DOCTOR OF PHILOSOPHY

Topological Ideas in Inverse Semigroup Theory

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Award date:
2004

Awarding institution:
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April 14, 2004

Acknowledgements

I would like to thank my supervisor Dr. Mark Lawson, who encouraged, guided and cajoled me towards completing this thesis. I am also grateful to Prof. Tim Porter for many helpful discussions. Thanks also go to my family and friends, without whose support I would not have finished. Finally, thanks are due to EPSRC who paid my fees and provided me with a maintenance grant.

This thesis was typeset using the LaTeX system, diagrams were produced using XY-pic, the bibliography was compiled using BibTeX.

Summary of Thesis

We introduce notions of homotopy and cohomology for ordered groupoids. We show that abstract homotopy theory can be used to define a suitable notion of homotopy equivalence for ordered groupoids (and hence inverse semigroups). As an application of our theory we prove a theorem which is the exact counterpart of the well-known result in topology which states that every continuous function can be factorised into a homotopy equivalence followed by a fibration. We show that this factorisation is isomorphic to the one constructed by Steinberg in his 'Fibration Theorem', originally proved using a generalisation of Tilson's derived category. We show that the cohomology of an ordered groupoid can be defined as the cohomology of a suitable small category, in doing so we generalise the cohomology of inverse semigroups due to Lausch. We define extensions of ordered groupoids and show that these provide an interpretation of low-dimensional cohomology groups.

Contents

Introduction

The aim of this thesis it to apply methods from algebraic topology to inverse semigroups.

An important feature of our theory is that it lives not in the category of inverse semigroups, but in the category of ordered groupoids. A formal definition of an ordered groupoid is given in Chapter 3, but roughly it is a groupoid in the sense of category theory equipped with an order on the set of morphisms. It is well-known that the category of inverse semigroups is isomorphic to a subcategory of the category of ordered groupoids (the construction is given in Section 3.2). The existence of an isomorphism may suggest that we should work in the category of inverse semigroups rather than ordered groupoids, since the former are better known and easier to handle than the latter. However, in [15], Lawson argues persuasively that many constructions involving inverse semigroups can best be carried out by working in the larger category of ordered groupoids. Thus this thesis is an application of the 'ordered groupoid approach' to inverse semigroups.

In Part I of this thesis we give the necessary background material from inverse semigroup theory and category theory. In Chapter 1 we quote the necessary definitions and results concerning inverse semigroups. Chapter 2 is devoted to background material on category theory. In Chapter 3, we examine the category of ordered groupoids and give details of the relationship between inverse semigroups and ordered groupoids.

In Part II of this thesis we set up the framework needed to define the homotopy theory of ordered groupoids. To show that our theory has some teeth we are easily able to reprove Steinberg's Fibration Theorem [28] using only ideas from homotopy theory. Two observations served as motivation for this part of the thesis:

- Steinberg's Fibration Theorem, which states that every ordered functor between ordered groupoids factorises into an enlargement followed by an ordered star surjective functor. This is strongly reminiscent of the result in topology which states that every continuous function can be factorised into a homotopy equivalence followed by a fibration (see Theorem 2.8.9 of [27]).

- Philip Higgins pioneered the idea of using groupoids in topology, and of interpreting groupoid-theoretic ideas in topological terms. An account of his work can be found in his book [7]. In particular, in Chapter 6, he introduces the idea of homotopy equivalence for groupoids, although this is nothing other than natural equivalence. Higgins approach was developed by Brown [4].

Putting these observations together, we show that Steinberg's Fibration Theorem is a consequence of the fact that the category of ordered groupoids can be endowed with a cocylinder in such a way that we can do homotopy.

In Chapter 4, we describe the relevant homotopy theory in its original topological setting. In Chapter 5, we outline the necessary homotopy theory of categories; we describe how to define a cylinder and a cocylinder on a category, and examine the cubical conditions a category must satisfy so that a sensible notion of homotopy exists. In Chapter 6, we develop the homotopy theory of ordered groupoids; we examine the notions of homotopy equivalence and fibrations of ordered groupoid, as an application we deduce Steinberg's Fibration Theorem.

The cohomology of inverse semigroups was introduced by Lausch [11], who used it to classify extensions of inverse semigroups. In [19], Loganathan showed that the cohomology of an inverse semigroup can be derived as the cohomology of a suitable small category. Independently, Renault [25] generalised the cohomology of groupoids to inverse semigroups, in doing so he obtained a cohomology different to that of Lausch and Loganathan.

In Part III of this thesis we develop a cohomology of ordered groupoids which generalises that of Lausch and Loganathan. Following Loganathan's approach, we show that we can define the cohomology of an ordered groupoid G to be the the cohomology of a small category $C(G)$. We introduce extensions of ordered groupoids and show that these may be used to characterise first and second cohomology groups. Finally, we generalise Renault's cohomology of inverse semigroups to ordered groupoids. We also generalise Renaults extensions of inverse semigroups and show that these are special types of the extensions already constructed. In this way we show that Renault's second cohomology groups are subgroups of those of Lausch.

Part III is divided into five chapters. In Chapter 7, we outline the cohomology of abelian categories that we will use. In Chapter 8 we show how to calculate the cohomology of small categories. In Chapter 9 we show that the cohomology of an ordered groupoid G can be defined as the cohomology of the small category $C(G)$. In Chapter 10, we define extensions of ordered groupoids. We define derivations and factor sets for ordered groupoids and show that these classify extensions, we then show that low dimensional cohomology groups can be interpreted in terms of extensions. In Chapter 11 we examine Renault's cohomology in an ordered groupoid setting, we show that his extensions are special extentions having an order-preserving transversal, in this way we show that Renault's second cohomology groups are subgroups of the cohomology groups obtained by Lausch.

Part I

Background on inverse semigroups, categories and groupoids

Chapter 1

Inverse semigroup theory

We begin by giving some basic definitions and results from inverse semigroup theory. Our principal sources are books by Howie [8], Lawson [15] and Petrich [23] to which we refer the reader for further details.

1.1 Inverse semigroups

A set S with a binary operation which we denote by concatenation is a *semigroup* if the operation is associative; that is if $(xy)z = x(yz)$, for all $x, y, z \in S$. An element $e \in S$ is called an *idempotent* if $e \cdot e = e$. We denote the set of idempotents by $E(S)$. If a semigroup S contains an element 1 such that $1x = x = x1$ for all $x \in S$, then S is called a *monoid.* A monoid S is a *group* if for all $x \in S$, there is an element $x^{-1} \in S$ such that $x^{-1}x = 1 = xx^{-1}$.

A subset T of a semigroup S is said to be *closed* under multiplication if $a, b \in T$ implies that $ab \in T$. A non-empty subset T of a semigroup T that is closed under multiplication is called a *subsemigroup* of S. If S is a monoid, then T is a *submonoid* of S if it contains the identity element of S. A subsemigroup of S which is a group with respect to the multiplication inherited from S is called a *subgroup* of S.

A function $\theta : S \longrightarrow T$, where S and T are semigroups, is called a *semigroup homomorphism* if $\theta(xy) = \theta(x)\theta(y)$, for all $x, y \in S$. If S and T are monoids, then θ is a *monoid homomorphism* if it satisfies the additional condition $\theta(1) = 1$. A homomorphism is called an *isomorphism* if the function θ is bijective. In this case, we say that S and T are *isomorphic.*

Wagner [29] first defined inverse semigroups (under the label of 'generalised groups') as follows:

Definition A semigroup S is an *inverse semigroup,* if

(IS1) S is *regular*, this means to each $s \in S$ there is an element $t \in S$, called an *inverse*, satisfying $s = sts$ and $t = tst$.

(IS2) The idempotents of S commute.

Shortly afterwards, Liber [18] showed that Wagner's definition is equivalent to the requirement of uniqueness of inverses. We refer the reader to Lawson [15] for proof of this result.

Theorem 1.1 *Let S be a regular semigroup, then the idempotents of S commute if, and only if, every element of S has a unique inverse.*

■

A subset of an inverse semigroup S is called an *inverse subsemigroup* of S if it is an inverse semigroup with respect to the multiplication inherited from S.

The relationship between groups and inverse semigroups is given by the following standard result.

Proposition 1.2 *Groups are precisely the inverse semigroups with exactly one idempotent.*

■

Let S be an inverse semigroup. If S has an identity which we wish to distinguish, then we say that S is an *inverse monoid.* If it has an element 0 such that $s0 = 0 = 0s$, for all $s \in S$, then we say that S is a *semigroup with zero.* Every (inverse) semigroup may be converted into an (inverse) monoid or an (inverse) semigroup with zero by adjoining an identity or a zero, as follows. Define $S^1 = S \cup \{1\}$ and extend the product to S^1 by defining $1s = s = s1$, for all $s \in S$, and $11 = 1$. Then S^1 is a monoid. Similarly, we may form a semigroup with zero $S^0 = S \cup \{0\}$, with extended product given by $0s = 0 = s0$, and $00 = 0$.

We now list a few properties of inverses in semigroups, for proofs see Proposition 1.4.1 of [15].

Proposition 1.3 *Let S be an inverse semigroup.*

(i) *For any $s \in S$, both $s^{-1}s$ and ss^{-1} are idempotents.*

(ii) *$(s^{-1})^{-1} = s$, for all $s \in S$.*

(iii) *For any $s \in S$ and $e \in E(S)$, $s^{-1}es$ is an idempotent.*

(iv) *If* $e \in E(S)$, *then* $e^{-1} = e$.

(v) $(s_1 \dots s_n)^{-1} = s_n^{-1} \dots s_1^{-1}$, *for all* $s_1, \dots, s_n \in S$.

■

Proposition 1.4 shows that homomorphisms between inverse semigroups are simply semigroup homomorphisms — we do not require any extra conditions. Proving these results is straightforward, but details can be found in [15].

Proposition 1.4 *Let* S *and* T *be inverse semigroups, and* $\theta : S \longrightarrow T$ *a semigroup homomorphism. Then*

(i) $\theta(s^{-1}) = \theta(s)^{-1}$, *for all* $s \in S$.

(ii) *If* $e \in E(S)$, *then* $\theta(e) \in E(T)$.

(iii) *If* $s \in S$ *and* $\theta(s)$ *is an idempotent, then there exists* $e \in E(S)$ *such that* $\theta(e) = \theta(s)$.

(iv) $\mathrm{Im}(\theta)$ *is an inverse subsemigroup of* T.

(v) *If* U *is an inverse subsemigroup of* T, *then* $\theta^{-1}(U)$ *is an inverse subsemigroup of* S.

■

We conclude this section by describing an example of an inverse semigroup.

Let G be a group and J a non-empty set. Define

$$B(G, J) = (J \times G \times J) \cup \{0\}.$$

Define a partial product on $B(J, G)$ as follows:

$$(i, g, j)(k, h, l) = (i, gh, l)$$

if $j = k$ and all other product are equal to 0. It is easy to show that $B(G, J)$ is an inverse semigroup. Note that

$$E(B(G, J)) = \{(i, 1, i) \mid i \in J\} \cup \{0\},$$

where 1 is the idempotent in G. The semigroups $B(G, J)$ are called *Brandt semigroups.* When J has cardinality n, we write $B(G, n)$ instead of $B(G, J)$, and when the group is trivial, we write $B_n = B(\{1\}, n)$.

1.2 The natural partial order

In this section we show that every inverse semigroup has an order structure. We begin by defining posets.

Let X be a set and $\leqslant$ a binary relation on X satisfying the following conditions:

1. The operation is reflexive; that is $a \leqslant a$, for all $a \in X$.

2. The operation is anti-symmetric; that is $a \leqslant b$ and $b \leqslant a$ implies $a = b$, for all $a, b \in X$.

3. The operation is transitive; that is $a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$, for all $a, b, c \in X$.

Then X is called a *partially ordered set* or *poset*, and $\leqslant$ is called a *partial order* on X. A partial order with the extra property that $a \leqslant b$ or $b \leqslant a$ for all a and b in X is called a *total order* on X. A poset is said to be *totally unordered* if the partial order is equality.

Any subset of a poset will be regarded as a poset with the induced order. If X is a partially ordered set and S is a subset of X such that, for all $x \in X$ and $s \in S$, $x \leqslant s$ implies that $x \in S$. Then we say that S is an *order ideal* of X.

Let X be a poset and $x, y \in X$. If $z \in X$ with $z \leqslant x$ and $z \leqslant y$, then z is said to be a *lower bound* of x and y. If the greatest lower bound of two elements exists, we denote it by $x \wedge y$ and call this element the *meet* of x and y. Similarly if $x \leqslant z$ and $y \leqslant z$, then z is an *upper bound* of x and y. We denote the least upper bound of x and y by $x \vee y$ (if it exists) and call it the *join* of x and y. A *meet semilattice* is a poset in which every pair of elements has a meet. A *join semilattice* is a poset in which every pair of elements has a join. A poset which is both a meet semilattice and a join semilattice is called a *lattice.*

Let X and Y be partially ordered sets. A function $f : X \longrightarrow Y$ is *order-preserving* if $a \leqslant b$ implies $f(a) \leqslant f(b)$, for all $a, b \in X$. A bijection of X onto Y is an *order isomorphism* if both f and f^{-1} are order-preserving.

We can define a partial order on every inverse semigroup S. Let $s, t \in S$. Define

$$s \leqslant t \quad \Longleftrightarrow \quad s = te$$

for some idempotent e. The following result is proved in [15], Proposition 1.4.7 and Proposition 1.4.8.

Proposition 1.5 *Let S be an inverse semigroup.*

(i) *The relation $\leqslant$ is a partial order on S.*

(ii) *For all $e, f \in E(S)$, $e \leqslant f$ if, and only if, $e = ef = fe$.*

(iii) *If $s \leqslant t$ and $u \leqslant v$, then $su \leqslant tv$.*

(iv) *If* $s \leqslant t$, *then* $s^{-1}s \leqslant t^{-1}t$ *and* $ss^{-1} \leqslant tt^{-1}$.

(v) $E(S)$ *is an order ideal of* S.

(vi) $E(S)$ *is a meet semilattice.*

■

The order defined above is called the *natural partial order* on S. The next result gives some different characterisations of the natural partial order. In particular the side on which the idempotent is written is irrelevant. Proof is given in [15] Lemma 1.4.6.

Lemma 1.6 *Let* S *be an inverse semigroup. The following are equivalent*

(i) $s \leqslant t$.

(ii) $s = ft$, *for some* $f \in E(S)$.

(iii) $s^{-1} \leqslant t^{-1}$.

(iv) $s = ss^{-1}t$.

(v) $s = ts^{-1}s$.

■

Chapter 2

Category Theory

In this section we outline the category theory that will be used throughout. Most of the theory that we require is given here, for further information we refer the reader to Mac Lane's book [21].

2.1 Categories

A 'category' comprises of 'objects' and 'morphisms' between objects. Morphisms can be composed and composition is associative. Furthermore, every object possesses an identity morphism. The following formal definition is taken from [22].

Definition Let $\mathbf{C}$ be a class of objects $A, B, C, \ldots$ together with two functions, as follows:

(i) A function which assigns to each pair of objects (A, B) of $\mathbf{C}$ a set $\hom_{\mathbf{C}}(A, B)$. An element f of $\hom_{\mathbf{C}}(A, B)$ is called a *morphism* of $\mathbf{C}$, with *domain* A and *range* B, we write $f : A \longrightarrow B$.

(ii) A function assigning to each triple of objects (A, B, C) of objects of $\mathbf{C}$ a function

$$\hom_{\mathbf{C}}(B, C) \times \hom_{\mathbf{C}}(A, B) \longrightarrow \hom_{\mathbf{C}}(A, C).$$

For morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$, this function is written as $(g, f) \longmapsto g \circ f$, and the morphism $g \circ f : A \longrightarrow C$ is called the *composite* of f and g.

$\mathbf{C}$ is called a *category* if the following axioms hold:

Associativity: If $f \in \hom_{\mathbf{C}}(A, B)$, $g \in \hom_{\mathbf{C}}(B, C)$ and $h \in \hom_{\mathbf{C}}(C, D)$ are morphisms in $\mathbf{C}$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Identity: For every object A of $\mathbf{C}$ there is a morphism $\mathrm{Id}_A : A \longrightarrow A$ called the *identity morphism* such that

$$\begin{aligned} f \circ \mathrm{Id}_A &= f \qquad \text{for any object } B \text{ and } f \in \hom_{\mathbf{C}}(A, B) \\ \mathrm{Id}_A \circ g &= g \qquad \text{for any object } C \text{ and } g \in \hom_{\mathbf{C}}(C, A). \end{aligned}$$

If there is no risk of confusion, we write $\hom(A, B)$ rather that $\hom_{\mathbf{C}}(A, B)$. We also denote composition by concatenation.

A category is *small* if the class of its objects is a set. This is a subtle distinction and we refer the reader to [21] for explanation. We shall mainly be concerned with small categories.

Examples

1. A category is *discrete* when every morphism is an identity. A discrete category is determined by the set of its objects, alternatively, given a set one can form a discrete category by attaching an identity morphism to each element. Thus, discrete categories are sets. All sets together with all functions form a category, denoted **Set**.

2. All (inverse) semigroups together with all homomorphisms of such form a category.

3. A monoid is a category with one object. All monoids together with all monoid homomorphisms form a category.

4. A group is a category with one object, in which every morphism has a (two-sided) inverse under composition. All groups together with all homomorphisms form a category, denoted **Grp**.

Let $\mathbf{C}$ and $\mathbf{D}$ be categories, we say that $\mathbf{D}$ is a *subcategory* of $\mathbf{C}$ if

(SC1) Each object of $\mathbf{D}$ is an object of $\mathbf{C}$.

(SC2) For all objects A and B in $\mathbf{D}$, $\hom_{\mathbf{D}}(A, B) \subseteq \hom_{\mathbf{C}}(A, B)$.

(SC3) Composition of morphisms in $\mathbf{D}$ is the same as that for $\mathbf{C}$.

(SC4) For each object A of $\mathbf{D}$, the identity in $\hom_{\mathbf{D}}(A, A)$ is the identity in $\hom_{\mathbf{C}}(A, A)$.

Given any category $\mathbf{C}$, there is a category $\mathbf{C}^{\mathrm{op}}$, called the *opposite category* obtained by reversing all the morphisms of $\mathbf{C}$, thus $\mathbf{C}^{\mathrm{op}}$ is defined by:

- $\mathbf{C}^{\mathrm{op}}$ has the same objects as $\mathbf{C}$.

- For each morphism $f : A \longrightarrow B$ of $\mathbf{C}$, there is a corresponding morphism $f^{\mathrm{op}} : B \longrightarrow A$ of $\mathbf{C}^{\mathrm{op}}$.
- The composite $f^{\mathrm{op}}g^{\mathrm{op}} = (gf)^{\mathrm{op}}$ is defined in $\mathbf{C}^{\mathrm{op}}$ exactly when the composite gf is defined in $\mathbf{C}$.

If a concept, definition or result involves purely categorical conditions and methods, then there is a dual concept obtained by reversing all morphisms, with valid dual results. For a formal statement of the duality principle see Section II.2 of MacLane [21].

Since the objects of a category correspond exactly to its identity morphisms, it is possible to dispense with the objects and deal only with the morphisms. The category $\mathbf{C}$ is then thought of as a set equipped with a partial binary operation. From this 'arrows-only' viewpoint a category is defined as follows:

Definition Let $\mathbf{C}$ be a set equipped with a partial binary operation, denoted by concatenation. If $x, y \in \mathbf{C}$ and the product xy is defined we write $\exists xy$. An element $e \in \mathbf{C}$ is called an *identity* if $\exists ex$ implies $ex = x$ and $\exists xe$ implies $xe = x$. $\mathbf{C}$ is a *category* if the following axioms hold:

(C1) $x(yz)$ exists if, and only if, $(xy)z$ exists, in which case they are equal.

(C2) $x(yz)$ exists if, and only if, xy and yz exist.

(C3) For each $x \in \mathbf{C}$ there exist identities e and f such that $\exists xe$ and $\exists fx$.

From axiom (C3), it follows that the identities e and f are uniquely determined by x. We write $e = \mathbf{d}(x)$ and call it the *domain* identity, and $f = \mathbf{r}(x)$, the *range* identity. Note that $\exists xy$ if, and only if, $\mathbf{d}(x) = \mathbf{r}(y)$. The set of identities of $\mathbf{C}$ is denoted $\mathbf{C}_o$, observe that $\mathbf{C}_o$ is a discrete subcategory of $\mathbf{C}$.

We do not use either definition of a category exclusively, instead at any time we will use whichever definition seems to suit our purpose best. The first definition is most appropriate when we are dealing with structures and the morphisms between them, whereas the arrows-only approach is best when we wish to regard categories as algebraic structures in their own right, generalising monoids.

We now define a few special types of objects and morphisms in categories. A morphism $m : A \longrightarrow B$ in $\mathbf{C}$ is *monic* if given any two morphisms $f, g : C \longrightarrow A$ as shown below

$$\begin{tikzcd} C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "m"] & B \end{tikzcd}$$

then $mf = mg$ implies $f = g$. In the category **Set** monics are precisely the injective functions. A morphism $e : A \longrightarrow B$ in $\mathbf{C}$ is *epi* if given any two morphisms $f, g : B \longrightarrow D$

as shown below

$$\begin{tikzcd} A \arrow[r, "e"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D \end{tikzcd}$$

then $fe = ge$ implies $f = g$. In the category **Set** epis are precisely surjections. If $g : A \longrightarrow B$ and $h : B \longrightarrow A$ are morphisms in a category $\mathbf{C}$ such that $hg = \mathrm{Id}_A$, then h is called a *split epimorphism* and g is called a *split monomorphism.* Morphisms f with the property that $f^2 = f$ are called *idempotents.*

An object T in $\mathbf{C}$ is a *weak terminal object* if for every object A in $\mathbf{C}$ there is at least one morphism $A \longrightarrow T$, if this morphism is unique, then T is said to be *terminal.* An object S is a *weak initial object* in $\mathbf{C}$ if for every object A, there is a morphism $S \longrightarrow A$, if this morphism is unique then S is *initial.* A *zero object* in $\mathbf{C}$ is an object which is both initial and terminal. If $\mathbf{C}$ has a zero object, then for any objects A and B in $\mathbf{C}$ there are unique morphisms $f : A \longrightarrow Z$ and $g : Z \longrightarrow B$, the composite gf is called the *zero morphism* from A to B, and is written 0^A_B or 0. Any composite with a zero morphism is itself a zero morphism.

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[rr, bend left, "0"] & Z \arrow[r, "g"'] & B \end{tikzcd}$$

If the category $\mathbf{C}$ has a zero object then a *kernel* of a morphism $f : B \longrightarrow C$ is a morphism $k : A \longrightarrow B$ such that $fk = 0$ and every morphism $h : A' \longrightarrow B$ with $fh = 0$ factors uniquely through k, as shown in the commutative diagram below

$$\begin{tikzcd} A \arrow[dr, "k"] \arrow[drr, bend left, "0"] & & \\ & B \arrow[r, "f"] & C \\ A' \arrow[uu, dotted, "\exists! h'"] \arrow[ur, "h"] \arrow[urr, bend right, "0"'] & & \end{tikzcd}$$

Dually, a *cokernel* of f is a morphism $i : B \longrightarrow D$ which is universal with respect to having $if = 0$. It is easy to see that every kernel is monic and every cokernel is epi.

2.2 Functors

Let $\mathbf{C}$ and $\mathbf{C}'$ be categories. A *functor* $\mathcal{T} : \mathbf{C} \longrightarrow \mathbf{C}'$ is a pair of functions:

- An *object function* which assigns to each object A of $\mathbf{C}$ an object $\mathcal{T}(A)$ of $\mathbf{C}'$.

- A *mapping function* which assigns to each morphism $f : A \longrightarrow B$ of $\mathbf{C}$, a morphism $\mathcal{T}(f) : \mathcal{T}(A) \longrightarrow \mathcal{T}(B)$ of $\mathbf{C}'$.

These functions must satisfy

$$\mathcal{T}(\mathrm{Id}_A) = \mathrm{Id}_{\mathcal{T}(A)} \qquad \text{for each identity morphism } \mathrm{Id}_A \text{ in } \mathbf{C},$$
$$\mathcal{T}(gf) = \mathcal{T}(g)\mathcal{T}(f) \qquad \text{whenever the composite } gf \text{ is defined in } \mathbf{C}.$$

It is easy to show that from the arrows-only viewpoint, functors are defined as follows: a function $\mathcal{T} : \mathbf{C} \longrightarrow \mathbf{D}$ between categories is a *functor* if

$$\mathcal{T}(\mathbf{d}(x)) = \mathbf{d}(\mathcal{T}(x)) \qquad \text{and} \qquad \mathcal{T}(\mathbf{r}(x)) = \mathbf{r}(\mathcal{T}(x))$$

for all $x \in \mathbf{C}$, and $\mathcal{T}(xy) = \mathcal{T}(x)\mathcal{T}(y)$ for all $x, y \in \mathbf{C}$ for which $\exists\ xy$.

All (small) categories together all functors between them form a category, denoted **Cat**.

A functor $\mathcal{T} : \mathbf{C} \longrightarrow \mathbf{C}'$ is an *isomorphism of categories* if both the object and mapping functions are bijective. Equivalently, a functor is an isomorphism if, and only if, it has a two sided inverse. A functor $\mathcal{T} : \mathbf{C} \longrightarrow \mathbf{C}'$ is said to be *full* if to every pair A, B of objects in $\mathbf{C}$ and every morphism $g : \mathcal{T}(A) \longrightarrow \mathcal{T}(B)$ of $\mathbf{C}'$, there is a morphism $f : A \longrightarrow B$ in $\mathbf{C}$ such that $g = \mathcal{T}(f)$. Clearly the composite of two full functors if full. A subcategory $\mathbf{D}$ of a category $\mathbf{C}$ is a *full subcategory* if the inclusion of $\mathbf{D}$ into $\mathbf{C}$ is a full functor, in which case, for all objects A and B of $\mathbf{D}$, we have $\hom_{\mathbf{D}}(A, B) = \hom_{\mathbf{C}}(A, B)$. A functor $\mathcal{T} : \mathbf{C} \longrightarrow \mathbf{C}'$ is *faithful* (or an *embedding*) if for every pair A, B of objects in $\mathbf{C}$ and to every pair $f_1, f_2 : A \longrightarrow B$ of parallel morphisms in $\mathbf{C}$, we have that $\mathcal{T}(f_1) = \mathcal{T}(f_2)$ implies $f_1 = f_2$. Clearly, composites of faithful functors are faithful. We say that a functor $\mathcal{T} : \mathbf{D} \longrightarrow \mathbf{C}$ is *dense* if, for every object A in $\mathbf{C}$, there is a morphism g in $\mathbf{D}$ with range A whose domain is an object in $\mathcal{T}(\mathbf{D})$. A subcategory $\mathbf{D}$ of a category $\mathbf{C}$ is called a *dense subcategory* if the inclusion functor $\iota : \mathbf{C} \longrightarrow \mathbf{D}$ is dense.

Let A be an object in the category $\mathbf{C}$, the set of all morphisms f of $\mathbf{C}$ which have A as domain is called the *star of* $\mathbf{C}$ *at* A, denoted $\mathrm{St}_{\mathbf{C}}(A)$. If $\mathcal{T} : \mathbf{C} \longrightarrow \mathbf{D}$ is a functor, then $\mathcal{T}$ induces a function $\mathcal{T}_A$ from $\mathrm{St}_{\mathbf{C}}(A)$ to $\mathrm{St}_{\mathbf{D}}(\mathcal{T}(A))$. We say that $\mathcal{T}$ is *star injective, star surjective, star bijective* according as $\mathcal{T}_A$ is injective, surjective, bijective, for all objects A of $\mathbf{C}$.

Important examples of functors that we shall use later are 'hom-functors'. Let $\mathbf{C}$ be a category with small hom-sets, thus every hom-set of $\mathbf{C}$ is an object in **Set**. For each object A of $\mathbf{C}$ there is a functor, called a *covariant hom-functor*

$$\hom_{\mathbf{C}}(A, -) : \mathbf{C} \longrightarrow \mathbf{Set}$$

which sends each object B to the set $\hom(A, B)$ and each morphism $k : B \longrightarrow B'$ to the function

$$\hom(A, k) : \hom(A, B) \longrightarrow \hom(A, B') \quad \text{defined by} \quad \hom(A, k) : f \longmapsto kf.$$

For each object B of $\mathbf{C}$, the *contravariant hom-functor* is a functor

$$\hom_{\mathbf{C}}(-, B) : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

which sends each object A to the set $\hom(A, B)$ and each morphism $g : A' \longrightarrow A$ of $\mathbf{C}$ to the function

$$\hom(g, B) : \hom(A, B) \longrightarrow \hom(A', B) \quad \text{defined by} \quad \hom(g, B) : f \longmapsto fg.$$

2.3 Natural transformations

Let $\mathcal{S}, \mathcal{T} : \mathbf{C} \longrightarrow \mathbf{C}'$ be two functors. A *natural transformation* $\tau : \mathcal{S} \longrightarrow \mathcal{T}$ is a function assigning to each object A of $\mathbf{C}$ a morphism $\tau_A : \mathcal{S}(A) \longrightarrow \mathcal{T}(A)$ in $\mathbf{C}'$ such that for any morphism $f : A \longrightarrow B$ in $\mathbf{C}$ we have $\mathcal{T}(f)\tau_A = \tau_B \mathcal{S}(f)$ as in the commutative diagram below.

$$\begin{tikzcd} A \arrow[d, "f"'] & \mathcal{S}(A) \arrow[r, "\tau_A"] \arrow[d, "\mathcal{S}(f)"'] & \mathcal{T}(A) \arrow[d, "\mathcal{T}(f)"] \\ B & \mathcal{S}(B) \arrow[r, "\tau_B"'] & \mathcal{T}(B) \end{tikzcd}$$

We also say that $\tau_A : \mathcal{S}(A) \longrightarrow \mathcal{T}(A)$ is *natural in A*.

Let $\mathbf{B}$ and $\mathbf{C}$ be categories. We describe the category whose objects are functors from $\mathbf{B}$ to $\mathbf{C}$ and whose morphisms are natural transformations.

Let $\mathcal{S}, \mathcal{T}, \mathcal{R} : \mathbf{B} \longrightarrow \mathbf{C}$ be functors and let $\tau : \mathcal{S} \longrightarrow \mathcal{T}$ and $\sigma : \mathcal{T} \longrightarrow \mathcal{R}$ be natural transformations. We define a composite $\sigma \circ \tau$ with component morphisms $(\sigma \circ \tau)_A = \sigma_A \tau_A$. To see that $\sigma \circ \tau$ is natural, let $f : A \longrightarrow B$ be a morphism in $\mathbf{B}$ and consider the diagram

$$\begin{tikzcd} \mathcal{S}(A) \arrow[r, "\mathcal{S}(f)"] \arrow[d, "\tau_A"] \arrow[dd, bend right=60, "(\sigma\circ\tau)_A"'] & \mathcal{S}(B) \arrow[d, "\tau_B"'] \arrow[dd, bend left=60, "(\sigma\circ\tau)_B"] \\ \mathcal{T}(A) \arrow[r, "\mathcal{T}(f)"] \arrow[d, "\sigma_A"] & \mathcal{T}(B) \arrow[d, "\sigma_B"'] \\ \mathcal{R}(A) \arrow[r, "\mathcal{R}(f)"'] & \mathcal{R}(B) \end{tikzcd}$$

Since τ and σ are natural, both the smaller squares commute, therefore the whole diagram commutes. We call $\sigma \circ \tau$ the *vertical composite* of σ and τ.

$$\begin{tikzcd} \mathbf{B} \arrow[r, bend left=40] \arrow[r, "\Downarrow\tau", "\Downarrow\sigma"'] \arrow[r, bend right=40] & \mathbf{C} \end{tikzcd}$$

Since the composition of morphisms in $\mathbf{C}$ is associative, this composition of natural transformations is associative. Also, for any identity functor $\mathcal{I} : \mathbf{B} \longrightarrow \mathbf{B}$, there is a natural transformation $1_{\mathcal{I}} : \mathcal{I} \longrightarrow \mathcal{I}$ with components $(1_{\mathcal{I}})_A = \mathrm{Id}_A$. Hence the set of functors from $\mathbf{B}$ to $\mathbf{C}$, together with the natural transformations between them form a category, called their *functor category*, it is denoted $\mathbf{C}^{\mathbf{B}}$.

We now look at a different way of composing natural transformations. Suppose we have three categories $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, with functors $\mathcal{S}, \mathcal{S}', \mathcal{T}, \mathcal{T}'$ and natural transformations τ and τ', as shown below.

$$\begin{tikzcd}
\mathbf{B} \arrow[r, shift left=2, "\mathcal{S}"] \arrow[r, shift right=2, "\mathcal{T}"'] \arrow[r, phantom, "\downarrow\tau"] & \mathbf{C} \arrow[r, shift left=2, "\mathcal{S}'"] \arrow[r, shift right=2, "\mathcal{T}'"'] \arrow[r, phantom, "\downarrow\tau'"] & \mathbf{D}
\end{tikzcd}$$

If A is an object of $\mathbf{B}$, the square below commutes since τ' is natural.

$$\begin{tikzcd}
\mathcal{S}'\mathcal{S}(\mathcal{A}) \arrow[r, "\tau'_{\mathcal{S}(A)}"] \arrow[d, "\mathcal{S}'(\tau_A)"'] & \mathcal{T}'\mathcal{S}(A) \arrow[d, "\mathcal{T}'(\tau_A)"] \\
\mathcal{S}'\mathcal{T}(A) \arrow[r, "\tau'_{\mathcal{T}(A)}"'] & \mathcal{T}'\mathcal{T}(A)
\end{tikzcd}$$

The *horizontal composite* $\tau' \cdot \tau$ of τ and τ' is then defined as this diagonal, that is

$$(\tau' \cdot \tau)_A = \mathcal{T}'(\tau_A)\tau'_{\mathcal{S}(A)} = \tau'_{\mathcal{T}(A)}\mathcal{S}'(\tau_A).$$

The proof that $\tau' \cdot \tau$ is indeed natural is well-known (see MacLane [21]), as is the verification of the following law

$$\tau' \cdot \tau = (\mathcal{T}' \cdot \tau) \circ (\tau' \cdot \mathcal{S}) = (\tau' \cdot \mathcal{T}) \circ (\mathcal{S}' \cdot \tau).$$

This law is in fact a special case of the *Godement interchange law*, relating horizontal and vertical composition of natural transformations, which states that given three categories and four natural transformations

$$\begin{tikzcd}
\mathbf{B} \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] \arrow[r, phantom, shift left=2, "\downarrow\tau"] \arrow[r, phantom, shift right=2, "\downarrow\sigma"] & \mathbf{C} \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] \arrow[r, phantom, shift left=2, "\downarrow\tau'"] \arrow[r, phantom, shift right=2, "\downarrow\sigma'"] & \mathbf{D}
\end{tikzcd}$$

we have that

$$(\sigma' \circ \tau') \cdot (\sigma \circ \tau) = (\sigma' \cdot \sigma) \circ (\tau' \cdot \tau).$$

2.4 Products and pullbacks

Suppose that $\mathbf{B}$ and $\mathbf{C}$ are categories, the *product category of* $\mathbf{B}$ *and* $\mathbf{C}$, denoted $\mathbf{B} \times \mathbf{C}$, is constructed as follows:

- The objects of $\mathbf{B} \times \mathbf{C}$ are ordered pairs (B, C), where B is an object of $\mathbf{B}$ and C is an object of $\mathbf{C}$.
- A morphism in $\mathbf{B} \times \mathbf{C}$ is a pair (f, g), where f is a morphism in $\mathbf{B}$ and g is a morphism in $\mathbf{C}$.

If (f, g) and (f', g') are morphisms in $\mathbf{B} \times \mathbf{C}$ such that

$$(f, g) : (B, C) \longrightarrow (B', C') \qquad \text{and} \qquad (f', g') : (B', C') \longrightarrow (B'', C'')$$

then the composite $(f', g')(f, g)$ is defined in terms of the composites within $\mathbf{B}$ and $\mathbf{C}$ as

$$(f', g')(f, g) = (f'f, g'g).$$

It is straightforward to show that $\mathbf{B} \times \mathbf{C}$ is a category.

There are two special functors $\mathcal{P}$ and $\mathcal{Q}$ on $\mathbf{B} \times \mathbf{C}$, called the *projections of the product*

$$\mathcal{P} : \mathbf{B} \times \mathbf{C} \longrightarrow \mathbf{B} \qquad \mathcal{Q} : \mathbf{B} \times \mathbf{C} \longrightarrow \mathbf{C}$$

such that for any object (B, C) of $\mathbf{B} \times \mathbf{C}$

$$\mathcal{P} : (B, C) \longmapsto B \qquad \text{and} \qquad \mathcal{Q} : (B, C) \longmapsto C$$

and for any morphism (f, g) in $\mathbf{B} \times \mathbf{C}$

$$\mathcal{P} : (f, g) \longmapsto f \qquad \text{and} \qquad \mathcal{Q} : (f, g) \longmapsto g.$$

Definition Given a pair of morphisms $f : A \longrightarrow B$ and $g : C \longrightarrow B$ in a category $\mathbf{C}$, a commutative square

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is called a *weak pullback* of (f, g) if given any other commutative square

$$\begin{tikzcd} D' \arrow[r, "h'"] \arrow[d, "k'"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

involving (f, g), there is a morphism $\Phi : D' \longrightarrow D$ such that $h\Phi = h'$ and $k\Phi = k'$.

If the morphism Φ is unique with this property, then the weak pullback is a *pullback*.

If in a category $\mathbf{C}$, any pair (f, g) of morphisms with common range has a (weak) pullback, then we say that $\mathbf{C}$ *has (weak) pullbacks.*

Lemma 2.1 *Pullbacks preserve split epimorphisms. Thus if g is a split epimorphism in the pullback square below, then so is k.*

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

PROOF. Since g is a split epimorphism there is a morphism $\gamma : B \longrightarrow C$ with $g\gamma = \mathrm{Id}_B$. Consider the square

$$\begin{tikzcd} A \arrow[r, "\gamma f"] \arrow[d, "\mathrm{Id}_A"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

which commutes since $g\gamma f = f$. Hence there is a unique morphism $\kappa : A \longrightarrow D$ such that $k\kappa = \mathrm{Id}_A$.

■

2.5 Adjoints

Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $\mathcal{S} : \mathbf{C} \longrightarrow \mathbf{D}$ and $\mathcal{T} : \mathbf{D} \longrightarrow \mathbf{C}$ be functors. An *adjunction* of $\mathcal{S}$ to $\mathcal{T}$ is a family of bijections

$$\alpha = \alpha_{A,B} : \hom_{\mathbf{D}}(\mathcal{S}(A), B) \cong \hom_{\mathbf{C}}(A, \mathcal{T}(B))$$

of the sets of morphisms, defined for all objects A of $\mathbf{C}$ and B of $\mathbf{D}$, which is natural in A and B. Given such an adjunction, the functor $\mathcal{S}$ is called a *left adjoint* of $\mathcal{T}$, and $\mathcal{T}$ is a *right adjoint* of $\mathcal{S}$.

This definition requires some explanation. Recall that for each object A of $\mathbf{C}$, $\hom_{\mathbf{C}}(A, -)$ and $\hom_{\mathbf{D}}(\mathcal{S}(A), -)$ are covariant hom-functors. Also, for each object B of $\mathbf{D}$, $\hom_{\mathbf{C}}(-, \mathcal{T}(B))$ and $\hom_{\mathbf{D}}(-, B)$ are contravariant hom-functors. Naturality of α means that

$$\begin{aligned} \alpha &: \hom_{\mathbf{D}}(\mathcal{S}(A), -) \longrightarrow \hom_{\mathbf{C}}(A, -) \\ \text{and} \quad \alpha &: \hom_{\mathbf{D}}(-, B) \longrightarrow \hom_{\mathbf{C}}(-, \mathcal{T}(B)) \end{aligned}$$

are natural transformations, for all objects A of $\mathbf{C}$ and B of $\mathbf{D}$. Thus for all morphisms $g : B' \longrightarrow B$ of $\mathbf{D}$ the diagram:

$$\begin{tikzcd}
B \arrow[d, "g"'] & \mathrm{hom}_{\mathbf{D}}(\mathcal{S}(A), B) \arrow[r, "\alpha_{A,B}"] \arrow[d, "\mathrm{hom}_{\mathbf{D}}(\mathcal{S}(A),g)"'] & \mathrm{hom}_{\mathbf{C}}(A, \mathcal{T}(B)) \arrow[d, "\mathrm{hom}_{\mathbf{C}}(A,\mathcal{T}(g))"] \\
B' & \mathrm{hom}_{\mathbf{D}}(\mathcal{S}(A), B') \arrow[r, "\alpha_{A,B'}"'] & \mathrm{hom}_{\mathbf{C}}(A, \mathcal{T}(B'))
\end{tikzcd}$$

must commute, and for all morphisms $f : A' \longrightarrow A$ the diagram below must commute.

$$\begin{tikzcd}
A & \mathrm{hom}_{\mathbf{D}}(\mathcal{S}(A), B) \arrow[r, "\alpha_{A,B}"] \arrow[d, "\mathrm{hom}_{\mathbf{D}}(\mathcal{S}(f),B)"'] & \mathrm{hom}_{\mathbf{C}}(A, \mathcal{T}(B)) \arrow[d, "\mathrm{hom}_{\mathbf{C}}(f,\mathcal{T}(B))"] \\
A' \arrow[u, "f"] & \mathrm{hom}_{\mathbf{D}}(\mathcal{S}(A'), B) \arrow[r, "\alpha_{A',B}"'] & \mathrm{hom}_{\mathbf{C}}(A', \mathcal{T}(B))
\end{tikzcd}$$

The most frequently cited example of an adjoint pair is the forgetful functor which takes a group to its underlying set, this has as right adjoint the functor which assigns to any set the free group on that set.

Chapter 3

Ordered groupoids

3.1 The category OG

In this section we shall examine the category of ordered groupoids. Ordered groupoids are important because they provide a categorical framework in which to study inverse semigroups. We shall therefore be treating each ordered groupoid as an algebraic structure and so the arrows-only definition is most appropriate.

Definition A category G is said to be a *groupoid* if for each $x \in G$ there is an element $x^{-1} \in G$ such that $x^{-1}x = \mathbf{d}(x)$ and $xx^{-1} = \mathbf{r}(x)$.

A groupoid with one identity is a group. If G is a groupoid, and e an identity in G, then $G(e) = \{x \in G \mid \mathbf{d}(x) = \mathbf{r}(x) = e\}$ is a group, called the *vertex group at e*.

A morphism between two groupoids is simply a functor. The category of groupoids is denoted **Grpd**.

Definition Let $(G, \cdot)$ be a groupoid, and let $\leqslant$ be a partial order defined on **G**. Then $(G, \cdot, \leqslant)$ is an *ordered groupoid* if the following axioms hold:

(OG1) $x \leqslant y$ implies $x^{-1} \leqslant y^{-1}$, for all $x, y \in G$.

(OG2) For all $x, y, u, v \in G$, if $x \leqslant y$, $u \leqslant v$, $\exists xu$ and $\exists yv$, then $xu \leqslant yv$.

(OG3) Let $x \in G$ and let e be an identity such that $e \leqslant \mathbf{d}(x)$. Then there exists a unique element $(x|e)$, called the *restriction* of x to e, such that $(x|e) \leqslant x$ and $\mathbf{d}(x|e) = e$.

(OG4) Let $x \in G$ and let e be an identity such that $e \leqslant \mathbf{r}(x)$. Then there exists a unique element $(e|x)$, called the *corestriction* of x to e, such that $(e|x) \leqslant x$ and $\mathbf{r}(e|x) = e$.

An ordered groupoid is said to be *inductive* if the partially ordered set of its identities forms a meet semilattice.

Proposition 3.1 provides some standard properties of ordered groupoids, for proofs see Lawson's book [15].

Proposition 3.1 *Let $(G, \cdot, \leqslant)$ be an ordered groupoid.*

(i) *If $x \leqslant y$, then $\mathbf{d}(x) \leqslant \mathbf{d}(y)$ and $\mathbf{r}(x) \leqslant \mathbf{r}(y)$.*

(ii) *If $x, y \in G$, with $x \leqslant y$, $\mathbf{d}(x) = \mathbf{d}(y)$ and $\mathbf{r}(x) = \mathbf{r}(y)$, then $x = y$.*

(iii) *Axiom* (OG4) *is a consequence of axioms* (OG1) *and* (OG3).

(iv) *The set of identities G_o is an order ideal of G.*

(v) *If $x, y \in G$ and $e \in G_o$ such that $\exists xy$ and $e \leqslant \mathbf{d}(y)$, then $(xy|e) = (x|\mathbf{r}(y|e))(y|e)$.*

■

Let $(G, \cdot, \leqslant)$ and $(H, \cdot, \leqslant)$ be ordered groupoids, with $\theta : G \longrightarrow H$ a functor. Then θ is said to be an *ordered functor* if for all $g_1, g_2 \in G$ with $g_1 \leqslant g_2$ we have that $\theta(g_1) \leqslant \theta(g_2)$. An ordered functor between two inductive groupoids is said to be *inductive* if it preserves the meet operation on the set of identities. It is easy to show that ordered groupoids together with ordered functors form a category, which is denoted **OG**. Inductive groupoids and ordered functors between them constitute a subcategory of **OG**, as does the category of inductive groupoids and inductive functors. If an ordered functor $\theta : G \longrightarrow H$ is just subset inclusion, then we say that G is an *ordered subgroupoid* of H. An *isomorphism* of ordered groupoids is a bijective ordered functor whose inverse is an ordered functor. Ordered functors can also be star injective, star surjective and star bijective.

We now provide some simple examples of ordered groupoids and ordered functors.

Examples

1. Every groupoid can be viewed as an ordered groupoid whose partial order is defined as the equality relation. An ordered functor between such groupoids is just a groupoid functor.

2. Given a partially ordered set X, we can form a discrete ordered groupoid X_D by defining a partial multiplication $\exists xy$ if and only if $x = y$, in which case $xx = x$. An ordered functor between such ordered groupoids corresponds to an order preserving

function between posets. Conversely, if G is an ordered groupoid, then G can be viewed as a partially ordered set by forgetting the multiplication, and we write G_D for the discrete ordered groupoid obtained from G viewed as a partially ordered set.

3. If G is an ordered groupoid with one identity then G is just a group (G is totally unordered since by Proposition 3.1(ii) the order restricted to hom-sets is trivial). An ordered functor between two such ordered groupoids is a group homomorphism.

The following useful result is proved in [15], Proposition 4.1.2.

Proposition 3.2 *Let $\theta : G \longrightarrow H$ be an ordered functor between ordered groupoids.*

(i) *If $(x|e)$ is defined in G, then $(\theta(x)|\theta(e))$ is defined in H and $\theta(x|e) = (\theta(x)|\theta(e))$.*

(ii) *If $(e|x)$ is defined in G, then $(\theta(e)|\theta(x))$ is defined in H and $\theta(e|x) = (\theta(e)|\theta(x))$.*

■

Let $\theta : G \longrightarrow H$ be an injective ordered functor such that

$$g_1 \leqslant g_2 \quad \Longleftrightarrow \quad \theta(g_1) \leqslant \theta(g_2),$$

for all $g_1, g_2 \in G$. Then θ is called an *ordered embedding.*

Definition Let G be an ordered subgroupoid of the ordered groupoid H. We say that H is an *enlargement* of G if the following axioms hold.

(GE1) G_o is an order ideal of H_o.

(GE2) If $x \in H$ and $\mathbf{d}(x), \mathbf{r}(x) \in G$, then $x \in G$.

(GE3) If $e \in H_o$, then there exists $x \in H$ such that $\mathbf{r}(x) = e$ and $\mathbf{d}(x) \in G$.

Enlargements were introduced by Lawson [13].

Note that the condition (GE2) is equivalent to the requirement that G is a full subgroupoid of H. The condition (GE3) is equivalent to saying that G is a dense subgroupoid of H.

Observe that if H is an enlargement of G and $g \in G$, $h \in H$ with $h \leqslant g$, then $\mathbf{d}(h) \leqslant \mathbf{d}(g)$ and $\mathbf{r}(h) \leqslant \mathbf{r}(g)$, by Proposition 3.1(i). So, by (GE1), $\mathbf{d}(h), \mathbf{r}(h) \in G_o$, but then $h \in G$, by (GE2). Hence G is an order ideal of H.

