in K^{n,n}$.

The identity linear automorphism also exists for infinite vector spaces, but since these are less amenable to matrix treatment, they have not been considered here.

2.2.3 Zero

Another general linear transformation is the zero map, which can be defined on any pair of vector spaces. This is the linear transformation $0 : K^n \to K^m$ such that $\mathbf{x} \mapsto \mathbf{0}$ for every $\mathbf{x} \in K^n$. Since the composite of a zero map with any other linear transformation is also a zero map, it follows that any pair (f_1, f_0) of linear automorphisms of suitable dimensions will be a chain automorphism in $\mathbf{Aut}(0)$. For homotopies, this gives the picture:

$$\begin{tikzcd}
K^n \arrow[r, shift left, "f_1"] \arrow[r, shift right, "g_1"'] \arrow[d, "0"'] & K^n \arrow[d, "0"] \\
K^m \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] \arrow[ur, "h'"] & K^m
\end{tikzcd} .$$

Since $h'0 = 0 = 0h'$, the chain homotopy conditions become $g_1 - f_1 = 0$ and $g_0 - f_0 = 0$, hence $f_1 = g_1$ and $f_0 = g_0$. In other words, while every pair of suitably dimensioned linear automorphisms is a chain automorphism, distinct chain automorphisms are never homotopic. Hence there is a distinct homotopy class for each chain automorphism and so $\pi_1\mathbf{Aut}(0) = \mathbf{Aut}(0)_1$. Any linear transformation $h' : K^m \to K^n$ is a suitable chain homotopy for any $f \simeq f$ and so the elements of $\pi_2\mathbf{Aut}(0)$ are of the form (h', id).

Of course, this situation can easily be translated into the language of matrices if needs be.

2.2.4 Inclusion

Suppose K^m is a subspace of K^n (i.e. $m \leqslant n$) and $\delta : K^m \hookrightarrow K^n$ the inclusion. Then K^n can be decomposed as $K^m \oplus K^p$, with $p = n - m$, so that the matrix corresponding to δ is $\Delta = \begin{pmatrix} I_m \\ 0 \end{pmatrix} \in K^{m+p,m}$ where I_m is the identity on $K^{m,m}$ and 0 is the zero matrix in $K^{p,m}$; thus, for any $\mathbf{x} \in K^m$, $\Delta\mathbf{x} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}$.

A chain automorphism in $\mathbf{Aut}(\Delta)_1$ will consist of invertible matrices $F_1 \in GL_m(K)$ and $F_0 \in GL_{m+p}(K)$ such that the chain map condition is satisfied, namely $F_0\Delta = \Delta F_1$. Suppose $F_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A \in GL_m(K), B \in K^{m,p}, C \in K^{p,m}$ and $D \in GL_p(K)$. Then $F_0\Delta = \begin{pmatrix} A \\ C \end{pmatrix}$ and $\Delta F_1 = \begin{pmatrix} F_1 \\ 0 \end{pmatrix}$, whence $A = F_1$ and $C = 0$. Therefore $F \in \mathbf{Aut}(\Delta)_1$ has the form (F_1, F_0) where

$$F_0 = \begin{pmatrix} F_1 & B \\ 0 & D \end{pmatrix},$$

with F_1 and D invertible and B an arbitrary matrix. Since $\mathbf{Aut}(\Delta)$ is a group, we should also consider its multiplication. Suppose F is as above and $G = (G_1, G_0)$ is another chain map, with $G_0 = \begin{pmatrix} G_1 & B' \\ 0 & D' \end{pmatrix}$. Then $F\#_0 G$ is the chain automorphism with $(F\#_0 G)_1 = F_1 G_1$ and

$$F_0 G_0 = \begin{pmatrix} F_1 G_1 & F_1 B' + BD' \\ 0 & DD' \end{pmatrix},$$

where $F_1 G_1 \in GL_m(K)$, $F_1 B' + BD' \in K^{m,p}$ and $DD' \in GL_p(K)$. Let $K^{m,p}$ have the structure of a left $GL_m(K)$-module and a right $GL_p(K)$-module. Then

$$\mathbf{Aut}(\Delta)_1 \cong GL_m(K) \ltimes K^{m,p} \rtimes GL_p(K)$$

with multiplication $(A, B, D)(A', B', D') = (AA', AB' + BD', DD')$.

Let F and G be the chain automorphisms described above and suppose that $(H, F) : F \simeq G$. The chain homotopy H will be an $m \times n$ matrix, and can initially be given as a block matrix $H = \begin{pmatrix} X & Y \end{pmatrix}$, where $X \in K^{m,m}$ and $Y \in K^{m,n-m}$. The chain homotopy conditions will dictate the actual possibilities for homotopy. Firstly, $X = H\Delta = G_1 - F_1$. Secondly,

$$\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} = \Delta H = G_0 - F_0 = \begin{pmatrix} G_1 - F_1 & B' - B \\ 0 & D' - D \end{pmatrix}.$$

Comparing coefficients, we deduce the following lemma:

Lemma 2.2.1. $F \simeq G \Leftrightarrow D = D'$

□

If $F \simeq G$ the unique chain homotopy is given by $\begin{pmatrix} G_1 - F_1 & B' - B \end{pmatrix}$, which is of course the top (block) row of $G_0 - F_0$. Thus we have computed the elements of $\mathbf{Aut}(\Delta)_2$; a description of the compositions is left to the reader, and can be easily obtained from the general case below (2.3).

From the lemma, the homotopy classes of chain automorphisms are characterised by the invertible $(m-n)$-square matrices D, hence $\pi_1\mathbf{Aut}(\Delta) = GL_p(K)$. The only chain homotopy id $\simeq$ id is the zero matrix, so $\pi_2\mathbf{Aut}(\Delta)$ is trivial.

2.2.5 Projection

Let $\delta : K^n \oplus K^m \to K^n$ be the projection of $K^n \oplus K^m$ onto one of its direct summands. K^n is a quotient space of $K^n \oplus K^m$ and so we can choose a basis $V =$

$\{\mathbf{v}_1, \dots \mathbf{v}_n, \mathbf{v}_{n+1}, \dots, \mathbf{v}_{n+m}\}$ for $K^n \oplus K^m$ and a basis $\bar{V} = \{\bar{\mathbf{v}}_1, \dots \bar{\mathbf{v}}_n\}$ for K^n where each $\bar{\mathbf{v}}_i = \mathbf{v}_i + K^m$ (for $i \leqslant n$). With these bases the map δ acts as follows:

$$\delta(\sum_{i=1}^{n+m} \alpha_i \mathbf{v}_i) = \sum_{i=1}^{n} \alpha_i \bar{\mathbf{v}}_i.$$

A chain automorphism $f = (f_0, f_1)$ on δ must satisfy $f_0 \circ \delta = \delta \circ f_1$. Switching to the matrix formulation with the bases described above, δ yields the matrix

$$\Delta = \begin{pmatrix} I_n & 0 \end{pmatrix} \in K^{n,n+m}.$$

To satisfy the chain map condition, matrix pairs must have the following structure. Suppose $F_0 \in GL_n(K)$ then $F_1 \in K^{n+m,n+m}$ has the block form:

$$F_1 = \begin{pmatrix} F_0 & 0 \\ C & D \end{pmatrix}$$

where $A \in K^{m,n}$ and $B \in GL_{n+m}(K)$. All the elements of $\mathbf{Aut}(\Delta)_1$ are of this form. Suppose G is another chain automorphism in matrix form, with

$$G_1 = \begin{pmatrix} G_0 & 0 \\ C' & D' \end{pmatrix}.$$

Then the multiplication on $\mathbf{Aut}(\Delta)_1$ yields the chain automorphism with matrices F_0G_0 and

$$F_1G_1 = \begin{pmatrix} F_0G_0 & 0 \\ CG_0 + DC' & DD' \end{pmatrix}.$$

If $K^{m,n}$ is given the structure of a left $GL_n(K)$, right $GL_m(K)$-module, this yields $\mathbf{Aut}(\Delta)_1 \cong GL_n(K) \ltimes K^{m,n} \rtimes GL_m(K)$.

A homotopy $F \simeq G$ consists of the pair (H, F) where $H : K^n \to K^{n+m}$ is a chain homotopy, given by an $(n+m) \times n$ matrix. Suppose initially that H has block form

$$H = \begin{pmatrix} X \\ Y \end{pmatrix}$$

where $X \in K^{n,n}$ and $Y \in K^{m,n}$. To satisfy the chain homotopy conditions requires that $X = \Delta H = G_0 - F_0$ and

$$\begin{pmatrix} X & 0 \\ Y & 0 \end{pmatrix} = H\Delta = \begin{pmatrix} G_0 - F_0 & 0 \\ C' - C & D' - D \end{pmatrix}.$$

We deduce that $F \simeq G$ if and only if $D = D'$; in other words the linear transformations must coincide on the lower right-hand block, so lemma 2.2.1 holds in this situation as well. When this condition is met, the unique corresponding chain homotopy is

$$H = \begin{pmatrix} G_0 - F_0 \\ C' - C \end{pmatrix}.$$

Again, these are the elements of the group $\mathbf{Aut}(\Delta)_2$; the composition can be recovered from the general case in 2.3.

The homotopy classes of chain automorphisms correspond to the distinct invertible matrices D, so $\pi_1\mathbf{Aut}(\Delta) \cong GL_m(K)$. Since there is only one homotopy for which the source and target are both the identity (namely, the one with zero chain homotopy), $\pi_2\mathbf{Aut}(\Delta)$ is trivial in this case as well.

Note that this example is a generalisation of the first, specific example (2.2.1).

2.3 The General Form of Aut(δ) Over A Vector Space

Having familiarised ourselves with various examples of $\mathbf{Aut}(\delta)$ for specific linear transformations, we can now turn to consider an arbitrary linear transformation of vector spaces. Note that, while the definition of $\mathbf{Aut}(\delta)$ itself works perfectly well if vector spaces are replaced by modules over a commutative ring, the calculations of this section (along with examples 2.2.4 and 2.2.5) require us to work with vector spaces, so that the direct sum decompositions are guaranteed to exist.

Given any linear transformation of vector spaces, $\delta : K^r \to K^s$, with $\ker\delta \cong K^m$, it is possible to rewrite δ in the form:

$$\delta : K^n \oplus K^m \to K^n \oplus K^p,$$

where $n = r - m$ and $p = s - n$, such that $\delta(\mathbf{x}, \mathbf{y}) = (\mathbf{x}', \mathbf{0})$ ($\mathbf{x} \in K^n$, etc.). Indeed, it is possible (in principle) to choose bases such that $\delta(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{0})$. Therefore, with a suitable choice of bases, δ can be expressed as a matrix

$$\Delta = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \in K^{n+p,n+m}.$$

Given such a Δ the usual conditions on chain automorphisms and homotopies apply. Although in practice it will usually be difficult to express a given linear transformation in this form, its generality makes it worth considering the structure of $\mathbf{Aut}(\Delta)$ in some detail. All the foregoing examples are, of course, special cases of this.

A chain map F consists of matrices F_1 and F_0 which commute with Δ. For a chain automorphism, we also require both F_1 and F_0 to be invertible, hence $F_1 \in GL_{n+m}(K)$ and $F_0 \in GL_{n+p}(K)$. Suppose

$$F_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } F_0 = \begin{pmatrix} U & V \\ W & X \end{pmatrix}$$

are such matrices, expressed in block form (so $A, U \in GL_n(K), B \in K^{n,m}$ etc.). Then

$$\Delta F_1 = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ and } F_0\Delta = \begin{pmatrix} U & 0 \\ W & 0 \end{pmatrix}.$$

Since $\Delta F_1 = F_0\Delta$, we deduce that $A = U$ while both B and W are zero matrices. Therefore the general block form of a chain automorphism (relabelling the blocks to conserve letters) is

$$F_1 = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, F_0 = \begin{pmatrix} A & D \\ 0 & E \end{pmatrix}$$

where A, C, E are invertible. This is a typical element of $\mathbf{Aut}(\Delta)_1$. Suppose G is another such element, with $G_1 = \begin{pmatrix} A' & 0 \\ B' & C' \end{pmatrix}$ and $G_0 = \begin{pmatrix} A' & D' \\ 0 & E' \end{pmatrix}$. Then the product $FG = F\#_0 G$ is the chain automorphism with

$$(F\#_0G)_1 = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} A' & 0 \\ B' & C' \end{pmatrix} = \begin{pmatrix} AA' & 0 \\ BA' + CB' & CC' \end{pmatrix}$$

and

$$(F\#_0G)_0 = \begin{pmatrix} A & D \\ 0 & X \end{pmatrix} \begin{pmatrix} A' & D' \\ 0 & E' \end{pmatrix} = \begin{pmatrix} AA' & AD' + DE' \\ 0 & EE' \end{pmatrix}.$$

With suitable module structures on $K^{m,n}$ and $K^{n,p}$ there are isomorphisms from the two levels of $\mathbf{Aut}(\Delta)_1$ to $GL_n(K) \ltimes K^{m,n} \rtimes GL_m(K)$ and $GL_n(K) \ltimes K^{n,p} \rtimes GL_p(K)$ respectively.

Suppose $(H, F) : F \simeq G$. The chain homotopy H is an element of $K^{n+m,n+p}$, so to start with assume it has form:

$$H = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

with $X \in K^{n,n}, Y \in K^{n,p}, Z \in K^{m,n}$ and $W \in K^{m,p}$. As usual, the chain homotopy conditions must apply. This gives

$$\Delta H = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} = G_0 - F_0 = \begin{pmatrix} A' - A & D' - D \\ 0 & E' - E \end{pmatrix},$$

from which we deduce that $E' = E,\ X = A' - A$ and $Y = D' - D$. Also

$$H\Delta = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X & 0 \\ Z & 0 \end{pmatrix} = G_1 - F_1 = \begin{pmatrix} A' - A & 0 \\ B' - B & C' - C \end{pmatrix},$$

from which $C' = C,\ Z = B' - B$ and, again, $X = A' - A$. These calculations prove the following:

Lemma 2.3.1. *Let F, G be chain automorphisms as given above. Then*

$$F \simeq G \Leftrightarrow C' = C \text{ and } E' = E.$$

In this case a chain homotopy H such that $(H, F) : F \simeq G$ is of the form

$$\begin{pmatrix} A' - A & D' - D \\ B' - B & W \end{pmatrix},$$

where $W \in K^{m,p}$ is arbitrary and the other blocks are determined by the source and target of the homotopy.

□

Thus a typical element of $\mathbf{Aut}(\Delta)_2$ is a homotopy (H, F) as described by the lemma. Once the source and chain homotopy (as encoded in the notation) are chosen, the target is fixed. For (H, F) using the notation above, the target is $G = (G_1, G_0)$ where

$$G_1 = \begin{pmatrix} A + X & 0 \\ B + Z & C \end{pmatrix}, \quad G_0 = \begin{pmatrix} A + X & D + Y \\ 0 & E \end{pmatrix}.$$

In the same way, once a chain homotopy and target are given, the source may be recovered straightaway. Unlike most of the special cases discussed above, the chain homotopy is not uniquely determined by the source and target of a homotopy, since there is the arbitrary block W. This fact has ramifications for the homotopy groups of the classifying space, which we shall return to after discussing the compositions on $\mathbf{Aut}(\Delta)_2$.

We turn first to the group operation of horizontal composition. Suppose H is the homotopy given above and $(\hat{H}, \hat{F}) : \hat{F} \simeq \hat{G}$ is another homotopy. Then the group operation on $\mathbf{Aut}(\Delta)_2$ is horizontal composition with $(H, F)\#_0(\hat{H}, \hat{F})$ the homotopy having source $F\#_0\hat{F}$ and chain homotopy $G_1\hat{H} + H\hat{F}_0$.

Compared to the horizontal composition of homotopies, vertical composition is straightforward. It is only defined where the target of one homotopy is the target of the next, and the chain homotopy is obtained by adding the chain homotopy matrices for the two homotopies. Suppose $(H, F) : F \simeq F'$ and $(H', F') : F' \simeq F''$. Then $(H', F')\#_1(H, F) = (H' + H, F) : F \simeq F''$.

It remains to examine the structure of $\pi_1\mathbf{Aut}(\Delta)$ and $\pi_2\mathbf{Aut}(\Delta)$. The first of these is the group of homotopy classes of chain automorphisms in $\mathbf{Aut}(\Delta)_1$. Lemma 2.3.1 shows that two chain automorphisms are homotopic precisely when the lower-right blocks of both levels (elements of $GL_m(K)$ and $GL_p(K)$ for top and bottom respectively) coincide. π_2 is the subgroup of $\mathbf{Aut}(\Delta)_2$ consisting of homotopies on the identity. For most of the examples we previously examined, π_2 was trivial since a homotopy was uniquely determined by its source and target. However, in the general case there is an arbitrary $m \times p$ block in the lower right-hand corner. These observations lead to the following theorem:

Theorem 2.3.2. *Let $\delta : K^n \oplus K^m \to K^n \oplus K^p$ be any linear transformation of K-vector spaces, expressed as the matrix*

$$\Delta = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}.$$

Then: (i)

$$\pi_1 \boldsymbol{Aut}(\Delta) \cong GL_m(K) \times GL_p(K).$$

(ii)

$$\pi_2 \boldsymbol{Aut}(\Delta) \cong K^{m,p}.$$

□

2.4 Linear Representations Defined

In section 1.2.3 we established that a cat^1-group is the same thing as a 2-group (which may be thought of as a graded set with 3 non-empty levels, the lowest of which is a singleton, and various graded maps). Therefore we may look for representations of a cat[1]-group $\mathfrak{C}$ as 2-functors into a suitable 2-category, taking elements of P to 1-cells and elements of $C \rtimes P$ to 2-cells, so as to preserve the structures (all the 1- and 2-cells will have the same object, $\star$, as their 0-source and target, even if the target category has many objects). By analogy with groups and groupoids, the target 2-category of a linear representation should involve vector spaces or modules. We have seen in section 1.3 that $\mathbf{Ch}_K^{(1)}$ is a 2-category which generalises $\mathbf{Vect}_K$, so this is suitable for our purpose. Although its ramifications will be far-reaching, the actual definition of a representation is fairly obvious.

Definition 2.4.1. A *linear representation* of the cat^1-group $\mathfrak{C}$ is a 2-functor

$$\phi : \mathfrak{C} \to \mathbf{Ch}_K^{(1)}.$$

As an abstract definition, this seems plausible enough, and there clearly exists a trivial representation sending everything to the identity. Also, it collapses to the ordinary notion of group representation in the special case of cat^1-groups with top group and base equal (see example 1.2.8). The best way to show that non-trivial representations exist is to find some. We shall postpone searching for representations until after the next chapter, in which we shall develop some more useful tools to help us, but for now we we may pause to consider what data is involved in specifying a representation.

Given $\mathfrak{C}$, the first step towards defining ϕ is to find a chain complex (i.e., linear transformation) to act as the implicit target object, $\delta = \phi(\star)$. The group algebra functor of section 1.4 provides a canonical way of getting from a group homomorphism to a linear transformation, although it will sometimes be useful to make a different choice. Once δ is chosen, the elements of the cat^1-group must be mapped to elements of $\mathbf{Ch}_K^{(1)}$, with elements of the base going to 1-cells (chain maps) and elements of the top group going to 2-cells (homotopies). For ϕ to be a functor, this mapping must preserve identities and composition. Therefore, the image of $\mathfrak{C}$ lies within $\mathbf{Aut}(\delta)$ (which is why we have studied it in such depth earlier in this chapter). This δ is clearly analogous to the representation space of a group representation; since it is a chain complex rather than a vector space it will be called the *representation complex* of the representation.