Let G be an ordered groupoid and let A be an ordered subgroupoid of G. Then A is called a *normal ordered subgroupoid* of G if the following conditions hold:

(i) $G_o = A_o$ and $\mathbf{d}(a) = \mathbf{r}(a)$ for all $a \in A$.

(ii) $gag^{-1} \in A$ for all $g \in G$ and $a \in A$ with $\mathbf{d}(g) = \mathbf{d}(a)$.

We describe the quotient groupoid G/A, following Higgins approach [7].

The elements of G/A are the equivalence classes of the equivalence relation given by

$$u \sim v \Longleftrightarrow u = avb \quad \text{for some } a, b \in A.$$

We denote the equivalence class containing u by $[u]$. For each a in A, $[a]$ is the group containing a and we can choose a unique identity representative for $[a]$. Clearly if $u \sim v$, then $\mathbf{d}(u) = \mathbf{d}(v)$, so write $\mathbf{d}[u] = [\mathbf{d}(u)]$, and similarly $\mathbf{r}[u] = [\mathbf{r}(u)]$. If $u, v \in G$ with $\mathbf{d}(u) = \mathbf{r}(v)$, then the product $[u][v]$ is defined an equal to $[uv]$. See [7] for proof that this is well defined and associative. Now G/A is a groupoid with $[u]^{-1} = [u^{-1}]$. Comparison with Section 3 of Lawson's paper [12] reveals that G/A is precisely G/ρ, where ρ is an identity-separating ordered congruence. By Theorem 11 of [12], $u \sim v$ if, and only if, uv^{-1} exists and is in A. Also proved in [12] is the fact that G/A is an ordered groupoid under the *Joubert order* defined as follows: $[u] \leqslant [v]$ if, and only if, for each $v' \in [v]$ there exists $u' \in [u]$ such that $u' \leqslant v'$.

An example of an ordered normal subgroupoid is the 'kernel' of an ordered functor. If $\theta : G \longrightarrow H$ is an ordered functor, and e is an identity of H. Then the *fibre* of θ at e is

$$\theta^{-1}(e) = \{g \in G \mid \theta(g) = e\}$$

which is easily seen to be an ordered subgroupoid of G. The *kernel* of θ is the disjoint union of the fibres $\theta^{-1}(e)$ taken over all $e \in H_o$ and so is an ordered subgroupoid of G. We denote the kernel of θ by $\mathrm{Ker}(\theta)$. It is straightforward to show that $\mathrm{Ker}(\theta)$ is normal.

Proposition 3.3 *The category* **OG**, *of ordered groupoids and ordered functors has all products of pairs of ordered groupoids and all pullbacks.*

PROOF. Let G and H be ordered groupoids. On the set $G \times H$ define

$$\mathbf{d}(g,h) = (\mathbf{d}(g), \mathbf{d}(h)) \quad \text{and} \quad \mathbf{r}(g,h) = (\mathbf{r}(g), \mathbf{r}(h))$$

and define a partial product by

$$(g,h)(g',h') = (gg', hh')$$

if $\mathbf{d}(g,h) = \mathbf{r}(g',h')$. One then has that $(g,h)^{-1} = (g^{-1}, h^{-1})$. It is easy to check that $G \times H$ is an ordered groupoid. Define a partial order on $G \times H$ as

$$(g,h) \leqslant (g',h') \quad \Longleftrightarrow \quad g \leqslant g' \text{ and } h \leqslant h'.$$

It is straightforward to show that $G \times H$ is an ordered groupoid. The projection functors

$$\pi_1 : G \times H \longrightarrow G \quad \text{and} \quad \pi_2 : G \times H \longrightarrow H$$

are ordered functors. It is routine to verify that $G \times H$ with the above definitions is the product of G by H.

Let G, H and K be ordered groupoids, and let $\theta : G \longrightarrow H$ and $\phi : K \longrightarrow H$ be ordered functors.

Define a subset of $G \times K$ by

$$G \boxtimes K = G_\theta \boxtimes_\phi K = \{(g, k) \in G \times K \mid \theta(g) = \phi(k)\}$$

Then it is easy to check that $G \boxtimes K$ is an ordered subgroupoid of $G \times K$. The only non-trivial verification being the condition (OG3): if $(g, k) \in G \boxtimes K$ and $(e, f) \in G_o \boxtimes K_o$ with $(e, f) \leqslant \mathbf{d}(g, k)$ then, by Proposition 3.2

$$\theta(g|e) = (\theta(g)|\theta(e)) = (\phi(k)|\phi(f)) = \phi(k|f)$$

and so $((g|e), (k|f)) \in G \boxtimes K$. It is routine to check that the restrictions of the projection functors $\pi_1 : G \boxtimes K \longrightarrow G$ and $\pi_2 : G \boxtimes K \longrightarrow K$ are such that

$$\begin{tikzcd}
G_\theta \boxtimes_\phi K \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & K \arrow[d, "\phi"] \\
G \arrow[r, "\theta"'] & H
\end{tikzcd}$$

commutes. It is straightforward to check that this is a pullback.

■

3.2 Inductive groupoids and inverse semigroups

We shall now describe the correspondence between inverse semigroups and inductive groupoids.

Let S be an inverse semigroup. We define a partial product $\cdot$ on S. Let $s, t \in S$. The *restricted product* $s \cdot t$ exists only when $s^{-1}s = tt^{-1}$, in which case it is equal to st. For $s \in S$, we write

$$\mathbf{d}(s) = s^{-1}s \quad \text{and} \quad \mathbf{r}(s) = ss^{-1}.$$

See Proposition 3.1.4 of [15] for proof of the following.

Proposition 3.4 *Every inverse semigroup S is a groupoid with respect to its restricted product, we call it the* associated groupoid *of S.* ■

Recall that for an inverse semigroup S, the natural partial order is defined as

$$s \leqslant t \Longleftrightarrow s = te, \quad \text{for some} \quad e \in E(S).$$

The importance of the restricted product and the natural partial order to the structure of an inverse semigroup leads to a generalisation of semigroup homomorphisms. A function $\theta : S \longrightarrow T$ between inverse semigroups is said to be a *prehomomorphism* if $\theta(st) \leqslant \theta(s)\theta(t)$ for all $s, t \in S$. The following result implies that every prehomomorphism between inverse semigroups induces a functor between their associated groupoids, see Theorem 3.1.5 of [15] for proof this result.

Theorem 3.5 *Let $\theta : S \longrightarrow T$ be a function between inverse semigroups. Then θ is a prehomomorphism if, and only if, it preserves the restricted product and the natural partial order; the composite of two prehomomorphisms is a prehomomorphism.* ■

Inverse semigroups together with prehomomorphims form a category.

The following result shows how the usual product can be reconstructed from the restricted product and the natural partial order (proof in [15], Theorem 3.1.2).

Theorem 3.6 *Let S be an inverse semigroup.*

(i) *Let $s \in S$ and $e \in E(S)$ such that $e \leqslant s^{-1}s$. Then $a = se$ is the unique element of S such that $a \leqslant s$ and $a^{-1}a = e$.*

(ii) *Let $s \in S$ and $e \in E(S)$ such that $e \leqslant ss^{-1}$. Then $a = es$ is the unique element of S such that $a \leqslant s$ and $aa^{-1} = e$.*

(iii) *Let $s, t \in S$, then $st = s' \cdot t'$ where $s' = se$, $t' = et$ and $e = s^{-1}stt^{-1}$.*

■

Proposition 3.4, Lemma 1.6, Proposition 1.5 and Theorem 3.6 can be combined to give the following result.

Proposition 3.7 *Let S be an inverse semigroup, then the groupoid associated with S is an inductive groupoid with respect to the natural partial order.*

■

The inductive groupoid associated with an inverse semigroup S is denoted by $\mathcal{G}(S)$.

As an example, consider the Brandt inverse semigroup $B_2 = B(\{1\}, 2)$, this has elements

$$e = (1,1), \quad u = (1,2), \quad u^{-1} = (2,1), \quad f = (2,2) \quad \text{and} \quad 0.$$

The ordered groupoid $\mathcal{G}(B_2)$ is the groupoid with two non-identity mutually inverse morphisms u and u^{-1}, and three identities e, f, 0 where $\mathbf{d}(u) = e$ and $\mathbf{r}(u) = f$. The order is given by $0 \leqslant e$, $0 \leqslant f$ and equality otherwise. Put $I = \mathcal{G}(B_2) \setminus \{0\}$, I is illustrated below

$$\begin{tikzcd} e \arrow[loop left] \arrow[r, bend left, "u"] & f \arrow[l, bend left, "u^{-1}"] \arrow[loop right] \end{tikzcd}$$

This groupoid will play a crucial rôle in the theory developed in Chapter 6.

Now let G be an ordered groupoid, and let $g, h \in G$ be such that $e = \mathbf{d}(g) \wedge \mathbf{r}(h)$ exists. Put

$$g \otimes h = (g|e)(e|h)$$

and call $g \otimes h$ the *pseudoproduct* of g and h. It is immediate from the definition that the pseudoproduct is everywhere defined in an inductive groupoid. See Proposition 4.1.7 of [15] for proof of the following result.

Proposition 3.8 *Let* $(G, \cdot, \leqslant)$ *be an inductive groupoid.*

(i) $(G, \otimes)$ *is an inverse semigroup, which we will denote by* $\mathcal{S}(G)$.

(ii) $\mathcal{G}(\mathcal{S}(G)) = G$.

(iii) *For any inverse semigroup* S *we have that* $\mathcal{S}(\mathcal{G}(S)) = S$.

$\blacksquare$

The following crucial result is proved by showing that $\mathcal{S}$ and $\mathcal{G}$ give rise to bijective functors, see [15] for details.

Theorem 3.9 (Ehresmann–Schein–Nambooripad) *The category of inverse semigroups and prehomomorphisms is isomorphic to the category of inductive groupoids and ordered functors; and the category of inverse semigroups and homomorphisms is isomorphic to the category of inductive groupoids and inductive functors.*

$\blacksquare$

Part II

The homotopy theory of inverse semigroups

Chapter 4

Classical homotopy theory

In this chapter we will outline some homotopy theory in its original topological setting. All the definitions and results in this section are standard, our main references are the books by Brown [3] and Spanier [27].

4.1 Some basic topology

Algebraic topology is largely concerned with finding structures to model the geometric properties of spaces.

Definition Let X be a non-empty set. A *topology on* X is a collection of subsets of X, called *open sets*, such that

(OS1) $\emptyset$ and X are open sets.

(OS2) The arbitrary union of open sets is open.

(OS3) The intersection of finitely many open sets is open.

The set X together with the collection of its open sets is called a *topological space.* The elements of X are called the *points* of the space.

Given a set X, there are two obvious ways of defining a topology on X:

- Only the sets $\emptyset$ and X are open.
- Every subset of X is defined to be open, this is the *discrete topology.*

The *usual topology* on the real line $\mathbb{R}$ is given by unions of the open sets

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\}.$$

If A is a subset of a topological space X, then a topology is induced on A by defining the open sets in A to be all the sets $A \cap U$, where U is an open set in X. This is the *subspace topology* on A and we say that A is a *subspace* of X. Let I denote the closed unit interval $[0, 1]$, we shall see later that the space I given the induced subspace topology plays a pivotal rôle in homotopy theory.

If X and Y are topological spaces, a function $f : X \longrightarrow Y$ is called a *continuous map* if, for every open subset V of Y, the set $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X. Topological spaces together with the continuous maps between them form a category, denoted **Top**.

Given a point x in a topological space X, we call a subset N of X a *neighbourhood of* x if there is an open set $U \subseteq N$ such that $x \in U$.

Let X and Y be topological spaces, a function $f : X \longrightarrow Y$ is called a *homeomorphism* if it is bijective and, for all $x \in X$, N is a neighbourhood of x if, and only if, $f(N)$ is a neighbourhood of $f(x)$.

Let $p : \widetilde{X} \longrightarrow X$ be a continuous map. An open subset U of X is said to be *evenly covered* by p if $p^{-1}(U)$ is the disjoint union of open subsets of $\widetilde{X}$ each of which is mapped homeomorphically onto U by p. We call p a *covering projection* if each point $x \in X$ has an open neighbourhood evenly covered by p, in which case $\widetilde{X}$ is called the *covering space* and X the *base space* of the covering projection.

Suppose that X is a topological space and S is a subset of X. A *cover* of S is a set of subsets $\mathcal{A} = \{U_j \mid j \in J\}$ of X such that $S \subseteq \bigcup_{j \in J} U_j$. If the indexing set J is finite then $\mathcal{A}$ is said to be a *finite cover*. If each U_j is an open subset of X then $\mathcal{A}$ is an *open cover*. If X is a topological space and $\mathcal{A}$ is a cover of a subset S of X, then a *subcover* of $\mathcal{A}$ is a subset $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}$ covers S. A subset S of a topological space X is said to be *compact* if every open cover of S has a finite subcover.

A standard result is that the unit interval $I \subseteq \mathbb{R}$ is compact. Proofs are given (for example) in [3, 10].

Let X and Y be topological spaces and let Y^X denote the set of continuous maps from X to Y. If A and B are subsets of X and Y respectively, then write

$$\langle A : B \rangle = \{f \in Y^X \mid f(A) \subseteq B\}$$

and

$$\mathcal{S} = \{\langle K : U \rangle \mid K \text{ is a compact subset of } X \text{ and } U \text{ is an open subset of } Y\}.$$

We define a topology on Y^X with open sets

$$\{V \subseteq Y^X \mid \text{if } f \in V, \text{ then } \exists\ F_1, F_2, \ldots, F_n \in \mathcal{S} \text{ such that } f \in F_1 \cap F_2 \cap \cdots \cap F_n \subseteq V\}$$

this is called the *compact-open topology* on Y^X, first introduced by Fox [6].

The problem is to find ways of classifying topological spaces and continuous maps up to some sort of 'topological equivalence'. An important ingredient in this is obtained by considering 'paths'.

Definition Let X be a topological space, a *path ω in X of length r* is a continuous map

$$\omega : [0, r] \longrightarrow X.$$

The *origin* of the path is the point $\omega(0)$ and the *end* of the path is the point $\omega(r)$.

One can use paths to make precise the intuitive idea of what it means for a space to be connected. A topological space X is *path connected* if for any x and y in X, there is a path in X with origin x and end y.

Let $\omega : [0, r] \longrightarrow X$ and $\omega' : [0, r'] \longrightarrow X$ be two paths, the *path sum* $\omega' + \omega$ is defined if and only if $\omega(r) = \omega'(0)$ in which case

$$(\omega' + \omega) : [0, r' + r] \longrightarrow X \qquad \text{with} \qquad (\omega' + \omega) : t \longmapsto \begin{cases} \omega(t) & \text{if } 0 \leqslant t \leqslant r, \\ \omega'(t - r) & \text{if } r \leqslant t \leqslant r + r'. \end{cases}$$

A point x in X determines a unique constant path of length r with value x, which we denote r_x. If $r = 0$ this path is called the *zero path at x*. If $\omega : [0, r] \longrightarrow X$ is a path, then $\omega + 0_{\omega(0)} = \omega$ and $0_{\omega(r)} + \omega = \omega$, so that the zero paths provide a set of left and right identities for the sum operation on paths.

If ω, ω' and ω'' are paths in X of length r, r' and r'' respectively. Then $\omega'' + (\omega' + \omega)$ is defined if and only if $(\omega'' + \omega') + \omega$ is defined, and both paths are given by

$$t \longmapsto \begin{cases} \omega(t) & \text{if } 0 \leqslant t \leqslant r, \\ \omega'(t - r) & \text{if } r \leqslant t \leqslant r + r', \\ \omega''(t - (r + r')) & \text{if } r + r' \leqslant t \leqslant r + r' + r''. \end{cases}$$

Hence, for any topological space X, there is a category of paths, denoted P_X.

It is usual to consider only paths of length 1. Since, given a path $\omega : [0, r] \longrightarrow X$, there is a path ω' of unit length, with $\omega'(t) = \omega(rt)$, $t \in [0, 1]$. Such a path is called *normalised*. Given two normalised paths, ω and ω' if their sum $\omega' + \omega$ is defined then it has length 2, however, we can define their *normalised sum* to be the normalised path $\omega' \oplus \omega$ with

$$(\omega' \oplus \omega) : t \longmapsto \begin{cases} \omega(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ \omega'(2t - 1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Clearly any interesting space will contain very many paths. It is therefore desirable to be able to classify paths.

Definition Let ω and ω' be (normalised) paths in a space X, with $\omega(0) = \omega'(0) = x$ and $\omega(1) = \omega'(1) = y$. A *fixed-end-point homotopy* from ω' to ω is a continuous map

$$\phi : I \times I \longrightarrow X$$

such that

$$\begin{array}{lll} \phi(s,0) = \omega'(s) & \phi(s,1) = \omega(s) & \text{for} \quad s \in I, \\ \phi(0,t) = x & \phi(1,t) = y & \text{for} \quad t \in I. \end{array}$$

We can think of ϕ as a 'deformation' of ω' into ω, see Figure 1.

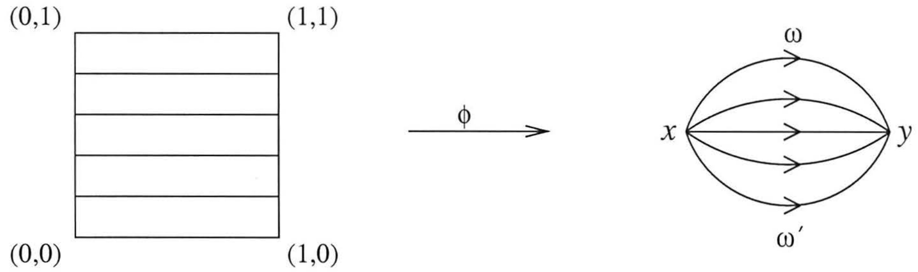


Figure 1

The concept of fixed-end-point homotopy arises out of the construction of the fundamental groupoid. One can show that fixed-end-point homotopy is an equivalence relation, the resulting equivalence classes are called *roads*. The fundamental groupoid of a space X has objects the points of X, and morphisms its roads, see Section 6.2 of Brown [3] for details.

4.2 Cylinders and cofibrations

So far we have looked at homotopies of paths, but more generally, one can also consider homotopies of maps.

Definition Let X and Y be topological spaces. A continuous map $\phi : X \times [0, q] \longrightarrow Y$ is called a *homotopy of length q*. Given such a map, there are maps

$$\begin{array}{llll} & f : X \longrightarrow Y & \text{with} & f : x \longmapsto \phi(x,0) \\ \text{and} & g : X \longrightarrow Y & \text{with} & g : x \longmapsto \phi(x,q) \end{array}$$

called the *initial* and *final maps of ϕ*, respectively.

We say that ϕ is a *homotopy from f to g*, and write $\phi : f \simeq g$.

Again, we think of a homotopy as 'deforming' one map into another: If we define $\phi_t : X \longrightarrow Y$ by $\phi_t(x) = \phi(x,t)$, then the homotopy ϕ gives a one-parameter family of continuous maps deforming f into g. One thinks of ϕ_t as describing the deformation at time t.

As before, it is sufficient to consider only homotopies of length 1, because it can be shown that there is a continuous surjection $\lambda : I \longrightarrow [0, q]$, and if we let $\theta = \phi(\mathrm{Id}_X \times \lambda)$ then θ is a homotopy of unit length with the same initial and final maps.

Note that fixed-end-point homotopies of paths are more restricted since the end points of the paths are fixed during the homotopy.

If $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ are maps such that $fg \simeq \mathrm{Id}_Y$, then we say that g is a *right homotopy inverse* of f, and that f is a *left homotopy inverse* of g. If g is both a left and right homotopy inverse of f, then g is called a *homotopy inverse* of f, and f is called a *homotopy equivalence*, we then write $f : X \simeq Y$.

Proposition 4.1 *The homotopy relation $\simeq$ is an equivalence relation.*

PROOF. To show that it is reflexive, let $f : X \longrightarrow Y$ be a continuous map and define

$$\phi : X \times I \longrightarrow Y \qquad \text{by} \qquad \phi : (x, t) \longmapsto f(x)$$

the constant homotopy on f. Clearly $\phi : f \simeq f$.

To show that $\simeq$ is symmetric, let ϕ be a homotopy from f to g, then define

$$\psi : X \times I \longrightarrow Y \qquad \text{by} \qquad \psi : (x, t) \longmapsto \phi(x, 1 - t)$$

it is then easy to check that $\psi : g \simeq f$.

To show that $\simeq$ is transitive, suppose $f, g, h : X \longrightarrow Y$ are continuous maps, and $\phi, \psi : X \times I \longrightarrow Y$ are homotopies with $\phi : f \simeq g$ and $\psi : g \simeq h$. Then define

$$\Phi : X \times I \longrightarrow Y \qquad \text{by} \qquad \Phi : (x, t) \longmapsto \begin{cases} \phi(x, 2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \psi(x, 2t - 1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

which is continuous, since $\phi(x, 1) = \psi(x, 0)$, and is a homotopy $\Phi : f \simeq h$.

■

Given a space X, we can think of $X \times I$ as a 'cylinder' with top face $(X, 1)$ and base $(X, 0)$. We can then define maps e_X^0 and e_X^1 which map X onto the base and top of $X \times I$ respectively, as follows:

$$\begin{array}{lllll} & e_X^0 : X \longrightarrow X \times I & \text{with} & e_X^0 : x \longmapsto (x, 0) \\ \text{and} & e_X^1 : X \longrightarrow X \times I & \text{with} & e_X^1 : x \longmapsto (x, 1). \end{array}$$

There is also a collapsing map from $X \times I$ to X

$$\sigma_X : X \times I \longrightarrow X \qquad \text{with} \qquad \sigma_X : (x, t). \longmapsto x.$$

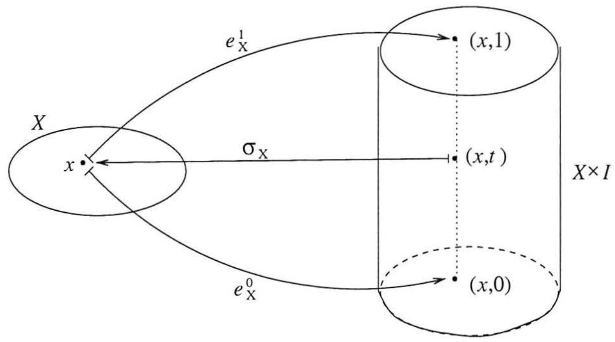


Figure 2

The situation is illustrated in Figure 2.

Given a continuous map $f : X \longrightarrow Y$ we can define a continuous map

$$f \times I : X \times I \longrightarrow Y \times I \qquad \text{by} \qquad f \times I : (x,t) \longmapsto (f(x), t).$$

It is straightforward to show that there is a functor

$$(\;) \times I : \mathbf{Top} \longrightarrow \mathbf{Top}$$

which assigns to a space X the space $X \times I$ and to a map f the map $f \times I$. Now if $x \in X$, then $e^0_Y f(x) = (f(x), 0)$ and $(f \times I)e^0_X(x) = (f \times I)(x,0) = (f(x),0)$, so the diagram below commutes.

$$\begin{tikzcd} X \arrow[r, "e^0_X"] \arrow[d, "f"'] & X \times I \arrow[d, "f\times I"] \\ Y \arrow[r, "e^0_Y"'] & Y \times I \end{tikzcd}$$

Thus the maps e^0_X give rise to a natural transformation

$$e^0 : \mathrm{Id} \longrightarrow (\;) \times I.$$

Similarly, there are natural transformations

$$e^1 : \mathrm{Id} \longrightarrow (\;) \times I \qquad \text{and} \qquad \sigma : (\;) \times I \longrightarrow \mathrm{Id}.$$

We shall use the notation

$$\mathbf{I} = ((\;) \times I, e^0, e^1, \sigma)$$

and call $\mathbf{I}$ a *cylinder* on **Top**. We can use the concept of a cylinder to redefine the notion of homotopy.

Definition If $f, g : X \longrightarrow Y$ are continuous maps of topological spaces, then f is *homotopic* to g, written $f \simeq g$, if there is a continuous map $\phi : X \times I \longrightarrow Y$ with $\phi e_X^0 = f$ and $\phi e_X^1 = g$. We call ϕ a *homotopy* between f and g.

Given a continuous map $f : X \longrightarrow Y$, the constant homotopy on f is given by $f\sigma_X : f \simeq f$.

We shall use the cylinder construction to describe some properties of subspaces. Let A be a subspace of a topological space X, and let $i : A \longrightarrow X$, with $i(x) = x$, be the inclusion map. We consider whether such a map has left, right or two-sided inverses.

A subspace A of X is called a *retract* of X if the inclusion map i has a left inverse in the category **Top**; that is a map $r : X \longrightarrow A$, such that $ri = \mathrm{Id}_A$. Such a map is called a *retraction of X to A*.

A subspace A of X is called a *weak retract* of X if the inclusion map i has a left homotopy inverse; that is a map $r : X \longrightarrow A$, such that $ri \simeq \mathrm{Id}_A$. Such a map is called a *weak retraction of X to A*.

Note that an inclusion map $i :\longrightarrow X$ never has a right inverse, except in the trivial case $A = X$. However, a subspace A of a space X is called a *weak deformation retract* of X if the inclusion map i is a homotopy equivalence. A subspace A is called a a *deformation retract* of X if there is a retraction r of X to A such that $\mathrm{Id}_X \simeq ir$.

Clearly, a weak retract need not be a retract, however these concepts do coincide when A is a suitable subspace of X. This occurs frequently enough to demand special consideration. In order to examine this situation we shall consider the conditions which A must satisfy in order that a homotopy on A can be extended to a homotopy on X.

Definition If A is a subspace of X, we say that (X, A) has the *homotopy extension property* with respect to a space Y if for all maps $f : X \longrightarrow Y$, any homotopy of the restriction $f|_A$ of f to A extends to a homotopy of f.

Let $u = f|_A : A \longrightarrow Y$, thus $u = fi$. A homotopy of u is a map $\phi : A \times I \longrightarrow Y$ such that $u = \phi e_A^0$. If A has the homotopy extension property with respect to Y, then there is a map $\Phi : X \times I \longrightarrow Y$ which is a homotopy of f, that is $f = \Phi e_X^0$, and $\Phi|_{A \times I} = \phi$.

Hence for given f and ϕ there exists Φ making the diagram below commute.

$$\begin{tikzcd}
 & A \times I \arrow[dr, "i\times I"] \arrow[drr, bend left, "\phi"] & & \\
A \arrow[ur, "e^0_A"] \arrow[dr, "i"'] & & X \times I \arrow[r, dotted, "\Phi"] & Y \\
 & X \arrow[ur, "e^0_X"'] \arrow[urr, bend right, "f"'] & &
\end{tikzcd}$$

Of particular importance is the case when (X, A) has the homotopy extension property with respect to any space, if this happens then we say that the inclusion map $i : A \longrightarrow X$ is a *cofibration*.

Proposition 4.2 *If (X, A) has the homotopy extension property with respect to any space, then A is a weak retract of X if, and only if, A is a retract of X.*

Proof. Let $i : A \longrightarrow X$ be the inclusion map and $r : X \longrightarrow A$ be a weak retraction. Then $ri \simeq \mathrm{Id}_A$. Let $\phi : A \times I \longrightarrow A$ be a homotopy from ri to Id_A; then $\phi e^0_A = ri$. Because (X, A) has the homotopy extension property with respect to A, there is a map $\Phi : X \times I \longrightarrow A$ such that $\Phi(i \times I) = \phi$ and $\Phi e^0_X = r$. If we define $r' : X \longrightarrow A$ by $r' = \Phi e^1_X$, we have $ir = \mathrm{Id}_A$. So r' is a retraction of X to A, and $\Phi : r \simeq r'$.

■

While it is true that the concept of a cofibration arises out of the examination of inclusion maps, it is not the case that only inclusions can be cofibrations. The definition can be generalised as follows: A continuous map $g : X' \longrightarrow X$ is a *cofibration* if the commutative square below is a weak pushout.

$$\begin{tikzcd}
X' \arrow[r, "e^0_{X'}"] \arrow[d, "g"'] & X' \times I \arrow[d, "g\times I"] \\
X \arrow[r, "e^0_X"'] & X \times I
\end{tikzcd}$$

4.3 Cocylinders and fibrations

In the previous section, we dealt with situations in which homotopies can be extended. In this section we consider whether homotopies can be 'lifted' along a map. Generally, if $p : E \longrightarrow B$ and $f : X \longrightarrow B$ are continuous maps. The *lifting problem* for f is to determine whether there is a map $g : X \longrightarrow Y$ such that $f = gp$. That is, whether the dotted arrow in the diagram below

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dotted, "g"] \arrow[r, "f"'] & B \end{tikzcd}$$

corresponds to a continuous map making the diagram commute. If such a map exists then we call g a *lifting* of f to E. If the map g is unique with this property, then p is said to have the *unique lifting property*.

Applying the lifting problem to homotopies, we obtain an analogue of the homotopy extension property called the 'homotopy lifting property' defined as follows.

Definition A map $p : E \longrightarrow B$ is said to have the *homotopy lifting property* with respect to a space X if, given maps $f : X \longrightarrow E$ and $\phi : X \times I \longrightarrow B$ such that $\phi e_X^0 = pf$, there is a map $\Phi : X \times I \longrightarrow E$ such that $\Phi e_X^0 = f$ and $p\Phi = \phi$. The situation is pictured below

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e_X^0"'] & E \arrow[d, "p"] \\ X \times I \arrow[ur, dotted, "\Phi"] \arrow[r, "\phi"'] & B \end{tikzcd}$$

The map p is a called *fibration* if it has the homotopy lifting property with respect to every space. For each $b \in B$, $p^{-1}(b)$ is called the *fibre over* b.

We have seen that paths play an important rôle in algebraic topology. There is a useful class of maps which have the property that they 'lift' paths.

Definition Let $p : E \longrightarrow B$ be a continuous map, then p has the *path lifting property* if for any path ω in B and point a in E with $\omega(0) = p(a)$, there is a path $\widetilde{\omega}$ in E with origin a such that $p\widetilde{\omega} = \omega$. Furthermore, p is said to have the *unique path lifting property* if given paths ω and ω' in E such that $p\omega = p\omega'$ and $\omega(0) = \omega'(0)$, then $\omega = \omega'$.

The next result shows that the path lifting property is a special case of the homotopy lifting property.

Proposition 4.3 *If a continuous map is a fibration, then it has the path lifting property.*

Proof. Suppose that $p : E \longrightarrow B$ is a fibration and that $\omega : I \longrightarrow B$ is a path in B such that $\omega(0) \in p(E)$. Let P denote a one-point space, and let $\iota : I \longrightarrow P \times I$ be the inclusion map. Then ω can be regarded as a homotopy $\omega^* : P \times I \longrightarrow B$, where $\omega = \omega^*\iota$. A point a of E such that $p(a) = \omega(0)$ corresponds to a map $f : P \longrightarrow E$ with $pf(P) = \omega^*(P, 0)$. It then follows from the fact that p has the homotopy lifting property with respect to P, that there is a map $\widetilde{\omega} : P \times I \longrightarrow E$ such that $e^0_P\widetilde{\omega} = f$ and $p\widetilde{\omega} = \omega^*$. Therefore $\widetilde{\omega}\iota : I \longrightarrow E$ is a path in E with $\widetilde{\omega}\iota(0) = \widetilde{\omega}(P, 0) = a$ and $p\widetilde{\omega}\iota = \omega^*\iota = \omega$. Hence p lifts ω to $\widetilde{\omega}\iota$.

■

The following result establishes the unique lifting property of covering projections, see Theorem 2.2.2 of Spanier [27] for proof.

Proposition 4.4 *Let $p : \widetilde{X} \longrightarrow X$ be a covering projection and let $f, g : Y \longrightarrow \widetilde{X}$ be continuous maps such that $pf = pg$. Thus f and g are both liftings of the same map. If Y is connected and f agrees with g on some point of Y, then $f = g$.*

■

It follows from the above result that a covering projection has unique path lifting. Proposition 4.4 is also used to prove the following result, see Theorem 2.2.3 of Spanier [27] for details.

Theorem 4.5 *Every covering projection is a fibration.*

■

From the above results, a covering projection is a fibration with unique path lifting. In Theorem II.4.10 of [27] it is shown that if the base space satisfies some 'mild hypotheses,' then any fibration with unique path lifting is a covering projection.

We have been examining the paths within a space, we shall now consider spaces which themselves consist of paths. Let X be a space, then X^I denotes the space consisting of paths in X equipped with the compact-open topology. We can construct maps ε^0_X and ε^1_X assigning to a path ω in X its origin and end points respectively

$$\begin{aligned} \varepsilon^0_X : X^I \longrightarrow X \qquad &\text{with} \qquad \varepsilon^0_X : \omega \longmapsto \omega(0) \\ \varepsilon^1_X : X^I \longrightarrow X \qquad &\text{with} \qquad \varepsilon^1_X : \omega \longmapsto \omega(1). \end{aligned}$$

For any map $f : X \longrightarrow Y$, we can define a map on path spaces

$$f^I : X^I \longrightarrow Y^I \qquad \text{with} \qquad f^I : \omega \longmapsto f\omega.$$

We thus obtain a functor $(\ )^I : \mathbf{Top} \longrightarrow \mathbf{Top}$. Given such a continuous map f, and an element ω of X^I, then $\varepsilon^0_Y f^I(\omega) = f\varepsilon^0_X(\omega) = f(\omega(0))$, and so if ε^0 is the function which assigns to each space X the map ε^0_X, then ε^0 is a natural transformation $\varepsilon^0 : (\ )^I \longrightarrow \mathrm{Id}$. Similarly there is a natural transformation $\varepsilon^1 : (\ )^I \longrightarrow \mathrm{Id}$. There is also a continuous map which takes a point in a space to the constant path at that point,

$$s_X : X \longrightarrow X^I \qquad \text{with} \qquad s_X : x \longmapsto 1_x.$$

If s is the function taking a space X to the map s_X then it is easy to show that s is a natural transformation $s : \mathrm{Id}_{\mathbf{Top}} \longrightarrow (\ )^I$. We write $\mathbf{P} = ((\ )^I, \varepsilon^0, \varepsilon^1, s)$ and call $\mathbf{P}$ the *cocylinder* on $\mathbf{Top}$.

We shall use the cocylinder to reformulate the definition of the path lifting property in terms of path spaces. Let $p : E \longrightarrow B$ be a map with the path lifting property. Then for any $\omega \in B^I$ with $\varepsilon^0_B(\omega) \in p(E)$, there exists $\widetilde{\omega} \in E^I$ such that $p\varepsilon^0_E(\widetilde{\omega}) = \varepsilon^0_B(\omega)$ and $\omega = p^I(\widetilde{\omega})$, that is $p\varepsilon^0_E(\widetilde{\omega}) = \varepsilon^0_B p^I(\widetilde{\omega})$. If P is a one-point space, then a map $\phi : P \longrightarrow B^I$ picks out an element ω of B^I, and a map $f : P \longrightarrow E$ picks out an element a of E. If $\omega(0) = a$; that is $\varepsilon^0_B\phi = pf$. Then the path lifting property requires that there exists $\widetilde{\omega} \in E^I$ such that $\varepsilon^0_E(\omega) = f(P)$ and $p^I(\widetilde{\omega}) = \phi(P)$. Thus the path lifting property is satisfied by p if there exists a map $\Phi : P \longrightarrow E^I$ picking out an element $\widetilde{\omega}$ of E^I such that the diagram below commutes

$$\begin{tikzcd}
 & & B^I \arrow[dr, "\varepsilon^0_B"] & \\
P \arrow[urr, bend left, "\phi"] \arrow[r, dotted, "\Phi"] \arrow[drr, bend right, "f"'] & E^I \arrow[ur, "p^I"] \arrow[dr, "\varepsilon^0_E"'] & & B \\
 & & E \arrow[ur, "p"'] &
\end{tikzcd}$$

This is a special case of an alternative description of the homotopy lifting property.

Given a continuous map $\phi : X \times I \longrightarrow Y$, for each $x \in X$ define

$$\phi_x : I \longrightarrow Y \quad \text{by} \quad \phi_x : t \longrightarrow \phi(x,t)$$

observe that there is a continuous map $1_x : I \longrightarrow X \times I$ with $1_x(t) = (x,t)$, and $\phi_x = \phi 1_x$; it follows that each ϕ_x is continuous. Thus each ϕ_x is an element of Y^I. We can therefore define a function

$$\alpha_\phi : X \longrightarrow Y^I \quad \text{by} \quad \alpha_\phi : x \longmapsto \phi_x.$$

By Theorem 11.1 of Rotman [26], α_ϕ is continuous. Now define

$$\alpha_{X,Y} : \hom(X \times I, Y) \longrightarrow \hom(X, Y^I) \quad \text{by} \quad \alpha : \phi \longmapsto \alpha_\phi$$

which is clearly injective. Furthermore, if $\psi : X \longrightarrow Y^I$ is a continuous map with $\psi(x) = \omega_x$, then the pair (x, t), where $t \in I$, determines a point $\omega_x(t)$ of Y. Hence obtain a function $X \times I \longrightarrow Y$ which has image ψ under $\alpha_{X,Y}$. Hence $\alpha_{X,Y}$ is bijective. The family of bijections α can be shown to provide an adjunction between $(\ ) \times I$ and $(\ )^I$.

If $\phi : X \times I \longrightarrow Y$, then $\phi e_X^0(x) = \phi(x, 0)$ and

$$\varepsilon_Y^0 \alpha_{X,Y}(\phi)(x) = \varepsilon_Y^0 \phi_x = \phi_x(0) = \phi(x, 0).$$

Similarly $\varepsilon_Y^1 \alpha_{X,Y}(\phi)(x) = \phi e_X^1(x) = \phi(x, 1)$. So the maps e_X^i and ε_Y^i are related by the formula

$$\varepsilon_Y^i \alpha_{X,Y}(\phi) = \phi e_X^i \qquad i \in \{0, 1\}.$$

Also we have that σ_X and s_Y are related by $\alpha_{X,Y}(f\sigma_X) = s_Y f$. If $f, g : X \longrightarrow Y$ are continuous maps with $\phi : f \simeq g$; that is

$$\phi e_x^0 = f \qquad \text{and} \qquad \phi e_X^1 = g.$$

There is a continuous map $\psi = \alpha(\phi) : X \longrightarrow Y^I$ such that

$$\varepsilon_Y^0 \psi = f \qquad \text{and} \qquad \varepsilon_Y^1 \psi = g.$$

We think of ψ as a homotopy from f to g with respect to the cocylinder on **Top**, we write $\psi : f \simeq g$.

We now use the adjunction α to reformulate the homotopy lifting property in the cocylinder.

Proposition 4.6 *A continuous map $p : E \longrightarrow B$ is a fibration if, and only if, the square pictured below is a weak pullback.*

$$\begin{tikzcd} E^I \arrow[r, "p^I"] \arrow[d, "\varepsilon_E^0"'] & B^I \arrow[d, "\varepsilon_B^0"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

PROOF. Let $p : E \longrightarrow B$ be a fibration and let $f : X \longrightarrow E$ and $\psi : X \longrightarrow B^I$ be continuous maps such that $pf = \varepsilon_B^0 \psi$. Since

$$\alpha = \alpha_{X,B} : \hom(X \times I, B) \longrightarrow \hom(X, X^I)$$

is a bijection, there is a unique continuous map $\phi : X \times I \longrightarrow B$ such that $\alpha(\phi) = \psi$. Furthermore $\varepsilon_B^0 \psi = \phi e_X^0$, since $\varepsilon_B^0 \alpha(\phi) = \phi e_X^0$. Now $pf = \phi e_X^0$, so the solid arrows in the diagram below commute.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "e_X^0"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "\phi"'] \arrow[ur, dotted, "\Phi"] & B
\end{tikzcd}$$

But p is a fibration so there is a continuous map $\Phi : X \times I \longrightarrow E$ such that $\Phi e_X^0 = f$ and $p\Phi = \phi$. Let

$$\Psi = \alpha(\Phi) : X \longrightarrow E^I.$$

Then $\varepsilon_E^0 \Psi = \varepsilon_E^0 \alpha(\Phi) = \Phi e_X^0 = f$. Now since α is natural $p^I \alpha_{X,E}(\Phi) = \alpha_{X,B}(p\Phi)$, but $\alpha(p\Phi) = \alpha(\phi) = \psi$, therefore $p^I \Psi = \psi$. Hence the diagram below commutes.

$$\begin{tikzcd}
X \arrow[dr, "\Psi"] \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "f"'] & & \\
& E^I \arrow[r, "p^I"'] \arrow[d, "\varepsilon_E^0"] & B^I \arrow[d, "\varepsilon_B^0"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}$$

Conversely, let $p : E \longrightarrow B$ be a continuous map and suppose that the square below is a weak pullback.

$$\begin{tikzcd}
E^I \arrow[r, "p^I"] \arrow[d, "\varepsilon_E^0"'] & B^I \arrow[d, "\varepsilon_B^0"] \\
E \arrow[r, "p"'] & B
\end{tikzcd}$$

We show that p is a fibration. Let $f : X \longrightarrow E$ and $\phi : X \times I \longrightarrow B$ be continuous maps making the diagram below commute

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "e_X^0"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "\phi"'] & B
\end{tikzcd}$$

Define $\psi = \alpha(\phi) : X \longrightarrow B^I$. Then

$$\psi \varepsilon_B^0 = \alpha(\phi)\varepsilon_B^0 = e_X^0 \phi = pf.$$

So the diagram below commutes,

$$
\begin{tikzcd}
X \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "f"'] & & \\
& E^I \arrow[r, "p^I"'] \arrow[d, "\varepsilon^0_E"] & B^I \arrow[d, "\varepsilon^0_B"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}
$$

but the inner square is a weak pullback, so there is a continuous map $\Psi : X \longrightarrow E^I$ such that $p^I\Psi = \psi$ and $\varepsilon^0_E\Phi = f$. Since α is bijective, there is a unique continuous map $\Phi : X \times I \longrightarrow E$ such that

$$\Phi e^0_X = \varepsilon^0_E\Psi = f$$

and

$$\alpha(p\Phi) = p^I\alpha(\Phi) = p^I\Psi = \psi = \alpha(\phi),$$

so $p\Phi = \phi$. Hence the diagram below commutes and p is a fibration.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "e^0_X"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "\phi"'] \arrow[ur, "\Phi"] & B
\end{tikzcd}
$$

■

Chapter 5

Abstract homotopy theory

In this chapter we describe cylinders, cocylinders, homotopies, cofibrations and fibrations in general categories. The main reference for this chapter is Kamps and Porter [9].

5.1 Homotopy theory in categories

Definition Let $\mathbf{C}$ be a category. A *cylinder* $\mathbf{I}$ on $\mathbf{C}$ consists of a functor

$$(\ ) \times I : \mathbf{C} \longrightarrow \mathbf{C}$$

called the *cylinder functor*. Together with three natural transformations

$$e^0, e^1 : \mathrm{Id}_{\mathbf{C}} \longrightarrow (\ ) \times I \qquad \text{and} \qquad \sigma : (\ ) \times I \longrightarrow \mathrm{Id}_{\mathbf{C}}$$

such that $\sigma e^0 = \sigma e^1 = \mathrm{Id}_{\mathbf{C}}$.

Given a cylinder on a category $\mathbf{C}$ we can then define a homotopy in $\mathbf{C}$

Definition If $f, g : X \longrightarrow Y$ are morphisms in $\mathbf{C}$ then f is *homotopic* to g, written $f \simeq g$, if there is a morphism $\phi : X \times I \longrightarrow Y$ in $\mathbf{C}$ with $\phi e^0_X = f$ and $\phi e^1_X = g$. We call ϕ a *homotopy* between f and g.

Note that for any morphism $f : X \longrightarrow Y$, the morphism $f\sigma_X : X \times I \longrightarrow Y$ is a homotopy, $f\sigma_X : f \simeq f$, called the *constant homotopy of f*. So $\simeq$ is reflexive, however it need not generally be symmetric or transitive. Later, in section 5.2, we shall see what conditions need to be imposed on a cylinder in order that $\simeq$ is an equivalence relation.

Definition A morphism $f : X \longrightarrow Y$ of $\mathbf{C}$ is a *homotopy equivalence* if there is a morphism $g : Y \longrightarrow X$ of $\mathbf{C}$ such that

$$gf \simeq \mathrm{Id}_X \qquad \text{and} \qquad fg \simeq \mathrm{Id}_Y$$

such a morphism g is called a *homotopy inverse* of f. If in addition, $gf = \mathrm{Id}_X$, then we say that g is a *deformation retraction* and that X is a *deformation retract* of Y.

Given a category with a cylinder, cofibrations are defined as those morphisms along which homotopies can be extended.

Definition Let $\mathbf{C}$ be a category with a cylinder $\mathbf{I} = ((\ ) \times I, e^0, e^1, \sigma)$. A morphism $i : A \longrightarrow X$ of $\mathbf{C}$ has the *homotopy extention property* with respect to an object Y of $\mathbf{C}$ if for any pair of morphisms of $\mathbf{C}$, $\phi : A \times I \longrightarrow Y$ and $f : X \longrightarrow Y$ such that $\phi e^0_A = fi$, there is a morphism $\Phi : X \times I \longrightarrow Y$ of $\mathbf{C}$ such that $\Phi(i \times I) = \phi$ and $\Phi e^0_X = f$.

$$\begin{tikzcd}
 & A \times I \arrow[dr, "i \times I"] \arrow[drr, bend left, "\phi"] & & \\
A \arrow[ur, "e^0_A"] \arrow[dr, "i"'] & & X \times I \arrow[r, dotted, "\Phi"] & Y \\
 & X \arrow[ur, "e^0_X"] \arrow[urr, bend right, "f"'] & &
\end{tikzcd}$$

A morphism of $\mathbf{C}$ which has the homotopy extention property with respect to any object of $\mathbf{C}$ is called a *cofibration*.

By dualising the notion of a cylinder on a category, we obtain the concept of a cocylinder. By definition, a cocylinder on a category $\mathbf{C}$ is a cylinder on the dual category $\mathbf{C}^{\mathrm{op}}$.

Definition Let $\mathbf{C}$ be a category. A *cocylinder* $\mathbf{P}$ on $\mathbf{C}$ consists of a functor

$$(\ )^I : \mathbf{C} \longrightarrow \mathbf{C}$$

called the *cocylinder functor*, together with three natural transformations

$$\varepsilon^0, \varepsilon^1 : (\ )^I \longrightarrow \mathrm{Id}_{\mathbf{C}} \qquad \text{and} \qquad s : \mathrm{Id}_{\mathbf{C}} \longrightarrow (\ )^I$$

such that $\varepsilon^0 s = \varepsilon^1 s = \mathrm{Id}_{\mathbf{C}}$.

Homotopy and homotopy equivalence in a cocylinder is defined in much the same way as it is in a cylinder.

Definition Given a cocylinder $\mathbf{P} = ((\ )^I, \varepsilon^0, \varepsilon^1, s)$ on a category $\mathbf{C}$, two morphisms $f, g : X \longrightarrow Y$ in $\mathbf{C}$ are said to be *homotopic*, written $f \simeq g$, if there is a morphism $\phi : X \longrightarrow Y^I$ in $\mathbf{C}$ with $\varepsilon^0_Y \phi = f$ and $\varepsilon^1_Y \phi = g$. We call ϕ a *homotopy* between f and g.

Dual to cofibrations (with respect to a cylinder) one has fibrations (with respect to a cocylinder) defined by the homotopy lifting property.