Recall that $\mathbf{Aut}(\delta)$ is itself a cat^1-group, whose elements are linear transformations. Therefore, another way of considering the representation ϕ is to take it as a cat^1-group morphism

$$\phi : \mathfrak{C} \to \mathbf{Aut}(\delta).$$

This is a similar situation to the group case, in which a linear representation group reformulates the original group as a group of linear transformations.

Chapter 3

Cat^1-Group Algebras

In which a cat^1 version of the group algebra functor is built, and modules over the cat^1-group algebra are defined and used to study representations of the cat^1-group.

A very common, and fruitful, approach to group representation theory is via modules over a group or an algebra (see for example [20], [24], or [26]). Linear representations of a group G are in one-to-one correspondence with modules over its group algebra, $K(G)$ (see section 1.4) . Whereas the representations, in their matrix form, are conducive to calculation, the module theoretic approach is more elegant and powerful for developing the theory. This provides motivation for exploring the corresponding notions of algebras and modules over a cat^1-group. Since a cat^1-group is a generalisation of a group, it is natural to ask whether there is a sensible notion of cat^1-group algebra. This should, reasonably, be a cat^1-algebra generated in a canonical way from the cat^1-group. Before constructing a definition of a cat^1-group algebra, we shall remind ourselves of the definition of general cat^1-algebras.

3.1 Cat^1-Algebras

Cat^1-algebras are well-known, at least as an analogue of cat^1-groups in another category. Their theory is not so well developed, however. A description of cat^1-algebras and their equivalence to crossed modules of algebras appears in Shammu's PhD thesis [68] and is implicit in more general expositions of cat^1-objects by Ellis [27] and Porter [62], but to fix notation and to keep our account reasonably self-contained we shall now pause to review the basic definition and some properties of cat^1-algebras.

3.1.1 Definition

Firstly, recall the definition of a cat^1-algebra. Just as a cat^1-group is equivalent to an internal category in the category of groups (**Gr**) [63], a cat^1-K-algebra is equivalent to an internal category in the category of K-algebras ($\mathbf{Alg}_K$), where K is a fixed commutative ring with identity[1]. Where there is no ambiguity, we shall refer to cat^1-K-algebras simply as cat^1-algebras.

Definition 3.1.1. A *cat^1-K-algebra* $\mathcal{A}$ consists of K-algebras A_0, A_1 and K-algebra morphisms $\sigma, \tau : A_1 \to A_0$, $\iota : A_0 \to A_1$ (called *structural morphisms*) satisfying

CA1 $\sigma\iota = \tau\iota = \mathrm{id}_{A_0}$,

CA2 $\ker\sigma.\ker\tau = 0, \quad \ker\tau.\ker\sigma = 0.$

This is very similar to the definition of a cat^1-group, except for the kernel conditions **CA1**, which are superficially quite different. Condition **CA2** states that $\mathcal{A}$ is a reflexive internal graph in $\mathbf{Alg}_K$. The kernel conditions ensure that this is an internal category [52] for composition defined as follows.

$\mathbf{Alg}_K$ is a complete category, so pullbacks exist. Define composition to be

$$\circ : A_1\,{}_\sigma\!\times_\tau A_1 \to A_1,$$

with

$$\alpha \circ \beta := \alpha - \iota\sigma\alpha + \beta,$$

where $\sigma\alpha = \tau\beta$. Note that, for any $\alpha \in A_1$, $\alpha - \iota\sigma\alpha \in \ker\sigma$, since $\sigma(\alpha - \iota\sigma\alpha) = \sigma\alpha - \sigma\iota\sigma\alpha = \sigma\alpha - \sigma\alpha = 0$. Similarly, $\beta - \iota\tau\beta \in \ker\tau$ for every $\beta \in A_1$.

In order for this definition to be useful, composition must be a K-algebra morphism. This is true if the interchange law for addition,

$$(\alpha \circ \beta) + (\gamma \circ \delta) = (\alpha + \gamma) \circ (\beta + \delta), \tag{3.1}$$

and the interchange law for multiplication

$$(\alpha \circ \beta)(\gamma \circ \delta) = \alpha\gamma \circ \beta\delta, \tag{3.2}$$

are satisfied whenever $\alpha \circ \beta$ and $\gamma \circ \delta$ are defined.

[1]As usual, K may be thought of as either $\mathbb{R}$ or $\mathbb{C}$ if desired.

Lemma 3.1.2. *Composition, $\alpha \circ \beta := \alpha - \iota\sigma\alpha + \beta$, is a K-algebra morphism.*

Proof:

Suppose $\alpha \circ \beta$ and $\gamma \circ \delta$ are defined. Thus, $\sigma\alpha = \tau\beta$ and $\sigma\gamma = \tau\delta$.

Now,

$$\begin{aligned}(\alpha \circ \beta) + (\gamma \circ \delta) &= \alpha - \iota\sigma\alpha + \beta + \gamma - \iota\sigma\gamma + \delta \\ &= \alpha + \gamma - \iota\sigma(\alpha + \gamma) + \beta + \delta \\ &= (\alpha + \gamma) \circ (\beta + \delta),\end{aligned}$$

so (3.1) is satisfied.

Also,

$$\begin{aligned}(\alpha \circ \beta)(\gamma \circ \delta) &= (\alpha - \iota\sigma\alpha + \beta)(\gamma - \iota\sigma\gamma + \delta) \\ &= (\alpha - \iota\sigma\alpha)\gamma + \beta\gamma + (\alpha - \iota\sigma\alpha)(\delta - \iota\tau\delta) + \beta(\delta - \iota\tau\delta) \\ &= \alpha\gamma - \iota\sigma\alpha\gamma + \beta\gamma + \alpha\delta - \iota\sigma\alpha\delta - \alpha\iota\tau\delta + \iota\sigma(\alpha\gamma) + \beta\delta - \beta\iota\tau\delta \\ &= \alpha\gamma \circ \beta\delta + A + B\end{aligned}$$

where

$$A = \alpha\delta - \iota\sigma\alpha\delta - \alpha\iota\tau\delta + \iota\sigma\alpha\iota\tau\delta = (\alpha - \iota\sigma\alpha)(\delta - \iota\tau\delta) \in \ker\sigma . \ker\tau$$

and

$$B = \beta\gamma - \iota\tau\beta\gamma - \beta\iota\sigma\gamma + \iota\tau\beta\iota\sigma\gamma = (\beta - \iota\tau\beta)(\gamma - \iota\sigma\gamma) \in \ker\tau . \ker\sigma.$$

However, the kernel conditions ensure that $A = B = 0$, whence (3.2) is also satisfied.

□

This lemma shows that, with the kernel conditions of our definition, the cat^1-algebra is an internal category in $\mathbf{Alg}_K$ as we would expect. Conversely, suppose we have an internal category in $\mathbf{Alg}_K$. This has a composition satisfying the interchange laws with addition and multiplication. Then

$$\begin{aligned}\alpha \circ \beta &= (\alpha + 0) \circ (\iota\sigma\alpha - \iota\sigma\alpha + \beta) \\ &= (\alpha \circ \iota\sigma\alpha) + (0 \circ (-\iota\sigma\alpha + \beta)) \\ &= \alpha - \iota\sigma\alpha + \beta.\end{aligned}$$

Thus composition is expressible in terms of addition, as in lemma 3.1.2. Since the interchange law for multiplication is satisfied, it follows that $(\alpha - \iota\sigma\alpha)(\delta - \iota\tau\delta) = 0$ and $(\beta - \iota\tau\beta)(\gamma - \iota\sigma\gamma) = 0$. But these are typical elements of the kernels of σ and τ, so the kernel conditions are satisfied. Hence every internal category in $\mathbf{Alg}_K$ is a cat^1-algebra.

If the elements of A_1 are pictured as 2-cells, then the composite $\alpha \circ \beta$ is defined when the 1-source of α coincides with the 1-target of β. This (vertical) composition over a 1-dimensional boundary could be written using the more suggestive notation $\#_1$ which we used for cat^1-groups. However, both addition and multiplication are defined for all the elements of both algebras A_1 and A_0 and both of these may be thought of as horizontal compositions. Rather than extending the notation $\#_0$ to distinguish between these operations, we shall retain the traditional notation of $+, .$ for these operations, and thus it makes sense to use $\circ$ for the vertical composition as well. There is, of course, also a scalar multiplication at each level, for which we shall use the usual notation of juxtaposition.

By analogy with the group case, a structure $\mathcal{A}$ satisfying **CA1** but not **CA2** will be called a *precat^1-algebra*. Morphisms of cat^1-algebras can also be defined by analogy with the group case. This leads to the category of cat^1-K-algebras and their morphisms, denoted $\mathbf{Cat1}_{\mathbf{Alg}_K}$; when necessary we shall write the category **Cat1** of cat^1-groups as $\mathbf{Cat1}_{\mathbf{Gr}}$ to distinguish it from this new category (or from any other categories of cat^1 objects we may have occasion to use). Note that if the multiplicative structures are ignored, $\mathcal{A}$ yields an abelian cat^1-group.

3.1.2 Involution

It is well-known (a fact observed by Duskin, and published by Brown and Spencer [16]) that an internal category in **Gr** is automatically an internal groupoid. The same result is true in $\mathbf{Alg}_K$, as we shall now demonstrate[2]. Let $\mathcal{A}$ be an internal category in $\mathbf{Alg}_K$, using the notation of 3.1.1. We have seen that composition is expressible in terms of addition, as $\alpha \circ \beta = \alpha - \iota\sigma\alpha + \beta$. Since addition is commutative and $\sigma\alpha = \tau\beta$ (for composition to be defined), we also have $\alpha \circ \beta = \beta - \iota\tau\beta + \alpha$.

[2]Analogous results hold in several other categories, and in fact results are known as to the types of categories for which such results are true. We shall not require this more general theory, however.

Theorem 3.1.3. *Every $\alpha \in A_1$ has an inverse for composition. Hence $\mathcal{A}$ is an internal groupoid in **Alg**$_K$.*

Proof:

Define $\alpha^* := \iota\sigma\alpha - \alpha + \iota\tau\alpha$. Then

$$\sigma\alpha^* = \sigma\iota\sigma\alpha - \sigma\alpha + \sigma\iota\tau\alpha = \tau\alpha$$

and

$$\tau\alpha^* = \tau\iota\sigma\alpha - \tau\alpha + \tau\iota\tau\alpha = \sigma\alpha.$$

Also,

$$\alpha \circ \alpha^* = \alpha - \iota\sigma\alpha + (\iota\sigma\alpha - \alpha + \iota\tau\alpha) = \iota\tau\alpha,$$

while

$$\alpha^* \circ \alpha = \alpha - \iota\tau\alpha + (\iota\sigma\alpha - \alpha + \iota\tau\alpha) = \iota\sigma\alpha.$$

Thus α^* is a 2-sided compositional inverse for α. Such an inverse exists for every $\alpha \in A_1$. □

The existence of inverses allows us to define an internal endofunctor on $\mathcal{A}$, which is an involution. For each $x \in A_0$ define $x^* := x$.

Proposition 3.1.4. $(\ )^* : \mathcal{A} \to \mathcal{A}$ *is a contravariant internal functor in **Alg**$_K$, with $(\alpha^*)^* = \alpha$ for each $\alpha \in A_1$.*

Proof:

A definition of internal functor is given by Borceux [7]. In **Alg**$_K$ it consists of a pair of K-algebra morphisms preserving identities and composition. A contravariant internal functor will reverse the composition.

For A_0, we have the identity morphism. For A_1 the mapping $\alpha \mapsto \alpha^*$ is used. We check that this is a K-algebra morphism. Let α, $\beta \in A_1$ and $\lambda\,, \mu \in K$.

$$\begin{aligned}(\lambda\alpha + \mu\beta)^* &= \iota\sigma(\lambda\alpha + \mu\beta) - (\lambda\alpha + \mu\beta) + \iota\tau(\lambda\alpha + \mu\beta)\\ &= \lambda\iota\sigma\alpha + \mu\iota\sigma\beta - \lambda\alpha - \mu\beta + \lambda\iota\tau\alpha + \mu\iota\tau\beta\\ &= \lambda(\iota\sigma\alpha - \alpha + \iota\tau\alpha) + \mu(\iota\sigma\beta - \beta + \iota\tau\beta)\\ &= \lambda\alpha^* + \mu\beta^*.\end{aligned}$$

Also,

$$\begin{aligned}
\alpha^*\beta^* &= (\iota\sigma\alpha - \alpha + \iota\tau\alpha)(\iota\sigma\beta - \beta + \iota\tau\beta) \\
&= \iota\sigma\alpha\iota\sigma\beta - \alpha\iota\sigma\beta + \iota\tau\alpha\iota\sigma\beta - \iota\sigma\alpha\beta + \alpha\beta - \iota\tau\alpha\beta + \iota\sigma\alpha\iota\tau\beta - \alpha\iota\tau\beta + \iota\sigma\alpha\iota\tau\beta \\
&= \iota\sigma(\alpha\beta) - \alpha\beta + \iota\tau\alpha\beta + \alpha\beta - \iota\sigma\alpha\beta - \alpha\iota\tau\beta + \iota\sigma\alpha\iota\tau\beta + \alpha\beta - \alpha\iota\sigma\beta \\
&\qquad - \iota\tau\alpha\beta + \iota\tau\alpha\iota\sigma\beta \\
&= (\alpha\beta)^* + (\alpha - \iota\sigma\alpha)(\beta - \iota\tau\beta) + (\alpha - \iota\tau\beta)(\beta - \iota\sigma\beta).
\end{aligned}$$

However, $(\alpha - \iota\sigma\alpha)(\beta - \iota\tau\beta) \in \ker\sigma.\ker\tau$ and $(\alpha - \iota\tau\beta)(\beta - \iota\sigma\beta) \in \ker\tau.\ker\tau$. Since $\mathcal{A}$ is a cat^1-algebra, the kernel conditions hold, whence $(\alpha\beta)^* = \alpha^*\beta^*$ as required.

Next, we check functoriality. Suppose $\sigma\alpha = \tau\beta$.

$$\begin{aligned}
\beta^* \circ \alpha^* &= \iota\sigma\beta - \beta + \iota\tau\beta - \iota\sigma(\iota\sigma\beta - \beta + \iota\tau\beta) + \iota\sigma\alpha - \alpha + \iota\tau\alpha \\
&= \iota\sigma\beta - \beta + \iota\tau\beta - \iota\sigma\beta + \iota\sigma\beta - \iota\tau\beta + \iota\sigma\alpha - \alpha + \iota\tau\alpha \\
&= \iota\sigma\beta - \beta + \iota\sigma\alpha - \alpha + \iota\tau\alpha \\
&= \iota\sigma\beta - (\alpha - \iota\sigma\alpha + \beta) + \iota\tau\alpha \\
&= \iota\sigma(\alpha \circ \beta) - \alpha \circ \beta + \iota\tau(\alpha \circ \beta) \\
&= (\alpha \circ \beta)^*.
\end{aligned}$$

Also, for any $x \in A_0$

$$(\iota x)^* = \iota\sigma\iota x - \iota x + \iota\tau\iota x = \iota x = \iota(x^*).$$

Finally, the functor is an involution, since

$$\begin{aligned}
(\alpha^*)^* &= \iota\sigma(\iota\sigma\alpha - \alpha + \iota\tau\alpha) - (\iota\sigma\alpha - \alpha + \iota\tau\alpha) + \iota\tau(\iota\sigma\alpha - \alpha + \iota\tau\alpha) \\
&= \iota\sigma\alpha - \iota\sigma\alpha + \iota\tau\alpha - \iota\sigma\alpha + \alpha - \iota\tau\alpha + \iota\sigma\alpha - \iota\tau\alpha + \iota\tau\alpha \\
&= \alpha.
\end{aligned}$$

□

We have used α^* to denote the inverse for α under composition in order to distinguish it from the multiplicative inverse α^{-1}. In fact, this distinction is unnecessary because, when the multiplicative inverse exists (i.e. for every non-zero element of A_1) it is the same as the compositional inverse (which always exists). Suppose $\alpha : x \Rightarrow y$. Then

$$1 = \alpha^{-1}\alpha = (1_x \circ \alpha^{-1})(\alpha \circ 1_{y^{-1}}) = (1_x\alpha) \circ (\alpha^{-1}1_{y^{-1}}) = \alpha \circ \alpha^{-1}.$$

Similarly, $\alpha \circ \alpha^{-1} = 1$, whence α^{-1} is a 2-sided inverse for α under composition. The uniqueness of such an inverse is a standard property. Note that the interchange law entails a flipping of terms when switching between multiplication and composition, i.e. $\alpha\alpha^{-1} = \alpha^{-1} \circ \alpha$.

3.2 The Cat^1-Group Algebra Construction

Just as any group G has an associated group algebra (with basis[3] indexed by G and multiplication induced from G), a cat^1-algebra can be constructed from any cat^1-group $\mathfrak{C}$. For groups (see 1.4), the construction is achieved via the *group algebra functor*, $K(\cdot) : \mathbf{Gr} \to \mathbf{Alg}_K$. A naive approach towards constructing a cat^1-group algebra is to apply this functor to the groups and homomorphisms making up $\mathfrak{C}$.

3.2.1 A First Approach

Suppose

$$\begin{tikzcd} C \rtimes P \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & P \arrow[l, bend left, "i"] \end{tikzcd}$$

is the cat^1-group $\mathfrak{C}$. Then, applying the group algebra functor gives

$$\begin{tikzcd} K(C \rtimes P) \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & K(P) \arrow[l, bend left, "\iota"] \end{tikzcd}$$

where $K(P)$ has basis $\{\mathbf{e}_p : p \in P\}$, $K(C \rtimes P)$ has basis $\{\mathbf{e}_{c,p} : c \in C,\ p \in P\}$ and the maps σ, τ, ι act on the basis elements as follows:

$$\begin{aligned} \sigma(\mathbf{e}_{c,p}) &= \mathbf{e}_p, \\ \tau(\mathbf{e}_{c,p}) &= \mathbf{e}_{\partial cp} \qquad (\partial = t\,|_{\ker\sigma}), \\ \iota(\mathbf{e}_p) &= \mathbf{e}_{1,p}. \end{aligned}$$

These maps extend linearly to the rest of $K(P)$ and $K(C \rtimes P)$.

[3]We may assume K is a field or an integral domain for simplicity here, hence "linear independence" and "basis" will make sense without complications. The algebraic calculations work in more generality however.

Since $K(\cdot)$ is a functor, condition **CA1** is induced from the equivalent condition on $\mathfrak{C}$. Thus, $K(\mathfrak{C})$ is certainly a precat^1-algebra.

It remains to examine the kernel conditions **CA2**. First of all, we shall need to find bases for $\ker\sigma$ and $\ker\tau$. For any $\mathbf{e}_{c,p} \in K(C \rtimes P)$, $\mathbf{e}_{c,p} - \mathbf{e}_{1,p} \in \ker\sigma$; define $\mathbf{v}_{c,p} := \mathbf{e}_{c,p} - \mathbf{e}_{1,p}$.

Lemma 3.2.1. *The set* $\{\mathbf{v}_{c,p} : c \neq 1\}$ *is a basis for* $\ker\sigma$.

Proof:
Clearly every $\mathbf{v}_{c,p}$ is in $\ker\sigma$. It suffices to show that these elements span $\ker\sigma$ and are linearly independent.

Suppose $\mathbf{v} \in \ker\sigma$. That is $\mathbf{v} = \sum_{p\in P}\sum_{c\in C} r_{c,p}\mathbf{e}_{c,p}$ with $\sigma(\mathbf{v}) = 0$. But $\sigma(\mathbf{v}) = \sum_p(\sum_c r_{c,p})\mathbf{e}_p$ and, since the $\mathbf{e}_p$ are a basis, this is zero iff $\sum_{c\in C} r_{c,p} = 0$ for each $p \in P$.

Now,

$$\mathbf{v} = \sum_p\sum_{c\neq 1} r_{c,p}(\mathbf{e}_{c,p} - \mathbf{e}_{1,p} + \mathbf{e}_{1,p}) + \sum_p r_{1,p}\mathbf{e}_{1,p} = \sum_p\sum_{c\neq 1} r_{c,p}\mathbf{v}_{c,p} + \sum_p\sum_c r_{c,p}\mathbf{e}_{1,p},$$

but

$$\sum_p\sum_c r_{c,p}\mathbf{e}_{1,p} = \sum_p(\sum_c r_{c,p})\mathbf{e}_{1,p} = 0,$$

because $\sum_{c\in C} r_{c,p} = 0$ for every $p \in P$. Hence $\mathbf{v} = \sum_p\sum_{c\neq 1} r_{c,p}\mathbf{v}_{c,p}$ for every $\mathbf{v} \in \ker\sigma$, so the $\mathbf{v}_{c,p}$ do indeed span $\ker\sigma$.

Now suppose that $\sum_p\sum_{c\neq 1} r_{c,p}\mathbf{v}_{c,p} = 0$. Then

$$\begin{aligned}\sum_p\sum_{c\neq 1} r_{c,p}(\mathbf{e}_{c,p} - \mathbf{e}_{1,p}) = 0 &\Leftrightarrow \sum_p\sum_{c\neq 1} r_{c,p}\mathbf{e}_{c,p} - \sum_p\sum_{c\neq 1} r_{c,p}\mathbf{e}_{1,p} = 0\\ &\Leftrightarrow \sum_p\sum_c r'_{c,p}\mathbf{e}_{c,p} = 0,\end{aligned}$$

where $r'_{c,p} = r_{c,p}$ when $c \neq 1$ and $r'_{1,p} = -\sum_{c\neq 1} r_{c,p}$. This is a linear combination of basis vectors $\mathbf{e}_{c,p}$ in $K(C \rtimes P)$, so $r'_{c,p} = 0$ for each c, p. In particular, this is true for $c \neq 1$, so every $r_{c,p} = 0$ and the $\mathbf{v}_{c,p}$ are linearly independent, as required. □

Similarly, $\mathbf{w}_{c,p} := \mathbf{e}_{c,p} - \mathbf{e}_{1,\partial cp} \in \ker\tau$ for every $c \in C,\ p \in P$.

Lemma 3.2.2. *The set* $\{\mathbf{w}_{c,p} : c \neq 1\}$ *is a basis for* $\ker\tau$.

Proof:
The inversion functor of proposition 3.1.4 may be applied to $K(\mathfrak{C})$.

First, observe that if $\alpha \in \ker\sigma$, then $\tau\alpha^* = \sigma\alpha = 1$, hence $\alpha^* \in \ker\tau$, and vice versa. Thus $(\ )^*$ interchanges the kernels of σ and τ. Further, it is evident that a basis for $\ker\tau$ can be constructed by taking the inverse of each element of a basis for $\ker\sigma$.

The set $\{\mathbf{v}_{c,p} : c \neq 1\}$ is a basis for $\ker\sigma$. Now,

$$\mathbf{e}^*_{c,p} = \iota\sigma\mathbf{e}_{c,p} - \mathbf{e}_{c,p} + \iota\tau\mathbf{e}_{c,p} = \mathbf{e}_{1,p} - \mathbf{e}_{c,p} + \mathbf{e}_{1,\partial cp},$$

while

$$\mathbf{e}^*_{1,p} = \mathbf{e}_{1,p} - \mathbf{e}_{1,p} + \mathbf{e}_{1,p} = \mathbf{e}_{1,p}.$$

Therefore

$$\begin{aligned}\mathbf{v}^*_{c,p} &= \mathbf{e}_{1,p} - \mathbf{e}_{c,p} + \mathbf{e}_{1,\partial cp} - \mathbf{e}_{1,p}\\ &= \mathbf{e}_{1,\partial cp} - \mathbf{e}_{c,p}\\ &= -\mathbf{w}_{c,p}\end{aligned}$$

Hence $\{-\mathbf{w}_{c,p} : c \neq 1\}$ is a basis for $\ker\tau$, and so (using an obvious change of basis) $\{\mathbf{w}_{c,p} : c \neq 1\}$ is also a basis, as required. □

To satisfy the kernel conditions, it would suffice that $\mathbf{v}_{c,p}.\mathbf{w}_{d,q} = 0$ and $\mathbf{w}_{d,q}.\mathbf{v}_{c,p} = 0$ for every $c, d \in C \setminus \{1_C\}$ and $p, q \in P$.

However,

$$\begin{aligned}\mathbf{v}_{c,p}.\mathbf{w}_{d,q} &= (\mathbf{e}_{c,p} - \mathbf{e}_{1,p})(\mathbf{e}_{d,q} - \mathbf{e}_{1,\partial dq})\\ &= \mathbf{e}_{c^pd,pq} - \mathbf{e}_{^pd,pq} - \mathbf{e}_{c,p\partial dq} + \mathbf{e}_{1,p\partial dq}.\end{aligned}$$

This is a linear combination of basis elements with non-zero coefficients, so $\mathbf{v}_{c,p}.\mathbf{w}_{d,q} \neq 0$ and the kernel condition fails. Likewise for the other kernel condition. Hence, $K(\mathfrak{C})$ is not a cat^1-algebra.

3.2.2 Fixing the Kernel Conditions

In order to construct a cat^1-algebra from $K(\mathfrak{C})$ it is necessary to impose some relations so that the kernel conditions are satisfied. Once suitable expressions are found, $K(\mathfrak{C})$

can be factored by the ideal they generate. This is analogous to the method given in [13] for producing a crossed module from a precrossed module by factoring out the Peiffer group.

Consider expressions of the form:

$$\mathbf{e}_{dc,p} - \mathbf{e}_{c,p} - \mathbf{e}_{d,\partial cp} + \mathbf{e}_{1,\partial cp} \qquad (c, d \in C,\, p \in P) \tag{3.3}$$

These can be represented pictorially as:

$$\begin{tikzcd}
& \mathbf{e}_{\partial cp} \arrow[loop above, "\mathbf{e}_{1,\partial cp}"] \arrow[dr, "\mathbf{e}_{d,\partial cp}"] & \\
\mathbf{e}_p \arrow[ur, "\mathbf{e}_{c,p}"] \arrow[rr, "\mathbf{e}_{dc,p}"'] & & \mathbf{e}_{\partial(dc)p}
\end{tikzcd}$$

Note that in this diagram (and those later in this section) the "composition" traced out is addition. The signs for the terms can be obtained by tracing in an anticlockwise direction, giving a plus sign to any term traced in the direction of the arrow and a minus sign to any term traced in the opposite direction. If $c = \mathsf{id}_C$, then (3.3) becomes $\mathbf{e}_{d,p} - \mathbf{e}_{1,p} - \mathbf{e}_{d,p} + \mathbf{e}_{1,p} = 0$; similarly if $d = \mathsf{id}_C$ (or both). In these cases, we can rewrite the condition as $\mathbf{e}_{dc,p} = \mathbf{e}_{c,p} - \mathbf{e}_{d,\partial cp} - \mathbf{e}_{1,\partial cp}$, so the diagram commutes.

We shall call expressions of the form (3.3) *cocycles*. Let J be the ideal

$$J = \langle \mathbf{e}_{dc,p} - \mathbf{e}_{c,p} - \mathbf{e}_{d,\partial cp} + \mathbf{e}_{1,\partial cp} : c, d \in C \setminus \{1_C\},\, p \in P \rangle$$

generated by the cocycles; J may be referred to as the *cocycle ideal*. Then both $\mathbf{v}_{c,p}.\mathbf{w}_{d,q}$ and $\mathbf{w}_{d,q}.\mathbf{v}_{c,p}$ are in J, as demonstrated by the following pictures:

$$\mathbf{v}_{c,p}.\mathbf{w}_{d,q} = \mathbf{e}_{c\,{}^pd,pq} - \mathbf{e}_{{}^pd,pq} - \mathbf{e}_{c,p\partial dq} + \mathbf{e}_{1,p\partial dq}$$

$$\begin{tikzcd}
& \mathbf{e}_{p\partial dq} \arrow[loop above, "\mathbf{e}_{1,p\partial dq}"] \arrow[dr, "\mathbf{e}_{c,p\partial dq}"] & \\
\mathbf{e}_{pq} \arrow[ur, "\mathbf{e}_{{}^pd,pq}"] \arrow[rr, "\mathbf{e}_{c\,{}^pd,pq}"'] & & \mathbf{e}_{\partial cp\partial dq}
\end{tikzcd}$$

and

$$\mathbf{w}_{d,q}.\mathbf{v}_{c,p} = \mathbf{e}_{d^qc,qp} - \mathbf{e}_{d,qp} - \mathbf{e}_{\partial d^qc,\partial dqp} + \mathbf{e}_{1,\partial dqp}$$

$$\begin{tikzcd}
& \mathbf{e}_{\partial dqp} \arrow[loop above, "\mathbf{e}_{1,\partial dqp}"] \arrow[dr, "\mathbf{e}_{\partial d^qc,\partial dqp}"] & \\
\mathbf{e}_{qp} \arrow[ur, "\mathbf{e}_{d,qp}"] \arrow[rr, "\mathbf{e}_{d^qc,qp}"'] & & \mathbf{e}_{\partial dq\partial cp}
\end{tikzcd}$$

$J \lhd K(C \rtimes P)$ is a two-sided ideal, so we can form the quotient algebra $K(C \rtimes P)/J$ with the natural epimorphism

$$\begin{aligned} \bar{\cdot} : K(C \rtimes P) &\twoheadrightarrow K(C \rtimes P)/J \\ \mathbf{e}_{c,p} &\mapsto \mathbf{e}_{c,p} + J. \end{aligned}$$

For notational convenience, write $\bar{\mathbf{e}}_{c,p} = \mathbf{e}_{c,p} + J$. From the structural morphisms σ, τ, ι we get induced maps $\bar{\sigma}, \bar{\tau} : K(C \rtimes P)/J \to K(P)$ and $\bar{\iota} : K(P) \to K(C \rtimes P)/J$. The basis elements of $\ker \sigma$ and $\ker \tau$ map to $\bar{\mathbf{v}}_{c,p} := \mathbf{v}_{c,p} + J$ and $\bar{\mathbf{w}}_{d,q} := \mathbf{w}_{d,q} + J$ respectively. The definition of $\mathbf{v}_{c,p}$ is preserved by factorisation since $\bar{\mathbf{v}}_{c,p} = \mathbf{v}_{c,p} + J = (\mathbf{e}_{c,p} - \mathbf{e}_{1,p}) + J = (\mathbf{e}_{c,p} + J) - (\mathbf{e}_{1,p} + J) = \bar{\mathbf{e}}_{c,p} - \bar{\mathbf{e}}_{1,p}$. Note that $\langle \bar{\mathbf{v}}_{c,p} : c \neq 1 \rangle = \ker \bar{\sigma}$ and $\langle \bar{\mathbf{w}}_{d,q} : d \neq 1 \rangle = \ker \bar{\tau}$, so that the $\bar{\mathbf{v}}_{c,p}$ and $\bar{\mathbf{w}}_{d,q}$ form generating sets for their respective kernels. These are not, however, bases, since the factorisation introduces linear dependencies as follows.

Expression (3.3) can be rewritten as

$$(\mathbf{e}_{dc,p} - \mathbf{e}_{1,p}) - (\mathbf{e}_{c,p} - \mathbf{e}_{1,p}) - (\mathbf{e}_{d,\partial cp} - \mathbf{e}_{1,\partial cp}) = \mathbf{v}_{dc,p} - \mathbf{v}_{c,p} + \mathbf{v}_{d,\partial cp} \in J.$$

Since this is in the ideal J, it will be killed off by factorisation. Thus, in $K(C \rtimes P)/J$ we get $\bar{\mathbf{v}}_{dc,p} - \bar{\mathbf{v}}_{c,p} - \bar{\mathbf{v}}_{d,\partial cp} = 0$, hence

$$\bar{\mathbf{v}}_{dc,p} = \bar{\mathbf{v}}_{c,p} + \bar{\mathbf{v}}_{d,\partial cp}. \tag{3.4}$$

There are redundancies among the $\bar{\mathbf{v}}_{c,p}$, so these do not form a basis. Similar relations hold for the $\mathbf{w}_{d,q}$.

In the same way, the basis $\{\mathbf{e}_{c,p}\}$ of $K(C \rtimes P)$ induces a generating set $\{\bar{\mathbf{e}}_{c,p}\}$ for the whole of $K(C \rtimes P)/J$. This also fails to be a basis, because of the linear dependencies

$$\bar{\mathbf{e}}_{dc,p} = \bar{\mathbf{e}}_{c,p} + \bar{\mathbf{e}}_{d,\partial cp} - \bar{\mathbf{e}}_{1,\partial cp} \tag{3.5}$$

induced by the factorisation. These relations will be called *cocycle relations*.

Define $\overline{K(\mathfrak{C})}$ to be the precat^1-algebra:

$$\begin{tikzcd} K(C \rtimes P)/J \arrow[r, shift left, "\bar{\sigma}"] \arrow[r, shift right, "\bar{\tau}"'] & K(P) \arrow[l, bend left, "\bar{\iota}"] \end{tikzcd}$$

induced from $K(\mathfrak{C})$. Condition **CA1** is satisfied by $\overline{K(\mathfrak{C})}$ since these properties are preserved by the quotient map.

Since, for every $c, d \in C$ and $p, q \in P$, the expressions $\mathbf{v}_{c,p}.\mathbf{w}_{d,q}$ and $\mathbf{w}_{d,q}.\mathbf{v}_{c,p}$ are in J, the kernel conditions **CA2** are satisfied in $\overline{K(\mathfrak{C})}$, hence:

Proposition 3.2.3. *$\overline{K(\mathfrak{C})}$ is a cat^1-K-algebra.* □

We are finally in a position to make our principal definition.

Definition 3.2.4. For any cat^1-group $\mathfrak{C} = (C \rtimes P, P, i, s, t)$, the *cat^1-group algebra* of $\mathfrak{C}$ is the cat^1-algebra $\overline{K(\mathfrak{C})} = (K(C \rtimes P)/J, K(P), \bar{\iota}, \bar{\sigma}, \bar{\tau})$, where J is the ideal

$$J = \langle \mathbf{e}_{dc,p} - \mathbf{e}_{c,p} - \mathbf{e}_{d,\partial cp} + \mathbf{e}_{1,\partial cp} : c, d \in C,\ p \in P \rangle.$$

$K(P)$ has a basis $\{\mathbf{e}_p : p \in P\}$ and $K(C \rtimes P)/J$ has a spanning set $\{\bar{\mathbf{e}}_{c,p}\}$ whose elements satisfy equation (3.5). The kernel of $\bar{\sigma}$ is spanned by the set $\{\bar{\mathbf{v}}_{c,p} : c \neq 1\}$, whose elements satisfy (3.4); $\ker \bar{\tau}$ is generated by $\{\bar{\mathbf{w}}_{c,p} : c \neq 1\}$ with a similar set of relations.

Example 3.2.5. The cat^1-group algebra construction can be applied to each of the examples in 1.2.11.

(i) $\mathfrak{C} = (C_2, I, 0, 0, i)$, where C_2 may be more strictly thought of as the isomorphic $C_2 \rtimes I$, gives $K(\mathfrak{C})$ as:

$$\begin{tikzcd} K^2 \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & K \arrow[l, bend left, "\iota"] \end{tikzcd}$$

where $\sigma = \tau$ is the map $(x, y) \mapsto x + y$ and $\iota(x) = (x, 0)$. In this case, there is only one $\mathbf{v}_{c,p}$ defined (namely $\mathbf{v}_{g,1}$, where g is the generator of C_2), so $\ker\sigma \cong K$ is singly generated. The only non-trivial generator of J is $\mathbf{e}_{1,1} - \mathbf{e}_{g,1} - \mathbf{e}_{g,1} + \mathbf{e}_{1,1}$, so the only cocycle relation introduced in $\overline{K(\mathfrak{C})}$ is

$$2\bar{\mathbf{e}}_{g,1} = 2\bar{\mathbf{e}}_{1,1}. \tag{3.6}$$

If K is a field of characteristic $\neq 2$, this implies that $\bar{\mathbf{e}}_{g,1} = \bar{\mathbf{e}}_{1,1}$, whence $K(C \rtimes P)/J \cong K$ and both σ and τ are the identity. If K is a field of characteristic 2 (for example, $\mathbb{Z}_2$) or a more general integral domain (e.g. $\mathbb{Z}$) this is not the case. Indeed, for $K = \mathbb{Z}_2$, relation (3.6) breaks down to the tautology $0 = 0$, so in fact no cocycle relations are introduced in $\mathbb{Z}_2(C_2)/J$ and $\mathbb{Z}_2(C_2) \rightrightarrows \mathbb{Z}_2$ is itself a cat^1-algebra. For $K = \mathbb{Z}$ we observe that (3.6) implies $2\bar{\mathbf{v}}_{g,1} = 0$, whence $\ker\bar{\sigma} \cong C_2$ (this is, of course, a $\mathbb{Z}$-module, since it is an abelian group).

(ii) $\mathfrak{C} = (C_3, I, 0, 0, i)$ (again C_3 is really $C_3 \rtimes I$), gives $K(\mathfrak{C})$ as:

$$\begin{tikzcd} K^3 \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & K \arrow[l, bend left, "\iota"] \end{tikzcd}$$

where $\sigma = \tau$ is the map $(x, y, z) \mapsto x + y + z$ and $\iota(x) = (x, 0, 0)$. Here, $\ker\sigma$ has two generators, $\mathbf{v}_{g,1}$ and $\mathbf{v}_{g^2,1}$ (with g the generator of C_3), while the non-trivial generators of J reduce to the cocycle relations

$$3\bar{\mathbf{e}}_{g,1} = 3\bar{\mathbf{e}}_{g^2,1} = 3\bar{\mathbf{e}}_{1,1}$$

in $\overline{K(\mathfrak{C})}$. Thus, for K a field of characteristic $\neq 3$, the three generators coincide and $K(C_3)/J \cong K$. For $\mathbb{Z}_3$ or $\mathbb{Z}$ the situation is also reminiscent of the previous example: $\mathbb{Z}_3(C_3)/J \cong \mathbb{Z}_3$ and $\mathbb{Z}(C_3)/J \cong C_3$.

(iii) $\mathfrak{C} = (C_3 \times C_2, C_2, s, s, i)$ gives:

$$\begin{tikzcd} K^6 \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & K^2 \arrow[l, bend left, "\iota"] \end{tikzcd}$$

with $\sigma = \tau$ sending $\mathbf{e}_{c,p}$ to $\mathbf{e}_p$ for each $c \in C_3$ and $p \in C_2$ and $\iota(\mathbf{e}_p) = \mathbf{e}_{1,p}$. There are four elements $\mathbf{v}_{c,p}$ with $c \neq 1$, so $\ker\sigma \cong K^4$.

3.2.3 The Cat^1-Group Algebra as a Functor

The group algebra construction provides a functor from **Gr** to $\mathbf{Alg}_K$; in the same way we may expect the cat^1-group algebra to give a functor from $\mathbf{Cat1}_{\mathbf{Gr}}$ to $\mathbf{Cat1}_{\mathbf{Alg}_K}$. For any cat^1-group $\mathfrak{C}$, definition 3.2.4 gives us a cat^1-group algebra $\overline{K(\mathfrak{C})}$, so it remains to define a mapping from cat^1-group morphisms to cat^1-algebra morphisms, and check that it is functorial.