Definition Let $\mathbf{C}$ be a category with cocylinder $\mathbf{P} = ((\ )^I, \varepsilon^0, \varepsilon^1, s)$. A morphism $p : E \longrightarrow B$ in $\mathbf{C}$ has the *homotopy lifting property* with respect to an object Y of $\mathbf{C}$ if for any pair of morphisms of $\mathbf{C}$

$$\phi : Y \longrightarrow B^I \qquad \text{and} \qquad f : Y \longrightarrow E$$

such that $\varepsilon^0_B \phi = pf$, there is a morphism $\Phi : Y \longrightarrow E^I$ of $\mathbf{C}$ such that $p^I \Phi = \phi$ and $\varepsilon^0_E \Phi = f$.

$$\begin{tikzcd}
 & & B^I \arrow[dr, "\varepsilon^0_B"] & \\
Y \arrow[urr, bend left, "\phi"] \arrow[r, dotted, "\Phi"] \arrow[drr, bend right, "f"'] & E^I \arrow[ur, "p^I"] \arrow[dr, "\varepsilon^0_E"'] & & B \\
 & & E \arrow[ur, "p"'] &
\end{tikzcd}$$

A morphism of $\mathbf{C}$ is a *fibration* if it has the homotopy lifting property with respect to any object of $\mathbf{C}$.

We have seen in the category **Top** that a category may possess both a cylinder and a cocylinder. In general, if a category $\mathbf{C}$ is equipped with both a cylinder $\mathbf{I}$ and a cocylinder $\mathbf{P}$ then there are potentially two notions of homotopy. We now describe a situation where these notions coincide

Definition Let $\mathbf{C}$ be a category, let $\mathbf{I} = ((\ ) \times I, e^0, e^1, \sigma)$ be a cylinder and let $\mathbf{P} = ((\ )^I, \varepsilon^0, \varepsilon^1, s)$ be a cocylinder on $\mathbf{C}$. We say that $\mathbf{I}$ is *left adjoint* to $\mathbf{P}$ if the functor $(\ ) \times I$ is left adjoint to the functor $(\ )^I$ and if the family of bijections

$$\alpha_{X,Y} : \hom_{\mathbf{C}}(X \times I, Y) \longrightarrow \hom_{\mathbf{C}}(X, Y^I)$$

which demonstrate the adjunction also satisfy the following two conditions: for all morphisms $\phi : X \times I \longrightarrow Y$ and $f : X \longrightarrow Y$, and for $i = 0, 1$ we have that

$$\varepsilon^i_Y \alpha_{X,Y}(\phi) = \phi e^i_X \quad \text{and} \quad \alpha_{X,Y}(f \sigma_X) = s_Y f.$$

We refer the reader to Kamps and Porter [9], Proposition II.3.6, for proof of the following result.

Theorem 5.1 *Let $\mathbf{C}$ be a category equipped with a cylinder $\mathbf{I}$ and a cocylinder $\mathbf{P}$ such that $\mathbf{I}$ is left adjoint to $\mathbf{P}$. Then two morphisms of $\mathbf{C}$ are homotopic with respect to $\mathbf{I}$ if, and only if, they are homotopic with respect to $\mathbf{P}$.*

■

Although we have defined fibrations with respect to a cocylinder, we have seen that in the category **Top** fibrations can be defined with respect to the cylinder and the notions are equivalent. In the abstract we have that following result (Proposition II.3.7 of [9]).

Theorem 5.2 *Let* $\mathbf{C}$ *be a category equipped with a cylinder* $\mathbf{I}$ *and a cocylinder* $\mathbf{P}$ *such that* $\mathbf{I}$ *is left adjoint to* $\mathbf{P}$. *Then a morphism* $p : E \longrightarrow B$ *is a fibration with respect to* $\mathbf{P}$ *if, and only if, the following homotopy lifting property with respect to the cylinder* $\mathbf{I}$ *holds: in each commutative diagram of solid arrows*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "e_Y^0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "\phi"'] \arrow[ur, dotted, "\Phi"] & B \end{tikzcd}$$

there is a morphism $\Phi : Y \times I \longrightarrow E$ *such that* $p\Phi = \phi$ *and* $\Phi e_Y^0 = f$.

■

We shall not make explicit all the relationships between cocylinders and cylinders. It will suffice to mention the heuristic principle known as *Eckmann-Hilton duality* which assigns to a theorem involving homotopy equivalences, cofibrations and fibrations a dual statement obtained as follows:

- Invert all arrows.
- Replace fibrations by cofibrations and *vice versa*.
- Leave homotopy equivalences unchanged.

Note that Eckmann-Hilton duality is not exactly equivalent to categorical duality. The Eckmann-Hilton dual of a result is not necessarily true, and if it is then a separate proof may be required.

The following result is the Eckmann-Hilton dual of Proposition I.2.7 of Kamps and Porter [9].

Proposition 5.3 *If in the pullback diagram in* $\mathbf{C}$,

$$\begin{tikzcd} Z \arrow[r, "v"] \arrow[d, "j"'] & X \arrow[d, "i"] \\ C \arrow[r, "u"'] & A \end{tikzcd}$$

i *is a fibration and* $(\ )^I$ *preserves pullbacks, then* j *is a fibration.*

PROOF. Suppose that we have morphisms $\phi : Y \longrightarrow C^I$ and $f : Y \longrightarrow Z$ with $\varepsilon^0_C\phi = jf$. Now f is a morphism to a pullback, therefore it is uniquely determined by its components jf and vf; that is if $g : Y \longrightarrow Z$ is a morphism such that $jg = jf$ and $vg = vf$, then $g = f$. Consider the diagram below

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\phi"'] & Z \arrow[r, "v"] \arrow[d, "j"'] & X \arrow[d, "i"] \\ C^I \arrow[r, "\varepsilon^0_C"'] & C \arrow[r, "u"'] & A \end{tikzcd}$$

Since ε^0 is natural we have $u\varepsilon^0_C = \varepsilon^0_A u^I$, so the diagram below commutes

$$\begin{tikzcd} Y \arrow[r, "vf"] \arrow[d, "u^I\phi"'] & X \arrow[d, "i"] \\ A^I \arrow[r, "\varepsilon^0_A"'] & A \end{tikzcd}$$

Since i is a fibration, there is a morphism $\Phi : Y \longrightarrow X^I$, such that $u^I\phi = i^I\Phi$ and $vf = \varepsilon^0_X\Phi$.

Now, since $(\ )^I$ is assumed to preserve pullbacks, the diagram below is a pullback

$$\begin{tikzcd} Z^I \arrow[r, "v^I"] \arrow[d, "j^I"'] & X^I \arrow[d, "i^I"] \\ C^I \arrow[r, "u^I"'] & A^I \end{tikzcd}$$

and since $u^I\phi = i^I\Phi$, we have that there is a unique morphism $\Psi : Y \longrightarrow Z^I$ such that $v^I\Psi = \Phi$ and $j^I\Psi = \phi$. Consider the diagram below

$$\begin{tikzcd} & & C^I \arrow[dr, "\varepsilon^0_C"] & \\ Y \arrow[r, "\Psi"] \arrow[urr, bend left, "\phi"] \arrow[drr, bend right, "f"'] & Z^I \arrow[ur, "j^I"] \arrow[dr, "\varepsilon^0_Z"'] & & C \\ & & Z \arrow[ur, "j"'] & \end{tikzcd}$$

to show that this commutes, it only remains to verify that $\varepsilon^0_Z\Psi = f$. However

$$(j\varepsilon^0_Z)\Psi = (\varepsilon^0_C j^I)\Psi = \varepsilon^0_C\phi = jf \quad \text{and} \quad (v\varepsilon^0_Z)\Psi = (\varepsilon^0_X v^I)\Psi = \varepsilon^0_X\Phi = vf.$$

Hence, by uniqueness of f, $\varepsilon^0_Z\Psi = f$ and j is a fibration.

■

5.2 Cubical enrichment and Kan conditions

In a category equipped with a cylinder (respectively, cocylinder) we have defined what it means for morphisms to be homotopic. In the category **Top** this is an equivalence relation, but in general all that we can assert is that this relation is reflexive. In order to do homotopy in a category with a cylinder (respectively, cocylinder) we therefore need extra structure. This is achieved by making explicit a cubical structure induced by a cylinder (respectively, cocylinder), we can then impose extra structure by demanding that certain conditions — called 'Kan conditions' — are satisfied. To explain this cubical enrichment we need to study cubical sets.

In the category **Top** consider an element $(t_1, t_2, \dots, t_{n-1})$ of I^{n-1} (so $t_i \in I$). There is a continuous map

$$0^i_n : I^{n-1} \longrightarrow I^n \qquad \text{with} \qquad 0^i_n : (t_1, \dots, t_{n-1}) \longmapsto (t_1, \dots, t_{i-1}, 0, t_i \dots, t_{n-1})$$

inserting a 0 in the i^{th} position. Similarly, there is a map 1^i_n which inserts a 1 in the i^{th} position. If X and Y are topological spaces, then for $\xi = 0, 1$ we can form maps

$$\xi : \hom(X \times I^n, Y) \longrightarrow \hom(X \times I^{n-1}, Y) \qquad \text{with} \qquad \xi : f \longmapsto f(\mathrm{Id}_X \times \xi^i_n).$$

There is also a continuous map

$$\widetilde{\zeta}^j_n : I^{n+1} \longrightarrow I^n \qquad \text{with} \qquad \widetilde{\zeta}^j_n : (t_1, \dots, t_{n+1}) \longmapsto (t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{n+1})$$

omitting the j^{th} coordinate, we can then form induced maps

$$\zeta^j_n : \hom(X \times I^n, Y) \longrightarrow \hom(X \times I^{n+1}, Y) \qquad \text{with} \qquad \zeta : f \longmapsto f(\mathrm{Id}_X \times \widetilde{\zeta}^j_n).$$

This example motivates the definition of a cubical set.

Definition A *cubical set* Q consists of a sequence of sets Q_n, where $n \in \mathbb{N}$, together with three families of functions

$$0^i_n, 1^i_n : Q_n \longrightarrow Q_{n-1} \qquad \text{where } i \in \{1, 2, \dots, n\}$$

called *face operators*, and

$$\sigma^j_n : Q_n \longrightarrow Q_{n+1} \qquad \text{where } j \in \{1, 2, \dots, n+1\}$$

called *degeneracy operators* such that, for ω, $e \in \{0, 1\}$, the following identities are satisfied:

$$\begin{array}{llcll}
\text{(Q1)} & e^i_{n-1}\omega^j_n & = & \omega^{j-1}_{n-1}e^i_n & i < j \\
\text{(Q2)} & \sigma^i_{n+1}\sigma^j_n & = & \sigma^{j+1}_{n+1}\sigma^i_n & i \leqslant j \\
\text{(Q3)} & e^i_{n+1}\sigma^j_n & = & \sigma^{j-1}_{n-1}e^i_n & i < j \\
\text{(Q4)} & e^i_{n+1}\sigma^i_n & = & \mathrm{Id}_{Q_n} & \\
\text{(Q5)} & e^i_{n+1}\sigma^j_n & = & \sigma^j_{n-1}e^{i-1}_n & i > j.
\end{array}$$

The elements of Q_n are called *n-cubes.*

It is easy to check that the continuous maps ξ and ζ above satisfy the axioms (Q1)–(Q5).

The following notation will be useful:

$$\begin{aligned} \phi_\nu &= \nu_1^1(\phi) \quad \text{where} \quad \phi \in Q_1, \quad \nu \in \{0,1\}, \\ \psi_e^i &= e_n^i(\psi) \quad \text{where} \quad \psi \in Q_n, \quad e \in \{0,1\}, \quad i \in \{1,2,\ldots,n\}. \end{aligned}$$

If $\psi \in Q_n$ is an n-cube, then

$$D\psi = (\psi_0^1, \psi_1^1, \psi_0^2, \psi_1^2, \ldots, \psi_0^n, \psi_1^n)$$

is called the *boundary* of ψ.

Let Q, Q' be cubical sets. A morphism $f : Q \longrightarrow Q'$ is a sequence of maps, $f_n : Q_n \longrightarrow Q'_n$, commuting with face and degeneracy operations, such a morphism is called a *cubical map.* Thus we get a category **Cub** of cubical sets.

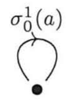
Cubical sets can best be regarded as expressing the geometric relationships between points, line segments, cubes, etc. To see how, let Q be a cubical set. The two maps

$$0_1^1, 1_1^1 : Q_1 \longrightarrow Q_0$$

mean that we can regard Q_1 as a set of edges and Q_0 as a set of vertices, thus obtaining a directed graph with source function 0_1^1 and target function 1_1^1. We represent $\phi \in Q_1$ pictorially as follows:

$$\phi_0 \bullet \xrightarrow{\ \phi\ } \bullet\ \phi_1$$

If a is a 0-cube, we can use the degeneracy operator σ_0^1 to form a 1-cube $\sigma_0^1(a)$. However condition (Q4) requires that $e_1^1\sigma_0^1(a) = a$ for $e \in \{0,1\}$, and so we think of $\sigma_0^1(a)$ as being a loop at a.



We call $\sigma_0^1(a)$ a *degenerate* 1*-cube.*

Let ψ be a 2-cube with boundary $(\psi_0^1, \psi_1^1, \psi_0^2, \psi_1^2)$. Then by (Q1) we have $(\psi_0^2)_1 = (\psi_1^1)_0$. Thus the source of ψ_1^1 is the target of ψ_0^2. In the same way $(\psi_0^2)_0 = (\psi_1^2)_0$, $(\psi_0^1)_1 = (\psi_1^2)_0$ and $(\psi_1^2)_1 = (\psi_1^1)_1$. Thus the 1-cubes that form the boundary of ψ fit together to form the edges of a square.

$$
\begin{tikzcd}
\bullet \arrow[r, "\psi_1^2"] & \bullet \\
\bullet \arrow[u, "\psi_0^1"] \arrow[r, "\psi_0^2"'] \arrow[ur, phantom, "\psi"] & \bullet \arrow[u, "\psi_1^1"']
\end{tikzcd}
$$

Consequently, we can regard the elements of Q_2 as being squares, and the four face maps

$$0_2^1, 1_2^1, 0_2^2, 1_2^2 : Q_2 \longrightarrow Q_1$$

as assigning to each square the edges on its boundary.

Let ϕ be a 1-cube. Then there is a *degenerate 2-cube* $\sigma_1^1(\phi)$. By (Q4), $e_2^1\sigma_1^1(\phi) = \phi$, for $e = 0, 1$, and by (Q5) $e_2^2\sigma_1^1(\phi) = \sigma_0^1(\phi_e)$, for $e \in \{0, 1\}$. So $\sigma_1^1(\phi)$ has boundary $(\phi, \phi, \sigma_0^1\phi_0, \sigma_0^1\phi_1)$, which we can depict as follows:

$$
\begin{tikzcd}
\bullet \arrow[r, "\sigma_0^1(\phi_1)"] & \bullet \\
\bullet \arrow[u, "\phi"] \arrow[r, "\sigma_0^1(\phi_0)"'] \arrow[ur, phantom, "\sigma_1^1(\phi)"] & \bullet \arrow[u, "\phi"']
\end{tikzcd}
$$

Similarly, there is a degenerate 2-cube $\sigma_2^1(\phi)$, as shown below.

$$
\begin{tikzcd}
\bullet \arrow[r, "\phi"] & \bullet \\
\bullet \arrow[u, "\sigma_0^1(\phi_0)"] \arrow[r, "\phi"'] \arrow[ur, phantom, "\sigma_1^2(\phi)"] & \bullet \arrow[u, "\sigma_0^1(\phi_1)"']
\end{tikzcd}
$$

Let $\mathbf{P} = ((\ )^I, \varepsilon^0, \varepsilon^1, s)$ be a cocylinder on a category $\mathbf{C}$. For each pair of objects X and Y in $\mathbf{C}$ we shall construct a cubical set $Q_{\mathbf{P}}(X,Y)_n$, making essential use of the cocylinder. First define the iterated cocylinder functor $(\ )^{I^n}$ by $Y^{I^0} = Y$, $Y^{I^1} = Y^I$ and $Y^{I^n} = (Y^{I^{n-1}})^I$. Let X and Y be a pair of objects in $\mathbf{C}$. Put

$$Q_{\mathbf{P}}(X,Y)_n = \hom(X, Y^{I^n}).$$

We shall show that the sequence of sets $Q_{\mathbf{P}}(X,Y) = \{Q_{\mathbf{P}}(X,Y)_n \mid n \in \mathbb{N}\}$ is a cubical set, to do this we need to define the face and degeneracy operators in $Q_{\mathbf{P}}(X,Y)$. For

$e = 0, 1$ and $i \in \{1, \ldots, n\}$ define functions

$$\begin{aligned} e_n^i(Y) : Y^{I^n} &\longrightarrow Y^{I^{n-1}} \quad \text{by} \quad e_n^i(Y) = (\varepsilon^e_{Y^{I^{n-i}}})^{I^{i-1}} \\ \text{and} \quad \sigma_n^j(Y) : Y^{I^n} &\longrightarrow Y^{I^{n+1}} \quad \text{by} \quad \sigma_n^j(Y) = (s_{Y^{I^{n-j+1}}})^{I^{j-1}}. \end{aligned}$$

In $Q_{\mathbf{P}}(X, Y)$ define

$$e_n^i : Q_{\mathbf{P}}(X, Y)_n \longrightarrow Q_{\mathbf{P}}(X, Y)_{n-1} \quad \text{by} \quad e_n^i(f) = e_n^i(Y)f$$

and

$$\sigma_n^j : Q_{\mathbf{P}}(X, Y)_n \longrightarrow Q_{\mathbf{P}}(X, Y)_{n+1} \quad \text{by} \quad \sigma_n^j(f) = \sigma_n^j(Y)f.$$

Proposition 5.4 *Let* $\mathbf{C}$ *be a category equipped with a cocylinder* $\mathbf{P}$, *and let* X *and* Y *be objects of* $\mathbf{C}$. *Then with the above definitions, the functions* e_n^i *and* σ_n^j *satisfy the axioms (Q1)-(Q5). Thus* $Q_{\mathbf{P}}(X, Y)$ *is a cubical set.*

PROOF. We need to show that e_n^i and σ_n^j satisfy the axioms (Q1)-(Q5), to do this, we use the Godement interchange law

(Q1): Suppose we are given the functors and natural transformations, shown below

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left=2, "(\ )^I"] \arrow[r, shift right=2, "\mathrm{Id}_{\mathbf{C}}"'] \arrow[r, phantom, "\downarrow\varepsilon^\omega"] & \mathbf{C} \arrow[r, shift left=2, "(\ )^I"] \arrow[r, shift right=2, "\mathrm{Id}_{\mathbf{C}}"'] \arrow[r, phantom, "\downarrow\varepsilon^e"] & \mathbf{C} \end{tikzcd}$$

By the interchange law, we have that for any object A of $\mathbf{C}$

$$\varepsilon^\omega_A \varepsilon^e_{A^I} = \varepsilon^e_A (\varepsilon^\omega_A)^I,$$

from this we have that, for $i < j$

$$(\varepsilon^\omega_{Y^{I^{n-j}}})^{I^{j-2}} (\varepsilon^e_{(Y^{I^{n-i-1}})^I})^{I^{i-1}} = (\varepsilon^e_{Y^{I^{n-i-1}}})^{I^{i-1}} ((\varepsilon^\omega_{Y^{I^{n-j}}})^{I^{j-2}})^I,$$

that is $(\varepsilon^\omega_{Y^{I^{n-j}}})^{I^{j-2}} (\varepsilon^e_{Y^{I^{n-i}}})^{I^{i-1}} = (\varepsilon^e_{Y^{I^{n-i-1}}})^{I^{i-1}} (\varepsilon^\omega_{Y^{I^{n-j}}})^{I^{j-1}}$ Hence $\omega^{j-1}_{n-1} e^i_n = e^i_{n-1} \omega^j_n$.

(Q2): Similar to (Q1).

(Q3): Applying the interchange law to the situation pictured below

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left=2, "\mathrm{Id}_{\mathbf{C}}"] \arrow[r, shift right=2, "(\ )^I"'] \arrow[r, phantom, "\downarrow s"] & \mathbf{C} \arrow[r, shift left=2, "(\ )^I"] \arrow[r, shift right=2, "\mathrm{Id}_{\mathbf{C}}"'] \arrow[r, phantom, "\downarrow\varepsilon^e"] & \mathbf{C} \end{tikzcd}$$

we have that, for $i < j$

$$(s_{Y^{I^{n-j+1}}})^{I^{j-2}} (\varepsilon^e_{Y^{I^{n-i}}})^{I^{i-1}} = (\varepsilon^e_{(Y^{I^{n-i}})^I})^{I^{i-1}} ((s_{Y^{I^{n-j+1}}})^{I^{j-2}})^I,$$

that is $(s_{Y^{I^{n-j+1}}})^{I^{j-2}} (\varepsilon^e_{Y^{I^{n-i}}})^{I^{i-1}} = (\varepsilon^e_{Y^{I^{n-i+1}}})^{I^{i-1}} (s_{Y^{I^{n-j+1}}})^{I^{j-1}}$. Hence $\sigma^{j-1}_{n-1} e^i_n = e^i_{n+1} \sigma^j_n$, and (Q3) holds.

(Q4): $(\varepsilon^{e}_{Y^{I^{n-i+1}}})^{I^{i-1}}(s_{Y^{I^{n-i+1}}})^{I^{i-1}} = (\varepsilon^{e}_{Y^{I^{n-i+1}}} s_{Y^{I^{n-i+1}}})^{I^{i-1}} = (\mathrm{Id}_{Y^{I^{n-i+1}}})^{I^{i-1}} = \mathrm{Id}_{Y^{I^n}}$ and so $e^i_{n+1}\sigma^i_n = \mathrm{Id}_{Q_{\mathbf{P}}(X,Y)_n}$.

(Q5): Similar to (Q3).

■

In a similar way it is possible to construct cubical sets $Q^{\mathbf{I}}(X,Y)$ on a category which has cylinder $\mathbf{I}$. This dual construction is described in Section I.5 of Kamps and Porter [9].

If a category $\mathbf{C}$ has a cocylinder $\mathbf{P}$, and $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$ are morphisms in $\mathbf{C}$ then define

$$Q_{\mathbf{P}}(f^{\mathrm{op}}, g)_n : Q_{\mathbf{P}}(X,Y)_n \longrightarrow Q_{\mathbf{P}}(X',Y')_n \qquad \text{by} \qquad Q_{\mathbf{P}}(f^{\mathrm{op}}, g)_n : \eta \longmapsto g^{I^n}\eta f^{\mathrm{op}}.$$

If X and Y are objects of $\mathbf{C}$, then clearly $\mathrm{Id}_{Q_{\mathbf{P}}(X,Y)} = Q_{\mathbf{P}}(\mathrm{Id}_X, \mathrm{Id}_Y)$. If $f_1 : X_1 \longrightarrow X_2$, $f_2 : X_2 \longrightarrow X_3$, $g_1 : Y_1 \longrightarrow Y_2$ and $g_2 : Y_2 \longrightarrow Y_3$ are morphisms in $\mathbf{C}$ and $\eta \in Q_{\mathbf{P}}(X_1,Y_1)_n$, then

$$\begin{aligned} Q_{\mathbf{P}}(f_2^{\mathrm{op}}, g_2)Q_{\mathbf{P}}(f_1^{\mathrm{op}}, g_1)(\eta) &= g_2^{I^n}(g_1^{I^n}\eta f_1^{\mathrm{op}})f_2^{\mathrm{op}} \\ &= (g_2g_1)^{I^n}\eta(f_2f_1)^{\mathrm{op}} \\ &= Q_{\mathbf{P}}((f_2f_1)^{\mathrm{op}}, g_2g_1)(\eta). \end{aligned}$$

Hence there is a functor

$$Q_{\mathbf{P}} : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{Cub}.$$

In what follows, we will mainly be concerned with 0-cubes, 1-cubes and 2-cubes. We make the above definitions explicit in these cases. The 0-cubes and 1-cubes in $Q_{\mathbf{P}}(X,Y)$ have simple interpretations. A 0-cube f is simply a morphism $f : X \longrightarrow Y$.

If $f \in Q_{\mathbf{P}}(X,Y)_1$ is a 1-cube, then

$$f_0 = 0^1_1(f) = \varepsilon^0_Y(f) \quad \text{and} \quad f_1 = 1^1_1(f) = \varepsilon^1_Y(f).$$

Thus a 1 cube f with boundary $Df = (f_0, f_1)$ is nothing other than a homotopy $f : X \longrightarrow Y^I$ from $f_0 = \varepsilon^0_Y(f)$ to $f_1 = \varepsilon^1_Y(f)$. Given a 0-cube, or morphism, $f : X \longrightarrow Y$, a degenerate 1-cube is the constant homotopy $s_Y f$.

If $f \in Q_{\mathbf{P}}(X,Y)_2$, then

$$\begin{aligned} f^1_0 &= 0^1_2(f) = \varepsilon^0_{Y^I}(f), \\ f^2_0 &= 0^2_2(f) = (\varepsilon^0_Y)^I(f), \\ f^1_1 &= 1^1_2(f) = \varepsilon^1_{Y^I}(f), \\ f^2_1 &= 1^2_2(f) = (\varepsilon^1_Y)^I(f). \end{aligned}$$

As illustrated below.

$$
\begin{tikzcd}
\bullet \arrow[r, "(\varepsilon^1_Y)^I(f)"] & \bullet \\
\bullet \arrow[u, "\varepsilon^0_{Y^I}(f)"] \arrow[r, "(\varepsilon^0_Y)^I(f)"'] \arrow[ur, phantom, "f"] & \bullet \arrow[u, "\varepsilon^1_{Y^I}(f)"']
\end{tikzcd}
$$

We shall be interested in those cylinders where the corresponding cubical sets satisfy certain conditions known as 'Kan conditions'. To define these conditions we need first the notion of a (n, ν, k)-box.

Definition Let Q be a cubical set and let (n, ν, k) be a triple of natural numbers with $\nu \in \{0, 1\}$ and $k \in \{1, 2, \ldots, n\}$. An (n, ν, k)-*box* γ in Q is a tuple

$$(\gamma_e^i \mid e = 0, 1, \ i = 1, 2, \ldots, n, \ (e, i) \neq (\nu, k))$$

of elements of Q_{n-1} satisfying

$$e_{n-1}^i \gamma_\omega^j = \omega_{n-1}^{j-1} \gamma_e^i, \quad \text{for} \quad i < j \quad \text{and} \quad (e, i), (\omega, j) \neq (\nu, k).$$

An (n, ν, k)-box γ will be denoted by

$$\gamma = (\gamma_0^1, \gamma_1^1, \ldots, -, \ldots, \gamma_0^n, \gamma_1^n),$$

where the '$-$' occurs in the (ν, k) position. We write $Q_{(n,\nu,k)}$ for the set of all (n, ν, k)-boxes in Q.

An n-cube $\lambda \in Q_n$ is called a *filler* of $\gamma \in Q_{(n,\nu,k)}$ if $\lambda_e^i = \gamma_e^i$ for all $(e, i) \neq (\nu, k)$. A cubical set Q is said to satisfy the *Kan condition* $E(n, \nu, k)$ if every (n, ν, k)-box has a filler. If Q satisfies $E(n, \nu, k)$ for a fixed n and all $\nu = 0, 1$ and $k \in \{1, \ldots, n\}$, then we say that Q satisfies the *Kan condition* $E(n)$.

We impose extra structure on a cocylinder $\mathbf{P}$ by demanding various Kan filler conditions on $Q_{\mathbf{P}}(X, Y)$. A cocylinder $\mathbf{P}$ on a category $\mathbf{C}$ is said to satisfy the *Kan condition* $E(n, \nu, k)$ ($\nu \in \{0, 1\}$ and $k \in \{1, \ldots, n\}$) if for any objects X and Y of $\mathbf{C}$, the cubical set $Q_{\mathbf{P}}(X, Y)$ satisfies the Kan condition $E(n, \nu, k)$.

The two results in the theorem below are obtained by dualising Propositions I.5.8 and I.5.6 respectively of [9].

Theorem 5.5 *Let* $\mathbf{P} = ((\ )^I, \varepsilon^0, \varepsilon^1, s)$ *be a cocylinder on a category* $\mathbf{C}$*. Then*

(i) *If the cocylinder satisfies the Kan condition* $E(2, 1, 1)$*, then the homotopy relation* $\simeq$ *is an equivalence relation.*

(ii) *If the cocylinder satisfies the Kan condition $E(2)$, then for each object Y in* $\mathbf{C}$ *the morphisms ε^0_Y and ε^1_Y are fibrations and s_Y is a homotopy equivalence.*

PROOF. (i) Any morphism is homotopic to itself via its constant homotopy. Thus $\simeq$ is reflexive.

To show that the $\simeq$ is symmetric. Let $f, g : X \longrightarrow Y$ be morphisms in $\mathbf{C}$ and $\phi : X \longrightarrow Y^I$ be a homotopy from f to g; that is $\varepsilon^0_Y \phi = f$ and $\varepsilon^1_Y \phi = g$. We show that

$$(s_Y f, -, \phi, s_Y f)$$

is a $(2,1,1)$-box. To do this we need $(s_Y f)_0 = (s_Y f)_1$ and $\phi_0 = (s_Y f)_0$. But

$$(s_Y f)_0 = \varepsilon^0_Y s_Y f = f, \quad (s_Y f)_1 = \varepsilon^1_Y s_Y f = f \quad \text{and} \quad \phi_0 = \varepsilon^0_Y \phi = f.$$

So $(s_Y f, -, \phi, s_Y f)$ is a $(2,1,1)$-box. Since $E(2)$ holds, this box has filler $\lambda : X \longrightarrow Y^{I^2}$. Put $\phi' = \lambda^1_1 = \varepsilon^1_{Y^I}(\lambda)$. We have

$$(\phi')_0 = 0^1_1 \lambda^1_1 = 0^1_1 1^1_2(\lambda) = 1^1_1 0^2_2(\lambda)$$

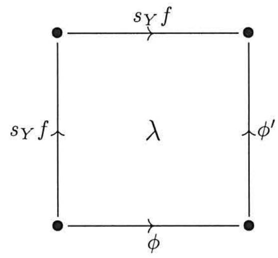
by the cubical condition (Q1), therefore

$$(\phi')_0 = 1^1_1 \lambda^2_0 = 1^1_1 \phi = \varepsilon^1_Y(\phi) = g,$$

as we have a box. Hence $(\phi')_0 = \phi_1 = \varepsilon^1_Y(\phi) = g$. Similarly,

$$(\phi')_1 = (s_Y f)_1 = \varepsilon^1_Y s_Y f = f.$$

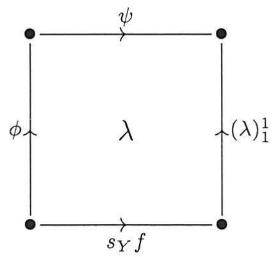
Thus ϕ' is a homotopy from g to f as required. The various maps involved are illustrated in the following diagram:



To show that $\simeq$ is transitive, let $f, g, h : X \longrightarrow Y$ be morphisms in $\mathbf{C}$ such that there is a homotopy ϕ from f to g and a homotopy ψ from g to h. It is easy to check that

$$(\phi, -, s_Y f, \psi)$$

is a $(2,1,1)$-box. Thus there is a filler $\lambda : X \longrightarrow Y^{I^2}$.



It is easy to check that $(\lambda_1^1)_0 = f$ and $(\lambda_1^1)_1 = h$. Thus λ_1^1 is a homotopy from f to h as required.

(ii) We prove explicitly that ε_Y^0 is a fibration; the proof that ε_Y^1 is a fibration is similar. Let $\phi, f : X \longrightarrow Y^I$ be morphisms such that $\varepsilon_Y^0\phi = \varepsilon_Y^0 f$. Then, together with $\varepsilon_Y^0(s_Y\varepsilon_Y^1 f) = \varepsilon_Y^1 f$, the 4-tuple

$$(f, -, \phi, s_Y\varepsilon_Y^1 f)$$

is a $(2,1,1)$-box in $Q_{\mathbf{P}}(X,Y)_1$. Thus by $E(2)$ there is a filler $\lambda : X \longrightarrow Y^{I^2}$ such that $\varepsilon_{Y^I}^0\lambda = f$ and $(\varepsilon_Y^0)^I\lambda = \phi$. It follows that ε_Y^0 is a fibration.

To show that s_Y is a homotopy equivalence, we only need to prove that $s_Y\varepsilon_Y^0 \simeq \mathrm{Id}_Y$. It is easy to check that

$$(\mathrm{Id}_{Y^I}, s_Y\varepsilon_Y^0, s_Y\varepsilon_Y^0, -)$$

is a $(2,1,2)$-box in $Q_{\mathbf{P}}(Y^I, Y)_1$. Thus there is a filler $\lambda : Y^I \longrightarrow Y^{I^2}$ such that $\varepsilon_{Y^I}^0\lambda = \mathrm{Id}_{Y^I}$ and $\varepsilon_{Y^I}^1\lambda = s_Y\varepsilon_Y^0$. Thus $s_Y\varepsilon_Y^0$ and Id_{Y^I} are homotopic as required.

■

5.3 The mapping cocylinder factorisation

In Theorem 5.7 we shall use the Kan conditions introduced in the previous section to examine properties of the 'mapping cocylinder factorisation' which we now define.

Let $\mathbf{C}$ be a category equipped with a cocylinder $\mathbf{P}$. Let $f : X \longrightarrow Y$ be a morphism in $\mathbf{C}$ and suppose that the pullback diagram of f and ε_Y^0 illustrated below exists

$$\begin{tikzcd} M^f \arrow[r, "\pi^f"] \arrow[d, "j^f"'] & Y^I \arrow[d, "\varepsilon_Y^0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Then M^f is called a *mapping cocylinder* of f. Now the diagram below commutes

$$\begin{tikzcd} X \arrow[r, "s_Y f"] \arrow[d, "\mathrm{Id}_X"'] & Y^I \arrow[d, "\varepsilon_Y^0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Consequently, there is a unique morphism $p^f : X \longrightarrow M^f$ such that $\pi^f p^f = s_Y f$ and $j^f p^f = \mathrm{Id}_X$. Put $i^f = \varepsilon_Y^1\pi^f$. Then $i^f p^f = \varepsilon_Y^1\pi^f p^f = \varepsilon_Y^1 s_Y f = f$. It follows that we have constructed the following factorisation of f

$$\begin{tikzcd} & M^f \arrow[dr, "i^f"] & \\ X \arrow[ur, "p^f"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

It is called the *mapping cocylinder factorisation.*

The Eckmann-Hilton dual of the mapping cocylinder is the 'mapping cylinder' which is constructed using pushouts. The dual of the following result can be found in [9] (Theorem I.5.11).

Proposition 5.6 *Let* $\mathbf{C}$ *be a category, with cocylinder* $\mathbf{P} = ((\ )^I, \varepsilon^0, \varepsilon^1, s)$, *and let* $f : X \longrightarrow Y$ *be a morphism of* $\mathbf{C}$ *which has the mapping cocylinder factorisation pictured above. Suppose that* $(\ )^I$ *preserves pullbacks and* $\mathbf{P}$ *satisfies* $E(2)$. *Then*

(i) p^f *is a homotopy equivalence with homotopy inverse* j^f; *in particular,* X *is a deformation retract of* M^f.

(ii) i^f *is a fibration.*

PROOF. (i) We show that $p^f j^f \simeq \mathrm{Id}_{M^f}$.

Define the following three elements of $Q_{\mathbf{P}}(M^f, Y)_1$

$$\gamma_0^1 = f^I s_X j^f, \quad \gamma_0^2 = s_Y f j^f \quad \text{and} \quad \gamma_1^1 = \pi^f$$

Since $f^I s_X = s_Y f$, we have that $f^I s_X j^f = s_Y f j^f$. Also

$$\begin{aligned}
(\gamma_0^2)_1 &= \varepsilon_Y^1(s_Y f j^f) = (\varepsilon_Y^1 s_Y) f j^f = f j^f \\
(\gamma_1^1)_0 &= \varepsilon_Y^0 \pi^f = f j^f \\
(\gamma_0^2)_0 &= \varepsilon_Y^0(s_Y f j^f) = (\varepsilon_Y^0 s_Y) f j^f = f j^f \\
(\gamma_0^1)_0 &= \varepsilon_Y^0(\gamma_0^1) = \varepsilon_Y^0(\gamma_0^2) = (\gamma_0^2)_0.
\end{aligned}$$

Put $\gamma = (\gamma_0^1, \gamma_1^1, \gamma_0^2, -) = (f^I s_X j^f, \pi^f, s_Y f j^f, -)$. Then we have shown that γ is a $(2,1,2)$-box in $Q_{\mathbf{P}}(M^f, Y)$. By assumption, $\mathbf{P}$ satisfies $E(2)$. Thus there is a filler $\lambda : M^f \longrightarrow Y^{I^2}$ such that

$$\lambda_0^1 = \varepsilon_{Y^I}^0 \lambda = f^I s_X j^f, \quad \lambda_1^1 = \varepsilon_{Y^I}^1 \lambda = \pi^f \quad \text{and} \quad \lambda_0^2 = (\varepsilon_Y^0)^I \lambda = s_Y f j^f = f^I s_X j^f.$$

By assumption $(\ )^I$ preserves pullbacks. Thus the inner square in the diagram below is a pullback. But it is easy to check that the solid arrows commute.

$$\begin{tikzcd}
M^f \arrow[dr, dotted, "\psi"] \arrow[drr, bend left, "\lambda"] \arrow[ddr, bend right, "s_X j^f"'] & & \\
& (M^f)^I \arrow[r, "(\pi^f)^I"] \arrow[d, "(j^f)^I"'] & Y^{I^2} \arrow[d, "(\varepsilon^0_Y)^I"] \\
& X^I \arrow[r, "f^I"'] & Y^I
\end{tikzcd}$$

Consequently, there is a unique morphism $\psi : M^f \longrightarrow (M^f)^I$ such that

$$(\pi^f)^I\psi = \lambda \qquad \text{and} \qquad (j^f)^I\psi = s_X j^f.$$

We shall prove that ψ is a homotopy from $p^f j^f$ to Id_{M^f}.

We prove first that $\psi_0 = p^f j^f$. Consider $j^f\psi_0 = j^f(\varepsilon^0_{M^f}\psi)$. The diagram below commutes since ε^0 is a natural transformation.

$$\begin{tikzcd}
(M^f)^I \arrow[r, "(j^f)^I"] \arrow[d, "\varepsilon^0_{M^f}"'] & X^I \arrow[d, "\varepsilon^0_X"] \\
M^f \arrow[r, "j^f"'] & X
\end{tikzcd}$$

Thus we have that

$$j^f\psi_0 = \varepsilon^0_X (j^f)^I\psi = \varepsilon^0_X s_X j^f = \mathrm{Id}_X j^f = j^f p^f j^f$$

Next consider $\pi^f\psi_0$. Again using the fact that ε^0 is a natural transformation we have that

$$\pi^f\psi_0 = \pi^f\varepsilon^0_{M^f}\psi = \varepsilon^0_{Y^I}(\pi^f)^I\psi = \varepsilon^0_{Y^I}\lambda = f^I s_X j^f = s_Y f j^f = \pi^f p^f j^f.$$

Hence the diagram below commutes.

$$\begin{tikzcd}
M^f \arrow[dr, "\psi_0"] \arrow[drr, bend left, "\pi^f p^f j^f"] \arrow[ddr, bend right, "j^f p^f j^f"'] & & \\
& M^f \arrow[r, "\pi^f"] \arrow[d, "j^f"'] & Y^I \arrow[d, "\varepsilon^0_Y"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

But the inner square is a pullback. Thus ψ_0 is unique making the diagram commute. Hence $\psi_0 = p^f j^f$, as required.

We now prove that $\psi_1 = \mathrm{Id}_{M^f}$. By definition $\psi_1 = \varepsilon^1_{M^f}\psi$. We calculate the composites $j^f\psi_1$ and $\pi^f\psi_1$ much as before. We have that

$$\begin{aligned} j^f\psi_1 &= j^f\varepsilon^1_{M^f}\psi = \varepsilon^1_X(j^f)^I\psi = \varepsilon^1_X s_X j^f = j^f \\ \text{and} \qquad \pi^f\psi_1 &= \pi^f\varepsilon^1_{M^f}\psi = \varepsilon^1_{Y^I}\lambda = \pi^f. \end{aligned}$$

Thus $\psi_1 : M^f \longrightarrow M^f$ is the unique morphism so that the pullback diagram below commutes.

$$\begin{tikzcd} M^f \arrow[dr, "\psi_1"] \arrow[drr, bend left, "\pi^f"] \arrow[ddr, bend right, "j^f"'] & & \\ & M^f \arrow[r, "\pi^f"] \arrow[d, "j^f"] & Y^I \arrow[d, "\varepsilon^0_Y"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

Hence $\psi_1 = \mathrm{Id}_{M^f}$ as required.

(ii) To prove that i^f is a fibration we now need to show that the diagram below is a weak pullback. We know that it commutes because ε^0 is a natural transformation.

$$\begin{tikzcd} (M^f)^I \arrow[r, "(i^f)^I"] \arrow[d, "\varepsilon^0_{M^f}"'] & Y^I \arrow[d, "\varepsilon^0_Y"] \\ M^f \arrow[r, "i^f"'] & Y \end{tikzcd}$$

Let V be an object in $\mathbf{C}$, and let $g : V \longrightarrow M^f$ and $\phi : V \longrightarrow Y^I$ be morphisms satisfying $\varepsilon^0_Y\phi = i^f g\ (= \phi_0)$. Let

$$\gamma^1_0 = \pi^f g, \quad \gamma^2_0 = f^I s_X j^f g \quad \text{and} \quad \gamma^2_1 = \phi.$$

Then it is easy to check that

$$\begin{aligned} (\gamma^1_0)_1 &= \varepsilon^1_Y(\pi^f g) = i^f g = \phi_0, \\ (\gamma^1_0)_0 &= \varepsilon^0_Y(\pi^f g) = f j^f g, \\ \text{and} \qquad (\gamma^2_0)_0 &= \varepsilon^0_Y(f^I s_X j^f g) = f j^f g. \end{aligned}$$

Consequently, $(\gamma^1_0, -, \gamma^2_0, \gamma^2_1)$ is a $(2,1,1)$-box in $Q_{\mathbf{P}}(V,Y)$. Since $\mathbf{P}$ satisfies $E(2)$, there is a filler $\lambda : V \longrightarrow Y^{I^2}$ satisfying

$$\begin{aligned} \lambda^2_0 &= (\varepsilon^0_Y)^I \lambda = \gamma^2_0 = f^I s_X j^f g, \\ \lambda^1_0 &= \varepsilon^0_{Y^I} \lambda = \gamma^1_0 = \pi^f g, \\ \text{and} \qquad \lambda^2_1 &= (\varepsilon^1_Y)^I \lambda = \gamma^2_1 = \phi. \end{aligned}$$

From the above results and the fact that $(\ )^I$ preserves pullbacks, the solid arrows in the diagram below commute.

$$\begin{tikzcd} V \arrow[dr, dotted, "\Phi"] \arrow[drr, bend left, "\lambda"] \arrow[ddr, bend right, "s_X j^f g"'] & & \\ & (M^f)^I \arrow[r, "(\pi^f)^I"] \arrow[d, "(j^f)^I"'] & Y^{I^2} \arrow[d, "(\varepsilon^0_Y)^I"] \\ & X^I \arrow[r, "f^I"'] & Y^I \end{tikzcd}$$

Thus there is a unique morphism $\Phi : V \longrightarrow (M^f)^I$ such that

$$(j^f)^I \Phi = s_X j^f g \quad \text{and} \quad (\pi^f)^I \Phi = \lambda.$$

We can now prove that Φ is the morphism we required. Using the fact that ε^0 is a natural transformation we obtain

$$\begin{aligned} \pi^f \varepsilon^0_{M^f} \Phi &= \varepsilon^0_{Y^I} (\pi^f)^I \Phi = \varepsilon^0_{Y^I} \lambda = \pi^f g \\ \text{and} \qquad j^f \varepsilon^0_{M^f} \Phi &= \varepsilon^0_X (j^f)^I \Phi = \varepsilon^0_X s_X j^f g = j^f g. \end{aligned}$$

By assumption, the following diagram is a pullback.

$$\begin{tikzcd} M^f \arrow[r, "\pi^f"] \arrow[d, "j^f"'] & Y^I \arrow[d, "\varepsilon^0_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

It is straightforward to show that $f j^f g = \varepsilon^0_Y \pi^f g$. It follows that $\varepsilon^0_{M^f} \Phi = g$. It is easy to check that

$$(i^f)^I \Phi = (\varepsilon^1_Y \pi^f)^I \Phi = (\varepsilon^1_Y)^I \lambda = \phi.$$

Consequently the diagram below commutes

$$\begin{tikzcd}
V \arrow[dr, "\Phi"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "g"'] & & \\
& (M^f)^I \arrow[r, "(i^f)^I"] \arrow[d, "\varepsilon^0_{M^f}"'] & Y^I \arrow[d, "\varepsilon^0_Y"] \\
& M^f \arrow[r, "i^f"'] & Y
\end{tikzcd}$$

Hence result.

■

Theorem 5.7 *Let* $\mathbf{C}$ *be a category which has a cocylinder* $\mathbf{P} = ((\ )^I, \varepsilon^0, \varepsilon^1, s)$ *satisfying the Kan condition* $E(2)$*. Suppose that* $\mathbf{C}$ *has pullbacks and that* $(\ )^I$ *preserves them. Then there is a functorial factorisation of each morphism of* $\mathbf{C}$ *into a homotopy equivalence followed by a fibration.*

PROOF. Since $\mathbf{C}$ has a cocylinder and pullbacks, every morphism f of $\mathbf{C}$ has mapping cocylinder factorisation $f = i^f p^f$. By Proposition 5.6, i^f is a fibration and p^f is a homotopy equivalence with homotopy inverse j^f. It remains to show that this factorisation is functorial. Let $f_1 : X_1 \longrightarrow Y_1$ and $f_2 : X_2 \longrightarrow Y_2$ be morphisms of $\mathbf{C}$ and suppose that there are morphisms u and v such that the diagram below commutes.

$$\begin{tikzcd}
X_1 \arrow[r, "u"] \arrow[d, "f_1"'] & X_2 \arrow[d, "f_2"] \\
Y_1 \arrow[r, "v"'] & Y_2
\end{tikzcd}$$

To show that the mapping cocylinder factorisation is functorial we shall construct a unique morphism γ from the mapping cocylinder of f_1 to the mapping cocylinder of f_2 making the diagram below commute

$$\begin{tikzcd}
& M^{f_1} \arrow[ddr, "i^{f_1}"] \arrow[drrr, dotted, "\gamma"] & & & & \\
& & & & M^{f_2} \arrow[ddr, "i^{f_2}"] & \\
X_1 \arrow[uur, "p^{f_1}"] \arrow[rr, "f_1"] \arrow[drrr, "u"'] & & Y_1 \arrow[drrr, "v"] & & & \\
& & & X_2 \arrow[uur, "p^{f_2}"] \arrow[rr, "f_2"'] & & Y_2
\end{tikzcd}$$

Now

$$f_2 u j^{f_1} = v f_1 j^{f_1} = v \varepsilon^0_{Y_1} \pi^{f_1} = \varepsilon^0_{Y_2} v^I \pi^{f_1}$$

and so the solid arrows in the diagram below commute

$$\begin{tikzcd}
M^{f_1} \arrow[dr, dotted, "\gamma"] \arrow[drr, bend left, "v^I \pi^{f_1}"] \arrow[ddr, bend right, "u j^{f_1}"'] & & \\
& M^{f_2} \arrow[r, "\pi^{f_2}"] \arrow[d, "j^{f_2}"'] & Y_2^I \arrow[d, "\varepsilon^0_{Y_2}"] \\
& X \arrow[r, "f_2"'] & Y_2
\end{tikzcd}$$

Since the inner square is a pullback there is a unique morphism $\gamma : M^{f_1} \longrightarrow M^{f_2}$ such that

$$j^{f_2} \gamma = u j^f \quad \text{and} \quad \pi^{f_2} \gamma = v^I \pi^{f_1}.$$

The second condition implies that $\varepsilon^1_{Y_2} \pi^{f_2} \gamma = \varepsilon^1_{Y_2} v^I \pi^{f_1}$ and so $i^{f_2} \gamma = v \varepsilon^1_{Y_1} \pi^{f_1} = v i^{f_1}$.