Suppose $\mathfrak{C}_i = (C_i \rtimes P_i, P_i, s_i, t_i, i_i)$ are cat^1-groups ($i = 1, 2, 3$), with $\phi : \mathfrak{C}_1 \to \mathfrak{C}_2$ and $\psi : \mathfrak{C}_2 \to \mathfrak{C}_3$.

$$\begin{tikzcd}
C_1 \rtimes P_1 \arrow[r, "\phi_C"] \arrow[d, "s_1"'] & C_2 \rtimes P_2 \arrow[r, "\psi_C"] \arrow[d, "s_2"'] & C_3 \rtimes P_3 \arrow[d, "s_3"] \\
P_1 \arrow[r, "\phi_P"'] & P_2 \arrow[r, "\psi_P"'] & P_3
\end{tikzcd}$$

Since both squares in the diagram commute (ϕ and ψ are cat^1-group morphisms) the outer rectangle also commutes. Similar diagrams hold with s_i replaced by t_i and by i_i.

Applying the group algebra functor to this setup gives us the following diagram of precat^1-algebras, where $K(\phi_C)(\mathrm{e}_{c,p}) = \mathrm{e}_{\phi(c,p)}$ etc.

$$\begin{tikzcd}
K(C_1 \rtimes P_1) \arrow[r, "K(\phi_C)"] \arrow[d, "\sigma_1"'] & K(C_2 \rtimes P_2) \arrow[r, "K(\psi_C)"] \arrow[d, "\sigma_2"'] & K(C_3 \rtimes P_3) \arrow[d, "\sigma_3"] \\
K(P)_1 \arrow[r, "K(\phi_P)"'] & K(P)_2 \arrow[r, "K(\psi_P)"'] & K(P)_3
\end{tikzcd}$$

Again, there are similar diagrams for τ and ι. Since $K(\cdot)$ is a functor, both squares and the rectangle commute here also.

To get from here to cat^1-algebras, we form the ideals

$$J_i = \langle \mathrm{e}_{d_i c_i, p_i} - \mathrm{e}_{c_i, p_i} - \mathrm{e}_{d_i, \partial c_i p_i} + \mathrm{e}_{1, \partial c_i p_i} : c_i, d_i \in C_i,\ p_i \in P_i \rangle.$$

Next factor each $K(C_i \rtimes P_i)$ by the corresponding J_i and replace σ_i (or τ_i, ι_i) by the induced $\bar{\sigma_i}$ ($\bar{\tau_i}$, $\bar{\iota_i}$). The factoring also induces maps $\bar{K}(\phi_C)$ and $\bar{K}(\psi_C)$, which are obviously well-defined. This gives us the diagram:

$$\begin{tikzcd}
K(C_1 \rtimes P_1)/J_1 \arrow[r, "\bar{K}(\phi_C)"] \arrow[d, "\bar{\sigma_1}"'] & K(C_2 \rtimes P_2)/J_2 \arrow[r, "\bar{K}(\psi_C)"] \arrow[d, "\bar{\sigma_2}"'] & K(C_3 \rtimes P_3)/J_3 \arrow[d, "\bar{\sigma_3}"] \\
K(P)_1 \arrow[r, "K(\phi_P)"'] & K(P)_2 \arrow[r, "K(\psi_P)"'] & K(P)_3
\end{tikzcd}$$

Again, commutativity is assured. Define $\bar{K}(\mathfrak{C}) := \overline{K(\mathfrak{C})}$ and let $\bar{K}(\phi)$ be the cat^1-algebra map with $\bar{K}(\phi_C)$ defined as above and $\bar{K}(\phi_P) := K(\phi_P)$.

Lemma 3.2.6. $\bar{K} : \boldsymbol{Cat1}_{\boldsymbol{Gr}} \to \boldsymbol{Cat1}_{\boldsymbol{Alg}_K}$ *is a functor.*

Proof:
The above discussion shows that $\bar{K}(\psi\phi) = \bar{K}(\psi)\bar{K}(\phi)$. It is easy to check that $\bar{K}$ also preserves the trivial morphism on a cat^1-group. □

One important property of the group algebra functor is that it is left adjoint to the unit group functor (see proposition 1.4.3). The claim that the cat^1-group algebra functor is a good generalisation of the group algebra functor will be strengthened if the cat^1-group algebra functor is also left adjoint to something. It is reasonable to suppose that this "something" should be a generalisation of the unit group functor.

Given a cat^1-group algebra, applying the unit group functor levelwise yields a cat^1-group, with structural homomorphisms given by restriction. This construction provides us with a unit cat^1-group functor $U : \mathbf{Cat1}_{\mathbf{Alg}_K} \to \mathbf{Cat1}_{\mathbf{Gr}}$.

It remains to verify the adjointness of U and $\bar{K}$. The following proposition, and its proof, is closely modelled on proposition 1.4.3.

Proposition 3.2.7. *The cat^1-group algebra functor* $\bar{K} : \boldsymbol{Cat1}_{\boldsymbol{Gr}} \to \boldsymbol{Cat1}_{\boldsymbol{Alg}_K}$ *is left adjoint to the unit cat^1-group functor* $U : \boldsymbol{Cat1}_{\boldsymbol{Alg}_K} \to \boldsymbol{Cat1}_{\boldsymbol{Gr}}$.

Proof:
For notational convenience we shall write $\mathbf{C}$ for $\mathbf{Cat1}_{\mathbf{Gr}}$ and $\mathbf{A}$ for $\mathbf{Cat1}_{\mathbf{Alg}_K}$.

Let $\mathfrak{C}$ be a cat^1-group, $\mathcal{A}$ a cat^1-algebra and $f : \mathfrak{C} \to U\mathcal{A}$ a homomorphism of cat^1-groups. Then $f \in \mathbf{C}(\mathfrak{C}, U\mathcal{A})$. For every such f, $\bar{K}(f) \in \mathbf{A}(\bar{K}\mathfrak{C}, \bar{K}U\mathcal{A})$. Clearly $\bar{K}U\mathcal{A} \subseteq \mathcal{A}$, so we may define

$$\theta_{\mathfrak{C},\mathcal{A}} = \mathsf{inc} \circ \bar{K} : \mathbf{C}(\mathfrak{C}, U\mathcal{A}) \to \mathbf{A}(\bar{K}\mathfrak{C}, \bar{K}U\mathcal{A}),$$

where inc is the inclusion. Since both inc and $\bar{K}$ are functors, $\theta_{\mathfrak{C},\mathcal{A}}$ is well-defined for every choice of $\mathfrak{C}$ and $\mathcal{A}$.

Now suppose $\phi : \bar{K}\mathfrak{C} \to \mathcal{A} \in \mathbf{A}(\bar{K}\mathfrak{C}, \mathcal{A})$. Then ϕ is completely determined by the images of the basis elements $\{\mathrm{e}_{c,p} : c \in C, p \in P\}$ and $\{\mathrm{e}_p : p \in P\}$ indexed by the top group and base of $\mathfrak{C}$. Further, for any (c, p) in the top group of $\mathfrak{C}$,

$$1_{\mathcal{A}} = \phi(\mathrm{e}_{c,p}\mathrm{e}_{(c,p)^{-1}}) = \phi(\mathrm{e}_{c,p})\phi(\mathrm{e}_{(c,p)^{-1}}),$$

whence each $\phi(\mathrm{e}_{c,p})$ is in $U\mathcal{A}$. Likewise for $\phi(\mathrm{e}_p)$. Define

$$\Theta_{\mathfrak{C},\mathcal{A}} : \mathbf{A}(\bar{K}\mathfrak{C}, \mathcal{A}) \to \mathbf{C}(\mathfrak{C}, U\mathcal{A})$$

by $\Theta_{\mathfrak{C},\mathcal{A}}(\phi) = \phi \mid_{\mathfrak{C}}$ (we may conveniently regard $\mathrm{e}_{c,p}$ and (c,p) to be the same element).

Now $\Theta_{\mathfrak{C},\mathcal{A}}\theta_{\mathfrak{C},\mathcal{A}}(f) = f$ and $\theta_{\mathfrak{C},\mathcal{A}}\Theta_{\mathfrak{C},\mathcal{A}}(\phi) = \phi$, so

$$\mathbf{C}(\mathfrak{C}, U\mathcal{A}) \cong \mathbf{A}(\bar{K}\mathfrak{C}, \mathcal{A}).$$

It remains to show that this bijection is natural in both $\mathfrak{C}$ and $\mathcal{A}$. This is analogous to the proof of 1.4.3, so we shall omit the details. □

3.3 Modules Over a Cat^1-Group Algebra

Armed with a definition of cat^1-group algebras, we may now begin to consider modules over them. We have seen in 1.4 that representations of a group G with representation space V (a K-vector space) are equivalent to $K(G)$-module structures on V. We would expect a similar result to be true for representations of a cat^1-group and modules over its group algebra.

A module over an algebra A is a vector space V together with an A-action on V. Since we are dealing with cat^1-algebras, it is reasonable to expect a module in this case to be a cat^1-vector space, or equivalently a 2-vector space, endowed with an action of the cat^1-algebra. Here we must be slightly careful, for there are at least two separate definitions of 2-vector spaces which describe quite different objects. The first, used by Kapranov and Voevodsky [42], is not relevant to our purposes but the second, which appears in recent work by Baez and Crans [4], ties in very closely with ideas used elsewhere in this thesis and is the definition we shall use.

Definition 3.3.1. A *2-vector space* is an internal category in $\mathbf{Vect}_K$.

That is, it consists of a vector space, V_0, of objects, a vector space V_1 of arrows, and morphisms (*structural morphisms*) selecting the source and target of each arrow and the identity arrow for each morphism, together with an associative composition which is a linear transformation of vector spaces. Internal functors in $\mathbf{Vect}_K$ are known as *linear functors*, and these are the morphisms in the category $\mathbf{2Vect}_K$ of 2-vector spaces over

K. This is in fact a 2-category, with 2-cells given by *linear natural transformations*, i.e. internal natural transformations in $\mathbf{Vect}_K$.

Furthermore, Baez and Crans show that $\mathbf{2Vect}_K$ is equivalent to the category they call **2Term** of 2-term chain complexes of K-vector spaces, which is none other[4] than our old friend $\mathbf{Ch}_K^{(1)}$. The nature of this equivalence is very reminiscent of the interplay between cat^1-groups and crossed modules discussed in section 1.2.2. Although we studied $\mathbf{Ch}_K^{(1)}$ in the case where K is a field, the definition would work for K merely a commutative ring with identity, although the terminology of vector spaces would in this case be replaced by that of K-modules. In the same way, we can define 2-K-modules to be internal categories in K-**Mod**. For the sake of convenience, we shall continue to speak in terms of vector spaces, but the results can be framed more generally. It should also be remembered that the terms "2-vector space" and "cat^1-vector space" are, to all intents and purposes, interchangable (the same is true for "2-groups"/"cat^1-groups" etc.).

Suppose we have a cat^1-algebra $\mathcal{A}$ and a 2-vector space $\mathcal{V}$, such that both V_0 and V_1 are left modules over their respective levels of $\mathcal{A}$. Hence there is a left action of A_1 on V_1 and a left action of A_0 on V_0 which provide scalar multiplication at both levels. These actions commute with the structural morphisms of $\mathcal{V}$ to define a left action of $\mathcal{A}$ on $\mathcal{V}$.

Definition 3.3.2. A *left $\mathcal{A}$-action* of a cat^1-algebra $\mathcal{A}$ on a 2-vector space $\mathcal{V}$ consists of a left action of A_1 on V_1 and a left action of A_0 on V_0 which commute with the structural morphisms of $\mathcal{V}$.

Such a V is called a *left $\mathcal{A}$-module.*

Of course, right $\mathcal{A}$-modules could be defined similarly if required. Because the cat^1-algebra module construction is essentially the algebra module construction applied levelwise, the correspondence between representations of a cat^1-group and modules over its cat^1-group algebra follows from the group case applied levelwise.

Theorem 3.3.3. *Let $\mathfrak{C}$ be a cat^1-group and $\mathcal{V}$ a 2-vector space equivalent to the chain complex $\delta \in \mathbf{Ch}_K^{(1)}$. Representations $\phi : \mathfrak{C} \to \mathbf{Aut}(\delta)$ are in bijective correspondence with $\overline{K(\mathfrak{C})}$-module structures on $\mathcal{V}$.*

Proof:

Apply theorem 1.4.7 to both levels of ϕ and $\mathbf{Aut}(\delta)$. □

[4]In fact, **2Term** uses chain homotopies as its 2-cells, but we have seen that these are equivalent to homotopies.

Chapter 4

Linear Representations of a Cat^1-Group

In which representations of cat^1-groups are found to exist, both in specific cases and in the general construction of regular representations, and the concept of faithfulness is explored.

Linear representations of cat^1-groups were earlier defined as 2-functors by analogy with the representations of groups. The definition is recalled in section 4.1. Some examples of cat^1-group representations are then introduced in the following sections, firstly for individual cat^1-groups and then via the general construction of a regular representation, which leads to a version of Cayley's theorem in section 4.2. Faithfulness is an important property of group representations, so section 4.3 explores the essence of this property and attempts to find a cat^1-group representation analogue. Finally, a direct description of representations from the point of view of crossed modules is considered in section 4.4.

4.1 Cat^1-Group Representations

The definition of a linear representation of a cat^1-group, $\mathfrak{C}$, was stated as definition 2.4.1 in chapter 2, namely that it is a 2-functor $\phi : \mathfrak{C} \to \mathbf{Ch}_K^{(1)}$. We have also seen that $\phi(\mathfrak{C})$ resides within $\mathbf{Aut}(\delta)$ where $\delta = \phi(\star)$, the image of the unique 0-cell in $\mathfrak{C}$, hence a representation can also be considered as a cat^1-group morphism $\phi : \mathfrak{C} \to \mathbf{Aut}(\delta)$. We are now ready to begin hunting for actual examples of representations.

As a first example, we shall attempt to construct a representation of the simplest cat^1-group that we encountered in examples 1.2.11, namely $\mathfrak{C} = (C_2, I, i, 0, 0)$, where I is the trivial group. We shall start by taking $\mathbb{C}$ as the base field (i.e. $K = \mathbb{C}$). Although we are at liberty to choose any δ as the image of $\star$ for our representation, we shall take one where the dimensions of the source and target are closely related to the orders of the top group and base of $\mathfrak{C}$. In this case, the δ of example 2.2.1 will do nicely. This, recall, is the linear transformation:

$$\begin{aligned} \delta : \mathbb{C}^2 &\to \mathbb{C} \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto x. \end{aligned}$$

We must now seek images for the elements of I and C_2 within, respectively, $\mathbf{Aut}(\delta)_1$ and $\mathbf{Aut}(\delta)_2$. Fortunately, our explorations in section 2.2.1 have furnished us with a reasonable amount of information about the structure of $\mathbf{Aut}(\delta)$. To specify a representation, ϕ with representation complex δ, we must designate images under ϕ for all the elements of $\mathfrak{C}$, and check that the mapping is functorial.

There is only one element of I to assign, and since this is the identity it must map to the identity in $\mathbf{Aut}(\delta)_1$. Hence $\phi(1_I) = \mathrm{id}_\delta$, the chain map consisting of the identity at both levels. For C_2 the situation is barely more complicated, since there are only 2 elements to worry about, one of which is the identity. The other element must also be a homotopy from id_δ to itself, since this is the only chain map available. The homotopies $\mathrm{id}_\delta \Rightarrow \mathrm{id}_\delta$ are precisely the elements of $\pi_2\mathbf{Aut}(\delta)$ and, for this choice of δ, $\pi_2\mathbf{Aut}(\delta)$ is trivial hence $\phi(g)$ is also the identity (where g is the generator of C_2). So the representation is trivial and to find a non-trivial representation we must look elsewhere for δ. In particular, we need $\pi_2\mathbf{Aut}(\delta)$ to be non-trivial so that we can find distinct images for the elements of C_2.

Bearing this last point in mind and looking back over the examples of chapter 2, we observe that $\mathbf{Aut}(0)$ has non-trivial π_2, where 0 is a zero linear transformation. We shall therefore take $\delta = 0 : \mathbb{C} \to \mathbb{C}$ to be the zero map ($z \mapsto 0$ for every $z \in \mathbb{C}$) and try to find suitable images for the elements of $\mathfrak{C}$. As before, the unique element of I must map to the identity chain map $\mathrm{id} \in \mathbf{Aut}(0)_1$. The identity element of C_2 must similarly map to the identity homotopy $(0, \mathrm{id})$. Whereas in the previous case, this was the only homotopy $\mathrm{id} \simeq \mathrm{id}$, we now have such a homotopy for every linear transformation $h' : \mathbb{C} \to \mathbb{C}$. Since the set of linear endomorphisms on $\mathbb{C}$ is isomorphic to $\mathbb{C}$ itself this gives a homotopy $(\alpha, \mathsf{id}) : \mathsf{id} \simeq \mathsf{id}$ for every $\alpha \in \mathbb{C}$. On the face of it, this would seem to give us plenty of choice for $\phi(g)$. However, ϕ must be a functor, so we require $\phi(g)\#_0\phi(g) = \phi(gg) = \phi(1) = (0, \mathsf{id})$. Now suppose $\phi(g) = (\alpha, \mathsf{id})$. Then the chain homotopy component of $\phi(g)\#_0\phi(g)$ is $\alpha + \alpha$, whence we require $2\alpha = 0$. Sadly, when $\alpha \in \mathbb{C}$ this implies that $\alpha = 0$ and functoriality forces us to make $\phi(g) = \phi(1)$ and end up with another trivial representation.

All is not lost, however. If we abandon $\mathbb{C}$ and work instead with $\mathbb{Z}_2$ (which, of course, is a field of characteristic 2), the same algebra enables us to set $\alpha = x$ (the non-zero element of $\mathbb{Z}_2$) and get $\phi(g) = (x, \mathsf{id}) \neq \phi(1) = (0, \mathsf{id})$ with $\phi(g)\#_0\phi(g) = (0, \mathsf{id})$ as required. This gives us a non-trivial representation $\phi : \mathfrak{C} \to \mathbf{Ch}^{(1)}_{\mathbb{Z}_2}$. In fact, this is a faithful representation, according to the definition we shall see in section 4.3.

Now that we have demonstrated the existence of non-trivial representations, we may attempt to find a more systematic method for constructing a representation of a given cat^1-group.

4.2 Regular Representations

The classic existence theorem for non-trivial group representations is Cayley's theorem, which explicitly constructs the *regular representation* of any group. Regular representations are a particularly important class of group representations, in which the elements act by multiplication. The *right regular permutation representation* is defined as $\rho : G^{\mathrm{op}} \to S_{|G|}$ with $\rho(g)(h) := hg$ (it is convenient to blur the distinction between an element $h \in G$ and the corresponding h in the underlying set, on which the permutation acts). The source of ρ is G^{op}, the opposite category to G, rather than G itself, since the natural definition of right multiplication forces ρ to be contravariant[1], i.e. $\rho(g_1g_2)(h) = h(g_1g_2) = (hg_1)g_2 = \rho(g_2)\rho(g_1)(h)$. Similarly, the left regular permutation representation is given by $\lambda : G \to S_{|G|}$ with $\lambda(g)(h) := gh$; λ is covariant.

Although usually stated for permutation representations (e.g. [6, 49]), it is easy to

[1]The choice of left or right regular representation depends largely on the notation employed - usually one or other of them is more natural to use. For the postfix function notation often employed by group theorists, the right representation is covariant (a homomorphism), while for the functional notation more common in other branches of mathematics (including category theory) it is contravariant (an antihomomorphism). Although most authors only deal with representations as homomorphisms, there is no intrinsic reason why antihomomorphisms cannot also be used.

reformulate Cayley's theorem for linear representations (for example, Serre [67] defines the (left) regular representation as a linear representation, though he does not mention Cayley's theorem). To the regular permutation representations of G there correspond *regular linear representations* $\rho : G^{\mathrm{op}} \to GL_{|G|}(K)$, $\lambda : G \to GL_{|G|}(K)$, with $\rho(g)(\mathrm{e}_h) := \mathrm{e}_{hg}$ and $\lambda(g)(\mathrm{e}_h) := \mathrm{e}_{gh}$ (where e_g are the basis vectors in the vector space $K(G)$ outlined in section 1.4, and $\rho(g), \lambda(g)$ are matrices). The right regular representation can be thought of as an action of G on the group algebra $K(G)$ by multiplication on the right, with $\rho(g)(\mathrm{e}_h) = {}^g\mathrm{e}_h = \mathrm{e}_{hg}$. The left regular representation is similarly a G-action on $K(G)$ by left multiplication. Regular representations of a group, then, reconstruct the elements of the group as automorphisms of its group algebra. From now on, we shall concentrate on right regular representations.

Regular representations exist for every group, and they are quite straightforward to define and use. An analogue of Cayley's theorem for cat^1-groups would be a desirable result, since it would enable us to construct a representation in a straightforward manner for any cat^1-group. It is to the search for such a theorem that we now turn our attention.

4.2.1 The Regular Representation of a Cat^1-Group

Suppose we have a cat^1-group $\mathfrak{C} = (C \rtimes P, P, s, t, i)$. By analogy with the group case, we would expect a regular representation to take elements of $\mathfrak{C}$ to automorphisms of its cat^1-group algebra. We have already studied both the cat^1-group algebra construction (section 3.2) and the automorphism cat^1-group of a linear transformation (chapter 2), so we may attempt to combine the two to get a definition of a regular representation.

The cat^1-group algebra of $\mathfrak{C}$, recall, is the cat^1-algebra

$$\overline{K(\mathfrak{C})} := \begin{tikzcd} K(C \rtimes P)/J \arrow[r, shift left, "\bar{\sigma}"] \arrow[r, shift right, "\bar{\tau}"'] & K(P) \arrow[l, bend left, "\bar{\iota}"] \end{tikzcd},$$

where J is the cocycle ideal and factorisation by J introduces the cocycle relations needed to make the kernel conditions work. We have not yet looked directly at the automorphisms of a cat^1-algebra, but we have studied $\mathbf{Aut}(\delta)$, the automorphism cat^1-group of a linear transformation δ. From $\overline{K(\mathfrak{C})}$ we can get a single linear transformation by defining $\delta : \ker\bar{\sigma} \to K(P)$ to be $\bar{\tau}\mid_{\ker\bar{\sigma}}$ (we can also construct δ using simplicial techniques, by observing that $\overline{K(\mathfrak{C})}$ is a heavily truncated simplicial algebra and constructing its Moore complex). We saw in section 3.2 that $\ker\bar{\sigma}$ has a generating set $\{\bar{\mathrm{v}}_{c,p} : c \neq 1\}$, where $\bar{\mathrm{v}}_{c,p} = \bar{\mathrm{e}}_{c,p} - \bar{\mathrm{e}}_{1,p}$.

Any representation of $\mathfrak{C}$ maps elements of P to chain automorphisms in $\mathbf{Aut}(\delta)_1$ and elements of $C \rtimes P$ to homotopies in $\mathbf{Aut}(\delta)_2$ for some representation complex δ. For the regular representation, ρ, we shall use the δ described above, which comes from the cat^1-group algebra. We can informally picture an action of $\mathfrak{C}$ on $\overline{K(\mathfrak{C})}$ by right multiplication. Note that this is a left action, and the elements of $\mathfrak{C}$ appear in the left on the pictures, while they appear on the right in the algebraic notation (hence the term *right* multiplication). In the following diagrams the dotted arrows denote elements of the cat^1-group, while the cells drawn with solid arrows are in its cat^1-group algebra.

A 1-cell in $\mathfrak{C}$ is an element $p \in P$. This can act both on the 1-cells and the 2-cells of $\overline{K(\mathfrak{C})}$. The action of p on a 1-cell is:

$$\begin{tikzcd} \cdot \arrow[r, dotted, "p"] & \cdot \arrow[r, "\mathbf{e}_q"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "\mathbf{e}_{q\#_0 p}"] & \cdot \end{tikzcd} \; . \tag{4.1}$$

The action on a 2-cell is similar:

$$\begin{tikzcd} \cdot \arrow[r, dotted, "p"] & \cdot \arrow[r, bend left=60, "\mathbf{e}_q"] \arrow[r, bend right=60, "\mathbf{e}_{\partial c.q}"'] \arrow[r, phantom, "\Downarrow \bar{\mathbf{e}}_{c,q}"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, bend left=60, "\mathbf{e}_{q\#_0 p}"] \arrow[r, bend right=60, "\mathbf{e}_{\partial c.q\#_0 p}"'] \arrow[r, phantom, "\Downarrow \bar{\mathbf{e}}_{c,q\#_0 p}"] & \cdot \end{tikzcd} \; . \tag{4.2}$$

A 2-cell in $\mathfrak{C}$ is an element $(c, p) \in C \rtimes P$. This acts on 1-cells of $\overline{K(\mathfrak{C})}$ by right multiplication:

$$\begin{tikzcd} \cdot \arrow[r, dotted, bend left=60, "p"] \arrow[r, dotted, bend right=60, "\partial c.p"'] \arrow[r, phantom, "\Downarrow (c,p)"] & \cdot \arrow[r, "\mathbf{e}_q"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, bend left=60, "\mathbf{e}_{q\#_0 p}"] \arrow[r, bend right=60, "\mathbf{e}_{q\#_0 \partial c p} = \mathbf{e}_{\partial({}^q c) q\#_0 p}"'] \arrow[r, phantom, "\Downarrow \bar{\mathbf{e}}_{{}^q c, q\#_0 p}"] & \cdot \end{tikzcd} \; . \tag{4.3}$$

We do not require the action of (c, p) on a 2-cell.

Although these pictures show an action of $\mathfrak{C}$ on $\overline{K(\mathfrak{C})}$, we actually want to define ρ as a map into $\mathbf{Aut}(\delta)$, so we need to do a little work. First of all, for each $p \in P$, $\rho(p) \in \mathbf{Aut}(\delta)_1$ must be a chain automorphism. From (4.1), we define

$$\rho(p)(\mathbf{e}_q) := \mathbf{e}_{q\#_0 p}$$

for the lower level (note that this should strictly be $(\rho(p))_0$ but it will be clear from the input into $\rho(p)$ which level we are working at, so the subscripts can be safely omitted; we shall also drop $\#_0$ for $\mathfrak{C}$ and write multiplication by juxtaposition from now on).

At the top level, (4.2) immediately suggests $\rho(p)(\bar{\mathbf{e}}_{c,q}) = \bar{\mathbf{e}}_{c,qp}$. However, the top level of $\mathbf{Aut}(\delta)$ is $\ker\bar{\sigma}$, which is generated by the elements $\bar{\mathbf{v}}_{c,q}$. Using (4.2) as a guide, $\rho(p)(\bar{\mathbf{v}}_{c,q}) = \rho(p)(\bar{\mathbf{e}}_{c,q} - \bar{\mathbf{e}}_{1,q}) = \bar{\mathbf{e}}_{c,qp} - \bar{\mathbf{e}}_{1,qp} = \bar{\mathbf{v}}_{c,qp}$, so define

$$\rho(p)(\bar{\mathbf{v}}_{c,q}) := \bar{\mathbf{v}}_{c,qp}.$$

It is straightforward to check that $\rho(p)$ is a chain map on δ and that the mapping is contravariantly functorial. So far we have essentially constructed the right regular group representation of P, and as a bonus we also have an action of P on the top level of $\mathbf{Aut}(\delta)$.

For every $(c,p) \in C \rtimes P$, there must be a homotopy $\rho(c,p) \in \mathbf{Aut}(\delta)_2$. Since ρ is to be a functor, it must preserve the source and target of each 2-cell, so we must have $\rho(c,p) : \rho(p) \simeq \rho(\partial cp)$. As a homotopy in $\mathbf{Aut}(\delta)$ we can specify $\rho(c,p)$ by its source and chain homotopy. The source, as we have seen, must be $\rho(p)$. The chain homotopy will be a map $\rho'(c,p) : K(P) \to \ker\bar{\sigma}$. We shall use diagram (4.3) to get a tentative definition for $\rho'(c,p)$ and then check that this is a functor and satisfies the chain homotopy conditions. As with the top level of $\rho(p)$, the diagram suggests $\rho'(c,p)(\mathbf{e}_q) = \bar{\mathbf{e}}_{{}^q c,qp}$ but we actually want a target in $\ker\bar{\sigma}$ so we shall define

$$\rho'(c,p)(\mathbf{e}_q) := \bar{\mathbf{v}}_{{}^q c,qp}.$$

The first thing we shall check is that the chain homotopy conditions are satisfied. At the lower level this check is quite straightforward:

$$\begin{aligned}\delta\rho'(c,p)(\mathbf{e}_q) &= \delta\bar{\mathbf{v}}_{{}^q c,qp} = \delta(\bar{\mathbf{e}}_{{}^q c,qp} - \bar{\mathbf{e}}_{1,qp}) = \mathbf{e}_{\partial({}^q c)qp} - \mathbf{e}_{qp} \\ &= \mathbf{e}_{q\partial cp} - \mathbf{e}_{qp} = \rho(\partial cp)(\mathbf{e}_q) - \rho(p)\mathbf{e}_q = [\rho(\partial cp) - \rho(p)](\mathbf{e}_q).\end{aligned}$$

For the top level, the idea is the same, but the algebra is somewhat more involved. We can start by working in from both ends.

$$\begin{aligned}\rho'(c,p)\delta(\bar{\mathbf{v}}_{d,q}) &= \rho'(c,p)\delta(\bar{\mathbf{e}}_{d,q} - \bar{\mathbf{e}}_{1,q}) = \rho'(c,p)(\mathbf{e}_{\partial dq}) - \rho'(c,p)(\mathbf{e}_q) \\ &= \bar{\mathbf{v}}_{\partial d^q c,\partial dqp} - \bar{\mathbf{v}}_{{}^q c,qp} = \bar{\mathbf{v}}_{d^q c d^{-1},\partial dqp} - \bar{\mathbf{v}}_{{}^q c,qp}\end{aligned} \tag{4.4}$$

and

$$\rho(\partial cp)(\bar{\mathbf{v}}_{d,q}) - \rho(p)(\bar{\mathbf{v}}_{d,q}) = \bar{\mathbf{v}}_{d,q\partial cp} - \bar{\mathbf{v}}_{d,qp}. \tag{4.5}$$

For the chain homotopy condition to be satisfied, we need (4.4) = (4.5). Equivalently, we may show that (4.5) − (4.4) = 0. Now,

$$(4.5) - (4.4) = \bar{\mathbf{v}}_{d,q\partial cp} - \bar{\mathbf{v}}_{d,qp} - \bar{\mathbf{v}}_{d^qcd^{-1},\partial dqp} + \bar{\mathbf{v}}_{^qc,qp}. \tag{4.6}$$

Since the $\bar{\mathbf{v}}_{c,p}$ are in $\ker\bar{\sigma}$ there are relations (given by equation (3.4) of chapter 3) which come from the cocycle relations induced by factoring $K(C \rtimes P)$ by J in the cat^1-group algebra. These relations in $\ker\bar{\sigma}$ will also be called cocycle relations. In particular, the relation

$$\bar{\mathbf{v}}_{^qc,qp} + \bar{\mathbf{v}}_{d,q\partial cp} = \bar{\mathbf{v}}_{d^qc,qp}$$

holds in $\ker\bar{\sigma}$, so we can rewrite the right-hand side of (4.6) as

$$\bar{\mathbf{v}}_{d^qc,qp} - \bar{\mathbf{v}}_{d^qcd^{-1},\partial dqp} - \bar{\mathbf{v}}_{d,qp},$$

which is equal to zero by another cocycle relation. Hence $\rho'(c,p)$ is a chain homotopy as required.

It remains to show that ρ is functorial at the level of homotopies. The most difficult condition to check is that ρ preserves horizontal composition. Since ρ is contravariant on the 1-cells, we expect this composition to be contravariant also. Therefore we require

$$\rho(c^pc', pp') = \rho[(c,p)(c',p')] = \rho(c',p')\#_0\rho(c,p). \tag{4.7}$$

It is clear that both ends of this chain of equations are homotopies with source $\rho(pp')$, so it remains to check that the chain homotopies coincide. We do this by checking their action on an element $\mathbf{e}_q \in K(P)$. Firstly,

$$\rho'(c^pc', pp')(\mathbf{e}_q) = \bar{\mathbf{v}}_{^q(c^pc'),qpp'}.$$

From the formula for the chain homotopy of a horizontal composite given in chapter 1, $\rho(c',p')\#_0\rho(c,p)$ has chain homotopy[2] $\rho'(c',p')\rho(\partial cp)_0 + \rho(p')_1\rho'(c,p)$, with

$$\begin{aligned}\rho'(c',p')\rho(\partial cp)_0 + \rho(p')_1\rho'(c,p)](\mathbf{e}_q) &= \rho'(c',p')(\mathbf{e}_{q\partial cp}) + \rho(p')(\bar{\mathbf{v}}_{^qc,qp}) \\ &= \bar{\mathbf{v}}_{^{q\partial c p}c',q\partial cpp'} + \bar{\mathbf{v}}_{^qc,qpp'}.\end{aligned}$$

[2]Note that here, contrary to our usual practice, we have included the subscripts on the levels of the chain map $\rho(p)$ for the sake of clarity.

Another cocycle relation in $\ker\bar{\sigma}$ gives

$$\bar{\mathbf{v}}_{q\partial c^p c',q\partial cpp'} + \bar{\mathbf{v}}_{^q c,qpp'} = \bar{\mathbf{v}}_{^q(c^p c'),qpp'},$$

so that (4.7) is satisfied.

Finally, let (c,p) and $(c',\partial cp) \in C \rtimes P$, so that the vertical composite is defined $((c',\partial cp)\#_1(c,p) := (c'c,p))$. Then

$$\rho'(c'c,p)(\mathbf{e}_q) = \bar{\mathbf{v}}_{^q c'c,qp}$$

while $\rho(c',\partial cp)\#_1\rho(c,p)$ has chain homotopy $\rho'(c',\partial cp) + \rho'(c,p)$, giving

$$[\rho'(c',\partial cp) + \rho'(c,p)](\mathbf{e}_q) = \bar{\mathbf{v}}_{^q c',q\partial cp} + \bar{\mathbf{v}}_{^q c,qp}.$$

There is another cocycle relation in $\ker\bar{\sigma}$ of the form

$$\bar{\mathbf{v}}_{^q c',q\partial cp} + \bar{\mathbf{v}}_{^q c,qp} = \bar{\mathbf{v}}_{^q c'c,qp},$$

whence

$$\rho[(c',\partial cp)\#_1(c,p)] = \rho(c',\partial cp)\#_1\rho(c,p)$$

as required.

Note that the functor ρ defined above is contravariant in the horizontal direction (at both levels) but covariant in the vertical direction. It should therefore be regarded as a functor $\mathfrak{C}^{\mathrm{op}} \to \mathbf{Ch}_K^{(1)}$, using the convention given in [44] that any 2-category $\mathcal{C}$ has a dual $\mathcal{C}^{\mathrm{op}}$ which reverses only the 1-cells (and hence affects both 1-cell and horizontal 2-cell composition) and another dual $\mathcal{C}^{\mathrm{co}}$ which reverses only the 2-cells (and hence affects vertical 2-cell composition).

This construction can be summarised in the following definition.

Definition 4.2.1. The *right regular representation* of a cat^1-group $\mathfrak{C}$ is the 2-functor $\rho : \mathfrak{C}^{\mathrm{op}} \to \mathbf{Ch}_K^{(1)}$ sending each $p \in P$ to the chain automorphism

$$\rho(p)(\mathbf{e}_q) := \mathbf{e}_{q\#_0 p}, \quad \rho(p)(\bar{\mathbf{v}}_{c,q}) := \bar{\mathbf{v}}_{c,qp}$$

and each $(c,p) \in C \rtimes P$ to the homotopy $\rho(c,p) : \rho(p) \to \rho(\partial cp)$ with chain homotopy

$$\rho'(c,p)(\mathbf{e}_q) := \bar{\mathbf{v}}_{^q c,qp},$$

where all chain automorphisms and homotopies reside in $\mathbf{Aut}(\delta)$ for the linear transformation $\delta := \bar{\tau}\mid_{\ker\bar{\sigma}}$ obtained from the cat^1-group algebra $\overline{K(\mathfrak{C})}$ of $\mathfrak{C}$.

Since the construction may be applied to any cat^1-group, it gives us a cat^1-group version of Cayley's theorem, in terms of linear regular representations.

Theorem 4.2.2 (Cayley). *For any cat^1-group $\mathfrak{C}$, the right regular representation, as defined in 4.2.1, exists.*

□

4.2.2 A Worked Example

In section 4.1 we found an example of a non-trivial representation by trying several possibilities until we found one which worked. Apart from proving that representations exist, Cayley's theorem gives us a recipe for constructing a representation for any given cat^1-group. We can now see how this recipe works in practice by plugging the same example $\mathfrak{C} = (C_2, I, i, 0, 0)$ into the machinery of Cayley's theorem.

As before, we shall start by taking $K = \mathbb{C}$. The first thing we need is δ, which is obtained by first forming the cat^1-group algebra $\overline{K(\mathfrak{C})}$ and then restricting the target map to the source of the kernel. We have already examined $\overline{K(\mathfrak{C})}$ for this particular $\mathfrak{C}$ in section 3.2 and have seen that, over $\mathbb{C}$, it leads to the rather trivial cat^1-algebra having $\mathbb{C}$ as both top group and base, with all maps the identity. Thus $\partial = \mathsf{id}$ and the regular representation is trivial. This shows an important difference from group representation theory, in that for cat^1-groups a regular representation need not be faithful (this particular example fails spectacularly).

If we instead take the base field to be $K = \mathbb{Z}_2$, the situation is quite different. In this case, the cocycle ideal is empty so that no relations are induced in the cat^1-group algebra. In this case $\bar{\sigma}(x, y) = x + y = \bar{\tau}(x, y)$ and $\ker \bar{\sigma} \cong \mathbb{Z}_2$. This gives $\delta : \mathbb{Z}_2 \to \mathbb{Z}_2$ as the zero map. The regular representation becomes $\rho(1) = \mathsf{id}$ on the chain map level, while $\rho(1) = (0, \mathsf{id})$ and $\rho(g) = (g, \mathsf{id})$ on the homotopy level (taking g as the generator of $\mathbb{Z}_2$). In this case, ρ is a faithful representation.

Since we have been throwing the term "faithful representation" around, we should pause to check that we know what it means for cat^1-group representations.

4.3 Faithful Representations

What makes a representation faithful? Is this a desirable property of representations? For groups, a representation is defined to be faithful if every element of the group G

has a distinct image; i.e. the representation is a monomorphism. In categorical terms, this means that the functor $\phi : G \to \mathbf{Vect}_K$ is faithful, so that for $g, h \in G$, we have $\phi(g) = \phi(h) \Rightarrow g = h$. Thus the terminologies of representation theory and category theory coincide for the notion of faithfulness.