It remains to show that $\gamma p^{f_1} = p^{f_2} u$. Now

$$v^I \pi^{f_1} p^{f_1} = v^I s_{Y_1} f_1 = s_{Y_2} v f_1 = s_{Y_2} f_2 u$$

and so $\varepsilon^0_{Y_2} v^I \pi^{f_1} p^{f_1} = \varepsilon^0_{Y_2} s_{Y_2} f_2 u = f_2 u$. It follows that the solid arrows in the diagram below commute

$$\begin{tikzcd}
X_1 \arrow[dr, dotted, "\mu"] \arrow[drr, bend left, "v^I \pi^{f_1} p^{f_1}"] \arrow[ddr, bend right, "u"'] & & \\
& M^{f_2} \arrow[r, "\pi^{f_2}"] \arrow[d, "j^{f_2}"'] & Y_2^I \arrow[d, "\varepsilon^0_{Y_2}"] \\
& X \arrow[r, "f_2"'] & Y_2
\end{tikzcd}$$

The inner square is a pullback and so there is a unique morphism

$$\mu : X_1 \longrightarrow M^{f_2}$$

such that

$$\pi^{f_2} \mu = v^I \pi^{f_1} p^{f_1} \quad \text{and} \quad j^{f_2} \mu = u.$$

However, $\pi^{f_2} \gamma p^{f_1} = v^I \pi^{f_1} p^{f_1}$ and $j^{f_2} \gamma p^{f_1} = u j^{f_1} p^{f_1} = u$, thus $\mu = \gamma p^{f_1}$. We have $\pi^{f_2} p^{f_2} = s_{Y_2} f_f$, therefore

$$\pi^{f_2} p^{f_2} u = \pi^{f_2} \gamma p^{f_1} = v^I \pi^{f_1} p^{f_1},$$

but $j^{f_2}p^{f_2}u = u$, hence $\mu = p^{f_2}u$. Therefore $\gamma p^{f_1} = p^{f_2}u$, as required.

■

The above result implies that in the category **Top** every continuous map can be factorised into a homotopy equivalence followed by a fibration (proved directly as Theorem 2.8.9 of [27]).

Since $j^f p^f = \mathrm{Id}_X$ we have that j^f is a split epimorphism and p^f is a split monomorphism. By Proposition 5.5(ii) ε_Y^0 is a fibration and since $(\ )^I$ is assumed to preserve pullbacks, j^f is a fibration by Proposition 5.3.

Chapter 6

The homotopy theory of ordered groupoids

In this chapter we shall apply some of the abstract homotopy introduced in the previous chapter to the category of ordered groupoids, thus obtaining notions of homotopy equivalence and fibration for inverse semigroups. We begin by constructing an adjoint cylinder and cocylinder satisfying the Kan condition $E(2)$. We then obtain some results about homotopy equivalence and fibrations in the category **OG** and consider some special cases and examples. Finally we construct the mapping cocylinder factorisation of an ordered functor and show that this is equivalent to the factorisation that appears in Steinberg's 'fibration theorem'.

6.1 The cylinder and cocylinder on the category of ordered groupoids

We have seen in Section 4 that in the category of topological spaces and continuous maps, the unit interval I can be used to construct an adjoint cylinder and cocylinder. We shall now show how the groupoid I, consisting of two identities 0, 1 and two non-identity mutually inverse morphisms u and u^{-1} with $\mathbf{d}(u) = 0$ and $\mathbf{r}(u) = 1$, plays the rôle of unit interval in **OG**, in that it can be used to define a cylinder and cocylinder on **OG**.

We begin with the cylinder. For each ordered groupoid G, we can form the direct product ordered groupoid $G \times I$ by Proposition 3.3. If $\theta : G \longrightarrow H$ is an ordered functor, then there is an ordered functor

$$\theta \times I : G \times I \longrightarrow H \times I \quad \text{given by} \quad \theta \times I : (g, i) \longmapsto (\theta(g), i).$$

It is now evident that

$$(\ ) \times I : \mathbf{OG} \longrightarrow \mathbf{OG}.$$

is a functor. Define ordered functors $e_G^0, e_G^1 : G \longrightarrow G \times I$ by

$$e_G^0 : g \longmapsto (g, \mathrm{Id}_0) \qquad\qquad e_G^1 : g \longmapsto (g, \mathrm{Id}_1).$$

It is easy to check that these functors give rise to natural transformations

$$e^0, e^1 : \mathrm{Id}_{\mathbf{OG}} \longrightarrow (\ ) \times I.$$

One can show that the projection onto the first factor $\sigma_G : G \times I \longrightarrow G$ yields a natural transformation $\sigma : (\ ) \times I \longrightarrow \mathrm{Id}_{\mathbf{OG}}$. It is straightforward to check that in this way we have constructed a cylinder $\mathbf{I} = ((\ ) \times I, e^0, e^1, \sigma)$.

We shall now describe a cocylinder on the category $\mathbf{OG}$. To do this we need to examine category G^I whose objects are the ordered functors from I to G, and whose morphisms are the natural transformations between such functors. There is a more convenient description of G^I. Each ordered functor $\theta : I \longrightarrow G$ is completely determined by its effect on u, and so determines an element of G. Conversely, every element of G determines an ordered functor from I to G. Thus the objects of G^I can be identified with the elements of G; that is each θ in G^I corresponds to the element

$$\theta(0) \xrightarrow{\theta(u)} \theta(1)$$

of G. Now let $\phi : I \longrightarrow G$ be another ordered functor, and let α be a natural transformation from θ to ϕ. Then $\alpha_1 \phi(u) = \theta(u)\alpha_0$. Conversely, any elements α_0 and α_1 of G such that $\alpha_1\phi(u) = \theta(u)\alpha_0$ determine a natural transformation α from ϕ to θ. Consequently, we can regard G^I as consisting of commutative squares

$$\begin{tikzcd} {} \arrow[d, "g_4"'] & {} \arrow[l, "g_2"'] \arrow[d, "g_1"] \\ {} & {} \arrow[l, "g_3"] \end{tikzcd}$$

which we represent by 4-tuples (g_4, g_3, g_2, g_1) satisfying the commutativity condition $g_4 g_2 = g_3 g_1$. Clearly

$$\mathbf{d}(g_4, g_3, g_2, g_1) = (g_1, \mathbf{r}(g_1), \mathbf{d}(g_1), g_1) \qquad \text{and} \qquad \mathbf{r}(g_4, g_3, g_2, g_1) = (g_4, \mathbf{r}(g_4), \mathbf{d}(g_4), g_4).$$

If $\mathbf{g} = (g_4, g_3, g_2, g_1)$ and $\mathbf{g}' = (g_4', g_3', g_2', g_1')$ are elements of G^I, satisfying $\mathbf{r}(\mathbf{g}) = \mathbf{d}(\mathbf{g}')$, then $g_1' = g_4$ and their product is

$$(g_4', g_3', g_2', g_1')(g_4, g_3, g_2, g_1) = (g_4', g_3' g_3, g_2' g_2, g_1)$$

which can be represented pictorially by

$$
\begin{tikzcd}
{} \arrow[d, "g_4'"'] & {} \arrow[l, "g_2'"'] \arrow[d, "g_1' = g_4"] & {} \arrow[l, "g_2"'] \arrow[d, "g_1"] \\
{} & {} \arrow[l, "g_3'"] & {} \arrow[l, "g_3"]
\end{tikzcd}
$$

Lemma 6.1 *Let G be an ordered groupoid. Then the functor category G^I is also an ordered groupoid.*

PROOF. Given an element (g_4, g_3, g_2, g_1) of G^I, we can form the commutative square $(g_1, g_3^{-1}, g_2^{-1}, g_4)$ as pictured below

$$
\begin{tikzcd}
{} \arrow[d, "g_1"'] & {} \arrow[l, "g_2^{-1}"'] \arrow[d, "g_4"] \\
{} & {} \arrow[l, "g_3^{-1}"]
\end{tikzcd}
$$

We may then form the products

$$(g_4, g_3, g_2, g_1)(g_1, g_3^{-1}, g_2^{-1}, g_4) = (g_4, g_3g_3^{-1}, g_2g_2^{-1}, g_4) = \mathbf{r}(g_4, g_3, g_2, g_1)$$

and

$$(g_1, g_3^{-1}, g_2^{-1}, g_4)(g_4, g_3, g_2, g_1) = (g_1, g_3^{-1}g_3, g_2^{-1}g_2, g_1) = \mathbf{d}(g_4, g_3, g_2, g_1).$$

Thus $(g_4, g_3, g_2, g_1)^{-1} = (g_1, g_3^{-1}, g_2^{-1}, g_4)$ and G^I is a groupoid.

Define an order on G^I by

$$(g_4, g_3, g_2, g_1) \leqslant (g_4', g_3', g_2', g_1')$$

if, and only if, $g_i \leqslant g_i'$ for $1 \leqslant i \leqslant 4$. It is easy to check that (OG1) and (OG2) hold. To prove that (OG3) holds, let (g_4, g_3, g_2, g_1) be an element of G^I and let (g, f, e, g) be an identity in G^I such that

$$(g, f, e, g) \leqslant \mathbf{d}(g_4, g_3, g_2, g_1) = (g_1, \mathbf{r}(g_1), \mathbf{d}(g_1), g_1).$$

Now e is an identity in G, and $e \leqslant \mathbf{d}(g_1) = \mathbf{d}(g_2)$. Thus $(g_2|e)$ is defined. Also $f \leqslant \mathbf{r}(g_1) = \mathbf{d}(g_3)$. Thus $(g_3|f) \leqslant g_3$ is defined. We can now form the square

$$
\begin{tikzcd}
{} \arrow[d, "(g_3|f)g(g_2|e)^{-1}"'] & {} \arrow[l, "(g_2|e)"'] \arrow[d, "g"] \\
{} & {} \arrow[l, "(g_3|f)"]
\end{tikzcd}
$$

which is commutative, is less than (g_4, g_3, g_2, g_1) and has domain (g, f, e, g). It is easy to see that it is the unique element of G^I with these properties.

■

Let $\theta : G \longrightarrow H$ be an ordered functor. Define

$$\theta^I : G^I \longrightarrow H^I \quad \text{by} \quad \theta^I(g_4, g_3, g_2, g_1) \longmapsto (\theta(g_4), \theta(g_3), \theta(g_2), \theta(g_1)).$$

It is easy to check that θ^I is a well-defined ordered functor. It follows that we have defined a functor

$$(\ )^I : \mathbf{OG} \longrightarrow \mathbf{OG}.$$

Let (g_4, g_3, g_2, g_1) be an element of G^I. Define the ordered functors

$$\varepsilon^0_G : G^I \longrightarrow G \quad \text{by} \quad \varepsilon^0_G : (g_4, g_3, g_2, g_1) \longmapsto g_2$$

and

$$\varepsilon^1_G : G^I \longrightarrow G \quad \text{by} \quad \varepsilon^1_G : (g_4, g_3, g_2, g_1) \longmapsto g_3.$$

It is straightforward to check that $\varepsilon^0, \varepsilon^1 : (\ )^I \longrightarrow \mathrm{Id}_{\mathbf{OG}}$ are natural transformations. Define the ordered functor

$$s_G : G \longrightarrow G^I \qquad \text{by} \qquad s_G : g \longmapsto (\mathbf{r}(g), g, g, \mathbf{d}(g)),$$

then $s : \mathrm{Id}_{\mathbf{OG}} \longrightarrow (\ )^I$ is a natural transformation. Hence we have constructed a cocylinder $\mathbf{P} = ((\ )^I, \varepsilon^0, \varepsilon^1, s)$ on $\mathbf{OG}$.

Proposition 6.2 *In the category of ordered groupoids and ordered functors, the cylinder* $\mathbf{I}$ *is left adjoint to the cocylinder* $\mathbf{P}$.

PROOF. Let G and H be ordered groupoids. We begin by constructing a bijection

$$\Omega_{G,H} : \hom(G \times I, H) \longrightarrow \hom(G, H^I).$$

Let $\theta : G \times I \longrightarrow H$ be an ordered functor. Then for each $g \in G$, we have that

$$(\mathbf{r}(g), u)(g, 0) = (g, u) = (g, 1)(\mathbf{d}(g), u).$$

It follows that we can define a function $\theta' : G \longrightarrow H^I$ by

$$\theta' : g \longmapsto \left(\begin{tikzcd} \cdot \arrow[d, "\theta(\mathbf{r}(g),u)"'] & \cdot \arrow[l, "\theta(g,0)"'] \arrow[d, "\theta(\mathbf{d}(g),u)"] \\ \cdot & \cdot \arrow[l, "\theta(g,1)"] \end{tikzcd} \right)$$

It is easy to check that θ' is an ordered functor. Define $\Omega_{G,H}(\theta) = \theta'$. It is clear that $\Omega_{G,H}$ is injective. To see that $\Omega_{G,H}$ is surjective, let $\phi : G \longrightarrow H^I$ be an ordered functor. Put $\phi(g) = (h_4^g, h_3^g, h_2^g, h_1^g)$ and define

$$\phi^* : G \times I \longrightarrow H \qquad \text{by} \qquad \phi^*(g,i) \longmapsto \begin{cases} h_2^g & \text{if } i = 0 \\ h_4^g h_2^g & \text{if } i = u \\ h_3^g & \text{if } i = 1 \\ h_2^g (h_1^g)^{-1} & \text{if } i = u^{-1} \end{cases}$$

We show that ϕ^* is a functor. Let $g \in G$, we shall show that $\mathbf{d}(\phi^*(g,i)) = \phi^*(\mathbf{d}(g,i))$ for all $i \in I$. Since $\phi(\mathbf{d}(g)) = (h_1^g, \mathbf{r}(h_1^g), \mathbf{d}(h_1^g), h_1^g)$, we have

$$\phi^*(\mathbf{d}(g), i) = \begin{cases} \mathbf{d}(h_1^g) & \text{if } i = 0 \\ h_1^g & \text{if } i = u \\ \mathbf{r}(h_1^g) & \text{if } i = 1 \\ h_1^g & \text{if } i = u^{-1}. \end{cases}$$

Consequently,

$$\phi^*(\mathbf{d}(g,i)) = \phi^*(\mathbf{d}(g), \mathbf{d}(i)) = \begin{cases} \mathbf{d}(h_1^g) & \text{if } i = 0, u \\ \mathbf{r}(h_1^g) & \text{if } i = 1, u^{-1}. \end{cases}$$

However

$$\begin{aligned} \mathbf{d}(\phi^*(g,i)) &= \begin{cases} \mathbf{d}(h_2^g) & \text{if } i = 0 \\ \mathbf{d}(h_2^g) & \text{if } i = u \\ \mathbf{d}(h_3^g) & \text{if } i = 1 \\ \mathbf{d}(h_3^g) & \text{if } i = u^{-1} \end{cases} \\ &= \begin{cases} \mathbf{d}(h_1^g) & \text{if } i = 0, u \\ \mathbf{r}(h_1^g) & \text{if } i = 1, u^{-1}. \end{cases} \end{aligned}$$

Similarly $\phi^*(\mathbf{r}(g,i)) = \mathbf{r}(\phi^*(g,i))$.

Next suppose that there exists $(g,i)(g',i')$ in $G \times I$, we shall show that $\phi^*(g,i)\phi^*(g',i') = \phi^*(gg', ii')$, these exist only if ii' is defined, in which case

$$\phi^*(g,i)\phi^*(g',i') = \begin{cases} h_2^g h_2^{g'} & \text{if } (i,i') = (0,0) \\ h_4^g h_2^g h_2^{g'} & \text{if } (i,i') = (u,0) \\ h_3^g h_4^{g'} h_2^{g'} & \text{if } (i,i') = (1,u) \\ h_2^g (h_1^g)^{-1} h_4^{g'} h_2^{g'} & \text{if } (i,i') = (u^{-1},u) \\ h_3^g h_3^{g'} & \text{if } (i,i') = (1,1) \\ h_2^g (h_1^g)^{-1} h_3^{g'} & \text{if } (i,i') = (u^{-1},1) \\ h_2^g h_2^{g'} (h_1^{g'})^{-1} & \text{if } (i,i') = (0,u^{-1}) \\ h_4^g h_2^g h_2^{g'} (h_1^{g'})^{-1} & \text{if } (i,i') = (u,u^{-1}) \end{cases}$$

$$= \begin{cases} h_2^g h_2^{g'} & \text{if } (i,i') = (0,0), (u^{-1}, u) \\ h_4^g h_2^g h_2^{g'} & \text{if } (i,i') = (u,0), (1,u) \\ h_3^g h_3^{g'} & \text{if } (i,i') = (1,1), (u,u^{-1}) \\ h_2^g h_2^{g'} (h_1^{g'})^{-1} & \text{if } (i,i') = (u^{-1},1), (0,u^{-1}). \end{cases}$$

However

$$\phi^*(gg', ii') = \begin{cases} h_2^g h_2^{g'} & \text{if } ii' = 0 \\ h_4^g h_2^g h_2^{g'} & \text{if } ii' = u \\ h_3^g h_3^{g'} & \text{if } ii' = 1 \\ h_2^g h_2^{g'} (h_1^{g'})^{-1} & \text{if } ii' = u^{-1}. \end{cases}$$

Thus ϕ^* is a functor as required. It is clear that ϕ^* is ordered, and that $\Omega_{G,H}(\phi^*) = \phi$.

To show that Ω is an adjunction, we need to prove that it is natural in G and in H. To show that $\Omega_{G,H}$ is natural in H we need the diagram below to commute for all ordered functors $\phi : H \longrightarrow H'$.

$$\begin{tikzcd} H \arrow[d, "\phi"'] & \hom(G \times I, H) \arrow[r, "\Omega_{G,H}"] \arrow[d, "\hom(G\times I,\phi)"'] & \hom(G, H^I) \arrow[d, "\hom(G,\phi^I)"] \\ H' & \hom(G \times I, H') \arrow[r, "\Omega_{G,H'}"'] & \hom(G, H'^I) \end{tikzcd}$$

However for any $\theta \in \hom(G \times I, H)$ and $g \in G$

$$(\Omega_{G,H'} \hom(G \times I, \phi)(\theta))(g) = (\Omega_{G,H'}(\phi\theta))(g) = \left(\begin{tikzcd} \cdot \arrow[d, "\phi\theta(\mathbf{r}(g),u)"'] & \cdot \arrow[l, "\phi\theta(g,0)"'] \arrow[d, "\phi\theta(\mathbf{d}(g),u)"] \\ \cdot & \cdot \arrow[l, "\phi\theta(g,1)"] \end{tikzcd} \right)$$

it is easy to verify that this is equal to $(\hom(G, \phi^I)\Omega_{G,H}(\theta))(g) = (\phi^I \Omega_{G,H}(\theta))(g)$. Hence Ω is natural in H. To show that Ω is natural in G we need to prove that for any ordered functor $\psi : G' \longrightarrow G$ the diagram below commutes.

$$\begin{tikzcd} G & \hom(G \times I, H) \arrow[r, "\Omega_{G,H}"] \arrow[d, "\hom(\psi\times I,H)"'] & \hom(G, H^I) \arrow[d, "\hom(\psi,H^I)"] \\ G' \arrow[u, "\psi"] & \hom(G' \times I, H) \arrow[r, "\Omega_{G',H}"'] & \hom(G', H^I) \end{tikzcd}$$

Now for $\theta \in \hom(G \times I, H)$ and $g \in G'$,

$$((\Omega_{G',H} \hom(\psi \times I, H))(\theta))(g) \quad = \quad (\Omega_{G',H}(\theta(\psi \times I)))(g)$$

$$= \left(\begin{tikzcd} {} \arrow[d, "\theta(\psi\times I)(\mathbf{r}(g),u)"'] & {} \arrow[l, "\theta(\psi\times I)(g,0)"'] \arrow[d, "\theta(\psi\times I)(\mathbf{d}(g),u)"] \\ {} & {} \arrow[l, "\theta(\psi\times I)(g,1)"] \end{tikzcd}\right)$$

$$= \left(\begin{tikzcd} {} \arrow[d, "\theta(\psi(\mathbf{r}(g)),u)"'] & {} \arrow[l, "\theta(\psi(g),0)"'] \arrow[d, "\theta(\psi(\mathbf{d}(g)),u)"] \\ {} & {} \arrow[l, "\theta(\psi(g),1)"] \end{tikzcd}\right)$$

and

$$(\hom(\psi, H^I)\Omega_{G,H}(\theta))(g) = ((\Omega_{G,H}(\theta))\psi)(g) = \left(\begin{tikzcd} {} \arrow[d, "\theta(\mathbf{r}(\psi(g)),u)"'] & {} \arrow[l, "\theta(\psi(g),0)"'] \arrow[d, "\theta(\mathbf{d}(\psi(g)),u)"] \\ {} & {} \arrow[l, "\theta(\psi(g),1)"] \end{tikzcd}\right)$$

so $\Omega_{G,H}$ is natural in H. Consequently Ω is an adjunction $\Omega : (\ ) \times I \dashv (\ )^I$.

It remains to verify that for $i \in \{0, 1\}$, and any ordered functors $\psi : G \times I \longrightarrow H$ and $\theta : G \longrightarrow H$, the function $\Omega_{G,H}$ satisfies

$$\varepsilon^i_H \Omega_{G,H}(\psi) = \psi e^i_G \quad \text{and} \quad \Omega_{G,H}(\theta\sigma_G) = s_H\theta.$$

If $g \in G$ then

$$\varepsilon^0_H \Omega_{G,H}(\psi)(g) = \varepsilon^0_H \left(\begin{tikzcd} {} \arrow[d, "\psi(\mathbf{r}(g),u)"'] & {} \arrow[l, "\psi(g,0)"'] \arrow[d, "\psi(\mathbf{d}(g),u)"] \\ {} & {} \arrow[l, "\psi(g,1)"] \end{tikzcd}\right) = \psi(g, 0) = \psi e^0_G(g)$$

and similarly $\varepsilon^1_H \Omega_{G,H}(\psi)(g) = \psi(g, 1) = \psi e^1_G(g)$. As for the second condition:

$$\Omega_{G,H}(\theta\sigma_G)(g) = \left(\begin{tikzcd} {} \arrow[d, "\theta\sigma_G(\mathbf{r}(g),u)"'] & {} \arrow[l, "\theta\sigma_G(g,0)"'] \arrow[d, "\theta\sigma_G(\mathbf{d}(g),u)"] \\ {} & {} \arrow[l, "\theta\sigma_G(g,1)"] \end{tikzcd}\right) = (\theta\mathbf{r}(g), \theta(g), \theta(g), \theta\mathbf{d}(g)) = s_H\theta(g).$$

■

It follows from the above result and Theorem 5.1, that the two notions of homotopy defined by the cylinder and cocylinder are equivalent. It follows from Theorem 5.2 that fibrations can be defined either by the cylinder or the cocylinder. Later we shall examine the notions of homotopy and fibration in the category of ordered groupoids, but first we need the following key result.

Proposition 6.3 *Let* $\mathbf{P} = ((\ )^I, \varepsilon^0, \varepsilon^1, s)$ *be the cocylinder on the category of ordered groupoids and ordered functors. Then*

(i) $\mathbf{P}$ *satisfies* $E(2)$.

(ii) *The cocylinder functor* $(\ )^I$ *preserves pullbacks.*

PROOF.

(i) We prove explicitly that every $(2,1,1)$-box has a filler; the remaining cases are proved similarly. We shall use the following notation: If $\theta : G \longrightarrow H^I$ is an ordered functor then write

$$\theta(g) = (h_4^\theta, h_3^\theta, h_2^\theta, h_1^\theta) = \mathfrak{h}^\theta.$$

Let $\theta, \phi, \psi : G \longrightarrow H^I$ be ordered functors such that $(\phi, -, \theta, \psi)$ is a $(2,1,1)$-box in $Q_{\mathbf{P}}(G, H)$. By definition

$$\varepsilon_H^0 \psi = \varepsilon_H^1 \phi \quad \text{and} \quad \varepsilon_H^0 \theta = \varepsilon_H^0 \phi,$$

as illustrated in the diagram below.

$$\begin{tikzcd}
\bullet \arrow[r, "\psi"] & \bullet \\
\bullet \arrow[u, "\phi"] \arrow[r, "\theta"'] & \bullet
\end{tikzcd}$$

From the definition of the cocylinder in $\mathbf{OG}$ it follows that

$$h_2^\psi = h_3^\phi \quad \text{and} \quad h_2^\theta = h_2^\phi,$$

and this is illustrated in the diagram below.

$$\begin{tikzcd}
\cdot \arrow[d, "h_3^\theta"'] \arrow[dr, phantom, "\mathfrak{h}^\theta"] & \cdot \arrow[l, "h_1^\theta"'] \arrow[r, "h_1^\phi"] \arrow[d, "h_2^\theta = h_2^\phi"] \arrow[dr, phantom, "\mathfrak{h}^\phi"] & \cdot \arrow[r, "h_1^\psi"] \arrow[d, "h_3^\phi = h_2^\psi"] \arrow[dr, phantom, "\mathfrak{h}^\psi"] & \cdot \arrow[d, "h_3^\psi"] \\
\cdot & \cdot \arrow[l, "h_4^\theta"] \arrow[r, "h_4^\psi"'] & \cdot \arrow[r, "h_4^\psi"'] & \cdot
\end{tikzcd}$$

We construct a filler for this box. Such a filler will be an ordered functor

$$\Theta : G \longrightarrow H^{I^2} \qquad \Theta : g \longmapsto (\mathfrak{h}_4, \mathfrak{h}_3, \mathfrak{h}_2, \mathfrak{h}_1) \qquad \text{where} \qquad \mathfrak{h}_i = (h_4^i, h_3^i, h_2^i, h_1^i) \in H^I$$

such that $\mathfrak{h}_4\mathfrak{h}_2 = \mathfrak{h}_3\mathfrak{h}_1$. It follows that Θ maps g into the commutative cube

$$\begin{tikzcd}
& {} \arrow[dl, "h_2^4"'] \arrow[dd, "h_1^4=h_4^2"] & & {} \arrow[ll, "h_2^2"'] \arrow[dl, "h_2^1"] \arrow[dd, "h_1^2=h_1^1"] \\
{} \arrow[dd, "h_4^4=h_4^3"'] & & {} \arrow[ll, "h_2^3"] \arrow[dd, "h_1^3=h_4^1"] & \\
& {} \arrow[dl, "h_3^4"] & & {} \arrow[ll, "h_3^2"] \arrow[dl, "h_3^1"] \\
{} & & {} \arrow[ll, "h_3^3"] &
\end{tikzcd}$$

in such a way that

$$\varepsilon^0_{H^I}\Theta = \phi, \quad (\varepsilon^0_H)^I\Theta = \theta \quad \text{and} \quad (\varepsilon^1_H)^I\Theta = \psi.$$

Now $\varepsilon^0_{H^I}\Theta(g) = \varepsilon^0_{H^I}(\mathfrak{h}_4, \mathfrak{h}_3, \mathfrak{h}_2, \mathfrak{h}_1) = \mathfrak{h}_2$, so let $\mathfrak{h}_2 = \phi(g)$, and

$$(\varepsilon^0_H)^I\Theta(g) = (\varepsilon^0_H)^I(\mathfrak{h}_4, \mathfrak{h}_3, \mathfrak{h}_2, \mathfrak{h}_1) = (\varepsilon^0_H(\mathfrak{h}_4), \varepsilon^0_H(\mathfrak{h}_3), \varepsilon^0_H(\mathfrak{h}_2), \varepsilon^0_H(\mathfrak{h}_1)) = (h_2^4, h_2^3, h_2^2, h_2^1)$$

so we want the top face of our cube to be $\theta(g)$. Similarly, $(\varepsilon^1_H)^I\Theta(g) = \psi(g)$ implies that the bottom face of the cube $\Theta(g)$ is $\psi(g)$.

It follows that we should define $\Theta(g)$ to be the cube

$$\begin{tikzcd}
& {} \arrow[dl, "h_4^\theta"'] \arrow[dd, "h_4^\phi"] & & {} \arrow[ll, "h_2^\theta=h_2^\phi"'] \arrow[dl, "h_1^\theta"] \arrow[dd, "h_1^\phi"] \\
{} \arrow[dd, "h"'] & & {} \arrow[ll, "h_3^\theta"] \arrow[dd, "h'"] & \\
& {} \arrow[dl, "h_4^\psi"] & & {} \arrow[ll, "h_3^\phi=h_2^\psi"] \arrow[dl, "h_1^\psi"] \\
{} & & {} \arrow[ll, "h_3^\psi"] &
\end{tikzcd}$$

where

$$h = h_4^\psi h_4^\phi (h_4^\theta)^{-1} \quad \text{and} \quad h' = h_1^\psi h_1^\phi (h_1^\theta)^{-1}.$$

To show that $\Theta(g)$ commutes, it is enough to show that the front face

$$\begin{tikzcd}
{} \arrow[d, "h_4^\psi h_4^\phi (h_4^\theta)^{-1}"'] & {} \arrow[l, "h_3^\theta"'] \arrow[d, "h_1^\psi h_1^\phi (h_1^\theta)^{-1}"] \\
{} & {} \arrow[l, "h_3^\psi"]
\end{tikzcd}$$

commutes. But

$$h_3^\psi(h_1^\psi h_1^\phi (h_1^\theta)^{-1})(h_3^\theta)^{-1} \quad = \quad (h_3^\psi h_1^\psi) h_1^\phi (h_3^\theta h_1^\theta)^{-1}$$

$$
\begin{aligned}
&= (h_4^{\psi} h_2^{\psi}) h_1^{\phi} (h_4^{\theta} h_2^{\theta})^{-1} \\
&= h_4^{\psi} (h_2^{\psi} h_1^{\phi} (h_2^{\theta})^{-1}) (h_4^{\theta})^{-1} \\
&= h_4^{\psi} (h_3^{\phi} h_1^{\phi} (h_2^{\phi})^{-1}) (h_4^{\theta})^{-1} \\
&= h_4^{\psi} h_4^{\phi} (h_4^{\theta})^{-1}.
\end{aligned}
$$

It is easy to check that Θ defined in this way is an ordered functor, and by construction it is a filler for the $(2,1,1)$–box $(\phi, -, \theta, \psi)$. In a similar way we can show that any two dimensional box has a filler, and so our cocylinder in **OG** satisfies $E(2)$.

(ii) That the ordered functor $(\ )^I$ preserves pullbacks is in fact immediate from Proposition 3.3, Proposition 6.2 and standard results in category theory (Theorem V.5.1 of [21]). However, we give a direct proof. Let $\theta : G \longrightarrow H$ and $\phi : K \longrightarrow H$ be ordered functors, and let

$$
\begin{tikzcd}
G_\theta \boxtimes_\phi K \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & K \arrow[d, "\phi"] \\
G \arrow[r, "\theta"'] & H
\end{tikzcd}
$$

be their pullback. The square

$$
\begin{tikzcd}
(G_\theta \boxtimes_\phi K)^I \arrow[r, "\pi_2^I"] \arrow[d, "\pi_1^I"'] & K^I \arrow[d, "\phi^I"] \\
G^I \arrow[r, "\theta^I"'] & H^I
\end{tikzcd}
$$

commutes since $(\ )^I$ is a functor. We prove that it too is a pullback. The ordered groupoid $(G_\theta \boxtimes_\phi K)^I$ has elements $((g_4,k_4),(g_3,k_3),(g_2,k_2),(g_1,k_1))$, where $\theta(g_i) = \phi(k_i)$, $g_i \in G$ and $k_i \in K$, such that $(g_4,k_4)(g_2,k_2) = (g_3,k_3)(g_1,k_1)$. It follows that $(g_4g_2, k_4k_2) = (g_3g_1, k_3k_1)$, therefore $(g_4,g_3,g_2,g_1) \in G^I$ and $(k_4,k_3,k_2,k_1) \in K^I$. We also have

$$\theta(g_4,g_3,g_2,g_1) = (\theta(g_4),\theta(g_3),\theta(g_2),\theta(g_1))$$

and

$$\phi(k_4,k_3,k_2,k_1) = (\phi(k_4),\phi(k_3),\phi(k_2),\phi(k_1))$$

Let $\alpha : L \longrightarrow G^I$ and $\beta : L \longrightarrow K^I$ be ordered functors such that $\theta^I\alpha = \phi^I\beta$. Write

$$\alpha(l) = (g_4^l, g_3^l, g_2^l, g_1^l) \quad \text{and} \quad \beta(l) = (k_4^l, k_3^l, k_2^l, k_1^l).$$

Thus $\theta(g_i^l) = \phi(k_i^l)$. Define

$$\psi : L \longrightarrow (G_\theta \boxtimes_\phi K)^I \quad \text{by} \quad \psi : l \longmapsto ((g_4^l, k_4^l), (g_3^l, k_3^l), (g_2^l, k_2^l), (g_1^l, k_1^l))$$

It is easy to check that ψ is an ordered functor satisfying $\pi_1^I \psi = \alpha$ and $\pi_2^I \psi = \beta$. Furthermore, it is straightforward to show that ψ is the unique such functor. It follows that $(\ )^I$ preserves pullbacks.

■

6.2 Homotopy equivalence of ordered groupoids

The rôle of Theorem 6.3 is to guarantee that we have the foundations for a reasonable homotopy theory of ordered groupoids. In this section we examine homotopies of ordered groupoids and construct a few examples.

Let $\phi, \theta : G \longrightarrow H$ be ordered functors, then ϕ is homotopic to θ with respect to the cylinder on **OG** if there is an ordered functor

$$\Phi : G \times I \longrightarrow H \quad \text{such that} \quad \Phi(g, 0) = \phi(g) \quad \text{and} \quad \Phi(g, 1) = \theta(g)$$

for all g in G.

Equivalently, a homotopy between θ and ϕ with respect to the cocylinder is given by an ordered functor

$$\Theta : G \longrightarrow H^I \quad \text{such that} \quad \Theta : g \longmapsto \left(\begin{tikzcd} {} \arrow[d, "g_4"'] & {} \arrow[l, "\phi(g)"'] \arrow[d, "g_1"] \\ {} & {} \arrow[l, "\theta(g)"] \end{tikzcd} \right)$$

for all $g \in G$ and some $g_4, g_1 \in H$ such that $g_4 \phi(g) = \theta(g) g_1$.

Theorem 5.5 and Theorem 6.3 guarantee that this notion of homotopy equivalence is an equivalence relation. We shall say that two ordered groupoids G and H are homotopy equivalent if there is a homotopy equivalence between them. By Theorem 3.9 the category of inverse semigroups and prehomomorphisms can be embedded as a full subcategory of **OG**, and therefore we can also talk about two inverse semigroups being homotopy equivalent.

In Proposition 6.4 we shall show how to characterise homotopy equivalence in a purely algebraic way, but first we need a definition.

Let G be an ordered groupoid. An order preserving function $\alpha : G_o \longrightarrow G$ is said to be $\mathbf{r}$-*coherent* if $\mathbf{r}(\alpha(e)) = e$ for each $e \in G_o$.

Proposition 6.4 *Let $\theta : G \longrightarrow G$ be an ordered functor on the ordered groupoid G. Then $\theta \simeq \mathrm{Id}_G$ if, and only if, there is an $\mathbf{r}$-coherent order-preserving function $\alpha : G_o \longrightarrow G$ such that $\theta(g) = \alpha(\mathbf{r}(g))^{-1} g \alpha(\mathbf{d}(g))$ for each $g \in G$.*

Proof. Suppose first that $\theta \simeq \mathrm{Id}_G$, then from the definition of the cocylinder on $\mathbf{OG}$, there is an ordered functor $\Phi : G \longrightarrow G^I$ such that $\varepsilon_G^0 \Phi = \theta$ and $\varepsilon_G^1 \Phi = \mathrm{Id}_G$. Let $g \in G$. Then

$$\Phi(g) = \left(\begin{array}{ccc} \cdot & \xleftarrow{\theta(g)} & \cdot \\ {\scriptstyle g_4}\Big\downarrow & & \Big\downarrow{\scriptstyle g_1} \\ \cdot & \xleftarrow[g]{} & \cdot \end{array} \right)$$

where $g_1, g_4 \in G$ are such that $g g_1 = g_4 \theta(g)$. For any identity e in G, write

$$\Phi(e) = (x_e, e, \theta(e), x_e).$$

Define

$$\alpha : G_o \longrightarrow G \quad \text{by} \quad \alpha(e) = x_e.$$

If $e \leqslant f$ then $\Phi(e) \leqslant \Phi(f)$ and so $x_e \leqslant x_f$. It follows that α is order preserving. It is immediate from the definition of G^I that $\mathbf{r}(x_e) = e$ and so $\mathbf{r}(\alpha(e)) = e$. Now since $\Phi(g) = \Phi(g\mathbf{d}(g)) = \Phi(g)\Phi(\mathbf{d}(g))$ we have that $\alpha(\mathbf{d}(g)) = g_1$, and similarly $\alpha(\mathbf{r}(g)) = g_4$. Therefore

$$\theta(g) = g_4^{-1} g g_1 = (\alpha(\mathbf{r}(g)))^{-1} g \alpha(\mathbf{d}(g)).$$

Hence α is the required $\mathbf{r}$-coherent function.

To prove the converse, let $\alpha : G_o \longrightarrow G$ be an order-preserving function such that $\theta(g) = \alpha(\mathbf{r}(g))^{-1} g \alpha(\mathbf{d}(g))$ for each $g \in G$. We prove that $\theta \simeq \mathrm{Id}_G$. Define

$$\Phi : G \longrightarrow G^I \quad \text{by} \quad \Phi : g \longmapsto (\alpha(\mathbf{r}(g)), g, \theta(g), \alpha(\mathbf{d}(g))).$$

This satisfies $\varepsilon_G^0 \Phi = \theta$ and $\varepsilon_G^1 \Phi = \mathrm{Id}_G$, so we only need to show that Φ is an ordered functor. It is immediate that ϕ is a well-defined function. The fact that Φ is order preserving follows from the fact that θ and α are order-preserving. It remains to show that Φ is a functor. Let $g \in G$, it is clear that

$$\begin{array}{rrcl} & \Phi(\mathbf{d}(g)) & = & (\alpha(\mathbf{d}(g)), \mathbf{d}(g), \theta(\mathbf{d}(g)), \alpha(\mathbf{d}(g))) = \mathbf{d}(\Phi(g)) \\ \text{and} & \Phi(\mathbf{r}(g)) & = & (\alpha(\mathbf{r}(g)), \mathbf{r}(g), \theta(\mathbf{r}(g)), \alpha(\mathbf{r}(g))) = \mathbf{r}(\Phi(g)) \end{array}$$

since θ is a functor. If gh is defined in G, then $\mathbf{d}(g) = \mathbf{r}(h)$ and so $\Phi(g)\Phi(h)$ is defined and is is easy to see that $\Phi(g)\Phi(h) = (\alpha(\mathbf{r}(g)), gh, \theta(gh), \alpha(\mathbf{d}(h))) = \Phi(gh)$.

■

The proof of the following is immediate from the above result and the definition of homotopy via the cocylinder on **OG**.

Proposition 6.5 *Let G and H be ordered groupoids. Then G and H are homotopy equivalent if, and only if, there are ordered functors*

$$\theta : G \longrightarrow H \quad \text{and} \quad \phi : H \longrightarrow G$$

and $\mathbf{r}$*-coherent order-preserving functions*

$$\alpha : G_o \longrightarrow G \quad \text{and} \quad \beta : H_o \longrightarrow H$$

such that for each $g \in G$ and each $h \in H$ we have that

$$\phi\theta(g) = \alpha(\mathbf{r}(g))^{-1} g \alpha(\mathbf{d}(g)) \quad \text{and} \quad \theta\phi(h) = \beta(\mathbf{r}(h))^{-1} h \beta(\mathbf{d}(h)).$$

■

The next result provides some simple necessary conditions for two ordered groupoids to be homotopy equivalent.

Proposition 6.6 *Let $\theta : G \longrightarrow H$ and $\phi : H \longrightarrow G$ be ordered functors such that $\phi\theta \simeq \mathrm{Id}_G$ and $\theta\phi \simeq \mathrm{Id}_H$. Then θ (respectively ϕ) is full, faithful and dense.*

Proof. By definition, $\theta\phi \simeq \mathrm{Id}_H$ means that there is an ordered functor $\Phi : H \longrightarrow H^I$ such that $\varepsilon^0_H \Phi = \theta\phi$ and $\varepsilon^1_H \Phi = \mathrm{Id}_H$. Let $h \in \hom_H(e, f)$. Then, using the same notation as in the proof of Proposition 6.4, we can write

$$\Phi(h) = (x_f, h, \theta\phi(h), x_e).$$

By the same reasoning $\phi\theta \simeq \mathrm{Id}_G$ means that there is an ordered functor $\Theta : G \longrightarrow G^I$ such that for each $g \in \hom_G(i, j)$, we can write

$$\Theta(g) = (y_j, g, \phi\theta(g), y_i).$$

The functor θ is faithful: to see this let $a, b \in \hom_G(i, j)$ and suppose that $\theta(a) = \theta(b)$. Then $\phi\theta(a) = \phi\theta(b)$. However $\Theta(a)$ is a commutative square and so

$$a = y_j \phi\theta(a) y_i^{-1}.$$

Likewise $\Theta(b)$ is a commutative square and so

$$b = y_j \phi\theta(b) y_i^{-1}.$$

Hence $a = b$ as required.

The functor θ is dense: to see this let $e \in H_o$. But then x_e is an element of H satisfying $\mathbf{d}(x_e) = \theta\phi(e)$ and $\mathbf{r}(x_e) = e$, as required.

By symmetry it is clear that ϕ is also faithful and dense.

The functor θ is full: to see this let $h \in \hom_H(\theta(e), \theta(f))$. We shall find an element in $\hom_G(e, f)$ which maps to h. Consider the element $x = x_f\phi(h)x_e^{-1}$ in G. The square $\Theta(x)$ commutes and so $x = x_f\phi\theta(x)x_e^{-1}$. It follows that $\phi(h) = \phi\theta(x)$. But ϕ is a faithful functor and so $h = \theta(x)$, as required.

■

Recall from Section 3 that the categories **Grpd** of groupoids and groupoid functors, and **Grp** of groups and group homomorphisms, are all full subcategories of the category **OG**. Hence the adjoint cylinder and cocylinder on **OG** give rise to notions of homotopy on the categories **Grpd** and **Grp**. In the following result we obtain some properties of homotopy equivalence in these special cases.

Proposition 6.7 (i) *Let G be a group regarded as an ordered groupoid with a unique identity and let $\theta : G \longrightarrow G$ be a group homomorphism. Then $\theta \simeq \mathrm{Id}_G$ if, and only if, θ is an inner automorphism.*

(ii) *Let G be a connected (unordered) groupoid, then G is homotopy equivalent to each of its vertex groups.*

PROOF. (i) Recall that an inner automorphism of a group G is the group homomorphism obtained by conjugating each element of G with an element $x \in G$

$$g \longmapsto x^{-1}gx.$$

Suppose that $\theta \simeq \mathrm{Id}_G$, then there is an ordered functor

$$\Phi : G \longrightarrow G^I \quad \text{with} \quad \Phi : g \longmapsto (g_4, g, \theta(g), g_1)$$

for some $g_4, g_1 \in G$ such that $g_4\theta(g) = gg_1$. Since $\mathbf{d}(g) = \mathbf{r}(g)$, we have $\mathbf{d}(\Phi(g)) = \mathbf{r}(\Phi(g))$ and so $g_1 = g_4 = x$, say. Hence $x\theta(g) = gx$, thus $\theta(g) = x^{-1}gx$. Now let g' be another element of G. By the above argument we have

$$\theta(g') = y^{-1}g'y$$

for some $y \in G$. To show that θ is an inner automorphism it remains to show that $x = y$. Since $g'g$ is defined it follows that $\Phi(g')\Phi(g)$ is defined, but this is only the case if $x = y$.

The converse is immediate.

(ii) This result is just Corollary 1 to Theorem 6.2 of [7]. However we give a direct proof.

Let e be an identity in G and let $\iota : G(e) \longrightarrow G$ be the inclusion functor. We shall construct its homotopy inverse. Since G is connected, for any identity $f \in G_o$ we can choose $g_f \in \hom(f, e)$, in particular $g_e = e$. Define

$$\rho : G \longrightarrow G(e) \quad \text{by} \quad \rho : g \longrightarrow g_{\mathbf{r}(g)} g (g_{\mathbf{d}(g)})^{-1}$$

which is clearly a functor and its restriction to $G(e)$ is $\mathrm{Id}_{G(e)}$. It remains to show that $\mathrm{Id}_G \simeq \iota\rho$, however, the functor

$$\Phi : G \longrightarrow G^I \quad \text{with} \quad \Phi : g \longmapsto (g_{\mathbf{r}(g)}, \rho(g), g, g_{\mathbf{d}(g)})$$

provides the required homotopy.

■

In the next result we obtain two further examples of homotopy equivalence. First we need a definition. In topology a space is called contractible if it is homotopic to a space with only one point (see Section I.3 of [27]). By analogy, we define an ordered groupoid G to be *contractible* if it is homotopic to an ordered groupoid consisting of just one identity. Thus contractible groupoids are homotopically as simple as possible.

Proposition 6.8 (i) *If an ordered groupoid is contractible, then it is connected and has trivial vertex groups.*

(ii) *The Brandt semigroup $B(G, J)$ is homotopic to G^0, the group G with a zero adjoined.*

PROOF. (i) Let G be an ordered groupoid which is homotopic to the ordered groupoid P consisting only of an identity $*$. Thus G is contractible and there is a homotopy equivalence $\theta : G \longrightarrow P$ with homotopy inverse $\phi : P \longrightarrow G$. Clearly $\theta(g) = *$ for all $g \in G$ and $\phi(*) = e$ for some identity e in G, therefore $\theta\phi = \mathrm{Id}_P$. Since $\phi\theta \simeq \mathrm{Id}_G$, there is an ordered functor

$$\Phi : G \longrightarrow G^I \quad \text{with} \quad \Phi : g \longmapsto (gg', g, e, g')$$

for some $g' \in G$. It is evident that G is connected. To see that it has trivial vertex groups, let f be an identity and suppose that $g \in G(f)$. Now $\Phi(f) = (h, f, e, h)$ for some $h \in \hom(e, f)$, and $\Phi(g) = (gh', g, e, h')$ for some $h' \in \hom(e, f)$. Since gf is defined $\Phi(gf) = \Phi(g)\Phi(f)$ is defined, and so $h = h'$. Similarly, since fg is defined,

$\Phi(fg) = \Phi(f)\Phi(g)$ implies $gh' = h$. Therefore $h' = gh'$ and so $g = \mathbf{r}(h') = f$. Hence $G(f)$ is trivial.

Note that by Proposition 6.7 the converse of this result holds in the category **Grpd**.

(ii) Recall from Section 1 that $B(G,J) = (J \times G \times J) \cup 0$, where J is a non-empty set and G is a group. The product of (i,g,j) and (k,h,l) is zero unless $j = k$ in which case it is (i,gh,l). Consider the natural partial order on $B(G,J)$. Suppose that (i,g,k) and (k,h,l) are elements of $B(G,J)$ such that $(i,g,j) \leqslant (k,h,l)$, then

$$(i,g,j) = (k,h,l)(l,1,l) = (k,h,l) \qquad \text{or} \qquad (i,g,j) = 0.$$

So the element 0 is beneath every element and the order is equality on $B(G,J)\setminus\{0\}$. The set of idempotents of G^0 is $E(G^0) = \{0,1\}$. Under the natural partial order 0 is beneath every element and the order is equality on G. We shall henceforth view $B(G,J)$ and G^0 as ordered groupoids under the natural partial order.

Choose $n \in J$ and keep it fixed. Define a function $\psi : G^0 \longrightarrow B(G,J)$ by

$$\psi(g) = (n,g,n) \quad \text{and} \quad \psi(0) = 0,$$

and define a function $\phi : B(G,J) \longrightarrow G^0$ by

$$\phi(i,g,j) = g \quad \text{and} \quad \phi(0) = 0.$$

It is straightforward to check that both ψ and ϕ are ordered functors. Observe that $\phi\psi = \mathrm{Id}_{G^0}$. We prove that $\psi\phi \simeq \mathrm{Id}_{B(G,J)}$. To do this we need to define an ordered functor $\Phi : B(G,J) \longrightarrow B(G,J)^I$ such that $\Phi : \psi\phi \simeq \mathrm{Id}_{B(G,J)}$. Let $(i,g,j) \in B(G,J)$, then $\psi\phi(i,g,j) = (n,g,n)$. It follows that we should define Φ by

$$\Phi : (i,g,j) \longmapsto \left(\begin{tikzcd} {} \arrow[d, "{(i,1,n)}"'] & {} \arrow[l, "{(n,g,n)}"'] \arrow[d, "{(j,1,n)}"] \\ {} & {} \arrow[l, "{(i,g,j)}"] \end{tikzcd}\right) \quad \text{and} \quad \Phi(0) = (0,0,0,0).$$

We show that Φ is an ordered functor. Let (i,g,j), $(j,h,k) \in B(G,J)$, then

$$\Phi((i,g,j)(j,h,k)) = \left(\begin{tikzcd} {} \arrow[d, "{(i,1,n)}"'] & {} \arrow[l, "{(n,g,n)}"'] \arrow[d, "{(j,1,n)}"] & {} \arrow[l, "{(n,h,n)}"'] \arrow[d, "{(k,1,n)}"] \\ {} & {} \arrow[l, "{(i,g,j)}"] & {} \arrow[l, "{(j,h,k)}"] \end{tikzcd}\right) = \left(\begin{tikzcd} {} \arrow[d, "{(i,1,n)}"'] & {} \arrow[l, "{(n,gh,n)}"'] \arrow[d, "{(k,1,n)}"] \\ {} & {} \arrow[l, "{(i,gh,k)}"] \end{tikzcd}\right)$$

Which is equal to $\Phi(i,gh,k)$. As for identities,

$$\Phi(\mathbf{d}(i,g,j)) = \Phi(j,1,j) = \left(\begin{tikzcd} {} \arrow[d, "{(j,1,n)}"'] & {} \arrow[l, "{(n,1,n)}"'] \arrow[d, "{(j,1,n)}"] \\ {} & {} \arrow[l, "{(j,1,j)}"] \end{tikzcd}\right) = \mathbf{d}(\Phi(i,g,j)),$$

and similarly $\Phi(\mathbf{r}(i,g,j)) = \mathbf{r}(\Phi(i,g,j))$. It is clear that Φ is order preserving. Hence Φ is the required homotopy. Thus $B(G,J)$ is homotopic to G^0.

■

The final result in this section can be deduced from Proposition 6.6 and Corollary 2 to Theorem 6.2 of [7]. However we give a direct proof.

Proposition 6.9 *If G and H are homotopy equivalent ordered groupoids, then there is a bijection between the sets of connected components of G and H such that corresponding components have isomorphic vertex groups.*

Proof. Denote by G^g the connected component of G containing the element g. Let G^C denote the set of connected components of G. An ordered functor $\theta : G \longrightarrow H$ induces a function

$$\theta^C : G^C \longrightarrow H^C \quad \text{with} \quad \theta^C(G^g) = H^{\theta(g)}$$

which is clearly well-defined since θ is a functor.

Suppose that θ is a homotopy equivalence from G to H with homotopy inverse ϕ. By definition of the cocylinder on $\mathbf{OG}$ there is an ordered functor

$$\Theta : G \longrightarrow G^I \quad \text{with} \quad \Theta(g) = (g_4, g, \phi\theta(g), g_1)$$

where $g_4, g_1 \in G$ are such that $gg_1 = g_4\phi\theta(g)$. Hence $G^g = G^{\phi\theta(g)}$. Similarly, $H^h = H^{\theta\phi(h)}$ for all $h \in H$.

We show that θ^C is bijective. To see that θ^C is injective, let $g, g' \in G$ be such that $\theta^C(G^g) = \theta^C(G^{g'})$. Thus $H^{\theta(g)} = H^{\theta(g')}$. Since ϕ^C is well-defined, we then have $\phi^C(H^{\theta(g)}) = \phi^C(H^{\theta(g')})$, that is $G^{\phi\theta(g)} = G^{\phi\theta(g')}$. Hence $G^g = G^{g'}$. To see that θ^C is surjective, let $h \in H$. Then $\phi^C(H^h) = G^{\phi(h)}$ and $\theta^C(G^{\phi(h)}) = H^{\theta\phi(h)} = H^h$.

We now consider the effect of θ and ϕ on vertex groups. Let e be an identity in G. The restriction of θ to $G(e)$ is a group homomorphism

$$G(e) \longrightarrow H(\theta(e)) \quad \text{with} \quad g \longmapsto \theta(g).$$

We show that this restriction of θ is bijective. One can perform a similar operation for ϕ. Using the same approach as for Proposition 6.7(ii), it is easy to show that for all $a \in G(e)$

$$\phi\theta(a) = x^{-1}ax$$

where $x \in \hom_G(\phi\theta(e), e)$ is fixed. We show first that $\theta|_{G(e)}$ is injective. Let $a, b \in G(e)$ be such that $\theta(a) = \theta(b)$. Then $\phi\theta(a) = \phi\theta(b)$, thus $x^{-1}ax = x^{-1}bx$ and so $a = b$. To see that

$\theta|_{G(e)}$ is surjective, let $h \in H(\theta(e))$. We know that $\phi(h) \in H(\phi\theta(e))$ and $\theta\phi(h) = y^{-1}hy$ for some fixed $y \in \hom_H(\theta\phi\theta(e), \theta(e))$. In particular $\theta\phi(\theta(e)) = y^{-1}\theta(e)y$, but

$$\theta(\phi\theta(e)) = \theta(x^{-1}ex) = \theta(x)^{-1}\theta(e)\theta(x).$$

Therefore $y = \theta(x)$. Put $g = x\phi(h)x^{-1} \in G(e)$, then

$$\begin{aligned} \theta(g) &= \theta(x)\theta\phi(h)\theta(x)^{-1} \\ &= \theta(x)(\theta(x))^{-1}h\theta(x)(\theta(x))^{-1} \\ &= h. \end{aligned}$$

Hence $\theta|_{G(e)}$ is bijective.

■

In the case of inverse semigroups S and T, the above result translates into the following: there is a bijection between $S/\mathcal{D}$ and $T/\mathcal{D}$ such that corresponding $\mathcal{D}$-classes have isomorphic $\mathcal{H}$-classes. See Howie [8] for the necessary definitions.

6.3 Fibrations of ordered groupoids

In this section we establish some properties of fibrations in the category of ordered groupoids, we also introduce ordered covering functors and obtain some properties of such. By Proposition 6.2 and Theorem 5.2, the cylinder definition of fibration is equivalent to that on a cocylinder, so for each result we shall use the definition which gives the easiest proof.

We saw in Section 4.3 that in the category of topological spaces and continuous maps, fibrations are defined using the homotopy lifting property, and the path lifting property is a special case of this.

If G is an ordered groupoid, then ordered functors $I \longrightarrow G$ are bijective with elements of G. By analogy with the topological case we think of ordered functors $I \longrightarrow G$ as being paths in G. Star surjectivity of an ordered functor $\theta : H \longrightarrow G$ can thus be interpreted as a path lifting property. The following result is therefore the counterpart of the topological result Proposition 4.3.

Proposition 6.10 *Every fibration in the category of ordered groupoids is star surjective.*

Proof. We shall use the cylinder definition of fibrations.

Let $\theta : G \longrightarrow H$ be an ordered functor which is a fibration. Let e be an identity in G and let $h \in H$ be such that $\theta(e) = \mathbf{d}(h)$. We shall show that there is an element $g \in G$ such that $\mathbf{d}(g) = e$ and $\theta(g) = h$.

Let P denote the ordered groupoid consisting of exactly one identity $*$. Clearly there is an isomorphism $\gamma : P \times I \cong I$. The set of ordered functors from P to G is in one to one correspondence with the set of identities G_0, since a functor $P \longrightarrow G$ picks out an identity in G. Similarly there is a bijection between the set of ordered functors $I \longrightarrow H$ and the non-identity elements of H. Hence there is a unique ordered functor $f : P \longrightarrow G$, with $f(*) = e$, and there is a unique ordered functor $\phi : I \longrightarrow H$ such that $\phi(u) = h$. The diagram of solid arrows below commutes.

$$\begin{tikzcd}
 & P \arrow[r, "f"] \arrow[dl, "e_P^0"'] \arrow[d, "\gamma e_P^0"] & G \arrow[d, "\theta"] \\
P \times I \arrow[r, "\cong"] & I \arrow[r, "\phi"'] \arrow[ur, dotted, "\Phi"] & H
\end{tikzcd}$$

Since θ is a fibration, there is an ordered functor $\Phi : I \longrightarrow G$ such that $\Phi(0) = e$ and $\theta\Phi(u) = \phi(u)$. Put $g = \Phi(u)$,then $\mathbf{d}(g) = \Phi(\mathbf{d}(u)) = e$ and $\theta(g) = h$. Therefore θ is star surjective as required.

■

In the following result we prove that the converse of Proposition 6.10 holds in the category of groupoids.

Proposition 6.11 *Let G and H be groupoids (regarded as ordered groupoids with trivial ordering) and let $\theta : G \longrightarrow H$ be functor. Then θ is a fibration if, and only if, it is star surjective.*

PROOF. This result is easiest to prove using the cocylinder definition of a fibration, see Proposition 2.1 of Brown [4] for a proof using the cylinder.

Suppose that θ is star surjective. Let X be a groupoid and let $\phi : X \longrightarrow H^I$ and $f : X \longrightarrow G$ be functors making the diagram below commute.

$$\begin{tikzcd}
 & & H^I \arrow[dr, "\varepsilon_H^0"] & \\
X \arrow[urr, bend left, "\phi"] \arrow[r, dotted, "\Phi"] \arrow[drr, bend right, "f"'] & G^I \arrow[ur, "\theta^I"] \arrow[dr, "\varepsilon_G^0"'] & & H \\
 & & G \arrow[ur, "\theta"'] &
\end{tikzcd}$$

We shall construct a functor Φ as illustrated above. Let $x \in X$ and write $\phi(x) = (h_4^x, h_3^x, h_2^x, h_1^x)$. Thus $\phi(\mathbf{d}(x)) = (h_1^x, \mathbf{r}(h_1^x), \mathbf{d}(h_1^x), h_1^x)$, so $\mathbf{d}(h_1^x) = \varepsilon_H^0\phi(\mathbf{d}(x))$, but since

$\theta f = \varepsilon_H^0 \phi$, we have $h_1^x \in \mathrm{St}_H(\theta f(\mathbf{d}(x)))$. Since θ is star surjective there is an element a_x of $\mathrm{St}_G(f(\mathbf{d}(x)))$ such that $\theta(a_x) = h_1^x$. Similarly, there is an element b_x of $\mathrm{St}_G(f(\mathbf{r}(x)))$ such that $\theta(b_x) = h_4^x$. Define $\Phi : X \longrightarrow G^I$, to be the function which assigns to each $x \in X$ the commutative square in G shown below.

$$\begin{tikzcd} {} \arrow[d, "b_x"'] & {} \arrow[l, "f(x)"'] \arrow[d, "a_x"] \\ {} & {} \arrow[l, "b_x f(x) a_x^{-1}"] \end{tikzcd}$$

The fact that Φ is a functor is immediate once we have noted that we can choose $a_{\mathbf{d}(x)} = a_x = b_{\mathbf{d}(x)}$ and $a_{\mathbf{r}(x)} = b_x = b_{\mathbf{d}(x)}$. It is also immediate that $\theta^I \Phi = \phi$ and $\varepsilon_G^0 \Phi = f$. Hence θ is a fibration.

■

On the basis of Proposition 6.11, the terms 'fibration' and 'star-surjective' are used synonymously in the category of groupoids. See [4] for a detailed examination of fibrations of groupoids. By analogy, Steinberg [28] uses these terms interchangeably in the category of ordered groupoids. However the functor Φ constructed in the proof of Proposition 6.11 need not be an ordered functor and therefore we see no reason to suppose that all star surjective ordered functors are fibrations.

Recall that in topology there is an important class of fibrations called coverings, and for spaces satisfying various connectivity conditions, these correspond exactly to fibrations with unique path lifting. By analogy we make the following definition in the category **OG**.

Definition Let $\theta : G \longrightarrow H$ be a fibration of ordered groupoids. We call θ an *ordered covering functor* if for any ordered functor $\omega : I \longrightarrow H$ and identity e in G with $\omega(0) = \theta(e)$, there is a unique ordered functor $\widetilde{\omega}$ such that $\widetilde{\omega}(0) = e$ and $\theta\widetilde{\omega} = \omega$. Thus θ lifts the 'path' ω to the unique path $\widetilde{\omega}$ in G.

$$\begin{tikzcd} & G \arrow[d, "\theta"] \\ I \arrow[ur, dotted, "\widetilde{\omega}"] \arrow[r, "\omega"'] & H \end{tikzcd}$$

Proposition 6.12 *An ordered functor is an ordered covering functor if, and only if, it is star-bijective.*

PROOF. Let $\theta : G \longrightarrow H$ be an ordered functor.

Suppose first that θ is an ordered covering. Thus θ is a fibration and by Theorem 6.10 it is star surjective. We show that θ is star injective. Let e be an identity in G and let $x, y \in \mathrm{St}_G(e)$ be such that $\theta(x) = \theta(y) = h$, say. Since the set of ordered functors from I to H is bijective with the set of elements of H, there is a unique ordered functor $\omega : I \longrightarrow H$ such that $\omega(u) = h$. Since θ has the unique path lifting property, there is a unique ordered functor $\widetilde{\omega} : I \longrightarrow G$ such that $\theta\widetilde{\omega}(u) = h$ and $\widetilde{\omega}(u) \in \mathrm{St}_G(e)$. But $\theta(x) = \theta(y) = h$ and so $x = y = \widetilde{\omega}(u)$. Thus θ is star bijective.

Now suppose that θ is star bijective, we show first that it is a fibration. This is easiest using the cocylinder definition of fibration, an alternative proof using the cylinder method is given as Proposition 4.8 of [17]. Let $\phi : X \longrightarrow H^I$ and $f : X \longrightarrow G$ be ordered functors such that $\theta f = \varepsilon_H^0 \phi$. Proceeding as in the proof of Proposition 6.11, we write $\phi(x) = (h_4^x, h_3^x, h_2^x, h_1^x)$ and since $h_1^x \in \mathrm{St}_H(\theta f(\mathbf{d}(x)))$, there is a element $a_x \in \mathrm{St}_G(f(\mathbf{d}(x)))$ such that $\theta(a_x) = h_1^x$. Similarly there is en element $b_x \in \mathrm{St}_G(f(\mathbf{r}(x)))$ such that $\theta(b_x) = h_4^x$. We have seen that

$$\Phi : X \longrightarrow G^I \quad \text{given by} \quad \Phi : x \longmapsto (b_x, b_x f(x) a_x^{-1}, f(x), a_x)$$

is a functor satisfying $\theta^I \Phi = \phi$ and $\varepsilon_G^0 \Phi = f$. To prove that θ is a fibration, it remains to show that Φ is ordered. This will be achieved by proving that if $x \leqslant y$ in G, then $a_x \leqslant a_y$ and $b_x \leqslant b_y$. Since ϕ is ordered, $\phi(x) \leqslant \phi(y)$ and thus $h_1^x \leqslant h_1^y$. Since f is ordered, $\mathbf{d}(a_x) = f(\mathbf{d}(x)) \leqslant f(\mathbf{d}(y)) = \mathbf{d}(a_y)$. Therefore by (OG3), there is a unique element $(a_y | f(\mathbf{d}(x))) \in G$ which is less than a_y and has domain $\mathbf{d}(a_x)$. By Proposition 3.2(i)

$$\theta(a_y | f(\mathbf{d}(x))) = (\theta(a_y) | \theta f(\mathbf{d}(x))) = (h_1^y | \theta f(\mathbf{d}(x))).$$

Thus, by uniqueness of restriction $(h_1^y | \theta f(\mathbf{d}(x))) = h_1^x$. Put $g_x = (a_y | f(\mathbf{d}(x)))$, then $a_x, g_x \in \mathrm{St}_G(f(\mathbf{d}(x)))$ and $\theta(g_x) = \theta(a_x) = h_1^x$. Therefore $a_x = g_x$, since θ is star-injective. But $g_x \leqslant a_y$, thus $a_x \leqslant a_y$. In a similar way we can show that $b_x \leqslant b_y$. Hence θ is a fibration.

It remains to show that θ has the unique path lifting property. Let $\omega : I \longrightarrow H$ be an ordered functor such that $\omega(0) = \theta(e)$ for some $e \in G_o$. Let P be the ordered groupoid consisting only of one identity $*$. We saw in Proposition 6.10 that there is a unique ordered functor $f : P \longrightarrow G$ with $f(*) = e$ and an obvious isomorphism $\gamma : P \times I \cong I$.

The diagram of solid arrows below commutes.

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\gamma e_P^0"'] & G \arrow[d, "\theta"] \\ I \arrow[r, "\omega"'] \arrow[ur, dotted, "\widetilde{\omega}"] & H \end{tikzcd}$$

We have seen that θ is a fibration so, by the definition of fibration in the cylinder, there is an ordered functor $\widetilde{\omega} : I \longrightarrow G$ such that $\widetilde{\omega}(0) = e$ and $\theta\widetilde{\omega} = \omega$. Suppose that $\alpha : I \longrightarrow G$ is an ordered functor such that $\alpha(0) = \widetilde{\omega}(0)$ and $\theta\alpha = \omega$. Then, $\alpha(u), \widetilde{\omega}(u) \in \mathrm{St}_G(e)$ and $\theta(\alpha(u)) = \theta(\widetilde{\omega}(u))$, so by star injectivity of θ, $\alpha(u) = \widetilde{\omega}(u)$, hence $\alpha = \widetilde{\omega}$. Therefore $\widetilde{\omega}$ is the unique lifting of ω.

■

On the basis of the above result the terms 'ordered covering' and 'ordered star bijection' are used synonymously.

The following result is taken from [4], Propositions 1.2 and 2.3.

Proposition 6.13 *A fibration is an ordered covering functor if, and only if, it has discrete kernel.*

PROOF. Let $\theta : G \longrightarrow H$ be a fibration.

Suppose first that θ is star injective (we know that it is star surjective by Proposition 6.10). Let f be an identity in H, and suppose that g is an element of G such that $\theta(g) = f$. We show that g is an identity. Clearly $\theta(\mathbf{d}(g)) = f$, thus $\theta(g)$ and $\theta(\mathbf{d}(g))$ are elements of $\mathrm{St}_H(f)$. By star injectivity $g = \mathbf{d}(g)$.

Conversely, suppose that θ has discrete kernel. By Proposition 6.10 we only need show that θ is star injective. Let e be an identity in G and suppose that a and b are elements of $\mathrm{St}_G(e)$ such that $\theta(a) = \theta(b)$. Then $(\theta(b))^{-1} = \theta(b^{-1})$, so $\theta(a)(\theta(b))^{-1} = \theta(ab^{-1})$ is an identity in H. Hence $ab^{-1} \in \ker\theta$. But $\ker\theta$ is discrete and thus ab^{-1} is an indentity in G. Therefore $a = b$.

■

6.4 Enlargements and deformation retracts

Enlargements of ordered groupoids were introduced by Lawson [14] as corresponding to enlargements of semigroups. In this section we examine the relationship between deformation retracts and enlargements in the category **OG**. Higgins [7] studied deformation retracts of unordered groupoids. In Theorem 6.5.13 of [3] Brown shows that deformation retracts of unordered groupoids correspond to full, dense subgroupoids, although his approach differs somewhat from ours.

Proposition 6.14 *Let $\theta : G \longrightarrow H$ and $\phi : H \longrightarrow G$ be ordered functors such that $\phi\theta = \mathrm{Id}_G$ and $\theta\phi \simeq \mathrm{Id}_H$. Thus G is a deformation retract of H. Then θ is an ordered embedding and H is an enlargement of $\theta(G)$.*

Proof. We begin by showing that θ is an ordered embedding. It is clear that θ is an injective ordered functor because $\phi\theta = \mathrm{Id}_G$. Suppose that $\theta(g) \leqslant \theta(g')$ for some $g, g' \in G$. Then $\phi\theta(g) \leqslant \phi\theta(g')$, since ϕ is an ordered functor. Thus $g \leqslant g'$, and so θ is an ordered embedding.

Before proving that H is an enlargement of $\theta(G)$ we make two observations.

Firstly, since θ is an ordered embedding, $\theta(G)$ is an ordered subgroupoid of H isomorphic to G. In particular, $\theta(G)$ is closed under restriction and corestriction. We prove the former; the proof of the latter is similar. Let $\theta(e)$ be an identity in $\theta(G)$ such that $\theta(e) \leqslant \mathbf{d}(\theta(g))$, for some $g \in G$. Then $\theta(e) \leqslant \theta(\mathbf{d}(g))$. Hence $e \leqslant \mathbf{d}(g)$, thus $(g|e)$ is defined. But by $\theta(g|e) \leqslant \theta(g)$ and $\mathbf{d}(\theta(g|e)) = \theta(e)$. It follows that $\theta(g|e) = (\theta(g)|\theta(e))$, as required.

Secondly, because $\theta\phi \simeq \mathrm{Id}_H$, there is an ordered functor $\Phi : H \longrightarrow H^I$ such that $\varepsilon_H^0\Phi = \theta\phi$ and $\varepsilon_H^1\Phi = \mathrm{Id}_H$. Let $h \in \hom(e, f)$, then we can write

$$\Phi : h \longmapsto \left(\begin{tikzcd} {} \arrow[d, "x_f"'] & {} \arrow[l, "\theta\phi(h)"'] \arrow[d, "x_e"] \\ {} & {} \arrow[l, "h"] \end{tikzcd} \right)$$

where $x_e, x_f \in G$ are such that $x_f\theta\phi(h) = hx_e$.

Now let $f \in G_o$ be any identity; we prove that $x_{\theta(f)} \in \theta(G)$. Note that since $\Phi(\theta(f))$ is an identity in H^I, we deduce immediately that $x_{\theta(f)}$ is a loop at $\theta(f)$. Now observe that $\Phi(x_{\theta(f)})$ is a commutative square and that writing down the maps in this square we obtain $\theta\phi(x_{\theta(f)}) = x_{\theta(f)}$, and so $x_{\theta(f)} \in \theta(G)$ as required.

We can now show that H is an enlargement of $\theta(G)$. Condition (GE2) holds because θ is a full functor by Proposition 6.6. Also by Proposition 6.6, the condition (GE3) holds since θ is dense. It remains to show that the condition (GE1) holds. Let $e \leqslant \theta(f)$, where $f \in G_o$. We shall prove that $e \in \theta(G)$. By assumption, Φ is an ordered functor, and so $\Phi(e) \leqslant \Phi(\theta(f))$. Thus $x_e \leqslant x_{\theta(f)}$. Also $\mathbf{d}(x_e) = \theta\phi(e)$. It follows that

$$x_e = (x_{\theta(f)}|\theta\phi(e)).$$

But we have proved that $x_{\theta(f)} \in \theta(G)$. Hence $x_e \in \theta(G)$. But $\mathbf{r}(x_e) = e$, and so $e \in \theta(G)$, as required. ∎

The next result shows that in the category **Grpd** the converse of the above result holds.

Proposition 6.15 *Let G and H be (unordered) groupoids. Then G is a deformation retract of H if, and only if, there is an injective functor $\theta : G \longrightarrow H$ such that H is an enlargement of $\theta(G)$.*

Proof. Suppose that θ is an injective functor and H is an enlargement of $\theta(G)$. To show that θ is a homotopy equivalence, we shall define its homotopy inverse. Let f be an identity in H, by (GE3) we can choose an element $x \in H$ with $\mathbf{d}(x) \in \theta(G)$ and $\mathbf{r}(x) = f$. Let $\Gamma : H_o \longrightarrow H$ be a function which assigns such an element to each identity in H, with the requirement that the restriction of Γ to the set $\theta(G_o)$ is the identity on $\theta(G_o)$. If $h \in H$ write $x_h = \Gamma(\mathbf{d}(h))$ and $y_h = \Gamma(\mathbf{r}(h))$. Thus $y_h^{-1}hx_h$ has domain $\mathbf{d}(x_h) \in \theta(G)$ and range $\mathbf{d}(y_h) \in \theta(G)$. Hence by (GE2), $y_h^{-1}hx_h \in \theta(G)$. Define

$$\psi : H \longrightarrow \theta(G) \qquad \text{by} \qquad \psi : h \longmapsto y_h^{-1}hx_h$$

note that the restriction of ψ to $\theta(G)$ is the identity on $\theta(G)$.

We show that ψ is a functor. If $h \in H$, then $\psi(\mathbf{d}(h)) = \mathbf{d}(x_h) = \mathbf{d}(\psi(h))$ and $\psi(\mathbf{r}(h)) = \mathbf{d}(y_h) = \mathbf{r}(\psi(h))$. If the composite $h'h$ is defined in H then $\mathbf{r}(h) = \mathbf{d}(h')$ and so $y_h = x_{h'}$, also $x_{h'h} = x_h$ and $y_{h'h} = y_{h'}$, it is now easy to show that $\psi(h')\psi(h) = \psi(h'h)$ Hence ψ is a functor. Since θ is injective, there is a functor $\theta^{-1} : \theta(G) \longrightarrow G$. Consider the composite $\theta^{-1}\psi : H \longrightarrow G$, we shall show that this is the homotopy inverse of θ. Now, for any $g \in G$, $\theta^{-1}\psi\theta(g) = \theta^{-1}\theta(g) = g$ so $(\theta^{-1}\psi)\theta = \mathrm{Id}_G$. Since $\theta(\theta^{-1}\psi) = \psi$, it remains to show that $\psi \simeq \mathrm{Id}_H$. The required homotopy is given by the ordered functor

$$\Psi : H \longrightarrow H^I \qquad \text{defined by} \qquad \Psi : h \longmapsto (x_h, \psi(h), h, y_h).$$

∎

The natural question to ask at this point is whether or not the above result holds in the general category of ordered groupoids. This problem was essentially considered by Lawson [12] without the topological framework provided here. The key point is that the functor ψ defined above need not be ordered, although in Lemma 16 (iv) of [12] some limited order-preserving properties were established.

6.5 The mapping cocylinder factorisation of an ordered functor

We can now prove the direct analogue of the theorem in topology which states that every continuous map can be factorised into a homotopy equivalence followed by a fibration (Theorem II.8.9 of [27]).

Theorem 6.16 *Let $\theta : G \longrightarrow H$ be an ordered functor between ordered groupoids. Then there is a functorial factorisation of θ*

$$\begin{tikzcd} & M^\theta \arrow[dr, "i^\theta"] & \\ G \arrow[ur, "p^\theta"] \arrow[rr, "\theta"'] & & H \end{tikzcd}$$

and an ordered functor $j^\theta : M^\theta \longrightarrow G$ such that $j^\theta p^\theta = \mathrm{Id}_G$, $p^\theta j^\theta \simeq \mathrm{Id}_{M^\theta}$ and i^θ is a fibration. In particular, M^θ is an enlargement of $p^\theta(G)$ and i^θ is an ordered star surjection.

PROOF. By Proposition 3.3, the category **OG** has all pullbacks. By Theorem 6.3, the cocylinder **P** satisfies the Kan condition $E(2)$ and the cocylinder functor preserves pullbacks. Thus by Proposition 5.6 and Theorem 5.7, every ordered functor has a mapping cocylinder factorisation $\theta = i^\theta p^\theta$ which is functorial, where $j^\theta p^\theta = \mathrm{Id}_G$, $p^\theta j^\theta \simeq \mathrm{Id}_{M^\theta}$ and i^θ is a fibration. By Proposition 6.14, M^θ is an enlargement of $p^\theta(G)$, and by Proposition 6.10, i^θ is star surjective.

■

We shall now describe the ordered functors involved in the mapping cocylinder factorisation. By definition, M^θ is the pullback of the ordered functors $\theta : G \longrightarrow H$ and $\varepsilon_H^0 : H^I \longrightarrow H$ in the diagram below.

$$\begin{tikzcd} M^\theta \arrow[r, "\pi^\theta"] \arrow[d, "j^\theta"'] & H^I \arrow[d, "\varepsilon_H^0"] \\ G \arrow[r, "\theta"'] & H \end{tikzcd}$$

Thus from the construction of the pullback given in Proposition 3.3, we have that

$$M^\theta = G_\theta \boxtimes_{\varepsilon_H^0} H^I = \Big\{ \Big(g, (h_4, h_3, h_2, h_1)\Big) \in G \times H^I \mid \theta(g) = h_2 \Big\}$$

with order and product inherited from $G \times H^I$, and we have that

$$\begin{array}{llll} & \pi^\theta : M^\theta \longrightarrow H^I & \text{is given by} & \pi^\theta : \Big(g, (h_4, h_3, \theta(g), h_1)\Big) \longmapsto (h_4, h_3, \theta(g), h_1) \\ \text{and} & j^\theta : M^\theta \longrightarrow G & \text{is given by} & j^\theta : \Big(g, (h_4, h_3, \theta(g), h_1)\Big) \longmapsto g. \end{array}$$

Define

$$\alpha : G \longrightarrow M^\theta \quad \text{by} \quad \alpha : g \longmapsto \Big(g, (\theta(\mathbf{r}(g)), \theta(g), \theta(g), \theta(\mathbf{d}(g)))\Big).$$

It is easy to check that α is an ordered functor, $j^\theta\alpha = \mathrm{Id}_G$ and $\pi^\theta\alpha = s_H\theta$. But these are the properties which characterise p^θ. Thus $\alpha = p^\theta$. By definition $i^\theta = \varepsilon_H^1\pi^\theta$. Thus

$$i^\theta\Big(g, (h_4, h_3, \theta(g), h_1)\Big) = \varepsilon_H^1(h_4, h_3, \theta(g), h_1) = h_3$$

6.6 Steinberg's construction

In Theorems 5.2 and 5.3 of [28], Steinberg proves the following result.

Theorem 6.17 (Fibration Theorem) *Let $\theta : G \longrightarrow H$ be an ordered functor, then there is an ordered enlargement $\iota : D \longrightarrow \mathrm{Der}(\theta) \rtimes H$ (with a right inverse) such that $\theta = \iota\psi$, where ψ is the semidirect product projection and is star surjective.*

■

In this section we show that this factorisation and the mapping cocylinder factorisation of an ordered functor described in the previous section are equivalent. We first need to recall some definitions from [28]. The reader should be alerted to the fact that whereas we compose functions from right-to-left, Steinberg composes from left-to-right. We have therefore modified his definitions and results accordingly.

Let G and H be ordered groupoids, a *left action* (π, A) of H on G consists of two ordered functors

$$\pi : G \longrightarrow H_0 \qquad \text{and} \qquad A : H_{D\mathbf{d}}\boxtimes_\pi G \longrightarrow G$$

where we write $A(h, g) = {}^h g$, and $\exists {}^h g$ if $\mathbf{d}(h) = \pi(g)$. These ordered functors must satisfy the following conditions:

(A1) If $\exists {}^h g$, then $\pi({}^h g) = \mathbf{r}(h)$.

(A2) If $\exists h_2 h_1$ and $\exists {}^{h_1} g$, then ${}^{h_2}({}^{h_1} g) = {}^{h_2 h_1} g$.

(A3) ${}^{\pi(g)} g = g$.

Given a left action (π, A) of an ordered groupoid H on an ordered groupoid G. Their *semidirect product* is defined as the set

$$G \rtimes H = G_{\pi}\boxtimes_{\mathbf{r}} H = \{(g,h) \in G \times H \mid \pi(g) = \mathbf{r}(h)\}.$$

There is a partial product on $G \rtimes H$ defined as follows: if (g_1, h_1), $(g_2, h_2) \in G \rtimes H$, and there exist h_2h_1 and $g_2({}^{h_2}g_1)$, then

$$(g_2, h_2)(g_1, h_1) = (g_2({}^{h_2}g_1), h_2h_1).$$

If $(g, h) \in G \rtimes H$ then it is straightforward to show that $\mathbf{d}(g, h) = ({}^{h^{-1}}(\mathbf{d}(g)), \mathbf{d}(h))$ and $\mathbf{r}(g, h) = (\mathbf{r}(g), \mathbf{r}(h))$. By Proposition 3.3 of [28] one has that $G \rtimes H$ is an ordered groupoid with the order inherited from $G \times H$.

Let $\theta : G \longrightarrow H$ be an ordered functor. The *derived ordered groupoid* $\mathrm{Der}(\theta)$ of θ is the set

$$\mathrm{Der}(\theta) = H_{\mathbf{r}}\boxtimes_{\theta\mathbf{d}} G = \{(h,g) \in H \times G \mid \mathbf{r}(h) = \mathbf{r}(\theta(g))\}$$

endowed with the following ordered groupoid structure (for proofs see [28]). The partial order is the one inherited from $H \times G$. We describe the groupoid structure. Let $(h, g) \in \mathrm{Der}(\theta)$, then

$$\mathbf{d}(h,g) = (h\theta(g), \mathbf{d}(g)) \quad \text{and} \quad \mathbf{r}(h,g) = (h, \mathbf{r}(g)).$$

So if $(h, g), (h', g') \in \mathrm{Der}(\theta)$ with $\mathbf{d}(h, g) = \mathbf{r}(h', g')$, then $h' = h\theta(g)$ and $\mathbf{r}(g') = \mathbf{d}(g)$. In which case the product is defined as

$$(h,g)(h',g') = (h, gg').$$

Let $\theta : G \longrightarrow H$ be an ordered functor. Steinberg defines a left action of H on the ordered groupoid $\mathrm{Der}(\theta)$ as follows:

$$\begin{array}{llll} & \pi : \mathrm{Der}(\theta) \longrightarrow H_o & \text{by} & \pi(h,g) = \mathbf{r}(h) \\ \text{and} & A : H_{D\mathbf{d}}\boxtimes_{\pi} \mathrm{Der}(\theta) \longrightarrow \mathrm{Der}(\theta) & \text{by} & A(h', (h,g)) = (h'h, g). \end{array}$$

As a result of this action, we can define a semidirect product ordered groupoid $\mathrm{Der}(\theta) \rtimes H$. The underlying set is

$$\begin{aligned} \mathrm{Der}(\theta) \rtimes H = \mathrm{Der}(\theta)_{\pi}\boxtimes_{\mathbf{r}} H &= \{((h,g),h') \mid \pi(h,g) = \mathbf{r}(h'), \mathbf{d}(h) = \mathbf{r}(\theta(g))\} \\ &= \{((h,g),h') \mid \mathbf{r}(h) = \mathbf{r}(h'), \mathbf{d}(h) = \mathbf{r}(\theta(g))\}. \end{aligned}$$

Thus elements of $\mathrm{Der}(\theta) \rtimes H$ can be illustrated as shown below.

$$\begin{tikzcd}
{} & {} \arrow[l, "g"'] \\
{} \arrow[d, "h"'] & {} \arrow[l, dotted, "\theta(g)"'] \\
{} & {} \arrow[l, "h'"]
\end{tikzcd}$$

We examine the product in $\mathrm{Der}(\theta) \rtimes H$. Let $((h_1, g_1), h_1')$ and $((h_2, g_2), h_2')$ be elements of $\mathrm{Der}(\theta) \rtimes H$. For $((h_2, g_2), h_2')((h_1, g_1), h_1')$ to be defined we need $\exists h_2' h_1'$ in which case $\mathbf{d}(h_2') = \mathbf{r}(h_1') = \mathbf{r}(h_1) = \pi(h_1, g_1)$ and so $A(h_2', (h_1, g_1)) = {}^{h_2'}(h_1, g_1)$ is defined. We also require $\exists (h_2, g_2)({}^{h_2'}(h_1, g_1))$, that is, $\exists (h_2, g_2)(h_2' h_1, g_1)$ in $\mathrm{Der}(\theta)$, for this we need $h_2' h_1 = h_2 \theta(g_2)$ and $\exists g_2 g_1$, in which case

$$(h_2, g_2)(h_2' h_1, g_1) = (h_2, g_2 g_1).$$

Thus

$$((h_2, g_2), h_2')((h_1, g_1), h_1') = ((h_2, g_2 g_1), h_2' h_1').$$

So given $((h_1, g_1), h_1')$ and $((h_2, g_2), h_2')$ in $\mathrm{Der}(\theta) \rtimes H$ such that the diagram below commutes

$$\begin{tikzcd}
{} & {} \arrow[l, "g_2"'] & {} \arrow[l, "g_1"'] \\
{} \arrow[d, "h_2"'] & {} \arrow[l, "\theta(g_2)"'] \arrow[d, "h_1"] & {} \arrow[l, "\theta(g_1)"'] \\
{} & {} \arrow[l, "h_2'"] & {} \arrow[l, "h_1'"]
\end{tikzcd}$$

then the composite is defined and is illustrated by the diagram below.

$$\begin{tikzcd}
{} & {} \arrow[l, "g_2 g_1"'] \\
{} \arrow[d, "h_2"'] & {} \arrow[l, dotted, "\theta(g_2 g_1)"'] \\
{} & {} \arrow[l, "h_2' h_1'"]
\end{tikzcd}$$

We now define three functions linked to this semidirect product as follows

$$\begin{aligned}
\iota &: G \longrightarrow \mathrm{Der}(\theta) \rtimes H \quad \text{with} \quad \iota(g) = ((\theta(\mathbf{r}(g)), g), \theta(g)), \\
\psi &: \mathrm{Der}(\theta) \rtimes H \longrightarrow H \quad \text{with} \quad \psi((h, g), h') = h', \\
\text{and} \quad \tau &: \mathrm{Der}(\theta) \rtimes H \longrightarrow G \quad \text{with} \quad \tau((h, g), h') = g.
\end{aligned}$$

Steinberg shows that ι is an ordered embedding and that $\mathrm{Der}(\theta) \rtimes H$ is an enlargement of $\theta(G)$ in Propositions 3.8 and 4.9 respectively, of [28]. In Proposition 4.7 Steinberg shows that the projection ψ is an ordered star surjective functor. Clearly $\tau\iota = \mathrm{Id}_G$ and $\theta = \psi\iota$, this is Steinberg's factorisation.

$$\begin{tikzcd}
& \mathrm{Der}(\theta) \rtimes H \arrow[dr, "\psi"] & \\
G \arrow[ur, "\iota"] \arrow[rr, "\theta"'] & & H
\end{tikzcd}$$

We now show that our factorisation is isomorphic to Steinberg's.

Proposition 6.18 *There is an isomorphism of ordered groupoids $\Gamma : \mathrm{Der}(\theta) \rtimes H \longrightarrow M^{\theta}$ such that*

$$\Gamma\iota = p^{\theta}, \quad i^{\theta}\Gamma = \psi \quad \textit{and} \quad j^{\theta}\Gamma = \tau$$

PROOF. Define

$$\Gamma : \mathrm{Der}(\theta) \rtimes H \longrightarrow M^{\theta} \quad \text{by} \quad \Gamma : ((h,g),k) \longmapsto \Big(g, (h, k, \theta(g), k^{-1}h\theta(g))\Big).$$

This function is pictured below.

$$\left(\begin{tikzcd} \cdot & \cdot \arrow[l, "g"'] \\ \cdot \arrow[loop left, dotted, "\theta(\mathbf{r}(g))"] \arrow[d, "h"'] & \\ \cdot & \cdot \arrow[l, "k"] \end{tikzcd}\right) \quad \stackrel{\Gamma}{\longmapsto} \quad \left(\begin{tikzcd} \cdot & \cdot \arrow[l, "g"'] \\ \cdot \arrow[d, "h"'] & \cdot \arrow[l, "\theta(g)"'] \arrow[d, "k^{-1}h\theta(g)"] \\ \cdot & \cdot \arrow[l, "k"] \end{tikzcd}\right)$$

It is easy to check that Γ is well-defined. We show that Γ is a functor. Let $((h,g),k) \in \mathrm{Der}(\theta) \rtimes H$, then

$$\begin{aligned}
\Gamma(\mathbf{d}((h,g),k)) &= \Gamma({}^{k^{-1}}(\mathbf{d}(h,g)), \mathbf{d}(k)) \\
&= \Gamma({}^{k^{-1}}(h\theta(g), \mathbf{d}(g)), \mathbf{d}(k)) \\
&= \Gamma((k^{-1}h\theta(g), \mathbf{d}(g)), \mathbf{d}(k)) \\
&= \Big(\mathbf{d}(g), (k^{-1}h\theta(g), \mathbf{d}(k), \theta(\mathbf{d}(g)), \mathbf{d}(k)^{-1}k^{-1}h\theta(g)\theta(\mathbf{d}(g)))\Big) \\
&= \Big(\mathbf{d}(g), (k^{-1}h\theta(g), \mathbf{d}(k), \theta(\mathbf{d}(g)), k^{-1}h\theta(g))\Big)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{d}(\Gamma((h,g),k)) &= \mathbf{d}\Big(g, (h, k, \theta(g), k^{-1}h\theta(g))\Big) \\
&= \mathbf{d}\Big(\mathbf{d}(g), (k^{-1}h\theta(g), \mathbf{d}(k), \mathbf{d}(\theta(g)), k^{-1}h\theta(g))\Big)
\end{aligned}$$

so $\mathbf{d}(\Gamma((h,g),k)) = \Gamma(\mathbf{d}((h,g),k))$. Now for the range identities,

$$\begin{aligned}
\Gamma(\mathbf{r}((h,g),k)) &= \Gamma(\mathbf{r}(h,g), \mathbf{r}(k)) \\
&= \Gamma((h, \mathbf{r}(g)), \mathbf{r}(k)) \\
&= \Big(\mathbf{r}(g), (h, \mathbf{r}(k), \theta(\mathbf{r}(g)), \mathbf{r}(k)^{-1}h\theta(\mathbf{r}(g)))\Big) \\
&= \Big(\mathbf{r}(g), (h, \mathbf{r}(k), \theta(\mathbf{r}(g)), h)\Big)
\end{aligned}$$

can easily be shown to be equal to $\mathbf{r}(\Gamma((h,g),k))$. Now examine the effect of Γ on composites. Let $((h_2,g_2),k_2)$ and $((h_1,g_1),k_1)$ be elements of $\mathrm{Der}(\theta) \rtimes H$, their composite exists if $\exists g_2g_1$, $\exists k_2k_1$ and $k_2h_1 = h_2\theta(g_2)$, in which case

$$\begin{aligned}
\Gamma\Big(((h_2,g_2),k_2)((h_1,g_1),k_1)\Big) &= \Gamma((h_2,g_2g_1)k_2k_1) \\
&= \Big(g_2g_1,(h_2,k_2k_1,\theta(g_2g_1),(k_2k_1)^{-1}h_2\theta(g_2g_1))\Big) \\
&= \Big(g_2g_1,(h_2,k_2k_1,\theta(g_2g_1),k_1^{-1}(k_2^{-1}h_2\theta(g_2))\theta(g_1))\Big) \\
&= \Big(g_2g_1,(h_2,k_2k_1,\theta(g_2g_1),k_1^{-1}h_1\theta(g_1))\Big),
\end{aligned}$$

but $\Gamma((h_2,g_2),k_2)\Gamma((h_1,g_1),k_1)$ is equal to

$$\Big(g_2,(h_2,k_2,\theta(g_2),k_2^{-1}h_2\theta(g_2))\Big)\Big(g_1,(h_1,k_1,\theta(g_1),k_1^{-1}h_1\theta(g_1))\Big).$$

Thus

$$\Gamma((h_2,g_2),k_2)\Gamma((h_1,g_1),k_1) = \Big(g_2g_1,(h_2,k_2k_1,\theta(g_2)\theta(g_1),k_1^{-1}h_1\theta(g_1))\Big)$$

which is equal to $\Gamma\Big(((h_2,g_2),k_2)((h_1,g_1),k_1)\Big)$. Therefore Γ is a functor. It is clearly ordered and injective. To see that Γ is surjective, let $\mathfrak{m} = \Big(g,(h_4,h_3,\theta(g),h_1)\Big) \in M^\theta$, then it is easy to check that $((h_4,g),h_3) \in \mathrm{Der}(\theta) \rtimes H$ with $\Gamma((h_4,g),h_3) = \mathfrak{m}$. Thus Γ has inverse

$$\Gamma^{-1} : M^\theta \longrightarrow \mathrm{Der}(\theta) \rtimes H \quad \text{with} \quad \Gamma^{-1} : \Big(g,(h_4,h_3,\theta(g),h_1)\Big) \longmapsto ((h_4,g),h_3),$$

which is also clearly an ordered functor. Consequently $\mathrm{Der}(\theta) \rtimes H$ and M^θ are order isomorphic.

To complete the proof, observe that

$$\begin{aligned}
\Gamma\iota(g) &= \Gamma((\theta(\mathbf{r}(g)),g)\theta(g)) \\
&= \Big(g,(\theta(\mathbf{r}(g)),\theta(g)\theta(g),\theta(g)^{-1}\theta(\mathbf{r}(g))\theta(g))\Big) \\
&= \Big(g,(\theta(\mathbf{r}(g)),\theta(g),\theta(g),\theta(\mathbf{d}(g)))\Big) \\
&= p^\theta(g),
\end{aligned}$$

for any $g \in G$. Also

$$i^\theta\Gamma((h,g),k) = i^\theta\Big(g,(h,k,\theta(g),k^{-1}h\theta(g))\Big) = k = \psi((h,g),k)$$

and

$$j^\theta\Gamma((h,g),k) = j^\theta\Big(g,(h,k,\theta(g),k^{-1}h\theta(g))\Big) = g = \tau((h,g),k),$$

for any $((h,g),k) \in \mathrm{Der}(\theta) \rtimes H$.

■

Part III

Cohomology of ordered groupoids

Chapter 7

Background on cohomology

Homology, like homotopy theory, arose out of attempts at constructing topological invariants for spaces, but was adapted to classify other mathematical objects such as groups. In this chapter we outline some of the theory we will use.

7.1 Abelian categories

We shall describe some homology in abelian categories. Later we shall apply these methods to the category of actions of ordered groupoids. For more on abelian categories we refer the reader to [5, 21, 24, 31].

In this section we shall think of categories in terms of objects and morphisms.

A category $\mathbf{A}$ is called an *Ab-category* if every hom set $\hom_{\mathbf{A}}(A, B)$ in $\mathbf{A}$ is an additive abelian group and if $f, f' : A \longrightarrow B$, $g, g' : B \longrightarrow C$ are morphisms in $\mathbf{A}$ then

$$(g + g')(f + f') = gf + gf' + g'f + g'f'.$$

An *additive functor* $\phi : \mathbf{A} \longrightarrow \mathbf{B}$ between Ab-categories $\mathbf{A}$ and $\mathbf{B}$ is a functor such that each induced function $\hom_{\mathbf{A}}(A, A') \longrightarrow \hom_{\mathbf{B}}(B, B')$ is a group homomorphism.

An object Z of a category $\mathbf{A}$ is called a *zero object* if for any objects A and B of $\mathbf{A}$, there are unique morphisms $f : A \longrightarrow Z$ and $g : Z \longrightarrow B$, the composite gf is called the *zero morphism* from A to B and is written 0^A_B or 0. Any composite with a zero morphism is itself a zero morphism.

An *additive category* is an Ab-category $\mathbf{A}$ with a zero object and direct sums, that is for every pair A, B of objects in $\mathbf{A}$ there is an object $A \oplus B$ and four morphisms forming a diagram

$$A \underset{p_1}{\overset{i_1}{\rightleftarrows}} A \oplus B \underset{p_2}{\overset{i_2}{\leftrightarrows}} B$$

with $p_1 i_1 = \mathrm{Id}_A$, $p_2 i_2 = \mathrm{Id}_B$ and $i_1 p_1 + i_2 p_2 = \mathrm{Id}_{A \oplus B}$. We say that A is *complete* if the product of any set of objects exists in $\mathbf{A}$.

In a category $\mathbf{A}$ which has a zero object, a *kernel* of a morphism $f : B \longrightarrow C$ is a morphism $k : A \longrightarrow B$ such that $fk = 0$, and k is universal with this property; that is, for any morphism $h : A' \longrightarrow B$ with $fh = 0$, there exists a unique morphism $h' : A' \longrightarrow A$ such that $h = kh'$, as shown in the commutative diagram below

$$\begin{tikzcd}
A \arrow[dr, "k"] \arrow[drr, bend left, "0"] & & \\
& B \arrow[r, "f"] & C \\
A' \arrow[uu, dotted, "\exists ! h'"] \arrow[ur, "h"'] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

Dually, a *cokernel* of f is a morphism $c : C \longrightarrow D$, which is universal with respect to $cf = 0$.