The fundamental purpose of a representation is to reconstruct an abstract group in more concrete terms, and faithful representations are particularly useful in this respect since they alone preserve the group structure completely. Let $\phi : G \to GL_n(K)$ be a linear representation of a group G. Then the fundamental homomorphism theorem tells us that

$$G/\ker\phi \cong \phi(G) \leqslant GL_n(K).$$

If ϕ is faithful, $\ker\phi = 1$ and so G itself is isomorphic to a subgroup of the general linear group. If ϕ is not faithful, the kernel is non-trivial and some collapsing occurs. In the extreme case, the trivial representation sends every element of G to the identity of K, and this gives us the trivial group as a representation. While considering group representations, it may finally be noted that the regular representation is always faithful. For, if $\rho(g)(h) := hg$ then

$$\rho(g)h = \rho(g')h \Rightarrow hg = hg' \Rightarrow g = g'.$$

Since a cat^1-group representation is given by a 2-functor, the obvious way to define a faithful representation is as a faithful 2-functor. This leads to the immediate question "what is a faithful 2-functor?". Again, the obvious answer would be a 2-functor ϕ for which $\phi(\alpha) = \phi(\beta) \Rightarrow \alpha = \beta$ for any 2-cells α, β in the source category. This definition would encapsulate the 1-functor idea of faithfulness, identifying 1-cells f and g with the 2-cells 1_f and 1_g.

Strangely, none of the standard references on 2-categories (for example, [7, 52]) seems to give a definition of faithful 2-functors. However, a definition can be pieced together from several sources. In their review of 2-category theory [44], Kelly and Street mention that a 2-category may be viewed as a **Cat**-category (that is, a category enriched over **Cat**), and that this definition determines the meanings of 2-functor and 2-natural transformation. Presumably, then, the definition of a faithful 2-functor should also come from enriched category theory.

Turning to the standard work on enriched category theory, also by Max Kelly [43], a $\mathcal{V}$-functor is defined as follows. Let $\mathcal{A}, \mathcal{B}$ be $\mathcal{V}$-categories, i.e. categories enriched over the monoidal category $\mathcal{V}$. Then the $\mathcal{V}$-functor $T : \mathcal{A} \to \mathcal{B}$ consists of an object function $T : |\mathcal{A}| \to |\mathcal{B}|$ together with maps $T_{A,A'} : \mathcal{A}(A, A') \to \mathcal{B}(TA, TA')$ for each pair of objects $A, A' \in |\mathcal{A}|$. These latter are compatible with composition (itself a $\mathcal{V}$-morphism) and identities. A $\mathcal{V}$-functor is *fully faithful* if each $T_{A,A'}$ is an isomorphism. Implicitly, a $\mathcal{V}$-functor is *faithful* when each $T_{A,A'}$ is a monomorphism; thus, every distinct 1- or 2-cell in $\mathcal{A}(A, A')$ has a distinct image in $\mathcal{B}(TA, TA')$. For $\mathcal{V} = \mathbf{Cat}$, this yields precisely the definition proposed earlier:

Definition 4.3.1. A 2-functor $\phi : \mathcal{A} \to \mathcal{B}$ is *faithful* if, for 2-cells $\alpha, \beta \in \mathcal{A}$,

$$\phi(\alpha) = \phi(b) \Rightarrow \alpha = \beta.$$

Armed with this definition, we may now define faithfulness of representations.

Definition 4.3.2. Let $\mathfrak{C}$ be a cat^1-group. A representation $\phi : \mathfrak{C} \to \mathbf{Ch}_K^{(1)}$ is *faithful* if it is faithful as a 2-functor.

An immediate question is whether the regular representation of section 4.2 is faithful. Since at the 1-cell level this reduces to the regular group representation, it is clear that at this level it must be faithful. However, the examples we considered in 4.2.2 provide a counterexample to the conjecture that the regular representation is always faithful on the homotopy level. Fortunately the same examples also show that faithful representations can exist and that the regular representation can be faithful for the right choice of base field.

4.4 Crossed Module Representations

In section 1.2.1 it was shown that cat^1-groups and crossed modules are equivalent, and both are ways of viewing 2-groups. We have a definition of representations for a cat^1-group, and this may also be considered as a representation of the corresponding crossed module. Therefore we might define a representation of the crossed module $\mathfrak{X}$ to be a representation of $\mathfrak{C}(\mathfrak{X})$ according to definition 2.4.1.

However, a direct definition of crossed module representations would also be useful since it is often more natural and convenient to work with crossed modules than with cat^1-groups. Since a crossed module is not a category, we should not expect this to be a functorial definition, but we might reasonably expect some kind of crossed module map where the target is a crossed module of algebras. An important criterion for a definition of crossed module representation is that it should be equivalent to a representation of the corresponding cat^1-group, as defined above.

One possible way of proceeding with a direct definition of a representation of the crossed module $\mathfrak{X}$ would be to first pass to the associated cat^1-group $\mathfrak{C}(\mathfrak{X})$ (as suggested above) and find a representation for this, whether the regular representation or another. Having obtained this representation, which would give us a mapping into the cat^1-group $\mathbf{Aut}(\delta)$ for our choice of δ, we could then pass back to the associated crossed module of $\mathbf{Aut}(\delta)$. In principle, this bidirectional exchange between crossed modules and cat^1-groups is straightforward. In practice, the notation and the definition of composition in $\mathbf{Aut}(\delta)$ (particularly at the top level) make it quite difficult, and further work is necessary to get a practical working definition of crossed module representations by this method.

Chapter 5

The Category of Representations

In which a category consisting of representations and their morphisms is defined, and properties of this category explored.

When faced with a collection of objects, it is natural for a category theorist to look for morphisms between those objects, in the hope of finding a new category. Therefore, since we are taking a categorical view of representations, an obvious question is whether there are morphisms between representations of a cat^1-group, and whether we can define a category of representations and their morphisms. As usual, the answer for the group case suggests an answer for the cat^1-group case, so we shall begin by reviewing the situation for groups.

5.1 The Group Case

A representation of a group G is a functor $G \to \mathbf{Vect}_K$, so it is natural to define morphisms of representations to be morphisms of functors; that is, natural transformations. This leads to:

Definition 5.1.1. The *category of (K-linear) representations* of a group G is the functor category $\mathbf{Rep}_G^K := (\mathbf{Vect}_K)^G$ whose objects are the functors $G \to \mathbf{Vect}_K$ and whose morphisms are the natural transformations between such functors.

We shall usually suppress explicit mention of K and just write $\mathbf{Rep}_G$ when this is unambiguous.

Suppose ϕ, ψ are K-linear representations of G with representation spaces V and V' respectively. Then a natural transformation $\sigma : \phi \to \psi$ is a linear transformation such that

$$\sigma\phi(g) = \psi(g)\sigma \tag{5.1}$$

for every $g \in G$, i.e. the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\sigma"] \arrow[d, "\phi g"'] & V' \arrow[d, "\psi g"] \\ V \arrow[r, "\sigma"'] & V' \end{tikzcd}$$

Such a σ is sometimes referred to in representation theory as an *intertwining operator* [52], so $\mathbf{Rep}_G$ may be thought of as the category whose objects are the K-linear representations of G and whose arrows are the intertwining operators between such representations.

If σ is a linear isomorphism, then (5.1) may be rewritten as:

$$\psi(g) = \sigma\phi(g)\sigma^{-1}. \tag{5.2}$$

By considering matrices for $\psi(g), \phi(g)$ and σ, it is clear that ψ and ϕ yield equivalent matrix representations. Hence, they are the same linear isomorphism, up to a change of basis.

5.2 Onwards and Upwards

Since a representation of a cat^1-group is a 2-functor, the obvious definition of a morphism of representations is as a 2-natural transformation. Therefore we may expect to get a functor category $(\mathbf{Ch}_K^{(1)})^{\mathfrak{C}}$. However, the 2 dimensional structure of $\mathbf{Ch}_K^{(1)}$ allows for arrows between the 2-natural transformations; these are *modifications*, which provide a 2-category structure for $(\mathbf{Ch}_K^{(1)})^{\mathfrak{C}}$. For the general theory of 2-functor 2-categories, a standard reference (e.g. [7, 52]) may be consulted. Following the definition of the 2-category of representations below, we shall consider the structure of the 2-functor 2-category in this special case.

Definition 5.2.1. The *2-category of (K-linear) representations of a cat^1-group* $\mathfrak{C}$ is the 2-functor 2-category $\mathbf{Rep}_{\mathfrak{C}}^K := (\mathbf{Ch}_K^{(1)})^{\mathfrak{C}}$ having as objects the 2-functors (representations) $\mathfrak{C} \to \mathbf{Ch}_K^{(1)}$, as 1-cells the 2-natural transformations between them, and as 2-cells the modifications between the 2-natural transformations.

As with group representations, we shall usually leave K implicit and simply write $\mathbf{Rep}_{\mathfrak{C}}$.

If $\phi, \psi : \mathfrak{C} \to \mathbf{Ch}_K^{(1)}$ are two representations of $\mathfrak{C}$ with representation complexes δ and δ' respectively, a 2-natural transformation $\sigma : \phi \Rightarrow \psi$ is a map $\sigma : \delta \to \delta'$ such that the diagram

$$\begin{tikzcd}
\delta \arrow[r, bend left, "\phi(p)"] \arrow[r, bend right, "\phi\partial cp"'] \arrow[r, phantom, "\Downarrow \phi(c,p)"] \arrow[d, "\sigma"'] & \delta \arrow[d, "\sigma"] \\
\delta' \arrow[r, bend left, "\psi(p)"] \arrow[r, bend right, "\psi\partial cp"'] \arrow[r, phantom, "\Downarrow \psi(c,p)"] & \delta'
\end{tikzcd}$$

commutes, i.e. $\psi(c,p)\sigma = \sigma\phi(c,p)$. Isomorphisms between representations are 2-natural isomorphisms (i.e. σ is invertible). These function as intertwining operators on the representations, just as natural isomorphisms are intertwining operators for group representations.

In the same way that intertwining operators exist for group representations precisely when there equivalent matrix representations, the 2-natural isomorphisms in $\mathbf{Rep}_{\mathfrak{C}}$ offer a way of defining equivalence for cat^1-group representations, namely:

Definition 5.2.2. Representations ϕ and ψ are said to be *equivalent* if there is an intertwining operator $\sigma \in (\mathbf{Rep}_{\mathfrak{C}})_1$ such that

$$\psi = \sigma\phi\sigma^{-1}.$$

Suppose $\sigma, \tau : \phi \Rightarrow \psi$ are 2-natural transformations (not necessarily invertible). Then a modification $\Xi : \sigma \Rrightarrow \tau$ has a single component[1], also denoted Ξ, which is a

[1]For a general modification, there is one such two-cell for each object in the source category of ϕ and ψ.

2-cell

$$
\begin{tikzcd}
\delta \arrow[r, bend left, "\sigma"] \arrow[r, bend right, "\tau"'] \arrow[r, phantom, "\Downarrow\Xi"] & \delta'
\end{tikzcd}
$$

which commutes with ϕ and ψ in the following way. Suppose (c, p) is a 2-cell in $\mathfrak{C}$, then

$$\Xi \#_0 \phi(c, p) = \psi(c, p) \#_0 \Xi. \tag{5.3}$$

This can be shown in the following diagram, although it is debatable whether this actually clarifies the situation:

$$
\begin{tikzcd}
& \delta' \arrow[rr, "\psi(p)"] \arrow[dd, equal] & & \delta' \arrow[dd, equal] \\
\delta \arrow[ur, "\sigma"] \arrow[rr, "\phi(p)"] \arrow[dd, equal] & & \delta \arrow[ur, "\sigma"'] \arrow[dd, equal] & \\
& \delta' \arrow[rr, "\psi(\partial cp)"] & & \delta' \\
\delta \arrow[ur, "\tau"'] \arrow[rr, "\phi(\partial cp)"'] & & \delta \arrow[ur, "\tau"'] &
\end{tikzcd}
$$

The top and bottom faces show that σ and τ respectively are natural transormations. The remaining faces are all 2-cells. Both the left and the right face are equal to Ξ (a 2-cell in $\mathbf{Rep}_{\mathfrak{C}}$), while the front and back are respectively $\phi(c, p)$ and $\psi(c, p)$. Condition (5.3) is interpreted as the commutativity of front and right with left and back faces.

The modifications in $\mathbf{Rep}_{\mathfrak{C}}$ may be thought of as homotopies between morphisms of representations.

5.3 Properties of $\mathbf{Rep}_{\mathfrak{C}}$

Upon discovering a new category, the category theorist will naturally ask questions about the properties of that category, viz. is it complete or cocomplete? is it additive, or abelian? is it monoidal? if so, is it strict and symmetric? is it cartesian closed? and so on.

The fact that $\mathbf{Rep}_{\mathfrak{C}}$ is a 2-functor 2-category enables us to use the general theory of functor categories to get a quick answer to several of these questions. Borceux [7, 8] gives some results for functor categories which generalise in a straightforward manner to 2-functor 2-categories. Several of these results require the source category (in our case $\mathfrak{C}$) to be a small category. In this situation, several properties of the target category ($\mathbf{Ch}_K^{(1)}$ for us) are inherited by the functor category ($\mathbf{Rep}_{\mathfrak{C}}$).

Proposition 5.3.1. *The category* $\mathbf{Ch}_K^{(1)}$ *is:*

(i) complete,

(ii) cocomplete,

(iii) abelian,

(iv) monoidal.

Proof:

First of all it may be remarked that these are all well-known properties of $\mathbf{Vect}_K$. By standard category theory [52], a category is complete if it has all equalisers and products, and cocomplete if it has all coequalisers and coproducts.

(i) The product of vector spaces is just the usual Cartesian product, so that the product of a family $V_1, V_2, \dots$ of vector spaces is the vector space $V_1 \times V_2 \times \dots$. This construction generalises immediately to $\mathbf{Ch}_K^{(1)}$. Suppose $\delta^C : C_1 \to C_0$ and $\delta^D : D_1 \to D_0$ are in $\mathbf{Ch}_K^{(1)}$. Then the product is $\delta^C \times \delta^D : C_1 \times D_1 \to C_0 \times D_0$. The product of an arbitary collection of chain complexes is defined likewise.

For two linear transformations $f, g : V \to W$, the equaliser is the subspace $\ker(f, g)$ of V given by $\ker(f, g) = \{\mathbf{v} \in V : f(\mathbf{v}) = g(\mathbf{v})\}$. Now suppose δ^C and δ^D are two elements of $\mathbf{Ch}_K^{(1)}$ as before, and $f, g : \delta^C \to \delta^D$ are chain maps. This gives the picture:

$$\begin{tikzcd}
\ker(f_1, g_1) \arrow[r, hook] \arrow[d, "\delta^C|_{\ker(f_1,g_1)}"'] & C_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "g_1"'] \arrow[d, "\delta^C"'] & D_1 \arrow[d, "\delta^D"] \\
\ker(f_0, g_0) \arrow[r, hook] & C_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] & D_0
\end{tikzcd}$$

Suppose $\mathbf{v} \in \ker(f_1, g_1) \subseteq C_1$. Then $f_1(\mathbf{v}) = g_1(\mathbf{v})$. Now $f_0\delta^C(\mathbf{v}) = \delta^D f_1(\mathbf{v}) = \delta^D g_1(\mathbf{v}) = g_0\delta^C(\mathbf{v})$, so $\delta^C(\mathbf{v}) \in \ker(f_0, g_0)$. This gives us an equaliser for $f, g \in \mathbf{Ch}_K^{(1)}$ (it is, of course, unique up to isomorphism).

Since $\mathbf{Ch}_K^{(1)}$ has products and equalisers, it follows that the category is complete.

(ii) Coproducts and coequalisers also exist for $\mathbf{Vect}_K$. These can be extended to coproducts and coequalisers for $\mathbf{Ch}_K^{(1)}$ in the same way as for their duals considered above. Therefore, $\mathbf{Ch}_K^{(1)}$ is also cocomplete.

(iii) $\mathbf{Vect}_K$ is abelian[2] since it has a zero object, products, coproducts, a kernel and a cokernel for every arrow, and every monomorphism is a kernel while every epimorphism is a cokernel.

We have seen above that products and coproducts can be constructed in $\mathbf{Ch}_K^{(1)}$ from those in $\mathbf{Vect}_K$ by doing the construction levelwise. Similarly, a zero object, kernels and cokernels can be constructed in $\mathbf{Ch}_K^{(1)}$. Since a monomorphism in $\mathbf{Ch}_K^{(1)}$ is a chain map for which both levels are monomorphisms in $\mathbf{Vect}_K$, it follows that every monomorphism is a kernel; similarly epimorphisms are always cokernels. Hence $\mathbf{Ch}_K^{(1)}$ is abelian as required.

(iv) $\mathbf{Vect}_K$ is a symmetric monoidal category with the usual tensor product. This construction can be applied levelwise to $\mathbf{Ch}_K^{(1)}$, so that this too is a symmetric monoidal category. $\square$

As remarked above, any functor category inherits the properties of (co)completeness, abelianity and monoidality from its target category, provided this is small (as $\mathbf{Ch}_K^{(1)}$ is). Therefore we get the following result for free:

Corollary 5.3.2. *$\mathbf{Rep}_{\mathfrak{C}}$ is complete, cocomplete, abelian and monoidal.* $\square$

5.4 Degree

Every finite group representation over a field is partially characterised by its degree, which is defined as the dimension of the representation space; equivalent representations must have the same degree, although the converse is false. It is natural to ask whether this concept generalises to representations of cat^1-groups. Before we turn to this question, however, let us consider why the degree is a useful tool in group representation theory, and which aspects of it we may hope to preserve for cat^1-group representations.

[2]We use the definition of abelian category given by Borceux [8] which at first sight is more general than that of MacLane [52] as it does not explicitly require the category to be preadditive. However, Borceux goes on to show that an abelian category by his definition can always be given an additive structure.

Although the degree offers only a partial characterisation of group representations, it can be quite useful. For one thing, as with topological invariants for homeomorphism, it provides a quick check whether two representations could possibly be equivalent. Also, the possible degrees of irreducible representations are constrained by the degree of the group they are representing. We shall return briefly to this fact in section 6.1, once we have seen the definition of an irreducible representation.

It is not immediately clear how the notion of degree could be defined for a cat^1-group representation, since there is no single concept of dimension for the representation complex. However, one possibility is to observe that a representation of a cat^1-group consists of representations of the base and top group (both ordinary group representations) tied together by some compatibility conditions, and therefore to consider the list of the degrees of the top group and base representations.

Definition 5.4.1. Let $\mathfrak{C} = (C \rtimes P, P, s, t, i)$ be a cat^1-group and $\phi : \mathfrak{C} \to \mathbf{Ch}_K^{(1)}$ a representation. The *degree list* of ϕ is defined to be the list $\langle \deg \phi(C \rtimes P), \deg \phi(P)\rangle$.

This is a very tentative definition, and it is not yet clear whether it provides a useful invariant for representations.

Chapter 6

The Structure of Representations

In which we break a representation down into bite-sized chunks, in order to examine its structure.

An important strand of the elementary theory of group representations is the notion of reducibility – breaking up a representation into smaller (and simpler) parts, much as an integer may be factorised. We shall remind ourselves of some of the basic definitions for group theory and then examine how these might be generalised to cat^1-groups. The ultimate goal of this chapter, which we shall not reach, is a cat^1-group analogue of Maschke's theorem. It is not yet completely clear exactly what form such a theorem should take, so here we shall limit ourselves to exploring the background and some tentative results leading to an understanding of the structure of 2-group representations.

6.1 Reducible and Irreducible Group Representations

The basic definitions come from module theory, and may be found in, for example, [24]. We repeat them here for convenience, and to fix our notation. Let R be a commutative unitary ring.

Definition 6.1.1. An R-module M is said to be *reducible* if it has a proper submodule $0 \neq N \neq M$. Otherwise, M is *irreducible*.

Remark: Irreducible modules are also called *simple* modules. The analogy with simple groups is clear. The irreducibility condition imposes strict limits on the maps that can be defined into or out of a module [38]; this leads to the result that the endomorphism ring $\mathrm{End}_R(M)$ of an irreducible module, M, is a division ring (Schur's Lemma).

As the foregoing remark suggests, the definition of irreducibility is actually a rather strong one. Another useful notion, and one which is somewhat easier for a module to satisfy, is indecomposability.

Definition 6.1.2. An R-module M is *decomposable* if it can be written as a direct sum of two non-trivial submodules. Otherwise M is *indecomposable.*

This definition also holds more generally for objects in any preadditive category [61] (Popescu's definition of *irreducible*, however, is quite different from the one adopted here).

Clearly, if M is an irreducible module it has no proper submodules and hence cannot be written as a direct sum of non-trivial submodules. Therefore any irreducible module is automatically indecomposable. However, the converse is false in general: there exist indecomposable modules which have proper submodules.

If a module is decomposable, it can be split up as a direct sum of submodules which may themselves be decomposable or not. A *Krull-Remak-Schmidt decomposition* (or K.R.S. decomposition, for short) of a module M is a direct sum $\bigoplus_i M_i \cong M$ where each M_i is an indecomposable submodule of M. In general, a K.R.S. decomposition need not be unique (unless the module is itself indecomposable, in which case the decomposition is trivial), but the *Krull-Schmidt theorem*[1] states that if both the ascending and descending chain conditions hold for modules over the ring R, then the K.R.S. decomposition of any R-module is unique.

If a module is decomposable it can be expressed as a direct sum of submodules. However, not every submodule is necessarily a direct summand. The following definition captures the stronger condition in which every submodule is a direct summand:

Definition 6.1.3. Let M be an R-module. Then M is *completely reducible* if every submodule $N < M$ is a direct summand of M, i.e. there exists a submodule $N' < M$ such that $M = N \oplus N'$.

Clearly any irreducible module is completely reducible, since it has no proper submodules to fail. A module which is both reducible and completely reducible is decomposable. However, a decomposable module need not be completely reducible.

[1] Sometimes this result is referred to as the Krull-Remak-Schmidt theorem, or some other permutation of the names of mathematicians who contributed to its discovery.

Remark: A module is completely reducible if and only if it is a (direct) sum of irreducible submodules. Because of this, a completely reducible module is also called *semisimple*.

These definitions, of course, make perfect sense when R is not just a ring but an algebra. In particular, we can speak of irreducible or reducible $K(G)$-modules, where $K(G)$ is the group algebra over a field K. Theorem 1.4.7 motivates the following:

Definition 6.1.4. A representation $\phi : G \to GL(V)$ is *reducible* if V is a reducible $K(G)$-module. The other terminology for modules (completely reducible, indecomposable etc.) is similarly extended to cover representations.

Clearly, the submodules of a module are important to the structure of the module. Since $K(G)$-modules correspond to representations of G, it is natural to look for a representation theoretic analogue of $K(G)$-submodules. Suppose $\phi : G \to GL(V)$ is a representation of G, and W is a $K(G)$-submodule of V. Then, since W is itself a $K(G)$-module, $\phi(g)(w) \in W$ for every $w \in W$.

Definition 6.1.5. Let $\phi : G \to GL(V)$ be a representation of G, and let W be a submodule of the $K(G)$-module V. Then $\phi_W : G \to GL(W)$, given by $\phi_W(g) = \phi(g) \mid_W$, is a *subrepresentation* of ϕ.

If ϕ is decomposable, then $V \cong W \oplus W'$ with $W < V$ and $W' \cong V/W$, so ϕ has a subrepresentation ϕ_W. Since W is a proper submodule, $\dim(W) < \dim(V)$, so the degree of the subrepresentation is smaller than the degree of the representation.

Turning to the matrix formulation of ϕ, let $\{v_1, \ldots, v_m v_{m+1}, \ldots, v_n\}$ be a basis for V such that $\{v_1, \ldots, v_m\}$ is a basis for W, and $\{v_{m+1}, \ldots, v_n\}$ is isomorphic to a basis for W' as a K-vector space. Then, for any $v_i \in W$ and $g \in G$,

$$\phi(g)(v_i) \in W$$

and hence the corresponding matrix $\Phi(g)$ must be of the form:

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A, B, C are respectively $m \times m$, $m \times (n-m)$ and $(n-m) \times (n-m)$ matrices, and 0 is the $(n-m) \times m$ zero matrix.

If ϕ is completely reducible, then so is $V = \oplus_{k=1}^{n} W_k$. A basis $\{v_{k_1} \ldots v_{k_{m_k}} : 1 \leqslant k \leqslant n\}$ can be found for V such that $\{v_{k_1} \ldots v_{k_{m_k}}\}$ is a basis for W_k for each k. Then the matrix for ϕ is block diagonal:

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_n \end{pmatrix}$$

Clearly, completely reducible representations are particularly handy, since they can be broken up completely into a series of subrepresentations. Therefore, the following result is not only very nice but also very useful:

Theorem 6.1.6 (Maschke). *Let G be a group and K a field of characteristic p which does not divide the order of G. Then every $K(G)$-module is semisimple.*

□

A sketch of the proof is given in [25]. Due to the correspondence between $K(G)$-modules and rings, Maschke's theorem can be reformulated directly in the language of representations. This is the version given (with proof) in [24], but it can also be derived as an easy corollary of theorem 6.1.6.

Corollary 6.1.7 (Maschke's Theorem - Representation Version). *Let G be a group and $\phi : G \to GL_n(K)$ a representation, where K is a field of characteristic $p \nmid |G|$. Then ϕ is completely reducible.*

□

While we are on the subject of reducible and irreducible group representations, we may now return to briefly consider the relationship between degree and irreducibility alluded to in section 5.4. The key result is the following.

Theorem 6.1.8. *Let G be a finite group and K an algebraically closed field of characteristic zero. Then the degree of any K-linear representation of G is a divisor of $|G|$.*

□

A proof of this theorem in a slightly more general form can be found in [24] (where it appears as theorem 33.7). The version given here is slightly simplified to avoid the need for introducing further definitions which we shall not need.

6.2 Towards a Maschke Theorem in Dimension 2

Given a cat^1-group $\mathfrak{C} = (G, P, i, s, t)$, a cat^1-subgroup of $\mathfrak{C}$ consists of a cat^1-group $\mathfrak{D} = (H, Q, i', s', t')$ where $H \leqslant G$, $Q \leqslant P$ and the structural morphisms i', s', t' are restrictions of the corresponding morphisms in $\mathfrak{C}$. In the same way, subobjects of cat^1-algebras, cat^1-modules and other cat^1 structures can be defined. We can then define reducible/irreducible and completely reducible cat^1-modules (ie. modules over a cat^1-ring) according to whether they have proper submodules or not, etc.

With these definitions in place, the definitions for reducible representations etc. should be immediate from the equivalent modules over the cat^1-group algebra. A version of Maschke's theorem should state that under fairly general conditions, such modules (and hence also representations) should be completely reducible. The exact nature of the conditions needs investigation, since the idea of group order doesn't immediately generalise to cat^1-groups (presumably the condition will be that the order of the base field is coprime to something, although it is not quite clear what).

As we have seen, notions such as reducibility can be interpreted in an abstract, module theoretic way or more concretely in terms of matrices. It should be possible also to derive a matrix formulation for the structure of cat^1-modules and representations. Judging by the pattern of the matrix formulation in chapter 2, we would expect matrices for both levels, with some compatibility conditions linking the two. The exact nature of these conditions will require further investigation.

Chapter 7

Further Directions

In which we survey the results accomplished, and consider possible further directions in which this research could continue.

In a finite amount of time, only a finite amount of work can get done, so this thesis represents in no way a complete account of every possible facet of 2-group representation theory. Instead it may be considered as an introductory survey of a new area of research in higher dimensional algebra. It is my aim in this chapter to consider what has been achieved so far (section 7.1), and to make some suggestions as to possible areas for further research (section 7.2). This latter section is not intended to be an exhaustive list, but only to highlight some of the questions raised by my work so far which I have not had time to explore fully as yet. Some open questions are posed, or hinted at, in earlier parts of this thesis (in particular, the question of degree in section 5.4 and the issue of reducibility in chapter 6), so here we shall concentrate on other possible avenues.

7.1 The Story So Far

In order to begin working on two-dimensional representation theory it was necessary to familiarise myself both with the classical one-dimensional representation theory of groups and with the techniques of higher-dimensional algebra. Indeed it was while exploring the latter, and considering groups as a way to understand 2-groups, that the idea for studying representations of 2-groups developed.

While cat^1-groups and their analogues are by now reasonably well-known and understood, there is a lack of detailed elementary presentations suitable for newcomers to the area who want more than just a definition of the structures. The early part of this thesis, therefore, has a strong emphasis on the exposition of this material as a way of bridging this gap, in the hope that it will be accessible to readers without a strong background in this area of algebra. In this way, the thesis may also be of interest to people whose main concern is not with representation theory. Similarly, the other material of chapter 1 and to a large extent the exposition of cat^1-algebras in chapter 3 is not groundbreaking new material but is difficult to find in a straightforward presentation elsewhere.

As remarked in section 1.3, the category $\mathbf{Ch}_K^{(1)}$ can be recovered from a construction given by Gabriel and Zisman [30]. It also crops up under a different name in a paper by Baez and Crans [4], which appeared sometime after my account of $\mathbf{Ch}_K^{(1)}$ was completed (see also section 3.3). The construction of $\mathbf{Aut}(\delta)$ detailed in chapter 2 is a subcategory of $\mathbf{Ch}_K^{(1)}$ and is what we really need for describing representations. However, the more general discussion of $\mathbf{Ch}_K^{(1)}$ (which was completed before the need for an explicit treatment of $\mathbf{Aut}(\delta)$ was realised) has been retained since this better illustrates the connection with vector spaces, and also generalises easily to the higher-dimensional case $\mathbf{Ch}$ which would be required for a more general representation theory of higher-dimensional n-groups.

The cat^1-group algebra is a new construction. The need to factor out the cocycle condition was unforeseen when the construction was first attempted. This adds an extra degree of subtlety and interest to the construction. The cocycle condition is necessary for the construction of the regular representation to work (in order for the top level of ρ to be a functor and a homotopy), and in fact it was in attempting to verify this construction that the need for the cocycle condition was first brought to light.

While many of the calculations contained within this thesis are at a fairly elementary level, they often contain a number of details to keep track of (and to trip up the unwary mathematician), as well as being susceptible to confusion between the different ways of viewing the various entities involved. Because of this, performing actual calculations with representations of cat^1-groups is still quite tedious, even for reasonably small examples.

The subject of group representation theory is a vast one, and there are many different directions in which to explore any generalisation to higher dimensions. In this thesis we have travelled a number of roads to a greater or lesser extent, but have certainly not reached the end of any of them.

7.2 The Road Ahead

From any crossed module $\mathfrak{X}$, there is an induced crossed module with the kernel as the top group and a coinduced crossed module with the image at the top (both share the same base as $\mathfrak{X}$). These are of the form of some of our first generic examples. Presumably the associated cat^1-groups should also be related to each other in the same way. It might be possible to reconstruct representations of $\mathfrak{X}$ (or $\mathfrak{C}(\mathfrak{X})$) from a knowledge of the induced and coinduced crossed modules on the kernel and image. Such a result would be particularly nice (and useful) if these generic example types have restricted possibilities for their representations. Perhaps the interplay between top and bottom groups imposes severe restrictions on the possible representations. It may even be that every representation can be built from representations of the kernel and image. If there is only a small list of possibilites for each, crossed module representations will be easily characterisable.

It was remarked in section 1.4 that a group algebra has a naturally occurring Hopf algebra structure. It seems plausible, therefore, that there should be some kind of cat^1-Hopf algebra structure on the cat^1-group algebra. Any associative algebra naturally gives rise to a Lie algebra by defining the bracket $[x, y] := xy - yx$ (see, for example, [36]). The same trick in the next dimension works to give a cat^1-Lie-algebra from any cat^1-algebra, including $\overline{K(\mathfrak{C})}$. In the group case, there are adjunctions between the categories of groups, Hopf algebras and Lie algebras (given in the first place by the group algebra functor and its right adjoint the unit group functor, and in the second place by the universal enveloping Hopf algebra of a Lie algebra together with its adjoint [65]). These adjunctions should carry through to the cat^1 case in much the same way as we earlier saw the adjointness of the group algebra/unit group functors (in 1.4) extending to the cat^1-group algebra (in 3.2.3). The details of the interactions between these three structures would be an interesting subject for further study.

As intimated in previous chapters, a desire to facilitate computation of actual examples was one of the main motivating factors behind this work. In order to do this practically, it would be necessary to enlist the help of computer algebra packages. The calculations developed in this thesis, and extensions of them, could be implemented in

GAP or another such package.

A recent paper by Baez and Lauda [5] details the theory of weak and coherent 2-groups. These are 2-groups in which inverses work only up to isomorphism. A nice theorem is that every weak 2-group is equivalent to some coherent 2-group, although these are not in general equivalent to strict 2-groups. Representation theory, which in this thesis is only developed for strict 2-groups, could be extended to cover coherent 2-groups as well. It seems reasonable to expect that in this case a weak version of $\mathbf{Ch}_K^{(1)}$ would be required, and lax functors would have to be considered.

As mentioned in the introduction, the representation theory could be extended both to still higher dimensions and to groupoids with more than one object. Also, permutation representations of 2-groups would be worth exploring. It is likely that, as with the group case, these could be modelled successfully by linear representations, but even so they might be of intrinsic interest. The 2-dimensional analogue of permutations on a set would quite probably be automorphisms of a graph. Some work on automorphisms of graphs has been done by R. Brown and his collaborators in [10, 15].

Appendix A

Matrices for $\mathbf{Ch}_K^{(1)}$

In which the matrix formulation for $\mathbf{Ch}_K^{(1)}$ is dissected with the aid of some examples.

This appendix contains some examples of matrices corresponding to the cells of $\mathbf{Ch}_K^{(1)}$, as discussed in section 1.3.6. For ease of computation, we shall work over $\mathbb{R}$, with the standard basis for each $\mathbb{R}^n$. Some of the following calculations have been given in detail; others are left as an easy (and hopefully pleasant) exercise for the reader.

Let A, B, C be the following 2×3 matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Then A, B, C are afforded by linear transformations $\mathbb{R}^3 \to \mathbb{R}^2$, which are objects in $\mathbf{Ch}_{\mathbb{R}}^{(1)}$. For the sake of convenience, we may slightly abuse language and refer to the matrices themselves as objects of $\mathbf{Ch}_{\mathbb{R}}^{(1)}$ — this presents no problems as long as the bases are fixed. Similarly, the 1- and 2-cells of $\mathbf{Ch}_{\mathbb{R}}^{(1)}$ are pairs of linear transformations, called chain maps and homotopies respectively, satisfying given conditions; the pairs of matrices which arise from these will also be referred to as chain maps and homotopies.

Chain maps between each pair of the objects A, B, C will be pairs consisting of a 3×3 matrix and a 2×2 matrix that satisfy the commutativity condition (1.8). For example, one chain map is $F : A \to B$ with

$$F_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The commutativity condition in this case is $F_0A = BF_1$. It is easy to check that this condition is satisfied and in fact $F_0A = BF_1 = B$. We could replace F_1 by the matrix

$$\tilde{F}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 5 \end{pmatrix},$$

or indeed by any matrix with the same top two rows and an arbitrary bottom row, and the commutativity condition would still be satisfied. Another chain map from A to B is F':

$$F_0' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_1' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This time we get $F_0'A = BF_1' = A$. A final example of a chain map $A \to B$, using slightly more adventurous coefficients, is F'':

$$F_0'' = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad F_1'' = \begin{pmatrix} 0 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here, $F_0''A = BF_1'' = \left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{smallmatrix}\right)$.

Some examples of chain maps $B \to C$ now follow:

$$G_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_0B = CG_1 = B.$$

$$G_0' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad G_1' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_0'B = CG_1' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

$$G_0' = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad G_1' = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_0'B = CG_1' = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 0 \end{pmatrix}.$$

Chain homotopies between these chain maps are linear transformations $\mathbb{R}^2 \to \mathbb{R}^3$, so the corresponding matrices are 3×2. These must satisfy the chain homotopy conditions of (1.9). A chain homotopy between F and F' is given by:

$$H = \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

It is straightforward to check that the homotopy conditions are satistied:

$$BH = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = F_0' - F_0; \quad HA = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = F_1' - F_1.$$

Therefore $H : F \simeq F'$, as claimed. Similarly, the following are chain homotopies between some of the other pairs of chain maps above:

$$H' = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad H' : F' \simeq F''.$$

$$K = \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad H' : G \simeq G'.$$

$$K' = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 0 & 0 \end{pmatrix}; \quad H' : G' \simeq G''.$$

Note that in order to fully specify a homotopy, it is necessary to give both the chain homotopy and its starting point. This is because the same chain homotopy may function for different pairs of chain maps. A trivial example of this is the identity homotopy $\mathrm{id}_f : f \Rightarrow f$ for any chain map f. In every case, the chain homotopy is the zero map (or its corresponding zero matrix).

We now have enough chain maps and homotopies to explore the matrix versions of the compositions. Composition of chain maps corresponds simply to multiplication of matrices at both levels:

$$G_0F_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G_1F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad G_0F_0A = CG_1F_1 = B.$$

$$G_0''F_0'' = \begin{pmatrix} 2 & 6 \\ 6 & 12 \end{pmatrix}, \quad G_1''F_1'' = \begin{pmatrix} 2 & 6 & 0 \\ 6 & 12 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad G_0''F_0''A = CG_1''F_1'' = \begin{pmatrix} 2 & 6 & 0 \\ 6 & 12 & 0 \end{pmatrix}.$$

These are both chain maps $A \to C$.

The vertical composite of two homotopies is found by adding the corresponding matrices. Thus:

$$H' + H = \begin{pmatrix} -1 & 3 \\ 2 & -1 \\ 0 & 0 \end{pmatrix}; \quad F_0'' - F_0 = B(H' + H), F_1'' - F_1 = (H' + H)A.$$

$$K' + K = \begin{pmatrix} 0 & 2 \\ 3 & 3 \\ 0 & 0 \end{pmatrix}; \quad G_0'' - G_0 = C(K' + K), G_1'' - G_1 = (K' + K)B.$$

These give, as we would expect, $(H' + H) : F \simeq F''$ and $(K' + K) : G \simeq G''$.

Prewhiskering corresponds to multiplying the homotopy with the bottom matrix of the chain map, while postwhiskering corresponds to multiplying the top matrix of the chain map with the homotopy. For example:

$$G_1H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$FF_0' = \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Both of these give homotopies, respectively $GF \simeq GF'$ and $GF' \simeq G'F'$. These homotopies may be added together (i.e. composed vertically) to give the horizontal composite of H and K:

$$G_1H + KF_0' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

To verify that this is a homotopy $GF \simeq G'F'$, check the homotopy conditions:

$$G_0'F_0' - G_0F_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = C(G_1H + KF_0');$$

$$G_1'F_1' - G_1F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (G_1H + KF_0')A.$$

The horizontal composite of H' and K' is formed similarly. The reader may like to check that:

$$G_1'H + KF_0'' = \begin{pmatrix} 2 & 5 \\ 5 & 11 \\ 0 & 0 \end{pmatrix}$$

and the homotopy conditions are satisfied.