Lemma 7.1 *Let $f : B \longrightarrow C$ be a morphism in a category C and let $k : A \longrightarrow B$ and $k' : A' \longrightarrow B$ be kernels for f. Then A and A' are isomorphic objects. The dual result holds for cokernels.*

PROOF. Since k is a kernel there exists a unique morphism $h : A' \longrightarrow A$ such that $k' = kh$, similarly there is a unique $h' : A \longrightarrow A'$ such that $k = k'h'$. Therefore $k = khh'$. But since k is a kernel Id_A is the unique morphism with $k = k\mathrm{Id}_A$. Hence $hh' = \mathrm{Id}_A$. Similarly $h'h = \mathrm{Id}_{A'}$. Hence A is isomorphic to A'.

The dual result for cokernels is proved similarly.

■

We now examine monics and epis an additive categories $\mathbf{A}$. Recall that a morphism $m : A \longrightarrow B$ is monic if given any two morphisms $f, g : C \longrightarrow A$, $mf = mg$ implies $f = g$. Clearly if m is monic, then

$$mf = 0 \implies f = 0.$$

Conversely, let $m : A \longrightarrow B$ be a morphism in $\mathbf{A}$, and suppose that $mf = 0$ implies $f = 0$. We show that m is monic. Let $f, g : A \longrightarrow B$, and suppose that $mf = mg$. Then $mf - mg = 0$. So $m(f - g) = 0$, since $\mathbf{A}$ is an Ab-category. Therefore $f - g = 0$, so m is monic.

Similarly, an epi $e : A \longrightarrow B$ in an additive category is a morphism such that $he = 0$ implies $h = 0$, for every morphism $h : B \longrightarrow B'$ of $\mathbf{A}$.

Monics and epis allow us to define sub- and quotient objects. Let $f : X \longrightarrow Y$ and $g : X' \longrightarrow Y$ be monics, we write $g \leqslant f$ if there exists a morphism $f' : X' \longrightarrow X$ such that $g = ff'$. If $g \leqslant f$ and $f \leqslant g$ then we write $f \equiv g$, it is easy to show that this is an equivalence relation. The corresponding equivalence classes are called *subobjects* of Y. For convenience we shall say that a particular equivalence class representative is a subobject. These subobjects correspond to the usual subobjects in **Ab**, **Grp**, **Grpd** etc. For example, let S be a subgroup of a group G, the inclusion $\iota : S \hookrightarrow G$ is monic. If $\mu : T \longrightarrow G$ is another monic and there exist homomorphisms θ and ϕ making the diagram below commute.

$$\begin{tikzcd} S \arrow[r, hook, "\iota"] \arrow[d, bend right, "\theta"'] & G \\ T \arrow[u, bend right, "\phi"'] \arrow[ur, hook, "\mu"'] & \end{tikzcd}$$

Then $\iota = \mu\theta = \iota\phi\theta$, but ι is a monomorphism so $\phi\theta = \mathrm{Id}_T$. Similarly $\theta\phi = \mathrm{Id}_S$. Therefore $S \cong T$, that is any subobject equivalent to ι has domain isomorphic to S. Dually, a *quotient* of an object X is the obvious equivalence class of epis having domain X. Let X be a subobject of Y, that is there is a monic $m : X \longrightarrow Y$. The *quotient* of X and Y is an epi $e : Y \longrightarrow Q$ such that $em = 0$, we call Q the *quotient object* and often write $Q = X/Y$. This definition is consistent with the familiar quotients in **Grp**, **Grpd**, etc.

Let **A** be an Ab-category. Suppose that $f : X \longrightarrow Y$ and $g : X' \longrightarrow Y$ are subobjects of an object Y of **A**. Suppose further that $g \leqslant f$. Let $q : Y \longrightarrow Q$ be a quotient for X and Y; that is an epi. We say that q is the *quotient of f and g* if $qg = 0$, we write $q = f/g$.

Definition An *abelian category* is an additive category **A** satisfying the following additional conditions:

1. Every morphism in **A** has a kernel and a cokernel.

2. Every monic in **A** is the kernel of its cokernel.

3. Every epi in **A** is the cokernel of its kernel.

The most commonly used abelian category is the category of abelian groups.

Proposition 7.2 *The category* **Ab** *of abelian groups is an abelian category.*

PROOF. Let A and B be abelian groups. Addition in $\hom(A, B)$ is defined pointwise. Thus for group homomorphisms $\theta, \phi : A \longrightarrow B$, the homomorphism

$$\theta + \phi : A \longrightarrow B \quad \text{is given by} \quad (\theta + \phi)(a) = \theta(a) + \phi(a).$$

It is immediate that this addition is associative and commutative. The homomorphism

$$\overline{0} : A \longrightarrow B \quad \text{given by} \quad \overline{0}(a) = 0_B, \quad \text{where } 0_B \text{ is the identity in } B$$

has the property that $\theta + \overline{0} = \theta = \overline{0} + \theta$. For each homomorphism $\theta : A \longrightarrow B$, define $-\theta : A \longrightarrow B$ by $(-\theta)(a) = -\theta(a)$. It is clear that $\theta - \theta = \overline{0} = -\theta + \theta$. Hence each $\hom(A, B)$ is an additive abelian group. For group homomorphisms, $\theta, \theta' : A \longrightarrow B$ and $\phi, \phi' : B \longrightarrow C$, and each $a \in A$, we have

$$\begin{aligned}
(\phi + \phi')(\theta + \theta')(a) &= (\phi + \phi')(\theta(a) + \theta'(a)) \\
&= \phi(\theta(a) + \theta'(a)) + \phi'(\theta(a) + \theta'(a)) \\
&= \phi\theta(a) + \phi\theta'(a) + \phi'\theta(a) + \phi'\theta'(a).
\end{aligned}$$

Hence **Ab** is an Ab-category.

The zero object in **Ab** is the group consisting only of an identity. The zero homomorphism from a group A to a group B is the homomorphism $\overline{0}$ defined above. The direct sum of two abelian groups A and B is the cartesian product $A \times B$ with the canonical inclusion and projection maps. Hence **Ab** is an additive category.

The kernel of a homomorphism $\theta : A \longrightarrow B$ is the inclusion $\kappa : \mathrm{Ker}(\theta) \longrightarrow A$ where, as usual, $\mathrm{Ker}(\theta)$ is the subgroup of A consisting of those elements which θ maps to the identity in B. The cokernel of θ is the projection $\rho : B \longrightarrow B/\mathrm{Im}(\theta)$. Let θ be monic. In **Ab**, monics are precisely the monomorphisms, therefore there is an isomorphism $\theta^* : A \cong \mathrm{Im}(\theta)$. But $\mathrm{Ker}(\rho) = \mathrm{Im}(\theta)$. Hence every monic is the kernel of its cokernel. A dual argument shows that every epi in **Ab** is the cokernel of its kernel.

■

The following important result is well-known.

Proposition 7.3 *Let* $\mathbf{A}$ *be an abelian category and* $\mathbf{C}$ *an arbitrary small category. The functor category* $\mathbf{A}^{\mathbf{C}}$ *is an abelian category.*

PROOF. Let θ and ϕ be functors from $\mathbf{C}$ to $\mathbf{A}$ and let $\mathrm{Nat}(\theta, \phi)$ denote the set of natural transformations $\alpha, \beta : \theta \longrightarrow \phi$. Let X be an object in $\mathbf{C}$. There are morphisms $\alpha_X, \beta_X : \theta(X) \longrightarrow \phi(X)$ in $\mathbf{A}$. Since $\mathbf{A}$ is abelian, $\hom(\theta(X), \phi(X))$ is an abelian group. We can therefore define

$$(\alpha + \beta)_X : \theta(X) \longrightarrow \phi(X) \quad \text{by} \quad (\alpha + \beta)_X = \alpha_X + \beta_X.$$

To see that $\alpha + \beta$ is a natural transformation, let $f : X \longrightarrow Y$ be a morphism in $\mathbf{C}$. Then

$$\phi(f)(\alpha + \beta)_X = \phi(f)(\alpha_X + \beta_X)$$

$$
\begin{aligned}
&= \phi(f)\alpha_X + \phi(f)\beta_X \\
&= \alpha_Y\theta(f) + \beta_Y\theta(f) \\
&= (\alpha_Y + \beta_Y)\theta(f).
\end{aligned}
$$

Hence the addition is well-defined in $\mathrm{Nat}(\theta, \phi)$. It is immediate that this addition is associative and commutative. The identity in $\mathrm{Nat}(\theta, \phi)$ is the natural transformation 0 which has each 0_X the identity in $\hom(\theta(X), \phi(X))$. For functors $\theta, \phi, \psi : \mathbf{C} \longrightarrow \mathbf{A}$, natural transformations $\alpha, \beta \in \mathrm{Nat}(\theta, \phi)$ and $\gamma, \delta \in \mathrm{Nat}(\phi, \psi)$, and any object X of $\mathbf{C}$, we have

$$
\begin{aligned}
((\gamma + \delta)(\alpha + \beta))_X &= (\gamma + \delta)_X(\alpha + \beta)_X \\
&= (\gamma_X + \delta_X)(\alpha_X + \beta_X) \\
&= \gamma_X\alpha_X + \gamma_X\beta_X + \delta_X\alpha_X + \delta_X\beta_X \\
&= (\gamma\alpha)_X + (\gamma\beta)_X + (\delta\alpha)_X + (\delta\beta)_X.
\end{aligned}
$$

Hence $(\gamma + \delta)(\alpha + \beta) = \gamma\alpha + \gamma\beta + \delta\alpha + \delta\beta$, making $\mathbf{A}^{\mathbf{C}}$ an Ab-category.

We now show that $\mathbf{A}^{\mathbf{C}}$ is an additive category. Let $\overline{0}$ denote the functor from $\mathbf{C}$ to $\mathbf{A}$ which maps every object of $\mathbf{C}$ to the zero object of $\mathbf{A}$, and every morphism of $\mathbf{C}$ to the identity at the zero object. It is easy to show that $\overline{0}$ is a zero object for $\mathbf{A}^{\mathbf{C}}$. To see that $\mathbf{A}^{\mathbf{C}}$ has direct sums, let $\theta, \phi : \mathbf{C} \longrightarrow \mathbf{A}$. Since $\mathbf{A}$ is abelian, we may define the sum of θ and ϕ component-wise, that is

$$
(\theta \oplus \phi)(X) = \theta(X) \oplus \phi(X)
$$

for every object X of $\mathbf{C}$ and

$$
(\theta \oplus \phi)(f)(X) = (\theta \oplus \phi)(f(X)) = \theta(f(X)) \oplus \phi(f(X))
$$

for every morphism $f : X \longrightarrow Y$ of $\mathbf{C}$. To see that $\theta \oplus \phi$ is a functor, let $g : Y \longrightarrow Z$ be another morphism in $\mathbf{C}$. Then

$$
\begin{aligned}
(\theta \oplus \phi)(gf)(X) &= (\theta \oplus \phi)(gf(X)) \\
&= \theta(gf(X)) \oplus \phi(gf(X)) \\
&= \theta(g)\theta(f)(X) \oplus \phi(g)\phi(f)(X) \\
&= (\theta \oplus \phi)(g)(\theta(f)(X) \oplus \phi(f)(X)) \\
&= (\theta \oplus \phi)(g)(\theta \oplus \phi)(f)(X).
\end{aligned}
$$

Hence $\mathbf{A}^{\mathbf{C}}$ is an additive category.

To show that $\mathbf{A}^{\mathbf{C}}$ is an abelian category we need to define kernels and cokernels for each natural transformation $\alpha \in \mathrm{Nat}(\theta, \phi)$. For each object X of $\mathbf{C}$, let $k_X : K_X \longrightarrow \theta(X)$

be a kernel for the morphism α_X. Let $f : X \longrightarrow Y$ be a morphism in $\mathbf{C}$. Since α is a natural transformation the diagram below commutes.

$$\begin{tikzcd}
K_X \arrow[r, "k_X"] & \theta(X) \arrow[r, "\alpha_X"] \arrow[d, "\theta(f)"'] & \phi(X) \arrow[d, "\phi(f)"] \\
K_Y \arrow[r, "k_Y"'] & \theta(Y) \arrow[r, "\alpha_Y"'] & \phi(Y)
\end{tikzcd}$$

Thus $\alpha_Y\theta(f)k_X = \phi(f)\alpha_X k_X$, but $\alpha_X k_X$ is the zero morphism, and every composite with a zero morphism is zero, therefore $\alpha_Y\theta(f)k_X = 0$. Since k_Y is a kernel for α_Y, it follows that there is a unique morphism $\kappa_f : K_X \longrightarrow K_Y$ such that $\theta(f)k_X = k_Y\kappa_f$. Define

$$\psi : \mathbf{C} \longrightarrow \mathbf{A} \quad \text{by} \quad \psi(X) = K_X \quad \text{and} \quad \psi(f) = \kappa_f$$

for objects X and morphisms f of $\mathbf{C}$. To see that ψ is a functor, let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms in $\mathbf{C}$. There is a unique morphism $\psi(gf) = \kappa_{gf}$ such that $\theta(gf)k_X = k_Z\kappa_{gf}$. Now

$$k_Z\kappa_g\kappa_f = \theta(g)k_Y\kappa_f = \theta(g)\theta(f)k_X = \theta(gf)k_X,$$

hence $\kappa_g\kappa_f = \kappa_{gf}$, that is $\psi(gf) = \psi(g)\psi(f)$. It is now clear that ψ is a functor. It is immediate that the morphisms $k_X : \psi(X) \longrightarrow \theta(X)$ define a natural transformation k from ψ to θ. We shall show that k is a kernel for α. Let $\psi' : \mathbf{C} \longrightarrow \mathbf{A}$ be a functor and suppose $\sigma \in \mathrm{Nat}(\psi', \theta)$ is such that $\alpha\sigma = 0$. For each object X of $\mathbf{C}$ the solid arrows in the diagram below commutes

$$\begin{tikzcd}
\psi'(X) \arrow[dr, "\sigma_X"'] \arrow[drr, bend left, "0"] \arrow[dd, dotted, "\exists!\sigma'_X"'] & & \\
 & \theta(X) \arrow[r, "\alpha_X"] & \phi(X) \\
\psi(X) \arrow[ur, "k_X"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

Since k_X is a kernel for α_X, there is a unique morphism $\sigma'_X : \psi'(X) \longrightarrow \psi(X)$ in $\mathbf{A}$ making the diagram commute. To see that the morphisms σ'_X define a natural transformation, let $f : X \longrightarrow Y$ be a morphism in $\mathbf{C}$. It is required to show that the left-hand square in the diagram below commutes

$$\begin{tikzcd}
\psi'(X) \arrow[r, "\sigma'_X"] \arrow[d, "\psi'(f)"'] \arrow[rr, bend left, "\sigma_X"] & \psi(X) \arrow[r, "k_X"] \arrow[d, "\psi(f)"'] & \theta(X) \arrow[r, "\alpha_X"] \arrow[d, "\theta(f)"] & \phi(X) \arrow[d, "\phi(f)"] \\
\psi'(Y) \arrow[r, "\sigma'_Y"] \arrow[rr, bend right, "\sigma_Y"'] & \psi(Y) \arrow[r, "k_Y"] & \theta(Y) \arrow[r, "\alpha_Y"'] & \phi(Y)
\end{tikzcd}$$

Now $\alpha_Y\theta(f)\sigma_X = \phi(f)\alpha_X\sigma_X = 0$. Since k_Y is a kernel for α_Y, there is a unique morphism $h : \psi'(X) \longrightarrow \psi(Y)$ making the diagram below commute

$$\begin{tikzcd}
\psi'(X) \arrow[d, dotted, "\exists! h"'] \arrow[dr, "\theta(f)\sigma_X"'] \arrow[drr, bend left, "0"] & & \\
& \theta(Y) \arrow[r, "\alpha_Y"] & \phi(Y) \\
\psi(Y) \arrow[ur, "k_Y"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

Now $k_Y\psi(f)\sigma'_X = \theta(f)k_X\sigma'_X = \theta(f)\sigma_X$, so $h = \psi(f)\sigma'_X$. But $k_Y\sigma'_Y\psi'(f) = \sigma_Y\psi'(f) = \theta(f)\sigma_X$. Hence $h = \psi(f)\sigma'_X = \sigma'_Y\psi'(f)$, so $\sigma' \in \mathrm{Nat}(\psi', \psi)$. It is clear that ψ' is unique, since each morphism ψ'_X is unique. We have therefore proved that k is a kernel for α.

Similarly, for each object X of $\mathbf{C}$ we can choose a cokernel $c_X : \phi(X) \longrightarrow C_X$ for α. A dual construction to that above shows that the morphisms c_X define a morphism in $\mathbf{A}^{\mathbf{C}}$ which is a cokernel for α.

We have therefore shown that every morphism α in $\mathbf{A}^{\mathbf{C}}$ has a kernel and a cokernel.

Now let $\eta \in \mathrm{Nat}(\theta, \phi)$ be monic in $\mathbf{A}^{\mathbf{C}}$ and let $\alpha \in \mathrm{Nat}(\psi, \theta)$ be such that $\eta\alpha$ is the zero morphism from ψ to ϕ. Then α is a zero morphism. But then for each object X of $\mathbf{C}$, $\eta_X\alpha_X = 0$ in $\mathbf{A}$, and $\alpha_X = 0$. Hence each η_X is monic in $\mathbf{A}$. Let $c \in \mathrm{Nat}(\phi, \phi')$ be a cokernel for η, that is each $c_X : \phi(X) \longrightarrow \phi'(X)$ is a cokernel for $\eta_X : \theta(X) \longrightarrow \phi(X)$ in $\mathbf{A}$. But $\mathbf{A}$ is an abelian category and η_X is monic, so η_X is a kernel for c_X. Hence η is a kernel for c in $\mathbf{A}^{\mathbf{C}}$. Similarly every epi in $\mathbf{A}^{\mathbf{C}}$ is the cokernel of its kernel.

■

Recall from Lemma 7.1 that kernels and cokernels are unique up to isomorphism. For every morphism $f : X \longrightarrow Y$ in an abelian category we pick a kernel which we denote by $\ker(f) : \mathrm{Ker}(f) \longrightarrow X$. Thus $\ker(f)$ denotes a morphism and $\mathrm{Ker}(f)$ an object. We use the similar notation $\mathrm{coker}(f) : Y \longrightarrow \mathrm{Coker}(f)$ for the cokernel of f.

The following result is well-known, and is proved in Proposition VIII.3.1 of [21].

Proposition 7.4 *In an abelian category* $\mathbf{A}$, *every morphism* $f : X \longrightarrow Y$ *has a factorisation* $f = me$ *with* m *monic and* e *epi; moreover,*

$$m = \ker(\operatorname{coker}(f)) \quad \textit{and} \quad e = \operatorname{coker}(\ker(f)).$$

Furthermore, such a factorisation is unique up to isomorphism.

■

From the above factorisation, the *image* of f is defined as $m = \operatorname{im}(f) : \operatorname{Im}(f) \longrightarrow Y$, similarly, e is called the *coimage* of f.

7.2 Exact functors

Let $f : X \longrightarrow Y$ be a morphism in an abelian category $\mathbf{A}$. In the previous section, we discussed the kernel $\ker(f)$, cokernel $\operatorname{coker}(f)$ and image $\operatorname{im}(f)$. Given another morphism $g : Y \longrightarrow Z$, we can form the sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

We say that such a sequence is *exact at* Y if $\ker(g) = \operatorname{im}(f)$. The following sequence is exact at X if, and only if, f is a monic

$$0 \longrightarrow X \xrightarrow{f} Y.$$

Dually, f is epi if, and only if, the following sequence is exact at Y

$$X \xrightarrow{f} Y \longrightarrow 0.$$

We shall say that the sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \tag{7.1}$$

is a *short exact sequence* if it is exact at each object of the sequence. Equivalently, if X is a subobject of Y and $Z = Y/X$.

Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be an additive functor between abelian categories, we say that F is an *exact functor* if for each short exact sequence (7.1) in $\mathbf{A}$, the sequence

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$$

is exact in $\mathbf{B}$.

Exact functors represent the 'ideal' against which we measure other functors that fail to be exact to some extent. We now define two examples of functors which are 'nearly exact'. An additive functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ between abelian categories is said to be *left exact* if, for each short exact sequence (7.1) in $\mathbf{A}$, the sequence

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

is exact. We say that F is *right exact* if the sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0.$$

is exact. Clearly an additive functor is exact precisely when it is both left and right exact.

The above definitions apply to covariant functors but can also be applied to contravariant functors. For example, if F is contravariant from $\mathbf{A}$ to $\mathbf{B}$, we say that F is exact if for each short exact sequence (7.1) in $\mathbf{A}$, the sequence

$$0 \longleftarrow F(X) \xleftarrow{F(f^{\mathrm{op}})} F(Y) \xleftarrow{F(g^{\mathrm{op}})} F(Z) \longleftarrow 0$$

is exact in $\mathbf{B}$.

The following result gives two examples of left exact functors, one covariant and the other contravariant. Proofs are given, for example, in Weibel [31] Proposition 1.6.8 and Corollary 1.6.9.

Proposition 7.5 *Let* $\mathbf{A}$ *be an abelian category.*

(i) *The covariant hom functor* $\hom(X, -)$ *is a left exact functor for every object* X *of* $\mathbf{A}$.

(ii) *The contravariant hom functor* $\hom(-, X)$ *is a left exact functor for every object* X *of* $\mathbf{A}$.

■

It can be proved that an additive functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ is left exact if whenever $0 \longrightarrow X \longrightarrow Y \longrightarrow Z$ is exact in $\mathbf{A}$, then $0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z)$ is exact in $\mathbf{B}$. (Exercise 1.6.3 of [31]). With a dual result for right exact functors.

7.3 Complexes

In the previous section we discussed pairs of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z. \tag{7.2}$$

over an abelian category $\mathbf{A}$, which are exact at Y. In particular, this means that the composite gf is the zero morphism. This leads us to study sequences (7.2) such that $gf = 0$ but which need not be exact. In this case $g\mathrm{im}(f) = 0$, so there is a unique morphism h such that $\mathrm{im}(f) = \ker(g)h$, that is $\mathrm{im}(f) \leqslant \ker(g)$. One can therefore form the quotient object $\mathrm{Ker}(g)/\mathrm{Im}(f)$.

In an abelian category $\mathbf{A}$, a *complex* $C^\bullet$ is a sequence of composable morphisms

$$\cdots\; C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \longrightarrow \cdots$$

with $d^{n+1}d^n = 0$. The morphisms $d^n : C^{n-1} \longrightarrow C^n$ are called *differentials.* The sequence need not be exact. A measure of how the complex fails to be exact at each object C^n is the n^{th} *cohomology object*

$$H^n(C) = \frac{\mathrm{Ker}(d^n)}{\mathrm{Im}(d^{n-1})}.$$

We call $\mathrm{Ker}(d^n)$ an *n-cocycle object* and $\mathrm{Im}(d^{n-1})$ an *n-coboundary object.* Observe that $C^\bullet$ is exact precisely when every $H^n(C) = 0$.

If $C^\bullet$ is a complex and $C^n = 0$ for all $n < 0$, then $C^\bullet$ is called a *positive complex.* If $C^n = 0$ for all $n > 0$, then $C^\bullet$ is called a *negative complex.* A positive complex is called a *cochain complex*, a negative complex is called a *chain complex.* For chain complexes the following notation is used:

$$C^{-n} = C_n, \qquad d^{-n} = d_n, \qquad 0 \leqslant n.$$

Let $C^\bullet$ and $D^\bullet$ be two complexes over an abelian category $\mathbf{A}$. We denote the differentials of C and D by d_C^n and d_D^n respectively. A *morphism of complexes* $u : C^\bullet \longrightarrow D^\bullet$ is a collection of morphisms $u^n : C^n \longrightarrow D^n$ such that $u^{n+1}d_C^n = d_D^n u^n$, for all n. That is, such that the diagram below commutes

$$\begin{tikzcd}
\cdots \arrow[r] & C^{-1} \arrow[r, "d_C^{-1}"] \arrow[d, "u^{-1}"'] & C^0 \arrow[r, "d_C^0"] \arrow[d, "u^0"] & C^1 \arrow[r, "d_C^1"] \arrow[d, "u^1"] & C^2 \arrow[r, "d_C^2"] \arrow[d, "u^2"] & C^3 \arrow[r] \arrow[d, "u^3"] & \cdots \\
\cdots \arrow[r] & D^{-1} \arrow[r, "d_D^{-1}"'] & D^0 \arrow[r, "d_D^0"'] & D^1 \arrow[r, "d_D^1"'] & D^2 \arrow[r, "d_D^2"'] & D^3 \arrow[r] & \cdots
\end{tikzcd}$$

If $u : C^\bullet \longrightarrow D^\bullet$ is a morphism between chain (respectively cochain) complexes, then we call u a *chain* (respectively *cochain*) *map.*

Complexes over an abelian category $\mathbf{A}$, together with their morphisms form a category, denoted $\mathbf{Ch}(\mathbf{A})$. It can be shown that this category is abelian (Theorem 1.2.3 of Weibel [31]).

We refer the reader to Section 7.1 of [5] for proof of the following.

Proposition 7.6 *Let $C^\bullet$ and $D^\bullet$ be complexes over an abelian category $\mathbf{A}$, and let $u : C^\bullet \longrightarrow D^\bullet$ be a morphism of complexes. For all n, there is a morphism $H^n(u) : H^n(C) \longrightarrow H^n(D)$ in $\mathbf{A}$, so that $H^n : \mathbf{Ch}(\mathbf{A}) \longrightarrow \mathbf{A}$ is a functor.*

■

In order to classify morphisms of complexes, there is a notion of homotopy. Let $u, v : C^\bullet \longrightarrow D^\bullet$ be morphisms in an abelian category $\mathbf{A}$. A *homotopy* s from u to v is a family of morphisms $s^n : C^n \longrightarrow D^{n-1}$, one for each dimension n, such that

$$d_D^{n-1}s^n + s^{n+1}d_C^n = u^n - v^n$$

or, dropping the superscripts for clarity, $ds + sd = u - v$. We write $s : u \simeq v$. The situation is illustrated below

$$\begin{tikzcd} C^{n-1} \arrow[r, "d_C^{n-1}"] & C^n \arrow[r, "d_C^n"] \arrow[dl, "s^n"'] \arrow[d, shift right, "v^n"'] \arrow[d, shift left, "u^n"] & C^{n+1} \arrow[dl, "s^{n+1}"] \\ D^{n-1} \arrow[r, "d_D^{n-1}"'] & D^n \arrow[r, "d_D^n"'] & D^{n+1} \end{tikzcd}$$

We say that a morphism of complexes $u : C^\bullet \longrightarrow D^\bullet$ is an *equivalence* if there is a morphism $\overline{u} : D^\bullet \longrightarrow C^\bullet$, and homotopies $s : \overline{u}u \simeq \mathrm{Id}_{C^\bullet}$, $t : u\overline{u} \simeq \mathrm{Id}_{D^\bullet}$. If such an equivalence exists, then we say that $C^\bullet$ and $D^\bullet$ are *homotopic*. See Proposition 7.1 of [5] for proof of the following.

Proposition 7.7 *Let $u, v : C^\bullet \longrightarrow D^\bullet$ be morphisms of complexes in some abelian category. If there is a cochain homotopy $s : u \simeq v$, then*

$$H^n(u) = H^n(v) : H^n(C) \longrightarrow H^n(D).$$

■

We let $0^\bullet$ denote the zero complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots.$$

Let $C^\bullet$ be a complex. Define $\mathbf{0} : C^\bullet \longrightarrow C^\bullet$, where

$$\mathbf{0}^n : C^n \longrightarrow C^n \quad \text{is given by} \quad \mathbf{0}^n(c) = 0,$$

for all $c \in C^n$.

We say that a complex $C^\bullet$ is *contractible* if it is homotopic to the zero complex $0^\bullet$, or equivalently, if $\mathrm{Id}_{C^\bullet} \simeq \mathbf{0}$. Thus $C^\bullet$ is contractible if there is a family of morphisms $s^n : C^n \longrightarrow C^{n-1}$ such that

$$d^{n-1}s^n + s^{n+1}d^n = \mathrm{Id}_{C^n}.$$

We call the homotopy s a *contracting homotopy.* The following is immediate from Proposition 7.7.

Proposition 7.8 *If a complex $C^\bullet$ is contractible, then it is exact.*

■

This result is useful because in order to show that a complex $C^\bullet$ is exact, it is sufficient to construct a contracting homotopy for $C^\bullet$.

7.4 Resolutions

Let $\mathbf{A}$ be an abelian category. An object P of $\mathbf{A}$ is called *projective* if it satisfies the following universal lifting property: given a morphism $\alpha : P \longrightarrow C$, and an epi $g : B \longrightarrow C$, there is at least one morphism $\beta : P \longrightarrow C$ such that $\alpha = g\beta$. The situation is illustrated below

$$\begin{tikzcd} & P \arrow[dl, dotted, "\exists\beta"'] \arrow[d, "\alpha"] & \\ B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

We say that the category $\mathbf{A}$ has *enough projectives* if for every object A, there is an epi $P \longrightarrow A$ with P projective. Dually an object I is said to be *injective* if, for every monic $f : A \longrightarrow B$, and morphism $\gamma : A \longrightarrow I$, there is a morphism $\delta : B \longrightarrow I$, such that $\delta f = \gamma$, as illustrated below

$$\begin{tikzcd} & I & \\ B \arrow[ur, dotted, "\exists\delta"] & A \arrow[l, "f"] \arrow[u, "\gamma"'] & 0 \arrow[l] \end{tikzcd}$$

We say that $\mathbf{A}$ has *enough injectives* if for every object A of $\mathbf{A}$, there is a monic $A \longrightarrow I$ with I injective.

The following is immediate.

Lemma 7.9 *Let I be an object of an abelian category $\mathbf{A}$. Then I is injective in $\mathbf{A}$ if, and only if, I is projective in $\mathbf{A}^{\mathrm{op}}$.* ■

See Example 2.3.13 of Weibel [31] for a proof of the following.

Proposition 7.10 *Let* $\mathbf{A}$ *be an abelian category and* $\mathbf{C}$ *a category. If* $\mathbf{A}$ *is complete and has enough injectives, then* $\mathbf{A}^{\mathbf{C}}$ *has enough injectives. A dual result holds for projectives.*

■

Let A be an object of $\mathbf{A}$. A *left resolution* of A is an exact sequence

$$\cdots \longrightarrow P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} A \longrightarrow 0.$$

If each P_i is projective then it is called a *projective resolution* of A. We write $P_\bullet \xrightarrow{\varepsilon} A$. Dually, a *right resolution* of an object A is an exact sequence

$$0 \longrightarrow A \xrightarrow{\delta} I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \longrightarrow \cdots$$

If each I^i is injective then it is called a *injective resolution* of A. We write $A \xrightarrow{\delta} I^\bullet$.

The first part of Lemma 7.11 is proved as Lemma 2.2.5 of [31], the second part is the dual case.

Lemma 7.11 *Let* $\mathbf{A}$ *be an abelian category.*

(i) *If* $\mathbf{A}$ *has enough injectives then every object in* $\mathbf{A}$ *has an injective resolution.*

(ii) *If* $\mathbf{A}$ *has enough projectives then every object in* $\mathbf{A}$ *has a projective resolution.*

■

Theorem 7.12 (Comparison Theorem for injective resolutions) *Let* $A \xrightarrow{\delta} I^\bullet$ *be an injective resolution of an object* A *in an abelian category* $\mathbf{A}$, *and let* $f' : A \longrightarrow B$ *be a morphism in* $\mathbf{A}$. *Then for every right resolution* $E^\bullet \xrightarrow{\eta} B$ *of* B, *there is a cochain map* $f^\bullet : E^\bullet \longrightarrow I^\bullet$ *lifting* f' *in the sense that* $\delta f' = f^0\eta$. *The map* $f^\bullet$ *is unique up to cochain homotopy equivalence.*

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\eta"] \arrow[d, "f'"'] & E^0 \arrow[r, "d"] \arrow[d, dotted, "\exists f^0"] & E^1 \arrow[r, "d"] \arrow[d, dotted, "\exists f^1"] & E^2 \arrow[r] \arrow[d, dotted, "\exists f^2"] & \cdots \\
0 \arrow[r] & A \arrow[r, "\delta"'] & I^0 \arrow[r, "d"'] & I^1 \arrow[r, "d"'] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

■

There is a dual result for projective resolutions. (See Theorem 2.2.6 of [31]).

7.5 Derived functors

Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a left exact functor between two abelian categories. We assume that $\mathbf{A}$ has enough injectives. For every object A of $\mathbf{A}$ we may, by Lemma 7.11, choose an injective resolution

$$0 \longrightarrow A \xrightarrow{\varepsilon_A} I_A^0 \xrightarrow{d} I_A^1 \xrightarrow{d} I_A^2 \longrightarrow \cdots.$$

Now $F(d^{n+1})F(d^n) = F(d^{n+1}d^n) = F(0) = 0$, hence the sequence

$$F(I_A^0) \xrightarrow{F(d)} F(I_A^1) \xrightarrow{F(d)} F(I_A^2) \longrightarrow \cdots$$

is a cochain complex, which we shall denote by $F(I_A)^\bullet$.

We construct a sequence of functors from the category $\mathbf{A}$ to the category $\mathbf{B}$. For each $n \geqslant 0$, define

$$(R^nF)(A) = H^n(F(I_A)).$$

We now show how to define R^nF on morphisms. Let $f' : A \longrightarrow B$ be a morphism in $\mathbf{A}$. By the Comparison Theorem, we can choose a cochain map $f^\bullet : I_A^\bullet \longrightarrow I_B^\bullet$. Hence there is a cochain map

$$F(f)^\bullet : F(I_A)^\bullet \longrightarrow F(I_B)^\bullet.$$

By Proposition 7.6, there are morphisms $H^n(F(f)) : H^n(F(I_A)) \longrightarrow H^n(F(I_B))$, for all $n \geqslant 0$. Thus we may define

$$(R^nF)(f') = H^n(F(f)) : (R^nF)(A) \longrightarrow (R^nF)(B).$$

If $g' : \mathbf{A} \longrightarrow \mathbf{B}$ is another morphism of A, then by the Comparison Theorem the cochain maps $f^\bullet$ and $g^\bullet$ are homotopic. It is then evident that the cochain maps $F(f)^\bullet$ and $F(g)^\bullet$ are homotopic. Thus by Proposition 7.7 $H^n(F(f)) = H^n(F(g))$; that is $(R^nF)(f') = (R^nF)(g')$.

It is immediate that $(R^nF)(uv) = (R^nF)(u)(R^nF)(u)$. Hence each R^nF is a functor from $\mathbf{A}$ to $\mathbf{B}$. These functors are called the *right derived functors* of F.

The definition of the functors R^nF appears to be dependent on the choice of injective resolution. However, with the help of the Comparison Theorem, it can be proved that if a different injective resolution is chosen, then each resulting functor $\overline{R}^nF$ is naturally isomorphic to R^nF. Thus the definition of R^nF is independent of the choice of injective resolution used.

Proposition 7.13 *Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a left exact functor and let $\mathbf{A}$ have enough injectives. Then*

(i) $R^0 F \cong F$.

(ii) *If Q is an injective object in* $\mathbf{A}$ *then* $(R^n F)(Q) = 0$, *for all* $n \geqslant 1$.

(iii) *If F is exact then* $(R^n F)(A) = 0$, *for all objects A of* $\mathbf{A}$ *and all* $n \geqslant 1$.

PROOF.
(i) By definition $(R^0 F)(A) = H^0(F(I_A)) = \mathrm{Ker}(F(d^0))/\mathrm{Im}(F(d^{-1}))$, but $F(I_A)^\bullet$ is a cochain complex, so $d^{-1} = 0$. Hence $(R^0 F)(A) = \mathrm{Ker}(F(d^0))$. Now

$$0 \longrightarrow A \xrightarrow{\varepsilon_A} I_A^0 \xrightarrow{d^0} I_A^1$$

is exact and F is a left exact functor. Hence the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(\varepsilon_A)} F(I_A^0) \xrightarrow{F(d^0)} F(I_A^1)$$

is exact. Therefore $\mathrm{Ker}(F(d^0)) = \mathrm{Im}(F(\varepsilon_A))$, but $\mathrm{Im}(F(\varepsilon_A)) \cong F(A)$. Hence $F(A) \cong (R^0 F)(A)$.
(ii) For an injective object Q one has that

$$0 \longrightarrow Q \xrightarrow{\mathrm{Id}_Q} Q \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

is an injective resolution of Q. It is immediate that this resolution yields $(R^n F)(Q) = 0$, for all $n \geqslant 1$.
(iii) If F is exact, then the whole sequence

$$0 \longrightarrow F(A) \xrightarrow{F(\varepsilon_A)} F(I_A^0) \xrightarrow{F(d)} F(I_A^1) \xrightarrow{F(d)} F(I_A^2) \longrightarrow \cdots$$

is exact, therefore every $\mathrm{Ker}(F(d^{n+1}))/\mathrm{Im}(F(d^n)) = 0$. ■

Dual to right derived functors are 'left derived functors'. Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a right exact functor between two abelian categories and suppose that $\mathbf{A}$ has enough projectives. For each object A of $\mathbf{A}$, choose a projective resolution $P_\bullet^A \xrightarrow{\varepsilon^A} A$. Applying F to $P_\bullet$ yields a chain complex $F(P^A)_\bullet$. Define

$$(L_n F)(A) = H_n(F(P^A)).$$

For each morphism $f' : A \longrightarrow B$ in $\mathbf{A}$, there are chain maps $f_\bullet : P_\bullet^A \longrightarrow P_\bullet^B$ and $F(f)_\bullet : F(P^A)_\bullet \longrightarrow F(P^A)_\bullet$. Define

$$(L_n F)(f') = H_n(F(f)) : (L_n F)(A) \longrightarrow (L_n F)(B).$$

These functors are called the *left derived functors* of F.

Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a left exact functor, where $\mathbf{A}$ has enough injectives. We can construct right derived functors R^nF. Now F also defines a right exact functor $F^{\text{op}} : \mathbf{A}^{\text{op}} \longrightarrow \mathbf{B}^{\text{op}}$, and $\mathbf{A}^{\text{op}}$ has enough projectives. Thus we can construct left derived functors L_nF^{op}. An injective resolution $A \longrightarrow I^\bullet$ of A in $\mathbf{A}$ is a projective resolution of A in $\mathbf{A}^{\text{op}}$. Thus

$$(R^nF)(A) = (L_nF^{\text{op}})^{\text{op}}(A).$$

Therefore all the results about right derived functors apply to left derived functors. In particular, the objects $(L_nF)(A)$ are independent of the choice of projective resolution, $L_0F \cong F$, and $(L_nF)(P) = 0$ whenever P is projective and $n \neq 0$.

Let $\mathbf{A}$ and $\mathbf{B}$ be abelian categories. For objects A and B of $\mathbf{A}$, we have seen that the covariant hom functor $\hom(A, -)$ is a left exact covariant functor and the contravariant hom functor $\hom(-, B)$ is a left exact contravariant hom functor. Let us assume $\mathbf{A}$ has enough injectives. We may construct the right derived functors

$$R^n \hom(A, -) \quad \text{and} \quad R^n \hom(-, B)$$

where $R^n \hom(A, -)(B)$ and $R^n \hom(-, B)(A)$ are isomorphic. (Remark: there is a precise sense in which this isomorphism is natural, see Weibel [31]). Define

$$\mathrm{Ext}^n(A, B) = R^n \hom(A, -)(B) \cong R^n \hom(-, B)(A).$$

We may compute $\mathrm{Ext}^n(A, B)$ in two ways:

1. via an injective resolution of B.

2. via a projective resolution of A; that is an injective resolution of A in $\mathbf{A}^{\text{op}}$.

7.6 Simplicial sets

One way of constructing cochain complexes is via simplicial sets.

Definition A *simplicial set* K is a family of sets K_n, $n \in \mathbb{N}$ together with functions

$$d_i : K_n \longrightarrow K_{n-1} \quad \text{and} \quad s_i : K_n \longrightarrow K_{n+1}$$

for each $1 \leqslant i \leqslant n$, which satisfy the following conditions:

$$\begin{array}{lll}
\text{(SS1)} & d_i d_j = d_{j-1} d_i & \text{for } i < j, \\
\text{(SS2)} & d_i s_j = s_{j-1} d_i & \text{for } i < j, \\
\text{(SS3)} & d_i s_j = \mathrm{Id}_{K_n} & \text{for } i = j \text{ or } i = j+1, \\
\text{(SS4)} & d_i s_j = s_j d_{i-1} & \text{for } i > j+1, \\
\text{(SS5)} & s_i s_j = s_{j+1} s_i & \text{for } i \leqslant j.
\end{array}$$

We picture K as shown below.

$$\begin{tikzcd} K_0 \arrow[r, shift left=1] & K_1 \arrow[l, shift left=2] \arrow[l, shift left=0] \arrow[r, shift left=1] \arrow[r, shift right=1] & K_2 \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] & \cdots \end{tikzcd}$$

The elements of K_n are called *n-simplices*. The functions d_i are called *face maps*, and the functions s_i are called *degeneracy maps*. If each of the K_n and each of the d_i and s_i are in a category C, then K is called a *simplicial object* over C.

We describe two examples of simplicial sets.

1. A simplicial set arising from a poset Let X be a partially ordered set, and let $\mathrm{PNer}_n(X)$ denote the set of totally ordered sequences of n elements of X. If $\mathbf{x} \in \mathrm{PNer}_n(X)$, we write

$$\mathbf{x} = (x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n).$$

The i^{th} face map $d_i : \mathrm{PNer}_n(X) \longrightarrow \mathrm{PNer}_{n-1}$ removes the element x_i from x; that is

$$d_i : (x_1 \leqslant \cdots \leqslant x_i \leqslant \cdots \leqslant x_n) \longmapsto (x_1 \leqslant \cdots \leqslant x_{i-1} \leqslant x_{i+1} \leqslant \cdots \leqslant x_n).$$

The i^{th} degeneracy map $s_i : \mathrm{PNer}_n(X) \longrightarrow \mathrm{PNer}_{n+1}(X)$ repeats the element x_i; that is

$$s_i : (x_1 \leqslant \cdots \leqslant x_i \leqslant \cdots \leqslant x_n) \longmapsto (x_1 \leqslant \cdots \leqslant x_i \leqslant x_i \leqslant \cdots \leqslant x_n).$$

2. The nerve of a category Let C be a small category. Define $\mathrm{Ner}_0(C) = C_o$ and

$$\mathrm{Ner}_n(C) = \{(a_n, a_{n-1}, \ldots, a_1) \mid a_i \in C \text{ and } \mathbf{d}(a_{i+1}) = \mathbf{r}(a_i)\}.$$

Thus $\mathrm{Ner}_n(C)$ is the set of composable sequences of n elements of C. An n-simplex in $\mathrm{Ner}_n(C)$ may be pictured as

$$\xleftarrow{a_n} \; \xleftarrow{a_{n-1}} \; \cdots \; \xleftarrow{a_2} \; \xleftarrow{a_1}$$

The face maps $d_0, d_1 : \mathrm{Ner}_1(C) \longrightarrow \mathrm{Ner}_0(C)$ on 1-simplices are given by

$$d_0(a) = \mathbf{r}(a) \quad \text{and} \quad d_1(a) = \mathbf{d}(a).$$

For $n > 1$, the i^{th} face map is

$$d_i : (a_n, a_{n-1}, \ldots, a_1) \longmapsto \begin{cases} (a_n, a_{n-1}, \ldots, a_2) & \text{for } i = 0, \\ (a_n, a_{n-1}, \ldots, a_{i+1}a_i, \ldots, a_1), & \text{for } i = 1, \ldots, n-1, \\ (a_{n-1}, \ldots, a_1) & \text{for } i = n. \end{cases}$$

The degeneracy maps $s_i : \mathrm{Ner}_n(C) \longrightarrow \mathrm{Ner}_{n+1}(C)$ are given by

$$s_i : (a_n, a_{n-1}, \ldots, a_1) \longmapsto \begin{cases} (a_n, a_{n-1}, \ldots, a_1, \mathbf{d}(a_1)) & \text{for } i = 0, \\ (a_n, a_{n-1}, \ldots, \mathbf{r}(a_i), a_i, \ldots, a_1) & \text{for } i = 1, \ldots, n. \end{cases}$$

7.7 A cochain complex from a simplicial set

Let K be a simplicial set. We shall construct a cochain complex over the category **Ab** of abelian groups. Let A be an abelian group. Let $X^n(K, A)$ denote the set of functions from K_n to A, where A is regarded as a set.

Consider the face maps $d_i : K_{n+1} \longrightarrow K_n$, $i = 0, \ldots, n$. These give rise to functions

$$d_i^n : X^n(K, A) \longrightarrow X^{n+1}(K, A) \quad \text{with} \quad d_i^n(\phi) = \phi d_i.$$

Define

$$d^n : X^n(K, A) \longrightarrow X^{n+1}(K, A) \quad \text{by} \quad \sum_{i=0}^{n} (-1)^i d_i^n(\phi).$$

Proposition 7.14 *For any abelian group A and simplicial set K, we have that $X^\bullet(K, A)$ is a cochain complex over* **Ab**.

PROOF. It is straightforward to show that each $X^n(K, A)$ is an abelian group under pointwise addition. We show that each d^n is a homomorphism. It is clear that $d^n(0_n) = 0_{n+1}$. Let $\phi, \psi \in X^n(K, A)$. Then for $i = 0, \ldots, n$, and $x \in K_{n+1}$

$$\begin{aligned} d_i^n(\phi + \psi)(x) &= (\phi + \psi)d_i(x) \\ &= \phi(d_i(x)) + \psi(d_i(x)) \\ &= d_i^n(\phi)(x) + d_i^n(\psi)(x) \\ &= (d_i^n(\phi) + d_i^n(\psi))(x). \end{aligned}$$

It follows that $d^n(\phi + \psi) = d^n(\phi) + d^n(\psi)$. To see that $d^{n+1}d^n = 0$, let $\phi \in X^n(K, A)$. Then

$$\begin{aligned} d^{n+1}d^n(\phi) &= d^{n+1}\Big(\sum_{i=0}^{n} (-1)^i d_i^n(\phi)\Big) \\ &= \sum_{j=0}^{n+1} (-1)^j d_j^{n+1}\Big(\sum_{i=0}^{n} (-1)^i d_i^n(\phi)\Big) \\ &= \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} d_j^{n+1} d_i^n(\phi) \\ &= \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} \phi d_i d_j \\ &= \sum_{j=i+1}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} \phi d_i d_j + \sum_{j=0}^{i} \sum_{i=0}^{n} (-1)^{i+j} \phi d_i d_j \\ &= \sum_{j=i+1}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} \phi d_{j-1} d_i + \sum_{j=0}^{i} \sum_{i=0}^{n} (-1)^{i+j} \phi d_i d_j \quad \text{by (SS1)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=i}^{n}\sum_{i=0}^{n}(-1)^{i+k+1}\phi d_k d_i + \sum_{j=0}^{i}\sum_{i=0}^{n}(-1)^{i+j}\phi d_i d_j \quad \text{where } k = j-1 \\
&= -\sum_{k=i}^{n}\sum_{i=0}^{n}(-1)^{i+k}\phi d_k d_i + \sum_{j=0}^{i}\sum_{i=0}^{n}(-1)^{i+j}\phi d_i d_j \\
&= 0_n
\end{aligned}
$$

■

Chapter 8

Cohomology of categories

In this chapter we shall describe the standard cohomology of categories. In Section 8.1, we define the cohomology of groups, this will motivate the definition of cohomology of categories given in Section 8.2.1.

8.1 Cohomology of groups

We use shall use the machinery outlined in Chapter 7 to define cohomology groups $H^n(G, A)$ where G is a group acting on the abelian group A.

Let G be a group. A *right action* of G on an abelian group A assigns to each pair $(a, g) \in A \times G$ an element $a \cdot g$ in such a way that the following conditions are satisfied.

(i) $a \cdot 1 = a$.

(ii) $a \cdot (gh) = (a \cdot g) \cdot h$.

(iii) $(a + b) \cdot g = a \cdot g + b \cdot g$.

Note that $0 \cdot g = (0 + 0) \cdot g = 0 \cdot g + 0 \cdot g$. Hence $g \cdot 0 = 0$. If G acts on abelian groups A and B, we say that a homomorphism $\alpha : A \longrightarrow B$ is a G-*morphism* if

$$\alpha(a \cdot g) = \alpha(a) \cdot g \quad \text{for all } a \in A.$$

It is easy to show that G-actions and G-morphisms form a category.