We may now verify that the interchange law holds for the four basic homotopies that we have so far defined. Composing first vertically, then horizontally:

$$G_1(H' + H) + (K' + K)F_0'' = \begin{pmatrix} 2 & 5 \\ 5 & 12 \\ 0 & 0 \end{pmatrix}.$$

Composing first horizontally, then vertically:

$$(G_1'H' + K'F_0'') + (G_1H + KF_0') = \begin{pmatrix} 2 & 5 \\ 5 & 12 \\ 0 & 0 \end{pmatrix}.$$

These two matrices are the same, just as we would expect.

Previously we took two chain maps and searched for a chain homotopy between them. Working this way round, we cannot guarantee finding a suitable matrix, for not all chain maps between a given pair of objects are homotopic to each other. To prove this claim, consider the following example. Let $F = (F_1, F_0) : A \to B$ be the chain map defined above and $E = (E_1, E_0) : A \to B$ the following chain map:

$$E_0 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 0 \\ 5 & 12 & 13 \end{pmatrix}.$$

Suppose[1] $H \in \mathbb{R}^{3,2}$ is a chain homotopy. Then we must have $AH = E_0 - F_0$, whence H is of the form:

$$H = \begin{pmatrix} 0 & 3 \\ 1 & 1 \\ a & b \end{pmatrix}$$

for some $a, b \in \mathbb{R}$. For any choice of a, b, we get:

$$HB = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 1 & 0 \\ a & b & 0 \end{pmatrix},$$

in which the lower right hand corner entry is 0. However, the lower right hand corner entry of $E_1 - F_1$ is 12, whence it is impossible to find a matrix H satisfying the homotopy conditions. Thus, $F \not\simeq E$.

Conversely, we may start with a chain map F (not neccesarily the one defined above) and a potential chain homotopy H and look for another chain map F' such that $(H, F) : F \simeq F'$ is a homotopy. In this case a suitable matrix may always be found. Suppose $F : A \to B$ with $A, B \in \mathbb{R}^{m,n}$. Put

$$F_1' = HA + F_1, \quad F_0' = BH + F_0.$$

These are both defined since all the matrices involved are the right shape. Rearranging these formulae, the chain homotopy conditions are automatically satisfied. It remains to verify that (F_1', F_0') is a chain map. Now,

$$\begin{aligned} F_0'A &= (BH + F_0)A = (BH)A + F_0A \\ &= B(HA) + BF_1 = B(HA + F_1) \\ &= BF_1', \end{aligned}$$

[1]Recall that $\mathbb{R}^{m,n}$ is notation for the vector space of all $m \times n$ matrices over $\mathbb{R}$.

as required. Hence every $n \times m$ matrix is a chain homotopy for a suitably chosen pair of chain maps.

We turn now to consider the question of inverses for chain maps and homotopies. It is clear from the definition that a chain map will be invertible precisely when the linear transformations at both level are invertible. In matrix terms, this means that $F = (F_1, F_0)$ is invertible if and only if both F_1 and F_0 (which are always square matrices) are non-singular. For homotopies, inverses always exist for vertical composition. Since this composition is simply addition in the matrix formulation, it is easy to see that the inverse for vertical composition of a chain homotopy matrix is the additive inverse of that matrix. For horizontal composition, (H, F) is invertible when both the source F and the target $F + \Delta H$ are invertible.

In our examples, most of the chain maps are not invertible, since the top level matrices include a column of zeros. The exception is the chain map E, for which both E_0 and E_1 are non-singular.

Appendix B

A Dictionary of the Analogy Between Group and Cat^1-Group Representations

In which we summarise some of the principal structures and results for group representation theory and their analogues in the next dimension.

The representation theory of cat^1-groups contained within this thesis has largely been developed by analogy with the classical theory of group representations. It is helpful to summarise the analogy between these two theories. The following table shows some of the principal structures involved in group representation theory and the corresponding structures in the cat^1 case. Throughout we shall assume that K is a field, although a commutative unitary ring could be substituted with minimal changes in the terminology and notation.

	Groups		Cat^1-groups
G	group	$\mathfrak{C}$	cat^1-group
V	K-vector space	$\mathcal{C} : C_1 \xrightarrow{\delta} C_0$	length 1 chain complex (a.k.a. linear transformation) of vector spaces
$\mathbf{Vect}_K$	category of vector spaces	$\mathbf{Ch}_K^{(1)}$	2-category of length 1 chain complexes

$GL(V)$	group of linear automorphisms on V	$\mathbf{Aut}(\delta)$	2-group of chain automorphisms on δ and homotopies between them
$\phi : G \to GL(V)$	linear representation of G (group morphism)	$\phi : \mathfrak{C} \to \mathbf{Aut}(\delta)$	linear representation of $\mathfrak{C}$ (cat^1-group morphism)
V	representation space of ϕ	δ	representation complex of ϕ
$\phi : G \to \mathbf{Vect}_K$	linear representation of G (functor)	$\phi : \mathfrak{C} \to \mathbf{Ch}_K^{(1)}$	linear representation of $\mathfrak{C}$ (2-functor)
$K(G)$	group algebra of G	$\overline{K(\mathfrak{C})}$	cat^1-group algebra of $\mathfrak{C}$

It will be observed that the representation ϕ, both for groups and cat^1-groups, appears twice in the table. It occurs initially as as a morphism of (cat^1-)groups, then as a (2-)functor (interpreting G as a category and $\mathfrak{C}$ as a 2-category, both with a single object and invertible morphisms). Similarly, V and δ both appear twice. The first time is their definition (note that we use δ and $\mathcal{C}$ interchangeably for the length 1 chain complex which is the cat^1 analogue of a vector space), while the second is their appearance in the representation, as the object on which the (cat^1-)group elements are realised as linear isomorphisms (chain isomorphisms/homotopies).

As well as the structures themselves, many of their properties have analogies between the two cases. For example, a faithful representation can be defined as one in which the (2-)functor ϕ is faithful. The degree of a group representation is defined as the dimension of the representation space, while the degree of a cat^1-group representation is related to the degrees of the terms in the representation complex, although this definition is not yet fully worked out.

Many of the results in group representation theory also have more-or-less direct analogues in cat^1-group representation theory. For example, in both cases, Cayley's theorem states that representations exist and provides a construction of right regular representations. The cat^1 analogue of Maschke's theorem, which we have not yet succeeded in establishing completely, will be a structure theorem along the lines of the classical Maschke theorem, and is likely to involve a similar condition on the character of the base field and the orders of the groups involved.

Of course, an analogy can only be taken so far. For example, there is nothing in the construction of the group algebra $K(G)$ to indicate the problem of the cocycles in the cat^1 case, which necessitates factoring out the cocycle ideal to get a cat^1-algebra.

List of Notation

The table below indicates some of the notation employed in this thesis, together with a short description and a reference to the place where the notation is defined (or the first page on which it is used, for things not defined here).

Notation	Description	Page
Cat	Category of (small) categories and functors	3
Set	Category of (small) sets and functions	3
$\mathbf{Vect}_K$	Category of K-vector spaces and linear transformations	3
$GL(V)$	Group of linear automorphisms of a vector space V	8
$GL_n(K)$	General linear group (invertible $n \times n$ matrices over K)	8
$\mathbb{N}$	Natural numbers $\{0, 1, 2, \dots\}$	33
$\mathbb{N}^+$	$\mathbb{N} \setminus \{0\} = \{1, 2 \dots\}$	9
$\mathbb{C}^\times$	$\mathbb{C} \setminus \{0\} \cong GL_1(\mathbb{C})$	9
$\mathfrak{X}$	Crossed module of groups (C, P, ∂, α)	12
$\nu : G \to G/N$	Natural map $g \mapsto g + N$ where $N \lhd G$ and $g \in G$ (also for rings, modules etc.)	13
$Z(G)$	$\{c \in G : cg = gc \ \forall g \in G\}$, centre of G	14
$\mathrm{Aut}(G)$	Group of automorphisms $G \to G$	14
XMod	Category of crossed modules (of groups)	16
$B\mathfrak{X}$	Classifying space of the crossed module $\mathfrak{X}$	16
$\mathfrak{C}$	Cat^1-group (G, P, i, s, t)	16
$\mathbf{Cat1} = \mathbf{Cat1}_{\mathbf{Gr}}$	Category of cat^1-groups	16
$\mathfrak{C}(\mathfrak{X})$	Cat^1-group associated to crossed module $\mathfrak{X}$	18
$\mathfrak{X}(\mathfrak{C})$	Crossed module associated to cat^1-group $\mathfrak{C}$	18
$B\mathfrak{C}$	Classifying space of the cat^1-group $\mathfrak{C}$	19
coker (f, g)	Coequaliser of morphisms f, g in a category	19
$\#_0$	(horizontal) composition across common 0-dimensional boundary	20

We generally adopt the following convention with regard to choice of letters for homomorphisms, linear transformations and matrices. Lowercase Roman letters are used for homomorphisms of groups, lowercase Greek letters for linear transformations and uppercase Greek letters for matrices. Where there is a correspondence between homomorphisms and linear transformations, or between linear transformations and matrices, we shall use the corresponding letters of the different alphabets. Suppose for example $f : G \to H$ is a group homomorphism, then the linear transformation $K(f) : K(G) \to K(H)$ is usually written as ϕ. A matrix afforded by ϕ would be written as Φ, or Φ_V if it is necessary to make the basis V explicit.

Bibliography

[1] M. ALP & C. D. WENSLEY, Enumeration of cat^1-groups of low order, *Internat. J. Algebra Comput.*, **10**(4):407–424, 2000.

[2] J. C. BAEZ, An introduction to n-categories, in *7th Conference on Category Theory and Computer Science* (eds. E. MOGGI & G. ROSOLINI), no. 1290 in Lecture Notes in Computer Science, pp. 1–33, Springer-Verlag, Berlin, 1997.

[3] —, Higher yang-mills theory, 2002. Available as hep-th/0206130.

[4] J. C. BAEZ & A. S. CRANS, Higher-dimensional algebra VI: Lie 2-algebras, 2003. Available as math.QA/0307263.

[5] J. C. BAEZ & A. D. LAUDA, Higher-dimensional algebra V: 2-groups, 2003. Available as math.QA/0307200.

[6] G. BIRKHOFF & S. MACLANE, *Algebra*, MacMillan, 1967.

[7] F. BORCEUX, *Handbook of Categorical Algebra 1*, Cambridge University Press, 1994.

[8] —, *Handbook of Categorical Algebra 2: Categories and Structures*, Cambridge University Press, 1994.

[9] R. BROWN, Higher dimensional group theory, in BROWN & THICKSTUN [17], pp. 215–238.

[10] —, Higher order symmetry of graphs, *Irish Math. Soc. Bull.*, **34**:46–59, 1994.

[11] —, Groupoids and crossed objects in algebraic topology, *Homology Homotopy Appl.*, **1**(1):1–78, 1999.

[12] —, Higher dimensional group theory, 1999. Web article, available at http://www.bangor.ac.uk/ mas010/hdaweb2.htm.

[13] R. Brown & J. Huebschmann, Identities among relations, in Brown & Thickstun [17], pp. 153–202.

[14] R. Brown & J.-L. Loday, Van Kampen theorems for diagrams of spaces, *Topology*, **26**:331–335, 1987.

[15] R. Brown, I. Morris, J. Shrimpton & C. D. Wensley, Graphs of morphisms of graphs, 2000. U.W.B. Mathematics Preprint 99.20.

[16] R. Brown & C. B. Spencer, $\mathcal{G}$-groupoids, crossed modules and the fundamental groupoid of a topological group, *Proc. Konink. Nederl. Akad. Wetensch.*, pp. 296–302, 1976.

[17] R. Brown & T. L. Thickstun (eds.), *Low Dimensional Topology*, no. 48 in LMS Lecture Note Series. Cambridge University Press, 1982.

[18] R. Brown & C. D. Wensley, Computation and homotopical applications of induced crossed modules, *J. Symbolic Comput.*, **35**:59–72, 2003.

[19] M. Bullejos, A. M. Cegarra & J. Duskin, On cat^n-groups and homotopy types, *J. Pure Appl. Algebra*, **86**:135–154, 1993.

[20] M. Burrow, *Representation Theory of Finite Groups*, Academic Paperbacks, 1965.

[21] P. Carrasco, *Complejos Hipercruzados: Cohomología y Extensiones*, Ph.D. thesis, Universidad de Granada, 1987.

[22] D. Conduché, Modules croisés généralisés de longeur 2, *J. Pure Appl. Algebra*, **34**:155–178, 1984.

[23] S. Crans, A tensor product for Gray-categories, *Theory Appl. Categ.*, **5**(2):12–69, 1999.

[24] C. W. Curtis & I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience, 1962.

[25] —, *Methods of Representation Theory*, vol. 1, Wiley, 1981.

[26] L. DORNHOFF, *Group Representation Theory*, Marcel Dekker, New York, 1971. In 2 parts.

[27] G. J. ELLIS, *Crossed Modules and their Higher Dimensional Analogues*, Ph.D. thesis, UCNW, Bangor, 1984.

[28] G. J. ELLIS & R. STEINER, Higher dimensional crossed modules and the homotopy groups of $(n+1)$-ads, *J. Pure Appl. Algebra*, **46**:117–136, 1987.

[29] M. E. FORRESTER-BARKER, Group objects and internal categories, Dec. 2002. Available as math.CT/0212065.

[30] P. GABRIEL & M. ZISMAN, *Calculus of Fractions and Homotopy Theory*, no. 35 in Ergebnisse der Mathematik und ihrer Grenzgebeite, Springer-Verlag, Berlin, 1967.

[31] A. R. GARZON & J. G. MIRANDA, Homotopy theory for (braided) cat-groups, *Cahiers Topologie Géom. Différentielle Catég.*, **38**(2):99–139, 1997.

[32] N. D. GILBERT, The low-dimensional homology of crossed modules, *Homology Homotopy Appl.*, **2**(4):41–50, 2000.

[33] R. GOODMAN & N. R. WALLACH, *Representations and Invariants of the Classical Groups*, Cambridge University Press, 1998.

[34] J. A. GREEN, *Sets and Groups*, Chapman and Hall, 2nd edn., 1988.

[35] T. L. HEATH (ed.), *Euclid - The Thirteen Books of the Elements*, vol. 1, Dover, 2nd edn., 1956.

[36] J. E. HUMPHREYS, *Introduction to Lie Algebras and Representation Theory*, no. 9 in Graduate Texts in Mathematics, Springer-Verlag, New York, 1972.

[37] G. JANELIDZE, T. PORTER & M. SOBRAL, Descent and its higher dimensional analogues, 2000. Unpublished.

[38] J. P. JANS, *Rings and Homology*, Holt, Rinehart and Winston, New York, 1964.

[39] K. H. KAMPS & T. PORTER, *Abstract Homotopy and Simple Homotopy Theory*, World Scientific, Singapore, 1997.

[40] —, 2-groupoid enrichments in homotopy theory and algebra, *K-Theory*, **25**:373–409, 2002.

[41] K. H. KAMPS, D. PUMPLÜN & W. THOLEN (eds.), *Category Theory (Gummersbach, 1981)*, no. 962 in Lecture Notes in Mathematics, Berlin, 1982, Springer-Verlag.

[42] M. KAPRANOV & V. VOEVODSKY, 2-categories and Zamelodchikov tetrahedra equations, *Proc. Symp. Pure Math.*, **56**:177–260, 1994.

[43] G. M. KELLY, *The Basic Concepts of Enriched Category Theory*, no. 64 in LMS Lecture Note Series, Cambridge University Press, 1982.

[44] G. M. KELLY & R. STREET, Review of the elements of 2-categories, in *Sydney Category Seminar (1972/3)*, no. 420 in Lecture Notes in Mathematics, Springer-Verlag, 1974.

[45] D. M. LATCH, The uniqueness of homology for the category of small categories, *J. Pure Appl. Algebra*, **9**:221–237, 1977.

[46] D. M. LATCH, R. W. THOMASON & W. S. WILSON, Simplicial sets from categories, *Math. Z.*, **164**(3):195–214, 1979.

[47] R. LAVENDHOMME & J. ROISIN, Cohomologie non-abélienne de structures algébriques, *J. Algebra*, **67**:355–414, 1980.

[48] W. LEDERMANN, *Introduction To Group Characters*, Cambridge University Press, 1977.

[49] W. LEDERMANN & A. J. WEIR, *Introduction To Group Theory*, Addison Wesley Longman, 2nd edn., 1996.

[50] T. LEINSTER, Topology and higher-dimensional category theory: The rough idea, 2001. Available as math.CT/0106240.

[51] J.-L. LODAY, Spaces with finitely many non-trivial homotopy groups, *J. Pure Appl. Algebra*, **24**:179–202, 1982.

[52] S. MACLANE, *Categories for the Working Mathematician*, no. 5 in Graduate Texts in Mathematics, Springer-Verlag, New York, 2nd edn., 1998.

[53] S. MAJID, *Foundations of Quantum Group Theory*, Cambridge University Press, 1995.

[54] A. A. MILNE, *Winnie the Pooh*, Methuen, 1926.

[55] —, *Domus Anguli Puensis*, Methuen, 1980. 'The House at Pooh Corner' translated into Latin by Brian Staples.

[56] I. MOERDIJK & J.-A. SVENSSON, Algebraic classification of equivariant homotopy 2-types, i, *J. Pure Appl. Algebra*, **89**:187–216, 1993.

[57] A. MUTLU & T. PORTER, Crossed squares and 2-crossed modules, 2002. Available as math.KT/0210462.

[58] K. J. NORRIE, *Crossed Modules and Analogues of Group Theorems*, Ph.D. thesis, King's College, London, 1987.

[59] D. S. PASSMAN, *Infinite Group Rings*, Marcel Dekker, New York, 1971.

[60] G. PÓLYA, *How To Solve It*, Penguin, 2nd edn., 1990.

[61] N. POPESCU, *Abelian Categories with Applications to Rings and Modules*, Academic Press, 1973.

[62] T. PORTER, Internal categories and crossed modules, in KAMPS *et al.* [41], pp. 249–255.

[63] —, Extensions, crossed modules and internal categories in categories of groups with operations, *Proc. Edinburgh Math. Soc.*, **30**:373–381, 1987.

[64] —, Interpretations of Yetter's notion of g-coloring: Simplicial fibre bundles and non-abelian cohomology, *J. Knot Theory Ramifications*, **5**(5):687–720, 1996.

[65] D. QUILLEN, Rational homotopy theory, *Ann. of Math. (2)*, **90**:205–295, 1969.

[66] J. J. ROTMAN, *An Introduction to the Theory of Groups*, Allyn and Bacon, 3rd edn., 1984.

[67] J.-P. SERRE, *Représentations Linéaires des Groupes Finis*, Hermann, Paris, 1971.

[68] N. M. SHAMMU, *Algebraic and Categorical Structure of Categories of Crossed Modules of Algebras*, Ph.D. thesis, UCNW, Bangor, 1992.

[69] I. STEWART, *Galois Theory*, Chapman and Hall, 2nd edn., 1989.

[70] R. STREET, The algebra of oriented simplexes, *J. Pure Appl. Algebra*, **49**:283–335, 1987.

[71] V. TURAEV, Homotopy field theory in dimension 2 and group algebras, 1999. Available as math.QA/9910010.

[72] J. H. C. WHITEHEAD, Combinatorial homotopy II, *Bull. Amer. Math. Soc.*, **55**:453–496, 1949.

12th March 2004

Index