Alternatively, an action of G on A can be thought of as a collection of homomorphisms $\theta(g) : A \longrightarrow A$, with $\theta(g)(a) = g \cdot a$. Note that $\theta(1) = \mathrm{Id}_A$. Thus an action θ is a functor from G to $\mathbf{Ab}$, where G is regarded as a category with one object $*$. Let $\theta, \phi : G \longrightarrow \mathbf{Ab}$ be functors. A natural transformation from θ to ϕ is a homomorphism $\alpha : \theta(*) \longrightarrow \phi(*)$

such that the diagram below commutes for all $g \in G$

$$\begin{tikzcd} \theta(*) \arrow[r, "\alpha"] \arrow[d, "\theta(g)"'] & \phi(*) \arrow[d, "\phi(g)"] \\ \theta(*) \arrow[r, "\alpha"] & \phi(*) \end{tikzcd}$$

That is $\alpha(\theta(g)(a)) = \phi(g)(\alpha(a))$, for all $a \in \theta(*)$. Hence $\alpha(g \cdot a) = g \cdot \alpha(a)$. Therefore for any group the category of G-actions and G-morphisms is the functor category $\mathbf{Ab}^G$, which we also denote by $\mathcal{A}$.

By Proposition 7.2, **Ab** is an abelian category, but then by Proposition 7.3, $\mathcal{A}$ is an abelian category.

In order to calculate cohomology groups in $\mathcal{A}$ we need to show that this category has enough injectives.

See Theorem 3.3 of Popescu [24] for proof of the following.

Proposition 8.1 *The category* **Ab** *has enough injectives.*

■

Hence $\mathcal{A}$ has enough injectives by Proposition 7.10.

We now construct a left exact functor from $\mathcal{A}$ to **Ab**. Let G act on A. Define

$$A^G = \{a \in A \mid a \cdot g = a, \forall g \in G\}.$$

It is immediate that A^G is a subgroup of A. Let $\alpha : A \longrightarrow B$ be a G-morphism. Define

$$\alpha^G : A^G \longrightarrow B^G$$

to be the restriction of α to A^G. This is clearly well-defined because for $a \in A^G$ and $g \in G$ we have

$$\alpha(a) \cdot g = \alpha(a \cdot g) = \alpha(a).$$

It is easy to check that $(-)^G$ is a functor. We show that it is left exact. Let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$. It is required to show that the sequence

$$0 \longrightarrow A^G \xrightarrow{\alpha^G} B^G \xrightarrow{\beta^G} C^G$$

is exact. Since α is a monomorphism, we have that α^G is a monomorphism, so the sequence is exact at A^G. We need to show that $\mathrm{Ker}(\beta^G) = \mathrm{Im}(\alpha^G)$. Let $b \in \mathrm{Im}(\alpha^G)$. Then $b = \alpha(a)$ for some $a \in A^G$, therefore $\beta(b) = \beta\alpha(a)$, but $\beta\alpha = 0$, hence $\beta(b) = 0$. Thus $\mathrm{Im}(\alpha^G) \subseteq \mathrm{Ker}(\beta^G)$. Now let $b \in \mathrm{Ker}(\beta^G)$. Then $\beta(b) = 0$, that is $b \in \mathrm{Ker}(\beta)$, hence $b \in \mathrm{Im}(\alpha)$. Therefore $b = \alpha(a)$ for some $a \in A$, it remains to show that $a \in A^G$. Let $g \in G$. Then

$$\alpha(a \cdot g) = \alpha(a) \cdot g = b \cdot g = b,$$

since $b \in B^G$. But $b = \alpha(a)$, hence $\alpha(a \cdot g) = \alpha(a)$. But α is a monomorphism, so $a \cdot g = a$. Hence $a \in A^G$. It follows that $\mathrm{Im}(\alpha^G) = \mathrm{Ker}(\beta^G)$. We have therefore constructed a left exact functor $(-)^G : \mathcal{A} \longrightarrow \mathbf{Ab}$.

We are now able to define cohomology groups of G. Let G act on A. Since $\mathcal{A}$ has enough injectives, there is an injective resolution $A \xrightarrow{\varepsilon} I_A^\bullet$. Applying the left exact functor $(-)^G$ to $I_A^\bullet$ yields a cochain complex which in turn yields cohomology groups. Hence we define

$$H^n(G, A) = (R^n(-)^G)(A)$$

to be the n^{th} *cohomology group of* G *with coefficients in* A.

By Proposition 7.13(i), $R^0(-)^G \cong (-)^G$. Therefore $H^0(G, A) \cong A^G$.

To explicitly compute cohomology groups one must construct appropriate resolutions. The standard resolution is the 'bar resolution'. The construction is given in [2, 20]. It can be proved [2] that:

- The first cohomology group $H^1(G, A)$ is isomorphic to the group of functions $d : G \longrightarrow A$ such that

  $$d(gh) = d(g) \cdot h + d(h), \quad \text{for all } g, h \in G,$$

  modulo the subgroup of functions $G \longrightarrow A$ of the form $g \longmapsto a - a \cdot g$, for some fixed $a \in A$.

- The second cohomology group is the group of functions $f : G \times G \longrightarrow A$ such that

  $$f(g, h) \cdot k + f(gh, k) = f(h, k) + f(g, hk) \quad \text{for all } g, h, k \in G,$$

  modulo the subgroup of functions $\delta c : G \times G \longrightarrow A$ of the form

  $$\delta c(g, h) = c(g) \cdot h + c(h) - c(gh),$$

  where $c : G \longrightarrow A$ is a function.

We shall now look at the definition of group cohomology from a different angle that will be useful in generalisation.

Let G be a group and A an abelian group. Let $T(A)$ denote the abelian group A with the trivial action of G on A, that is $a \cdot g = a$ for all $g \in G$ and $a \in A$. For any homomorphism $\theta : A \longrightarrow B$ define

$$T(\theta) : T(A) \longrightarrow T(B) \quad \text{by} \quad T(\theta)(a) = \theta(a).$$

It is immediate that $T : \mathbf{Ab} \longrightarrow \mathcal{A}$ is a functor. Note that $(T(A))^G = A$.

Proposition 8.2 *The functor $(-)^G$ is right adjoint to T.*

PROOF. Let A be an abelian group and B a G-module. Let $\alpha : T(A) \longrightarrow B$ be a G-morphism. For $a \in A$ and $g \in G$ we have

$$\alpha(a) \cdot g = \alpha(a \cdot g) = \alpha(a),$$

so $\alpha(a) \in B^G$. Hence we can define a function

$$\tau_{A,B} : \hom_{\mathcal{A}}(T(A), B) \longrightarrow \hom_{\mathbf{Ab}}(A, B^G) \quad \text{by} \quad \tau_{A,B}(\alpha)(a) = \alpha(a).$$

It is immediate that $\tau_{A,B}$ is bijective. It remains to show that τ is natural in A and B. Let $\beta : B \longrightarrow C$ be a G-morphism and $\theta : A \longrightarrow A'$ a group homomorphism. It is straightforward to show that the conditions

$$\tau_{A,C}(\beta\alpha) = \beta^G \tau_{A,B}\alpha \quad \text{and} \quad \tau_{A',B}(\alpha T(\theta)) = \tau_{A,B}(\alpha)\theta$$

hold. It follows that τ is natural.

■

If we put A equal to the group of integers $\mathbb{Z}$ with the trivial G-action. Then

$$\hom_{\mathcal{A}}(\mathbb{Z}, B) \simeq \hom_{\mathbf{Ab}}(\mathbb{Z}, B^G) \simeq B^G.$$

Hence we obtain the following result.

Proposition 8.3 *The left exact functor $(-)^G$ is naturally isomorphic to* $\hom_{\mathcal{A}}(\mathbb{Z}, -)$.

■

8.2 Cohomology of categories

8.2.1 Definition of the cohomology functor

In this section we generalise the cohomology of groups given in the previous section to obtain a cohomology of categories. The cohomology of small categories is due to Watts [30], for more on the cohomology of categories, we refer the reader to [1]. Let $\mathbf{J}$ be a small category. We can form the functor category $\mathbf{Ab}^{\mathbf{J}}$, since $\mathbf{Ab}$ is abelian so too is $\mathbf{Ab}^{\mathbf{J}}$ by Proposition 7.3. Consider the diagonal functor

$$\Delta : \mathbf{Ab} \longrightarrow \mathbf{Ab}^{\mathbf{J}}$$

which sends each abelian group A to the constant functor $\Delta(A)$; the functor which has the value A at each object of $\mathbf{J}$ and the value Id_A at each morphism of $\mathbf{J}$. A *limit* for a functor $F : \mathbf{J} \longrightarrow \mathbf{Ab}$ consists of an abelian group $L = \mathrm{Lim}(F)$ together with a natural transformation $\nu : \Delta(L) \longrightarrow F$ which is universal among natural transformations. Since $\Delta(L) : \mathbf{J} \longrightarrow \mathbf{Ab}$ is the functor with constant value L, the natural transformation ν assigns to each object J of $\mathbf{J}$ a homomorphism $\nu_J : L \longrightarrow F(J)$ so that the triangle below commutes

$$\begin{tikzcd}
L \arrow[r, "\nu_J"] \arrow[dr, "\nu_{J'}"'] & F(J) \arrow[d, "F(j)"] \\
& F(J')
\end{tikzcd}$$

for any morphism $j : J \longrightarrow J'$. The universal property means that for any other abelian group A and natural transformation $\tau : \Delta(A) \longrightarrow F$, there is unique homomorphism u such that $\nu_J u = \tau_J$, for all objects J of $\mathbf{J}$. The situation is pictured below.

$$\begin{tikzcd}
L \arrow[r, "\nu_J"] \arrow[dr, "\nu_{J'}"'] & F(J) \arrow[d, "F(j)"] \\
A \arrow[u, "u"] \arrow[ur, "\tau_J"] \arrow[r, "\tau_{J'}"'] & F(J')
\end{tikzcd}$$

We now show how to define Lim on the morphisms of $\mathbf{Ab}^{\mathbf{J}}$. Let $F, G : \mathbf{J} \longrightarrow \mathbf{Ab}$ be functors and $\alpha : F \longrightarrow G$ a natural transformation. There are universal natural transformations $\nu : \Delta(\mathrm{Lim}(F)) \longrightarrow F$ and $\eta : \Delta(\mathrm{Lim}(G)) \longrightarrow G$ such that the diagram below commutes

$$\begin{tikzcd}
\mathrm{Lim}(F) \arrow[r, "\nu_J"] \arrow[dr, "\nu_{J'}"'] & F(J) \arrow[r, "\alpha_J"] \arrow[d, "F(j)"] & G(J) \arrow[d, "G(j)"'] & \mathrm{Lim}(G) \arrow[l, "\eta_J"'] \arrow[dl, "\eta_{J'}"] \\
& F(J') \arrow[r, "\alpha_{J'}"'] & G(J') &
\end{tikzcd}$$

for any morphism $j : J \longrightarrow J'$ of $\mathbf{J}$. Now $\alpha\nu : \Delta(\mathrm{Lim}(F)) \longrightarrow G$ is a natural transformation, so by the universal property of η there is a unique homomorphism $\mathrm{Lim}(\alpha)$: $\mathrm{Lim}(F) \longrightarrow \mathrm{Lim}(G)$. By uniqueness of $\mathrm{Lim}(\alpha)$ it is easy to show that $\mathrm{Lim} : \mathbf{Ab}^{\mathbf{J}} \longrightarrow \mathbf{Ab}$ is a functor. Furthermore Lim is right adjoint to Δ, see page 88 of MacLane [21] for details. By the dual of Theorem IV.1.3 of [21], Lim is an additive functor. By Theorem 2.6.1 of Weibel [31], Lim is left exact. Hence we arrive at the following result.

Proposition 8.4 $\mathrm{Lim} : \mathbf{Ab}^{\mathbf{J}} \longrightarrow \mathbf{Ab}$ *is a left exact additive functor which is right adjoint to the diagonal functor* Δ.

■

Let A be an abelian group and $F : \mathbf{J} \longrightarrow \mathbf{Ab}$ a functor. By adjointness we have

$$\hom_{\mathbf{Ab}^{\mathbf{J}}}(\Delta(A), F) \cong \hom_{\mathbf{Ab}}(A, \mathrm{Lim}(F)).$$

Let $A = \mathbb{Z}$. Then

$$\hom_{\mathbf{Ab}^{\mathbf{J}}}(\Delta(\mathbb{Z}), F) \cong \hom_{\mathbf{Ab}}(\mathbb{Z}, \mathrm{Lim}(F)) \cong \mathrm{Lim}(F).$$

Hence we have the following result.

Proposition 8.5 *The left exact functor* $\mathrm{Lim}(-)$ *is naturally isomorphic to* $\hom_{\mathbf{Ab}^{\mathbf{J}}}(\Delta(\mathbb{Z}), -)$..

■

Proposition 8.5 generalises the group case, Proposition 8.3.

Definition Let C be a small category. A *left C-module* is a (covariant) functor from C to $\mathbf{Ab}$. A *right C-module* is a contravariant functor from C to $\mathbf{Ab}$; that is, a functor from C^{op} to $\mathbf{Ab}$. So a right C-module is a left C^{op}-module. If $\mathcal{A}$ and $\mathcal{B}$ are two left (right) C-modules, a *left (right) C-morphism* α from $\mathcal{A}$ to $\mathcal{B}$ is a natural transformation $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$. Let $\mathrm{Mod}_{\mathrm{L}}(C)$ denote the category of left C-modules and left C-morphisms and let $\mathrm{Mod}_{\mathrm{R}}(C)$ denote the category of right C-modules and right C-morphisms. Thus $\mathrm{Mod}_{\mathrm{L}}(C)$ is the abelian category $\mathbf{A}^C$ and $\mathrm{Mod}_{\mathrm{R}}(C)$ is the abelian category $\mathbf{A}^{C^{\mathrm{op}}}$.

The functor $\mathrm{Lim} : \mathbf{A}^{C^{\mathrm{op}}} \longrightarrow \mathbf{Ab}$ is an additive left exact functor. We define

$$H^n(C, A) = (R^n\mathrm{Lim})(A)$$

to be the n^{th} *cohomology group of* C *with coefficients in* A.

8.2.2 Free right C-modules

In this section we shall define free objects in the category $\mathbf{Ab}^{C^{\mathrm{op}}}$ of right C-modules.

Let X be a set viewed as the discrete category. An X-*set* is a functor.

$$T : X \longrightarrow \mathbf{Set}.$$

We can view T as a disjoint union of sets

$$T = \bigsqcup_{x \in X} T(x).$$

If $T, R : X \longrightarrow \mathbf{Set}$ are two X-sets, an X-*morphism* from T to R is a natural transformation

$$\alpha : T \longrightarrow R.$$

Since X is a discrete category, α simply assigns to each element x of X a function $\alpha_x : T(x) \longrightarrow R(x)$. The category of X-sets and X-morphisms is the category $\mathbf{Set}^X$.

For a small category C, any (right or left) C-module is clearly a C_o-set. Given a C_o-set T the problem is to construct C-module which is 'free' over T in some sense.

We begin with free objects in $\mathbf{Set}^{C^{\mathrm{op}}}$. The starting point is the forgetful functor

$$\mathbb{U} : \mathbf{Set}^{C^{\mathrm{op}}} \longrightarrow \mathbf{Set}^{C_o}$$

which assigns to each C-module, its underlying C_o-set. We need to define a functor in the opposite direction which is left adjoint to $\mathbb{U}$. Let $T : C_o \longrightarrow \mathbf{Set}$ be a C_o-set. We shall construct a functor $\mathbb{F}(T) : C^{\mathrm{op}} \longrightarrow \mathbf{Set}$. For each $e \in C_o$ define

$$\mathbb{F}(T)(e) = \{(t, x) \mid x \in C \text{ such that } \mathbf{d}(x) = e, \text{ and } t \in T(\mathbf{r}(x))\},$$

and for each element $f \xrightarrow{a} e$ of C, define a function

$$\mathbb{F}(T)(a) : \mathbb{F}(T)(e) \longrightarrow \mathbb{F}(T)(f) \quad \text{by} \quad \mathbb{F}(T)(a) : (t, x) \longmapsto (t, xa).$$

This is a well-defined function.

Proposition 8.6 $\mathbb{F}(T) : C^{\mathrm{op}} \longrightarrow \mathbf{Set}$ *is a functor, for any* C_o*-set* T.

Proof. If $e \in C_o$ and $(t,x) \in \mathbb{F}(T)(e)$, then $\mathbb{F}(T)(e)(t,x) = (t, xe) = (x,t)$, so $\mathbb{F}(T)(e)$ is an identity. If $\exists ab$ in C and $\mathbf{r}(b) = e$, then

$$\mathbb{F}(T)(a)\mathbb{F}(T)(b)(t,x) = \mathbb{F}(T)(a)(t,xb) = (t,xba) = \mathbb{F}(T)(ba)(t,x).$$

So $\mathbb{F}(T) : C^{\text{op}} \longrightarrow \mathbf{Set}$ is a functor. ■

Let T be a C_o-set. We shall define a C_o-morphism η_T from T to $\mathbb{F}(T)$, where we regard $\mathbb{F}(T)$ as a C_o-set. Define

$$\eta_T : T \longrightarrow \mathbb{F}(T) \quad \text{by} \quad \eta_T(t) = (t,e)$$

where $t \in T(e)$.

Theorem 8.7 *Let $F : C^{\text{op}} \longrightarrow$ **Set** be a functor together with a C_o-function ι from T to F, where F is regarded as a C_o-set. Then there is a unique natural transformation γ from $\mathbb{F}(T)$ to F such that the following diagram commutes in the category of C_o-sets.*

$$\begin{tikzcd}
\mathbb{F}(T) \arrow[r, "\gamma"] & F \\
T \arrow[u, "\eta_T"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

Proof. For each $e \in C_o$ define a function

$$\gamma_e : \mathbb{F}(T)(e) \longrightarrow F(e) \quad \text{by} \quad \gamma_e(t,x) = F(x)(\iota_{\mathbf{r}(x)}(t)).$$

It is immediate that each γ_e is well-defined. We show that γ is a natural transformation. Let $f \xrightarrow{a} e$ be an element of C, we need to show that the diagram below commutes.

$$\begin{tikzcd}
e & & \mathbb{F}(T)(e) \arrow[r, "\gamma_e"] \arrow[d, "\mathbb{F}(T)(a)"'] & F(e) \arrow[d, "F(a)"] \\
f \arrow[u, "a"] & & \mathbb{F}(T)(f) \arrow[r, "\gamma_f"'] & F(f)
\end{tikzcd}$$

Let $(t,x) \in \mathbb{F}(T)(e)$, then

$$F(a)\gamma_e(t,x) = F(a)F(x)(\iota_{\mathbf{r}(x)}(t)) = F(xa)(\iota_{\mathbf{r}(x)}(t))$$

and

$$\gamma_f \mathbb{F}(T)(a)(t,x) = \gamma_f(t,xa) = F(xa)(\iota_{\mathbf{r}(x)}(t)).$$

Hence γ is a natural transformation. Furthermore, for $e \in C_o$ and $t \in T_e$ we have

$$\gamma_e \eta_T(t) = \gamma(t, e) = F(e)(\iota_e(t)) = \iota_e(t),$$

in the category of C_o-sets. Hence $\gamma\eta_T = \iota$.

It remains to show that γ is unique. Let $\sigma : \mathbb{F}(T) \longrightarrow F$ be a natural transformation such that $\sigma\eta_T = \iota$. Let $e \xrightarrow{x} f$ in C and $t \in T(f)$; that is $(t, x) \in \mathbb{F}(T)(e)$, also $\eta_T(t) = (t, f) \in \mathbb{F}(T)(f)$ and $\mathbb{F}(T)(x)(t, f) = (t, x)$. The diagram below commutes since σ is a natural transformation.

$$\begin{tikzcd} f & \mathbb{F}(T)(f) \arrow[r, "\sigma_f"] \arrow[d, "\mathbb{F}(T)(x)"'] & F(f) \arrow[d, "F(x)"] \\ e \arrow[u, "x"] & \mathbb{F}(T)(e) \arrow[r, "\sigma_e"'] & F(e) \end{tikzcd}$$

Thus $F(x)\sigma_f(t, f) = \sigma_e\mathbb{F}(T)(x)(t, f) = \sigma_e(t, x)$. But $\sigma_f(t, f) = \sigma_f\eta_T(t) = \iota_f(t)$, by assumption. Therefore

$$\sigma_e(t, x) = F(x)\sigma_f(t, f) = \mathbb{F}(T)(x)(\iota_f(t)) = \gamma_e(t, x).$$

■

By part (ii) of Theorem IV.1.2 of MacLane [21], it follows that $\mathbb{F}$ is the object part of a functor

$$\mathbb{F} : \mathbf{Set}^{C_o} \longrightarrow \mathbf{Set}^{C^{\mathrm{op}}}$$

which is left adjoint to $\mathbb{U}$. The functor $\mathbb{F}$ is defined on morphisms as follows. Let T and R be C_o-sets and $\beta : T \longrightarrow R$ a C_o-function. The natural transformation

$$\mathbb{F}(\beta) : \mathbb{F}(T) \longrightarrow \mathbb{F}(R) \quad \text{is given by } C\text{-morphisms} \quad \mathbb{F}(\beta)_e : (t, x) \longmapsto (\beta(t), x).$$

Now consider the forgetful functor

$$\mathbb{V} : \mathbf{Ab}^{C^{\mathrm{op}}} \longrightarrow \mathbf{Set}^{C^{\mathrm{op}}}.$$

This has a left adjoint $\mathbb{G}$ which assigns to each $e \in C_o$ the free abelian group on $F(e)$, where $F : C^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a functor. For a morphism $e \xrightarrow{x} f$ of C, $\mathbb{G}(F)(x)$ is the homomorphism induced by $\mathbb{F}(x)$ and for a natural transformation $\alpha : F \longrightarrow G$ in $\mathbf{Set}^{C^{\mathrm{op}}}$, each $\mathbb{G}(\alpha)_e : \mathbb{G}(F)_e \longrightarrow \mathbb{G}(G)(e)$ is the homomorphism induced by α_e.

We thus have the following diagram of forgetful functors $\mathbb{U}$, $\mathbb{V}$ and left adjoints $\mathbb{F}$, $\mathbb{G}$.

$$\begin{tikzcd} \mathbf{Ab}^{C^{\mathrm{op}}} \arrow[r, shift left, "\mathbb{V}"] & \mathbf{Set}^{C^{\mathrm{op}}} \arrow[l, shift left, "\mathbb{G}"] \arrow[r, shift left, "\mathbb{U}"] & \mathbf{Set}^{C_o} \arrow[l, shift left, "\mathbb{F}"] \end{tikzcd}$$

Let

$$\mathbb{W} = \mathbb{U}\mathbb{V} : \mathbf{Ab}^{C^{\mathrm{op}}} \longrightarrow \mathbf{Set}^{C_o} \quad \text{and} \quad \mathbb{Z} = \mathbb{G}\mathbb{F} : \mathbf{Set}^{C_o} \longrightarrow \mathbf{Ab}^{C^{\mathrm{op}}},$$

by Theorem IV.8.1 of MacLane $\mathbb{Z}$ is left adjoint to $\mathbb{W}$.

Definition Let $F : C^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ be a functor and $T : C_o \longrightarrow \mathbf{Set}$ a C_o-set. Let $\eta : T \longrightarrow F$ be a C_o-morphism, regarded as a C_o-set. We say that F is a *free right C-module on T* if for any functor $F' : C^{\mathrm{op}} \longrightarrow \mathbf{Ab}$, and C_o-morphism η' from T to F', there is a unique natural transformation $\alpha : F \longrightarrow F'$ such that the diagram below commutes in the category of C_o-sets.

$$\begin{tikzcd} F \arrow[r, "\alpha"] & F' . \\ T \arrow[u, "\eta"] \arrow[ur, "\eta'"'] & \end{tikzcd}$$

Let T be a C_o-set. The functor $\mathbb{Z}(T) = \mathcal{G}\mathcal{F}(T) : C^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ assigns to each identity e of C the free abelian group generated by the set $\mathbb{F}(T)(e)$, and to each x in C the homomorphism induced by the function $\mathbb{F}(T)(x)$.

The following is immediate from Theorem 8.7 and properties of free groups.

Proposition 8.8 *The functor $\mathbb{Z}(T) : C^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ is a free right C-module over T, for any C_o-set T.*

■

Corollary 8.8.1 *Let C be a category, T a C_o-set, F a C-module and $\beta : T \longrightarrow F$ a C_o-morphism. The unique C-morphism $\alpha : \mathbb{Z}(T) \longrightarrow F$ such that $\alpha\eta = \beta$ is defined on the generators of $\mathbb{Z}(T)$ by*

$$\alpha(t, x) = F(x)(\beta(t)).$$

■

Corollary 8.8.2 *Let C be a category, T a C_o-set and F a C-module. Define*

$$\omega : \hom_{\mathbf{Ab}^{C^{\mathrm{op}}}}(\mathbb{Z}(T), F) \longrightarrow \hom_{\mathbf{Set}^{C^{\mathrm{op}}}}(T, F)$$

to be the function which assigns to each C-morphism $\alpha : \mathbb{Z}(T) \longrightarrow F$, the C_o-morphism

$$\omega(\alpha) : T \longrightarrow F \quad \textit{given by} \quad \omega(\alpha) : t \longmapsto \alpha(t, e)$$

where $t \in T(e)$. Then ω is bijective.

PROOF. The C_o-morphism

$$\eta_T : T \longrightarrow \mathbb{Z}(T) \quad \text{is defined by} \quad \eta_T(t) = (t, e),$$

where $t \in T(e)$. So $\omega(\alpha) = \alpha\eta$. Since $\mathbb{Z}(T)$ is free over T, every C-morphism $\alpha : \mathbb{T}(T) \longrightarrow F$ is uniquely determined by $\omega(\alpha)$, so that the diagram below commutes

$$\begin{tikzcd} \mathbb{Z}(T) \arrow[r, "\alpha"] & F . \\ T \arrow[u, "\eta"] \arrow[ur, "\omega(\alpha)"'] & \end{tikzcd}$$

With this information it is straightforward to show that ω is injective and surjective.

■

Proposition 8.9 *Free right C-modules are projective in $\mathbf{Ab}^{\mathbf{C}^{\mathrm{op}}}$.*

PROOF. Let P be a free C module over a C_o-set T, and let $\eta : T \longrightarrow P$ be a C_o-morphism, where P is regarded as a C_o-set. Suppose that $A, B : C^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ are right C-modules, $g : A \longrightarrow B$ is a C-epimorphism and $\gamma : P \longrightarrow A$ a C-morphism. We need to construct a C-morphism from P to A. Since g is a C-epimorphism, the functions $g_e : A_e \longrightarrow B_e$ are surjective for every $e \in C_o$. Therefore we can choose functions $\varepsilon_e : B_e \longrightarrow A_e$ such that $g_e\varepsilon_e = \mathrm{Id}_{B_e}$. Hence there is a C_o-morphism $\varepsilon : B \longrightarrow A$ in the category of C_o-sets, with $g\varepsilon = \mathrm{Id}_B$. Let $\eta' = \varepsilon\gamma\eta$ which is a C_o-morphism from T to A, regarded as a C_o-set. Since P is free over T there is a unique C-morphism $\alpha : P \longrightarrow A$ such that $\alpha\eta = \eta'$. It remains to show that the diagram below commutes in $\mathbf{Ab}^{C^{\mathrm{op}}}$

$$\begin{tikzcd} & P . \arrow[dl, "\alpha"'] \arrow[d, "\gamma"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

Since P is free over T, γ is the unique C-morphism from P to B making the diagram below commute in $\mathbf{Set}^{C_o}$

$$\begin{tikzcd} P \arrow[r, "\gamma"] & B . \\ T \arrow[u, "\eta"] \arrow[ur, "\gamma\eta"'] & \end{tikzcd}$$

But

$$g\alpha\eta = g\eta' = g\varepsilon\gamma\eta = \gamma\eta,$$

hence $\gamma = g\alpha$. ■

8.2.3 Computing the cohomology of categories

The aim of this section is to construct a cochain complex for a category C, which will enable us to calculate cohomology groups.

An example of a C-module is the *constant module over* $\mathbb{Z}$, given by the diagonal functor $\Delta\mathbb{Z} : C^{\mathrm{op}} \longrightarrow \mathbb{Z}$. This has value $\mathrm{Id}_{\mathbb{Z}}$ at every element of C. Thus $\Delta\mathbb{Z}(e) \cong \mathbb{Z}$ for all $e \in C_o$. We write n_e for elements of $\Delta\mathbb{Z}(e)$, and for each element x of C, we have

$$\Delta\mathbb{Z}(x) : \Delta\mathbb{Z}(\mathbf{r}(x)) \longrightarrow \Delta\mathbb{Z}(\mathbf{d}(x)) \quad \text{given by} \quad \Delta\mathbb{Z}(x)(n_{\mathbf{r}(x)}) = n_{\mathbf{d}(x)}.$$

For any small category C, we can form the simplicial set $\mathrm{Ner}(C)$, introduced in Section 7.6. The set $\mathrm{Ner}_0(C)$, is the set of identities of C. For $n \geqslant 1$, elements of $\mathrm{Ner}_n(C)$ are composable sequences of elements of C, denoted $(x_n, x_{n-1}, \ldots, x_1)$. For each integer $n \geqslant 0$, $\mathrm{Ner}_n(C)$ is a C_o-set with

$$\mathrm{Ner}_n(C)(e) = \{(x_n, \ldots, x_1) \in \mathrm{Ner}_n(C) \mid \mathbf{d}(x_1) = e\}.$$

We shall construct the free C-module on the C_o-set $\mathrm{Ner}_n(C)$. The functor

$$\mathbb{F}(\mathrm{Ner}_n(C)) : C^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

is defined on objects by

$$\mathbb{F}(\mathrm{Ner}_n(C))(e) = \{((x_n, \ldots, x_1), y) \in \mathrm{Ner}_n(C) \times C \mid \mathbf{d}(y) = e,\ \mathbf{d}(x_1) = \mathbf{r}(y)\}$$

but then $(x_n, \ldots, x_1, y) \in \mathrm{Ner}_{n+1}(C)(e)$, so $\mathbb{F}(\mathrm{Ner}_n(C))(e) = \mathrm{Ner}_{n+1}(C)(e)$. We shall write elements of $\mathbb{F}(\mathrm{Ner}_n(C))(e)$ as $(x_n, \ldots, x_1, x_0)$. For $e \xrightarrow{y} f$ in C, we have

$$\mathbb{F}(\mathrm{Ner}_n(C))(y) : \mathbb{F}(\mathrm{Ner}_n(C))(f) \longrightarrow \mathbb{F}(\mathrm{Ner}_n(C))(e)$$

given by

$$\mathbb{F}(\mathrm{Ner}_n(C))(y) : (x_n, \ldots, x_1, x_0) \longmapsto (x_n, \ldots, x_1, x_0 y).$$

$\mathbb{F}(\mathrm{Ner}_n(C))$ is a functor with the universal property of Theorem 8.7. The functor

$$\mathbb{Z}(\mathrm{Ner}_n(C)) : C^{\mathrm{op}} \longrightarrow \mathbf{Ab}$$

assigns to each $e \in C_o$ the free abelian group generated by $\mathbb{F}(\mathrm{Ner}_n(C))(e)$, and to each $e \xrightarrow{y} f$ the homomorphism induced by $\mathbb{F}(\mathrm{Ner}_n(C))(y)$. The C_o-morphism

$$\eta_n : \mathrm{Ner}_n(C) \longrightarrow \mathbb{Z}(\mathrm{Ner}_n(C)) \quad \text{is given by} \quad \eta_n : (x_n, \ldots, x_1) \longmapsto (x_n, \ldots, x_1, \mathbf{d}(x_1)),$$

where $\mathbb{Z}(\mathrm{Ner}_n(C))$ is regarded as a C_o-set.

For each integer $n \geqslant 0$, the face maps $\mathrm{Ner}_{n+1}(C) \longmapsto \mathrm{Ner}_n(C)$ give rise to C_o-functions $\overline{d}_n : \mathrm{Ner}_{n+1}(C) \longrightarrow \mathbb{Z}(\mathrm{Ner}_n(G))$ given by

$$\begin{aligned}\overline{d}_n : (x_{n+1}, \ldots, x_1) \longmapsto (x_{n+1}, \ldots, x_1) \;&+\; \sum_{j=1}^{n}(-1)^j(x_{n+1}, \ldots, x_{j+1}x_j, \ldots, x_1, \mathbf{d}(x_1)) \\ &+\; (-1)^{n+1}(\mathbf{d}(x_{n+1}), x_n, \ldots, x_1, \mathbf{d}(x_1))\end{aligned}$$

Since $\mathbb{Z}(\mathrm{Ner}_{n+1}(C))$ is freely generated by $\mathrm{Ner}_{n+1}(C)$, there is a unique C-morphism d_n from $\mathbb{Z}(\mathrm{Ner}_{n+1}(C))$ to $\mathbb{Z}(\mathrm{Ner}_n(C))$ such that the diagram

$$\begin{tikzcd} \mathbb{Z}(\mathrm{Ner}_{n+1}(C)) \arrow[r, "d_n"] & \mathbb{Z}(\mathrm{Ner}_n(C)) \\ \mathrm{Ner}_{n+1}(C) \arrow[u, "\eta_n"] \arrow[ur, "\overline{d}_n"'] & \end{tikzcd}$$

commutes in the category of C_o-sets. By Corollary 8.8.1, d_n is defined on the generators of the free abelian group $\mathbb{Z}(\mathrm{Ner}_n(C))$ by

$$\begin{aligned} d_n(x_{n+1}, \ldots, x_1, x_0) &= \mathbb{Z}(\mathrm{Ner}_{n+1}(C))(x_0)(\overline{d}(x_{n+1}, \ldots, x_1)) \\ &= (x_{n+1}, \ldots, x_1x_0) + \sum_{j=1}^{n}(-1)^j(x_{n+1}, \ldots, x_{j+1}x_j, \ldots, x_1, x_0) \\ &\quad + (-1)^{n+1}(\mathbf{d}(x_{n+1}), x_n, \ldots, x_1, x_0). \end{aligned}$$

That is

$$d_n(x_{n+1}, \ldots, x_0) = \sum_{j=1}^{n}(-1)^j(x_{n+1}, \ldots, x_{j+1}x_j, \ldots, x_0) + (-1)^{n+1}(\mathbf{d}(x_{n+1}), x_n, \ldots, x_0).$$

Define a C_o-function

$$\overline{\varepsilon} : \mathrm{Ner}_0(C) \longrightarrow \Delta\mathbb{Z} \quad \text{by} \quad \overline{\varepsilon} : e \longmapsto 1_e.$$

This gives rise to a C-morphism $\varepsilon : \mathbb{Z}(\mathrm{Ner}_0(C)) \longrightarrow \Delta\mathbb{Z}$ such that $\varepsilon\eta_n = \overline{\varepsilon}$.

Proposition 8.10 *For any category C, the sequence*

$$\cdots \longrightarrow \mathbb{Z}(\mathrm{Ner}_2(C)) \xrightarrow{d_1} \mathbb{Z}(\mathrm{Ner}_1(C)) \xrightarrow{d_0} \mathbb{Z}(\mathrm{Ner}_0(C)) \xrightarrow{\varepsilon} \Delta\mathbb{Z} \longrightarrow 0 \qquad (8.1)$$

is a projective resolution of $\Delta\mathbb{Z}$ in the category of right C-modules.

PROOF. The C-modules $\mathbb{Z}(\mathrm{Ner}_n(C))$ are free by construction, so by Proposition 8.9, each $\mathbb{Z}(\mathrm{Ner}_n(C))$ is a projective object in $\mathbf{Ab}^{C^{\mathrm{op}}}$. It is routine to show that $\overline{d}_n\overline{d}_{n+1} = 0$; the calculation is similar to that given in the proof of Proposition 7.14. By uniqueness of the C-morphisms d_n, it is immediate that $d_n d_{n+1} = 0$. Hence 8.1 is a chain complex.

We need to show that the sequence 8.1 is exact. By Proposition 7.8, to show that a sequence is exact, it is enough to construct a contracting homotopy. Define C_o-morphisms $\overline{\sigma}_n : \mathbb{F}(\mathrm{Ner}_n(C)) \longrightarrow \mathbb{Z}(\mathrm{Ner}_{n+1}(C))$ by

$$\overline{\sigma}_n : (x_n, \ldots, x_1, x_0) \longmapsto (x_n, \ldots, x_1, x_0, \mathbf{d}(x_0)).$$

Now $\mathbb{F}(\mathrm{Ner}_n(C))$ generates $\mathbb{Z}(\mathrm{Ner}_n(C))$ freely as an abelian group. Hence $\overline{\sigma}_n$ extends uniquely to a C_o-morphism $\sigma_n : \mathbb{Z}(\mathrm{Ner}_n(C)) \longrightarrow \mathbb{Z}(\mathrm{Ner}_{n+1}(C))$, such that each $\sigma|(\mathbb{Z}(\mathrm{Ner}_n(C))(e))$ is a group homomorphism.

In addition, we define a C-morphism

$$\tau : \Delta\mathbb{Z} \longrightarrow \mathbb{Z}(\mathrm{Ner}_0(C)) \quad \text{by} \quad \tau : n_e \longmapsto n(e).$$

To show that σ and τ define a contracting homotopy, it is required to show that

$$\varepsilon\tau = \mathrm{Id}_{\Delta\mathbb{Z}}, \quad \tau\varepsilon + d_0\sigma_0 = \mathrm{Id}_{\mathbb{Z}(\mathrm{Ner}_0(C))} \quad \text{and} \quad d_n\sigma_n + \sigma_{n-1}d_{n-1} = \mathrm{Id}_{\mathbb{Z}(\mathrm{Ner}_n(C))},$$

for all $n \geqslant 1$. The first condition is immediate. Let (x) be a generator for $\mathbb{Z}(\mathrm{Ner}_0(C))$. Then

$$(\tau\varepsilon + d_0\sigma_0)(x) = \tau(1_{\mathbf{d}(x)}) + d_0(x, \mathbf{d}(x)) = (\mathbf{d}(x)) + (x\mathbf{d}(x)) - (\mathbf{d}(x)) = (x).$$

So $\tau\varepsilon + d_1\sigma_0 = \mathrm{Id}_{\mathbb{Z}(\mathrm{Ner}_0(C))}$. Let $(x_n, \ldots, x_0)$ be a generator for $\mathbb{Z}(\mathrm{Ner}_n(C))$. Then

$$\begin{aligned}
\sigma_{n-1}d_{n-1}(x_n, \ldots, x_0) &= \sigma\Big(\sum_{i=0}^{n-1}(-1)^i(x_n, \ldots, x_{i+1}x_i, \ldots, x_0) \\
&\quad +(-1)^n(\mathbf{d}(x_n), x_{n-1}, \ldots, x_0)\Big) \\
&= \sum_{i=0}^{n-1}(-1)^i(x_n, \ldots, x_{i+1}x_i, \ldots, x_0, \mathbf{d}(x_0)) \\
&\quad +(-1)^n(\mathbf{d}(x_n), x_{n-1}, \ldots, x_0, \mathbf{d}(x_0))
\end{aligned}$$

and

$$\begin{aligned}
d_n\sigma_n(x_n, \ldots, x_0) &= d_n(x_n, \ldots, x_0, \mathbf{d}(x_0)) \\
&= (x_n, \ldots, x_1, x_0\mathbf{d}(x_0)) \\
&\quad +\sum_{i=0}^{n-1}(-1)^{j+1}(x_n, \ldots, x_{i+1}x_i, \ldots, x_0, \mathbf{d}(x_0))
\end{aligned}$$

$$
\begin{aligned}
& \quad +(-1)^{n+1}(\mathbf{d}(x_n), x_{n-1}, \ldots, x_0, \mathbf{d}(x_0)) \\
= & \quad (x_n, \ldots, x_0) \\
& \quad - \sum_{i=0}^{n-1} (-1)^j (x_n, \ldots, x_{i+1}x_i, \ldots, x_0, \mathbf{d}(x_0)) \\
& \quad -(-1)^n (\mathbf{d}(x_{n-1}, \ldots, x_0, \mathbf{d}(x_0)) \\
= & \quad (x_n, \ldots, x_0) - \sigma_{n-1} d_n (x_n, \ldots, x_0).
\end{aligned}
$$

Hence $d_n \sigma_n + \sigma_{n-1} d_{n-1} = \mathrm{Id}_{\mathbb{Z}(\mathrm{Ner}_n(C))}$.

■

Let $\mathcal{A}$ be a right C-module. We apply the right exact functor $\hom(-, \mathcal{A})$ to the projective resolution constructed above. This yields a cochain complex

$$
K^0(C, \mathcal{A}) \xrightarrow{d_*^0} K^1(C, \mathcal{A}) \xrightarrow{d_*^1} K^2(C, \mathcal{A}) \longrightarrow \cdots
$$

where $K^n(C, \mathcal{A}) = \hom(\mathbb{Z}(\mathrm{Ner}_n(C)), \mathcal{A})$ and d_*^n is defined by $d_*^n(\phi) = \phi d_n$.

The n^{th} cohomology group of the category C with coefficients in $\mathcal{A}$ is defined to be the group

$$
H^n(C, \mathcal{A}) = \frac{\mathrm{Ker}(d_*^n)}{\mathrm{Im}(d_*^{n-1})}.
$$

Chapter 9

Cohomology of ordered groupoids

In this chapter we show that the cohomology of an ordered groupoid can be defined as the cohomology of a small category.

9.1 Actions of inverse semigroups

The material in this section provides motivation for the definition of actions of ordered groupoids.

Let S be a semigroup. A *right action* of S on a set X is a function

$$X \times S \longrightarrow X \quad \text{denoted} \quad (x, s) \longmapsto x \cdot s$$

satisfying $x \cdot (st) = (x \cdot s) \cdot t$, for all $s, t \in S$ and $x \in X$. If S happens to be an inverse semigroup, then we get left actions of inverse semigroups. However this definition is often not sufficient for inverse semigroups because it doesn't respect all the extra structure; in particular, we require the action to take account of the natural partial order and the structure of the associated groupoid. In this section we define a special class of left actions which will be the basis for our definition of left actions of ordered groupoids.

Definition Let $(X, \leqslant)$ be a poset regarded as a category in which there is an arrow $x \longrightarrow y$ if $x \leqslant y$. A *presheaf* on X with values in a category $\mathbf{C}$ is a functor

$$\phi : X^{\mathrm{op}} \longrightarrow \mathbf{C}.$$

If $x \leqslant y$ then we write

$$\phi_x^y : \phi(y) \longrightarrow \phi(x).$$

Clearly, $\phi_e^e = \mathrm{Id}_{X(e)}$ and $\phi_w^x \phi_x^y = \phi_w^y$, for $w \leqslant x \leqslant y$.

Let S be an inverse semigroup. Then $E(S)$ is a meet semilattice with respect to the natural partial order. Let F be a presheaf of sets over $E(S)$. Put

$$X = \bigsqcup_{e \in E(S)} F(e) = \bigsqcup_{e \in E(S)} X_e.$$

For $e, k \in E(S)$ with $k \leqslant e$ there is a function $\phi^e_k : X_e \longrightarrow X_k$. A *partial right action* of S on the presheaf X is a family of functions

$$f_s : X_{\mathbf{r}(s)} \longrightarrow X_{\mathbf{d}(s)},$$

one for each $s \in S$, satisfying the following conditions

(SA1) If $e \in E(S)$, then $f_e = \mathrm{Id}_{X_e}$.

(SA2) If $s, t \in S$ with $\mathbf{d}(s) = \mathbf{r}(t)$, then $f_t f_s = f_{st}$.

(SA3) If $s, t \in S$ with $t \leqslant s$ then $\phi^{\mathbf{d}(s)}_{\mathbf{d}(t)} f_s = f_t \phi^{\mathbf{r}(s)}_{\mathbf{r}(t)}$.

If we write $x \cdot s = f_s(x)$, then the conditions (SA1) and (SA2) become

$$x \cdot e = x \quad \text{and} \quad x \cdot (st) = (x \cdot s) \cdot t$$

respectively. The condition (SA3) amounts to the square pictured below being commutative,

$$\begin{tikzcd} X_{\mathbf{d}(s)} \arrow[d, "\phi^{\mathbf{d}(s)}_{\mathbf{d}(t)}"'] & X_{\mathbf{r}(s)} \arrow[l, "f_s"'] \arrow[d, "\phi^{\mathbf{r}(s)}_{\mathbf{r}(t)}"] \\ X_{\mathbf{d}(t)} & X_{\mathbf{r}(t)} \arrow[l, "f_t"] \end{tikzcd}$$

that is $\phi^{\mathbf{d}(s)}_{\mathbf{d}(t)}(x \cdot s) = \phi^{\mathbf{r}(s)}_{\mathbf{r}(t)}(x) \cdot t$. This condition shows that the action is compatible with the natural partial order.

It is important to notice that each element of S acts partially on X. In fact $f_s : X_{\mathbf{r}(s)} \longrightarrow X_{\mathbf{d}(x)}$ is bijective, this is because

$$f_s f_{s^{-1}} = f_{\mathbf{d}(s)} = \mathrm{Id}_{X_{\mathbf{d}(x)}} \quad \text{and} \quad f_{s^{-1}} f_s = f_{\mathbf{r}(s)} = \mathrm{Id}_{X_{\mathbf{r}(x)}}.$$

Hence each f_s is a partial bijection of X.

Our definition does not look much like a semigroup action. However, it can be extended so that each $s \in S$ acts on all of X. Let $x \in X$ be arbitrary, say $x \in X_e$. Put $f = \mathbf{r}(s) \wedge e = \mathbf{r}(s)e$, then $\mathbf{r}(fs) = f$ and $\phi^e_f(x) \in X_f$. We can therefore define

$$x \circ s = \phi^e_f(x) \cdot (fs),$$

note that $fs = es$.

Proposition 9.1 *Given a partial right action of a semigroup S on a presheaf X, the action of S on X defined above is a right action in the usual semigroup sense.*

PROOF. We show that $(x \circ s) \circ t = x \circ (st)$. Let $x \in X_e$. Then

$$x \circ s = \phi^e_{e\mathbf{r}(s)}(x) \cdot (es),$$

an element of $X_{\mathbf{d}(es)}$. Now $x \circ (st) = \phi^e_{e\mathbf{r}(st)}(x) \cdot (est)$, and

$$(x \circ s) \circ t = \phi^{\mathbf{d}(es)}_{\mathbf{d}(es)\mathbf{r}(t)}(\phi^e_{\mathbf{r}(es)}(x) \cdot (es)) \cdot \mathbf{d}(es)t. \tag{9.1}$$

Observe that

$$\begin{aligned}
\phi^{\mathbf{d}(es)}_{\mathbf{d}(es)\mathbf{r}(t)}(\phi^e_{\mathbf{r}(es)}(x) \cdot (es)) &= \phi^{\mathbf{d}(es)}_{\mathbf{d}(es\mathbf{r}(t))}(\phi^e_{\mathbf{r}(es)}(x) \cdot (es)) \\
&= \phi^{\mathbf{r}(es)}_{\mathbf{r}(es\mathbf{r}(t))}(\phi^e_{\mathbf{r}(es)}(x)) \cdot (es\mathbf{r}(t)) \quad \text{by (SA3)} \\
&= \phi^e_{\mathbf{r}(es\mathbf{r}(t))}(x) \cdot (es\mathbf{r}(t)) \quad \text{since } \phi \text{ defines a presheaf,}
\end{aligned}$$

now

$$\begin{aligned}
\mathbf{r}(es\mathbf{r}(t)) &= (es\mathbf{r}(t))(es\mathbf{r}(t))^{-1} \\
&= es\mathbf{r}(t)\mathbf{r}(t)s^{-1}e \\
&= estt^{-1}s^{-1} \\
&= e\mathbf{r}(st),
\end{aligned}$$

hence

$$\phi^{\mathbf{d}(es)}_{\mathbf{d}(es)\mathbf{r}(t)}(\phi^e_{\mathbf{r}(es)}(x) \cdot (es)) = \phi^e_{e\mathbf{r}(st)}(x) \cdot (es\mathbf{r}(t)).$$

We substitute this into (9.1) to get

$$\begin{aligned}
(x \circ s) \circ t &= (\phi^e_{e\mathbf{r}(st)}(x) \cdot (es\mathbf{r}(t))) \cdot \mathbf{d}(es)t \\
&= \phi^e_{e\mathbf{r}(st)}(x) \cdot (es\mathbf{r}(t)\mathbf{d}(es)t) \\
&= \phi^e_{\mathbf{r}(est)}(x) \cdot (est) \\
&= x \circ (st),
\end{aligned}$$

as required.

■

Thus actions of inverse semigroups on presheaves of sets are special kinds of actions on sets in the usual sense.

Example Let S be an inverse semigroup. Recall that the $\mathcal{L}$*-class* at an idempotent e of S is the set

$$L_e = \{x \in S \mid \mathbf{d}(x) = e\}.$$

We write $L = \bigsqcup_{e \in E(S)} L_e$. If $e, f \in E(S)$ with $f \leqslant e$ and $x \in L_e$, then $xf \in L_f$. So we can define

$$\phi^e_f : L_e \longrightarrow L_f \quad \text{by} \quad \phi^e_f : x \longmapsto xf.$$

This makes the set of $\mathcal{L}$-classes a presheaf over $E(S)$. We shall define a left action of S on L. Let $s \in S$ and $x \in L_{\mathbf{r}(s)}$. Then $xs \in L_{\mathbf{d}(s)}$, we can therefore define

$$f_s : L_{\mathbf{r}(s)} \longrightarrow L_{\mathbf{d}(s)} \quad \text{by} \quad f_s : x \longrightarrow xs.$$

It is immediate that this action satisfies the conditions (SA1) and (SA2), to see that (SA3) holds let $t \leqslant s$ and $x \in L_{\mathbf{d}(s)}$, then

$$\begin{aligned} f_t \phi^{\mathbf{r}(s)}_{\mathbf{r}(t)}(x) &= f_t(x\mathbf{r}(t)) = x\mathbf{r}(t)t = xt \\ \text{and} \quad \phi^{\mathbf{d}(s)}_{\mathbf{d}(t)} f_s(x) &= \phi^{\mathbf{d}(s)}_{\mathbf{d}(t)}(xs) = xs\mathbf{d}(t) = xt. \end{aligned}$$

Hence S acts on the presheaf L. We now examine the corresponding global action. Let $x \in L_e$, and $s \in S$ be arbitrary. The action of s on x is

$$x \circ s = \phi^e_{e\mathbf{r}(s)}(x) \cdot (es) = (x\mathbf{r}(s))es = xs.$$

Therefore the global action is simply right multiplication.

9.2 The category $C(G)$

In [16], Lawson proved the following result. From each inverse semigroup (with zero) S a left cancellative category $C(S)$ can be constructed together with a right action of $C(S)$ on a set X which satisfies a number of conditions. From the pair $(C(S), X)$ an isomorphic copy of S can be constructed. Thus inverse semigroup theory becomes the study of certain types of category actions. In [19], Loganathan showed that a cohomology of an inverse semigroup S can be calculated as the cohomology of the category $C(S)$. We shall therefore examine the category $C(S)$ from an ordered groupoid perspective.

Let G be an ordered groupoid. Put

$$C(G) = \{(e, g) \in G_o \times G \mid \mathbf{r}(g) \leqslant e\}.$$

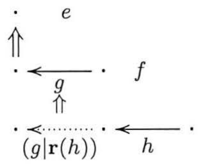
We shall define a partial product making $C(G)$ into a category. Define

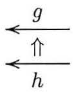
$$\mathbf{d}(e, g) = (\mathbf{d}(g), \mathbf{d}(g)) \quad \text{and} \quad \mathbf{r}(e, g) = (e, e)$$

for all $(e, g) \in C(G)$. If $(e, g), (f, h) \in C(G)$ with $\mathbf{d}(e, g) = \mathbf{r}(f, h)$, that is $\mathbf{d}(g) = f$ then define the product

$$(e, g)(f, h) = (e, (g|\mathbf{r}(h))h)$$

this is illustrated below.



where  indicates that $h \leqslant g$.

Proposition 9.2 *$C(G)$ is a left-cancellative category. If G has a maximal identity, then $C(G)$ has a weak terminal object*

PROOF. It is easy to see that $C(G)$ is a category, and that the identities are as indicated. To show that it is a left cancellative category, suppose that

$$(e, g)(f, h) = (e, g)(f', h').$$

Thus $\mathbf{d}(g) = f = f'$ and $(e, g \otimes h) = (e, g \otimes h')$, so $g \otimes h = g \otimes h'$. That is

$$(g|\mathbf{r}(h))h = (g|\mathbf{r}(h'))h'.$$

It follows that $\mathbf{r}(g|\mathbf{r}(h)) = \mathbf{r}(g|\mathbf{r}(h'))$, therefore

$$(\mathbf{r}(g|\mathbf{r}(h))|g) = (g|\mathbf{r}(h)) = (g|\mathbf{r}(h')),$$

by uniqueness of corestriction. Hence $(g|\mathbf{r}(h))h = (g|\mathbf{r}(h))h'$, and so $h = h'$, as required.

Now suppose that G has a maximal identity 1. Let $e \in G_o$. Then $(1, e)$ is an element of $C(G)$ with

$$\mathbf{d}(1, e) = (e, e) \quad \text{and} \quad \mathbf{r}(1, e) = (1, 1).$$

Thus $(1, 1)$ is a weak terminal object in $C(G)$.

■

9.3 Abelian ordered groupoids

Let A be an abelian ordered groupoid. For all $a \in A$, $\mathbf{d}(a) = \mathbf{r}(a)$, thus each connected component of A is an abelian group.

Abelian ordered groupoids are examples of presheaves which we discussed in Section 9.1.

Proposition 9.3 *Let $(X, \leqslant)$ be a poset. Presheaves on X with values in the category of additive abelian groups are precisely those abelian ordered groupoids whose poset of identities is X.*

PROOF. Let $\phi : X \longrightarrow \mathbf{Ab}$ be a presheaf on X, having values in $\mathbf{Ab}$. For each $x \in X$, let A_x denote the abelian group $\phi(x)$ and put

$$A = \bigsqcup_{x \in X} A_x$$

Let $x, y \in X$ with $x \leqslant y$. Define an order on A as follows: if $a \in A_x$ and $b \in A_y$, then

$$a \leqslant b \quad \Longleftrightarrow \quad a = \phi^y_x(b).$$

We show that this relation is indeed a partial order. Since ϕ is a functor $\phi^x_x = \mathrm{Id}_{A_x}$, so the relation is reflexive. It is clearly anti-symmetric. For $x, y, z \in X$ with $x \leqslant y \leqslant z$, we have $\phi^y_x \phi^z_y = \phi^z_x$, since ϕ is a functor, this condition implies that the order relation is transitive. To show that A is an ordered groupoid, it remains to show that the conditions (OG1)–(OG3) of Section 3 hold. The conditions (OG1) and (OG2) follow from the fact that each ϕ^y_x is a homomorphism. It is easy to see that (OG3) holds with $(a|y) = \phi^y_x(a)$, for any $x \leqslant y$ and $a \in A_y$.

Conversely given an abelian ordered groupoid A, define for each $e \in A_o$ $\phi(e) = \{a \in A \mid \mathbf{d}(a) = e\}$, for each $e \in A_o$. For $e, f \in G_o$ with $e \leqslant f$, define

$$\phi(f, e) : \phi_f \longrightarrow \phi_e \quad \text{by} \quad \phi(f, e)(a) = (a|e).$$

It is immediate that $\phi : G_o \longrightarrow \mathbf{Ab}$ is a functor. Hence ϕ is a presheaf of abelian groups over G_o.

We have therefore shown that every presheaf of abelian groups determines, and is determined by an abelian ordered groupoid. It is straightforward to show that this correspondence is bijective.

■

Let X be a poset and A an abelian group. We let $\Delta(A)$ denote the presheaf of abelian groups given by $\Delta(A)(x) = A$, for all $x \in X$, and $\Delta(A)^x_y(a) = a$ for all $x \leqslant y$ in X. We call $\Delta(A)$ a *constant abelian ordered groupoid.*

9.4 Actions of ordered groupoids

Let G be an ordered groupoid and A an abelian ordered groupoid, such that there is an order-isomorphism $\theta : G_o \cong A_o$. The components of A are abelian groups, one for each identity in G. We write $A_e = \{a \in A \mid \mathbf{d}(a) = \theta(e)\}$. We call A a *(right) G-module* if for each pair $(a, g) \in A \times G$ such that $\mathbf{d}(a) = \theta(\mathbf{r}(g))$, there is an element $a \cdot g$ of A with $\mathbf{d}(a \cdot g) = \theta(\mathbf{d}(g))$. This operation must satisfy the following axioms:

(GM1) If $\exists gh$ in G and $\exists a \cdot g$, then $(a \cdot g) \cdot h = a \cdot (gh)$.

(GM2) If a and b are elements of the same component of A, and $\exists a \cdot g$, then $(a + b) \cdot g = (a \cdot g) + (b \cdot g)$.

(GM3) If e is an identity in G, then $a \cdot e = a$, for all $a \in A_e$.

(GM4) For all g in G, $\theta(\mathbf{r}(g)) \cdot g = \theta(\mathbf{d}(g))$.

(GM5) If $\exists a \cdot g$, $b \cdot h$ where $a \leqslant b$ and $g \leqslant h$, then $a \cdot g \leqslant b \cdot h$

Comparing this definition with the inverse semigroup actions defined in Section 9.1, it is clear that ordered groupoid actions generalise action of inverse semigroups.

Where there is no risk of confusion we write 0_e instead of $\theta(e)$.

The following properties of G-modules will be useful.

Lemma 9.4 *Let A be a G-module.*

(i) *If $\exists a \cdot g$, then $\exists (-a) \cdot g$ and $(-a) \cdot g = -(a \cdot g)$.*

(ii) *If $\exists a \cdot g$ and $h \leqslant g$, then $(a \cdot g | 0_{\mathbf{d}(h)}) = (a | 0_{\mathbf{r}(h)}) \cdot h$.*

PROOF. If $\exists a \cdot g$, then $\mathbf{d}(a) = 0_{\mathbf{r}(g)}$, but A is abelian so $\mathbf{d}(a) = \mathbf{r}(a) = \mathbf{d}(-a)$. Hence $\mathbf{d}(-a) = 0_{\mathbf{r}(g)}$ therefore $\exists (-a) \cdot g$. Now $(a - a) \cdot g = a \cdot g + (-a) \cdot g$ by (GM2), but $a - a = 0_{\mathbf{r}(g)}$, so $(a - a) \cdot g = 0_{\mathbf{r}(g)} \cdot g = 0_{\mathbf{d}(g)}$ by (GM4). Hence $a \cdot g + (-a) \cdot g = 0_{\mathbf{d}(g)}$, that is $(-a) \cdot g = -(a \cdot g)$.

We now show that (ii) holds. Since $\exists a \cdot g$, we know that $\mathbf{d}(a) = \mathbf{r}(a) = 0_{\mathbf{r}(g)}$. We also know that $0_{\mathbf{r}(h)} \leqslant 0_{\mathbf{r}(g)}$, because $h \leqslant g$. Therefore there exist $(a | 0_{\mathbf{r}(h)})$ and $(a | 0_{\mathbf{r}(h)}) \cdot h$. Since $(a | 0_{\mathbf{r}(h)}) \leqslant a$, we have $(a | 0_{\mathbf{r}(h)}) \cdot h \leqslant a \cdot g$, by (GM5). The result follows by uniqueness of restriction.

■

Let G be an ordered groupoid, let A and B be G-modules with order-isomorphisms $\theta : G_o \cong A_o$ and $\phi : G_o \cong B_o$, respectively. An ordered functor $\alpha : A \longrightarrow B$ is called a *G-morphism* if is satisfies the following conditions:

(i) $\alpha(\theta(e)) = \phi(e)$ for all $e \in G_o$.

(ii) $\alpha(a \cdot g) = \alpha(a) \cdot g$, for all $a \in A$ and $g \in G$ with $\mathbf{d}(a) = \theta(\mathbf{r}(g))$.

Note that the condition (i) implies the existence of $\alpha(a) \cdot g$ in (ii).

It is straightforward to show that G-modules together with G-morphisms from a category, which we denote by $\mathbf{Mod}(G)$.

Recall from Section 9.2 that each ordered groupoid G gives rise to a small category $C(G)$. The following correspondence between G-modules and right modules of the small category $C(G)$ is the generalisation to ordered groupoids of Lemma 2.6 of Loganathan [19].

Theorem 9.5 *Let G be an ordered groupoid. There is an isomorphism between the category of left G-modules, and the category of right $C(G)$-modules.*

PROOF. Let A be a left G-module. Define $\mathcal{A} : C(G)^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ as follows: for $e \in G_o$, $\mathcal{A}(e) = A_e$, for each $(e, g) \in C(G)$ define

$$\mathcal{A}(e,g) : \mathcal{A}(e) \longrightarrow \mathcal{A}(\mathbf{d}(g)) \quad \text{by} \quad \mathcal{A}(e,g) : a \longmapsto (a|0_{\mathbf{r}(g)}) \cdot g.$$

We show that $\mathcal{A}(e, g)$ is a homomorphism.

$$\mathcal{A}(e,g)(0_e) = (0_e|0_{\mathbf{r}(g)}) \cdot g = 0_{\mathbf{r}(g)} \cdot g = 0_{\mathbf{d}(g)}$$

and for $a, b \in A_e$

$$\mathcal{A}(e,g)(a+b) = (a+b|0_{\mathbf{r}(g)}) \cdot g = (a|0_{\mathbf{r}(g)}) \cdot g + (b|0_{\mathbf{r}(g)}) \cdot g = \mathcal{A}(e,g)(a) + \mathcal{A}(e,g)(b).$$

We now prove that $\mathcal{A}$ is a contravariant functor. Let $(e, g) \in C(G)$ with $\mathbf{d}(g) = f$, for $a \in A_f$ we have

$$\mathcal{A}(\mathbf{d}(e,g))(a) = \mathcal{A}(f,f)(a) = (a|0_f) \cdot f = a$$

so $\mathcal{A}(\mathbf{d}(e,g)) = \mathrm{Id}_{A_f}$. Similarly $\mathcal{A}(\mathbf{r}(e,g)) = \mathrm{Id}_{A_e}$. Now let $(e,g),(f,h) \in C(G)$ with $\mathbf{d}(g) = f$, for $a \in A_e$ we have

$$\mathcal{A}((e,g)(f,h))(a) = \mathcal{A}(e, g \otimes h)(a) = (a|0_{\mathbf{r}(g \otimes h)}) \cdot (g \otimes h),$$

and

$$\begin{aligned}
\mathcal{A}(f,h)\mathcal{A}(e,g)(a) &= \mathcal{A}(f,h)((a|0_{\mathbf{r}(g)}) \cdot g) \\
&= (((a|0_{\mathbf{r}(g)}) \cdot g)|0_{\mathbf{r}(h)}) \cdot h \\
&= ((a|0_{\mathbf{r}(g|\mathbf{r}(h))}) \cdot (g|\mathbf{r}(h))) \cdot h \quad \text{by Lemma 9.4 (ii)} \\
&= (a|0_{\mathbf{r}(g|\mathbf{r}(h))}) \cdot ((g|\mathbf{r}(h))h) \quad \text{by (GM1).} \\
&= (a|0_{\mathbf{r}(g \otimes h)}) \cdot (g \otimes h).
\end{aligned}$$

Hence $\mathcal{A}$ is a contravariant functor; that is $\mathcal{A}$ is a right $C(G)$-module.

Conversely, let $\mathcal{A} : C(G)^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ be a right $C(G)$ module. We shall construct a left G-module A. For each $e \in G_o$ define

$$\rho(e) = \mathcal{A}(e).$$

If $e, f \in G$ with $e \leqslant f$, define

$$\rho_e^f : \rho(f) \longrightarrow \rho(e) \quad \text{by} \quad \rho_e^f : a \longmapsto \mathcal{A}(f, e)(a).$$

It is immediate that $\rho : G_o \longrightarrow \mathbf{Ab}$ is a functor, where the poset G_o is regarded as a small category. That is, ρ is a presheaf of abelian groups. By Proposition 9.3, ρ defines an abelian ordered groupoid A where $A_e = \rho(e) = \mathcal{A}(e)$ for $e \in G_o$, and

$$(a|0_e) = \mathcal{A}(f, e)(a), \qquad \text{where } a \in A_f \text{ and } e \in G \text{ with } e \leqslant f.$$

To make A into a G-module we need to define an action of G on A. Let $a \in A$ and $g \in G$ with $\mathbf{d}(a) = 0_{\mathbf{r}(g)}$. Define

$$a \cdot g = \mathcal{A}(\mathbf{r}(g), g)(a).$$

The conditions (GM1)–(GM4) are immediate from the fact that $\mathcal{A}$ is a functor. To show that A is a G-module it therefore remains to show that (GM5) holds. Let $a, b \in A$ and $g, h \in G$ such that $\exists g \cdot a, h \cdot b$. Suppose that $b \leqslant a$ and $h \leqslant g$. From the definition of the order on A we have $b = \mathcal{A}(\mathbf{r}(g), \mathbf{r}(h))(a)$. Now

$$\begin{aligned}
b \cdot h &= \mathcal{A}(\mathbf{r}(h), h)(b) \\
&= \mathcal{A}(\mathbf{r}(h), h)(\mathcal{A}(\mathbf{r}(g), \mathbf{r}(h))(a)) \\
&= \mathcal{A}((\mathbf{r}(g), \mathbf{r}(h))(\mathbf{r}(h), h))(a) \quad \text{since } \mathcal{A} \text{ is a contravariant functor} \\
&= \mathcal{A}(\mathbf{r}(g), h)(a) \\
&= \mathcal{A}((\mathbf{r}(g), g)(\mathbf{d}(g), \mathbf{d}(h)))(a) \\
&= \mathcal{A}(\mathbf{d}(g), \mathbf{d}(h))(\mathcal{A}(\mathbf{r}(g), g)(a)) \\
&= \mathcal{A}(\mathbf{d}(g), \mathbf{d}(h))(a \cdot g)
\end{aligned}$$

that is $b \cdot h \leqslant a \cdot g$. Thus A is a G-module.

We have therefore shown that every G-module determines and is determined by a $C(G)$-module. Let $\Gamma : \mathrm{Mod}(G) \longrightarrow \mathrm{Mod}(C(G))$ assign to each G-module A the corresponding $C(G)$-module $\mathcal{A}$ constructed above. Let $\alpha : A \longrightarrow B$ be a G-morphism, and let $\mathcal{A} = \Gamma(A)$ and $\mathcal{B} = \Gamma(B)$. Define

$$\Gamma(\alpha) : \mathcal{A} \longrightarrow \mathcal{B} \quad \text{by} \quad \Gamma(\alpha) : a \longmapsto \alpha(a).$$

We show that $\Gamma(\alpha)$ is a $C(G)$-morphism. Let $a \in \mathcal{A}(e)$ and let $(e, g) \in C(G)$. Then

$$\mathcal{B}(e,g)(\Gamma(\alpha)(a)) = \mathcal{B}(e,g)(\alpha(a)) = (\alpha(a)|0_{\mathbf{r}(g)}) \cdot g,$$

and

$$\begin{aligned}
\Gamma(\alpha)(\mathcal{A}(e,g)(a)) &= \Gamma(\alpha)((a|0_{\mathbf{r}(g)}) \cdot g) \\
&= \alpha((a|0_{\mathbf{r}(g)}) \cdot g) \\
&= \alpha(a|0_{\mathbf{r}(g)}) \cdot g \\
&= (\alpha(a)|0_{\mathbf{r}(g)}) \cdot g.
\end{aligned}$$

Hence the diagram below commutes

$$\begin{tikzcd}
\mathcal{A}(e) \arrow[r, "\Gamma(\alpha)"] \arrow[d, "\mathcal{A}(e,g)"'] & \mathcal{B}(e) \arrow[d, "\mathcal{B}(e,g)"] \\
\mathcal{A}(\mathbf{d}(g)) \arrow[r, "\Gamma(\alpha)"'] & \mathcal{B}(\mathbf{d}(g))
\end{tikzcd}$$

so $\Gamma(\alpha)$ is a $C(G)$-morphism. It is easy to check that $\Gamma : \mathbf{Mod}(G) \longrightarrow \mathbf{Mod}(C(G))$ is a functor.

Conversely, for each $C(G)$-module $\mathcal{A}$, let $\Gamma'(\mathcal{A}) = A$, the G-module constructed above. Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a $C(G)$-morphism, and let $A = \Gamma'(\mathcal{A})$ and $B = \Gamma'(\mathcal{B})$. Define

$$\Gamma'(\phi) : A \longrightarrow B \quad \text{by} \quad \Gamma'(\phi) : a \longmapsto \phi(a).$$

We show that $\Gamma'(\phi)$ is a G-morphism. Suppose that $e, f \in G_o$ with $e \leqslant f$, let $a \in A_e$ and $b \in A_f$ with $a \leqslant b$; that is $a = \mathcal{A}(f, e)(b)$ in $\mathcal{A}$. Then

$$\Gamma'(\phi)(a) = \phi(\mathcal{A}(f,e)(b)) = \mathcal{B}(f,e)(\phi(b)),$$

that is $\Gamma'(\phi)(a) \leqslant \Gamma'(\phi(b))$. It is immediate that $\Gamma'(\phi(0_e)) = 0_e$, for all $e \in G_o$. Now let $g \in G$ and $a \in A_{\mathbf{d}(g)}$. Then

$$(\Gamma'(\phi)(a)) \cdot g = \mathcal{A}(\mathbf{r}(g), g)(\phi(a)) = \phi(\mathcal{B}(\mathbf{r}(g), g)(a)) = \Gamma'(\phi)(a \cdot g).$$

Hence $\Gamma'(\phi)$ is a G-morphism.

It is straightforward to show that $\Gamma' : \mathbf{Mod}(C(G)) \longrightarrow \mathbf{Mod}(G)$ is a functor, and that it is inverse to $\Gamma : \mathbf{Mod}() \longrightarrow \mathbf{Mod}(C(G))$.

■

9.5 The cohomology of an ordered groupoid

By Proposition 9.5, the category of G-modules is isomorphic to the abelian category $\mathbf{Mod}(C(G)) = \mathbf{Ab}^{C(G)^{\mathrm{op}}}$, of right $C(G)$-modules. In Section 8.2 we showed how to compute the cohomology of modules of small categories. We therefore define the cohomology of a G-module to be the cohomology of the corresponding $C(G)$-module.

Explicitly, let A be a G-module and let $\mathcal{A}$-denote the corresponding $C(G)$-module. In Section 8.2.2, we constructed projective modules of categories. Therefore, by Lemma 7.11, we can construct a projective resolution of $\mathcal{A}$. Also there is a left exact functor $\mathrm{Lim} : \mathbf{Ab}^{C(G)^{\mathrm{op}}} \longrightarrow \mathbf{Ab}$ which is is naturally isomorphic to $\hom(\Delta(\mathbb{Z}), -)$. Applying this functor to a projective resolution of $\mathcal{A}$ enables us to compute cohomology groups $H^n(\mathcal{A}, C(G))$. We define the n^{th} cohomology group of the G-module A to be the n^{th} cohomology group of the corresponding $C(G)$-module $\mathcal{A}$; that is

$$H^n(G, A) = H^n(C(G), \mathcal{A}).$$

Chapter 10

Extensions of ordered groupoids

In this section we shall work towards an interpretation of the first and second cohomology groups of an ordered groupoid.

10.1 Definitions

We introduce extensions of ordered groupoids which generalise the notion of extensions of inverse semigroups due to Lausch [11].

We say that an ordered functor $\phi : G \longrightarrow H$ is an *identity-separating* if it induces an order-isomorphism between G_o and H_o.

Definition Let G be an ordered groupoid and A an abelian ordered groupoid. An *extension of A by G* is a triple $\mathcal{E} = (\iota, U, \sigma)$ where

- U is an ordered groupoid.
- $\sigma : U \longrightarrow G$ is a surjective ordered functor, which is identity-separating.
- $\iota : A \longrightarrow U$ is an injective ordered functor, such that the image of ι is isomorphic to the kernel of σ; that is $\iota(A) \cong \mathrm{Ker}(\sigma)$.

We picture the extension $\mathcal{E}$ as shown below

$$A \overset{\iota}{\hookrightarrow} U \overset{\sigma}{\twoheadrightarrow} G.$$

It is clear that $\iota(A)$ is a wide subgroupoid of U. Furthermore $\iota(A)$ is a normal ordered subgroupoid of U because if $a \in A$ and $u \in U$ with $\mathbf{d}(\iota(a)) = \mathbf{d}(u)$, then

$$\sigma(u\iota(a)u^{-1}) = \sigma(u)\sigma\iota(a)\sigma(u)^{-1} = \sigma(u)\sigma(u)^{-1},$$

because $\sigma(a) \in G_o$. Thus $u\iota(a)u^{-1} \in \mathrm{Ker}(\sigma)$.

We can therefore form the quotient groupoid $U/\iota(A)$. Recall that the elements of $U/\iota(A)$ are the equivalence classes of the equivalence relation given by

$$u \sim v \iff u = \iota(a)v\iota(b) \quad \text{for some } a, b \in A.$$

If $\exists uv$ in U, then $\exists[u][v]$ in $U/\iota(A)$, and $[u][v] = [uv]$. The order in $U/\iota(A)$ is defined as $[u] \leqslant [v]$ if, and only if, for each $v' \in [v]$ there exists $u' \in [u]$ such that $u' \leqslant v'$.

Proposition 10.1 *Let $A \overset{\iota}{\hookrightarrow} U \overset{\sigma}{\twoheadrightarrow} G$ be an extension of an abelian ordered groupoid A by an ordered groupoid G.*

(i) *The groupoid $\iota(A)$ is an ordered normal subgroupoid of U.*

(ii) *There is a unique isomorphism $\psi : U/\iota(A) \cong G$ such that $\sigma = \psi\rho^\natural$, where $\rho^\natural : U \longrightarrow U/\iota(A)$ is the natural map defined by $\rho^\natural(u) = [u]$.*

We have already dealt with (i).

Condition (ii) can be obtained from the First Isomorphism Theorem for ordered groupoids (Theorem 7 of [12]), but we give a direct proof. It is straightforward to show that $\rho^\natural$ is a functor. To see that $\rho^\natural$ is ordered, let $u, v \in U$ with $u \leqslant v$ and let $v' \in [v]$; that is $v' = \iota(a)v\iota(b)$, for some $a, b \in A$. Then $(v'|\mathbf{d}(u)) \leqslant v'$ and

$$(v'|\mathbf{d}(u)) = (\iota(a)|\mathbf{r}(u))(v|\mathbf{d}(u))(\iota(b)|\mathbf{d}(u)) = (\iota(a)|\mathbf{r}(u))u(\iota(b)|\mathbf{d}(u))$$

so $(v'|\mathbf{d}(u)) \sim u$, hence $[u] \leqslant [v]$. It is straightforward to show that $\rho^\natural$ is identity-separating and surjective. Define $\psi : U/\iota(A) \longrightarrow G$ by $\psi[u] = \sigma(u)$. This is well-defined because if $u' = \iota(a)u\iota(b)$ for some $a, b \in A$, then $\sigma(u') = \sigma\iota(a)\sigma(u)\sigma\iota(b) = \sigma(u)$. To see that ψ is injective, suppose that $\sigma(u) = \sigma(v)$ for some $u, v \in U$. Then $\sigma(\mathbf{d}(u)) = \sigma(\mathbf{d}(v))$, but σ is identity-separating, so $\mathbf{d}(u) = \mathbf{d}(v)$. Thus uv^{-1} is defined and

$$\sigma(uv^{-1}) = \sigma(u)\sigma(v^{-1}) = \sigma(v)\sigma(v^{-1}) = \sigma(\mathbf{r}(v))$$

hence $uv^{-1} \in A$ and so $[u] = [v]$. Since σ is surjective so also is ψ. The condition $\psi\rho^\natural = \sigma$ is immediate.

■

Let A be an abelian ordered groupoid and G an ordered groupoid. Two extensions $\mathcal{E} = (\iota, U, \sigma)$ and $\mathcal{E}' = (\iota', U', \sigma')$ are said to be *congruent* if there is an ordered functor $\mu : U \longrightarrow U'$ such that $\mu\iota = \iota'$ and $\sigma = \sigma'\mu$ as shown in the commutative diagram below.

$$\begin{tikzcd} & U \arrow[dd, "\mu"] \arrow[dr, two heads, "\sigma"] & \\ A \arrow[ur, hook, "\iota"] \arrow[dr, hook, "\iota'"'] & & G \\ & U' \arrow[ur, two heads, "\sigma'"'] & \end{tikzcd}$$

We write $\mu : \mathcal{E} \cong \mathcal{E}'$.

Lemma 10.2 *If $\mu : \mathcal{E} \cong \mathcal{E}'$ as above, then μ is an isomorphism.*

PROOF. To see that μ is injective, let $u, v \in U$ and assume $\mu(u) = \mu(v)$. Then $\sigma'\mu(u) = \sigma'\mu(v)$, and so $\sigma(u) = \sigma(v)$. Since σ is identity separating $\mathbf{d}(u) = \mathbf{d}(v)$, so $\exists uv^{-1}$. Now

$$\sigma(uv^{-1}) = \sigma(u)\sigma(v)^{-1} = \sigma(u)\sigma(u)^{-1},$$

so $uv^{-1} \in \mathrm{Ker}(\sigma)$, thus $uv^{-1} = \iota(a)$ for some $a \in A$. Now

$$\mu(uv^{-1}) = \mu(u)\mu(v)^{-1} = \mu(u)\mu(u)^{-1} = uu^{-1},$$

so $\mu(\iota(a))$ is an identity. But $\mu(\iota(a)) = \iota'(a)$, so $a \in A_o$ since ι' is an isomorphism. It follows that $\iota(a) \in U_o$, that is $uv^{-1} \in U_o$, hence $u = v$.

To see that μ is surjective, let $u' \in U'$ and write $g = \sigma'(u')$. Since σ is surjective, there is an element u of U with $\sigma(u) = g$. Clearly $\sigma(\mathbf{d}(u)) = \mathbf{d}(g) = \sigma'(\mathbf{d}(u'))$. Since $\mathbf{d}(u)$ is an identity, there is an $e \in A_o$ such that $\iota(e) = \mathbf{d}(u)$, then $\iota'(e) = \mu\iota(e) = \mu(\mathbf{d}(u))$. Now $\sigma'\iota'(e) = \sigma'\mu(\mathbf{d}(u)) = \sigma(\mathbf{d}(u))$, but then $\sigma'(\mathbf{d}(u')) = \sigma'(\iota'(e))$. So $\mathbf{d}(u') = \iota'(e)$ since σ' is identity separating, thus $\mathbf{d}(u') = \mu(\mathbf{d}(u))$. Hence $\exists \mu(u)u'^{-1}$. Now

$$\sigma'(\mu(u)u'^{-1}) = \sigma'\mu(u)\sigma'(u')^{-1} = \sigma(u)\sigma'(u')^{-1} = gg^{-1},$$

so $\mu(u)u'^{-1} \in \iota'(A)$. Write $\iota'(a) = \mu(u)u'^{-1}$, then

$$u' = \iota'(a^{-1})\mu(u) = \mu(u)\mu\iota(a^{-1}) = \mu(u\iota(a^{-1})).$$

Hence μ is surjective as required.

■

Corollary 10.2.1 *Congruence of extensions of ordered groupoids is an equivalence relation.*

■

Let $\mathcal{E} = (\iota, U, \sigma)$ be an extension of an abelian ordered groupoid A by an ordered groupoid G. Since σ is surjective, for each g in G we can choose an element $l(g)$ of U with $\sigma(l(g)) = g$. Thus for each extension $\mathcal{E} = (\iota, U, \sigma)$ of A by G, we can pick a function $l : G \longrightarrow U$ satisfying

$$\sigma l = \mathrm{Id}_G.$$

We call such a function a *transversal* for the extension.

The following results on transversals will be useful.

Lemma 10.3 *Let l be a transversal for an extension $\mathcal{E} = (\iota, U, \sigma)$ of A by G. Suppose that $g, h \in G$ with $\mathbf{r}(h) \leqslant \mathbf{d}(g)$. Then $\mathbf{r}(l(h)) \leqslant \mathbf{d}(l(g))$.*

PROOF. Consider

$$\sigma(\mathbf{d}(l(g))) = \mathbf{d}(\sigma l(g)) = \mathbf{d}(g) \quad \text{and} \quad \sigma(\mathbf{r}(l(h))) = \mathbf{r}(\sigma l(h)) = \mathbf{r}(h).$$

Therefore $\sigma(\mathbf{d}(l(g))) \geqslant \sigma(\mathbf{r}(l(h)))$. But σ is an order-isomorphism on identities, so $\mathbf{d}(l(g)) \geqslant \mathbf{r}(l(h))$.

■

Corollary 10.3.1 *If $\exists gh$ in G, then $\exists l(g)l(h)$.*

■

Lemma 10.4 *Let $\mathcal{E} = (\iota, U, \sigma)$ be an extension of an abelian ordered groupoid A by an ordered groupoid G, and let $l : G \longrightarrow U$ be a transversal for $\mathcal{E}$. Let $u \in U$ and put $g = \sigma(u)$. Then there is a unique $a \in A$ such that $u = l(g)\iota(a)$.*

PROOF. Let $g = \sigma(u)$. Consider

$$\sigma(\mathbf{r}(l(g))) = \mathbf{r}(\sigma l(g)) = \mathbf{r}(g) \quad \text{and} \quad \sigma(\mathbf{r}(u)) = \mathbf{r}(\sigma(u)) = \mathbf{r}(g),$$

but σ is identity-separating, so $\mathbf{r}(u) = \mathbf{r}(l(g))$, hence $\exists l(g)^{-1}u$. Now

$$\sigma(l(g)^{-1}u) = (\sigma l(g))^{-1}\sigma(u) = g^{-1}g = \mathbf{d}(g),$$

so $l(g)^{-1}u \in \mathrm{Ker}(\sigma)$. Thus $\exists a \in A$, such that $\iota(a) = l(g)^{-1}u$. Clearly a is unique, since ι is injective.

■

Proposition 10.5 *Let $\mathcal{E} = (\iota, U, \sigma)$ be an extension of an abelian ordered groupoid A by an ordered groupoid G, and let $l : G \longrightarrow U$ be a transversal for $\mathcal{E}$. For each element (g, a) of $G \times A$ satisfying $\sigma\iota(\mathbf{d}(a)) = \mathbf{r}(g)$, define $a \cdot g$ by*

$$\iota(a \cdot g) = l(g)^{-1}\iota(a)l(g).$$

Then this gives A the structure of a G-module which is independent of the choice of transversal l.

PROOF. If $\sigma\iota(\mathbf{d}(a)) = \mathbf{r}(g)$, then $\iota(\mathbf{d}(a)) = \mathbf{r}(l(g))$, by Corollary 10.3.1. So the groupoid product $l(g)^{-1}\iota(a)l(g)$ exists. Define

$$\theta : G_o \longrightarrow A_o \quad \text{by} \quad \theta(e) = \iota^{-1}(\sigma|U_o)^{-1}(e).$$

It is clear that θ is an order-isomorphism, since σ is identity-separating. If $a \in A$ and $g \in G$ with $\mathbf{d}(a) = \theta(\mathbf{r}(g))$, then $l(g)^{-1}\iota(a)l(g)$ is an element of $\iota(A)$, since $\iota(A)$ is an ordered normal subgroupoid of U by Proposition 10.1 (ii). To prove that the definition is independent of the choice of transversal, let $l' : G \longrightarrow U$ be another transversal. By Lemma 10.4, there is a unique element b of A such that $l(g) = l'(g)\iota(b)$, so

$$l(g)^{-1}\iota(a)l(g) = (l'(g)\iota(b))^{-1}\iota(a)l'(g)\iota(b) = \iota(b)^{-1}l'(g)^{-1}\iota(a)l'(g)\iota(b),$$

but $l'(g)^{-1}\iota(a)(l'(g)) \in \iota(A)$ by normality, and A is abelian so

$$l(g)^{-1}\iota(a)(l(g)) = \iota(b)^{-1}\iota(b)l'(g)^{-1}\iota(a)(l'(g)) = l'(g)^{-1}\iota(a)(l'(g)).$$

To see that (GM1) holds, let $g, h \in G$ such that $\exists gh$. Let $a \in A$ with $\iota(\mathbf{d}(a)) = \theta(\mathbf{r}(g))$. By Corollary 10.3.1, $l(g)l(h)$ is defined. Now $\sigma(l(g)l(h)) = gh = \sigma l(gh)$, so by Lemma 10.4, there is a unique element $f(g,h)$ of A with $l(g)l(h) = l(gh)\iota f(g,h)$. Hence

$$\begin{aligned}
\iota((a \cdot g) \cdot h) &= (l(g)l(h))^{-1}\iota(a)l(h)l(g) \\
&= (l(gh)\iota f(g,h))^{-1}\iota(a)l(gh)\iota f(g,h) \\
&= \iota f(g,h)^{-1}l(gh)^{-1}\iota(a)l(gh)\iota f(g,h) \\
&= \iota f(g,h)^{-1}\iota f(g,h)l(gh)^{-1}\iota(a)l(gh) \\
&= l(gh)^{-1}\iota(a)l(gh) \\
&= \iota(a \cdot gh).
\end{aligned}$$

To see that (GM2) holds, let $g \in G$ and $a, b \in A$ such that $(a+b)\cdot g$ exists. Then

$$\begin{aligned}
\iota((a+b)\cdot g) = l(g)^{-1}\iota(a+b)l(g) &= l(g)^{-1}\iota(a)\iota(b)l(g) \\
&= l(g)^{-1}l(g)l(g)^{-1}\iota(a)\iota(b)l(g) \\
&= l(g)^{-1}\iota(a)l(g)l(g)^{-1}\iota(b)l(g) \\
&= \iota(a\cdot g)\iota(b\cdot g) \\
&= \iota(a\cdot g + b\cdot h)
\end{aligned}$$

To see that (GM3) holds, let $a \in A$ and write $e = \sigma(\iota(\mathbf{d}(a)))$. Then

$$\iota(a \cdot a) = \theta(e)^{-1}\iota(a)\theta(e) = \iota(a).$$

To see that(GM4) holds, Let $g \in G$. Then

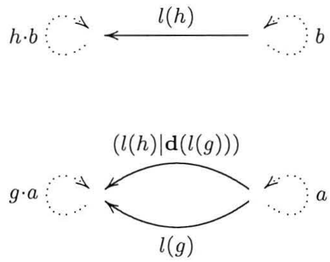
$$\iota(\theta(\mathbf{d}(g)) \cdot g) = l(g)^{-1}(\sigma|U_o)^{-1}(\mathbf{d}(g))l(g) = \mathbf{d}(l(g)).$$

So $\theta(\mathbf{d}(g)) \cdot g = \theta(\mathbf{d}(g))$.

To see that (GM5) holds, let $g, h \in G$ with $g \leqslant h$. Let $a, b \in A$, such that $a \leqslant b$, $\exists a \cdot g$ and $\exists b \cdot h$. We need to show that $a \cdot g \leqslant b \cdot h$. We cannot assume that $l(g)$ is beneath $l(h)$, however, there is an element $(l(h)|\mathbf{d}(l(g))) \leqslant l(h)$. Furthermore

$$\sigma(\mathbf{r}(l(h)|\mathbf{d}(l(g)))) = \mathbf{r}(\sigma l(h)|\sigma(\mathbf{d}(l(g)))) = \mathbf{r}(h|\mathbf{d}(g)) = \mathbf{r}(h),$$

and $\sigma(\mathbf{r}(l(g))) = \mathbf{r}(g)$. Hence $\mathbf{r}(l(h)|\mathbf{d}(l(g))) = \mathbf{r}(l(g))$, since σ is identity-separating. The situation is illustrated below



Define a function

$$l' : G \longrightarrow U \quad \text{by} \quad l' : x \longmapsto \begin{cases} l(x) & \text{if } x \neq g \\ (l(h)|\mathbf{d}(l(g))) & \text{if } x = g \end{cases}$$

which is clearly a transversal for $\mathcal{E}$. Since the action is independent of the choice of transversal, we have

$$\iota(a \cdot g) = l'(g)^{-1}\iota(a)l'(g) = (l(h)|\mathbf{d}(l(g)))^{-1}\iota(a)(l(h)|\mathbf{d}(l(g))) \leqslant l(h)^{-1}bl(h) = \iota(b \cdot h).$$

■

For each $e \in G_o$ we write $\theta(e) = 0_e$ as before.

The following result is essentially Proposition 7.2 of Lausch, generalised to ordered groupoids.

Proposition 10.6 *Let $\mathcal{E} = (\iota, U, \sigma)$ and $\mathcal{E}' = (\iota', U', \sigma')$ be congruent extensions of an abelian ordered groupoid A by an ordered groupoid G. The arising G-module structures defined above are identical.*

PROOF. Since $\mathcal{E}$ and $\mathcal{E}'$ are congruent, there is an order-isomorphism $\mu : U \cong U'$. Let l and l' be transversals for the extensions $\mathcal{E}$ and $\mathcal{E}'$ respectively. Let $g \in G$ and write $u = l(g)$, $u' = l'(g)$. Now $\sigma'(u') = g = \sigma(u) = \sigma'(\mu(u))$. Therefore $\mathbf{d}(\mu(u)) = \mathbf{d}(u')$, since σ' is identity-separating. So $\exists u'\mu(u)^{-1}$. Furthermore, $\sigma'(u'\mu(u)^{-1}) = \sigma'(u')\sigma(u)^{-1} = gg^{-1}$, thus $u'^{-1}\mu(u) \in \iota'(A)$. Let $a \in A_{\mathbf{r}(g)}$, we will show that $\mu(u^{-1}\iota(a)u) = u'\iota'(a)u'$. Now

$$\begin{aligned}
\mu(u^{-1}\iota(a)u) &= \mu(u)^{-1}\iota'(a)\mu(u) \\
&= u'^{-1}u'\mu(u)^{-1}\iota'(a)\mu(u) \\
&= u'^{-1}\iota'(a)u'\mu(u)^{-1}\mu(u) \\
&= u'^{-1}\iota'(a)u',
\end{aligned}$$

as required.

■

Now let $\mathcal{E} = (\iota, U, \sigma)$ be an extension of an abelian ordered groupoid A by an ordered groupoid G. Let $l : G \longrightarrow A$ be a transversal for $\mathcal{E}$. The following result tells us that the ordered groupoid U can be recovered from A, G and l.

Proposition 10.7 *Let $\mathcal{E} = (\iota, U, \sigma)$ be an extension of A by G and let l be a transversal. Define*

$$G * A = \{(g, a) \in G \times A \mid \mathbf{d}(a) = 0_{\mathbf{r}(g)}\}.$$

Then there is a bijection

$$G * A \longrightarrow U \quad \textit{given by} \quad (g, a) \longrightarrow l(g)\iota(a).$$

If $g, h \in G$ with $\mathbf{r}(h) \leqslant \mathbf{d}(g)$, then there is a unique $\zeta(g, h) \in A$ such that

$$\iota\zeta(g, h) = l(g \otimes h)^{-1} \otimes l(g) \otimes l(h).$$

Furthermore

(i) *If $u = l(g)\iota(a)$, $v = l(h)\iota(b)$ and $\exists uv$. Then*

$$uv = l(gh)\iota(\zeta(g, h) + a \cdot h + b).$$

(ii) *Let $u = l(g)\iota(a)$ and $v = l(h)\iota(b)$. Then $v \leqslant u$ if, and only if,*

$$h \leqslant g \quad \textit{and} \quad b = (a|0_{\mathbf{d}(h)}) + \zeta(g, \mathbf{d}(h)) - \zeta(\mathbf{d}(h), \mathbf{d}(h)).$$

PROOF. By Lemma 10.4, every $u \in U$ can be uniquely written as a groupoid product $u = l(g)\iota(a)$, where $(g, a) \in G * A$. Therefore the assignment $(g, a) \longmapsto l(g)\iota(a)$ is a bijection from $G * A$ to U.

Now if $g, h \in G$ and $\mathbf{r}(h) \leqslant \mathbf{d}(g)$, then $\exists g \otimes h$ and, by Lemma 10.3, $\exists l(g) \otimes l(h)$. Now

$$\sigma l(g \otimes h) = g \otimes h = \sigma(l(g) \otimes l(h)).$$

So by Lemma 10.4, there is a unique $\zeta(g, h) \in A$ such that $l(g) \otimes l(h) = l(g \otimes h)\iota(\zeta(g, h))$.

To prove (i), suppose that $u = l(g)\iota(a)$, $v = l(h)\iota(b)$ and $\exists uv$; that is $\exists gh$. Then

$$\begin{aligned}
uv &= l(g)\iota(a)l(h)\iota(b) \\
&= l(g)l(h)l(h)^{-1}\iota(a)l(h)\iota(b) \\
&= l(g)l(h)\iota(a \cdot h)\iota(b) \\
&= l(g)l(h)\iota(a \cdot h + b) \\
&= l(gh)\iota(\zeta(g, h))\iota(a \cdot h + b) \\
&= l(gh)\iota(\zeta(g, h) + a \cdot h + b).
\end{aligned}$$

So (i) holds.

It remains to show that (ii) holds. Let $u = l(g)\iota(a)$, $v = l(h)\iota(b)$ and $v \leqslant u$. Then $\sigma(v) \leqslant \sigma(u)$; that is $h \leqslant g$. Furthermore,

$$l(h)\iota(b) = (l(g)\iota(a)|\mathbf{d}(l(h))) = (l(g)|\mathbf{d}(l(h)))(\iota(a)|\mathbf{d}(l(h))).$$

So $\iota(b)(\iota(a)|\mathbf{d}(l(h)))^{-1} = l(h)^{-1}(l(g)|\mathbf{d}(l(h)))$; that is

$$\begin{aligned}
\iota(b - (a|0_{\mathbf{d}(h)})) &= l(h)^{-1} \otimes l(g) \\
&= l(g \otimes \mathbf{d}(h))^{-1} \otimes l(g) \\
&= ((l(g) \otimes l(\mathbf{d}(h)))\iota(\zeta(g, \mathbf{d}(h)))^{-1})^{-1} \otimes l(g) \\
&= \iota(\zeta(g, \mathbf{d}(h))) \otimes l(\mathbf{d}(h))^{-1} \otimes l(g)^{-1} \otimes l(g) \\
&= \iota(\zeta(g, \mathbf{d}(h))) \otimes l(\mathbf{d}(h))^{-1}.
\end{aligned}$$

But $l(\mathbf{d}(h)) \otimes l(\mathbf{d}(h)) = l(\mathbf{d}(h))\iota(\zeta(\mathbf{d}(h), \mathbf{d}(h)))$, so $\iota(\zeta(\mathbf{d}(h), \mathbf{d}(h))) = l(\mathbf{d}(h))$. Hence

$$\iota(b - (a|0_{\mathbf{d}(h)})) = \iota(\zeta(g, \mathbf{d}(h)) - \zeta(\mathbf{d}(h), \mathbf{d}(h)),$$

so (ii) holds.

■

10.2 Split extensions

In this section we study the simplest class of extensions, but we begin by reviewing semidirect products of ordered groupoids.

Let A be a G-module. We construct the semidirect product of A by G.

Put $G \ltimes A = \{(g, a) \in G \times A \mid \mathbf{d}(a) = 0_{\mathbf{d}(g)}\}$. Let $(a, g), (b, h) \in G \ltimes A$. If $\exists gh$, then define the product

$$(g, a)(h, b) = (gh, a \cdot h + b).$$

Proposition 10.8 *Let G be an ordered groupoid and A a G-module. Then $G \ltimes A$ is an ordered groupoid with the above product and order inherited from $G \times A$.*

PROOF. We begin by locating the identities of $G \ltimes A$. Suppose that $(g, a) \in G \ltimes A$ and $\exists (g, a)(g, a) = (g, a)$; that is $(gg, a \cdot g + b) = (a, g)$. Then g is an identity, thus $a \cdot g + a = a$, that is $a + a = a$. So the identities of $G \ltimes A$ are of the form $(e, 0_e)$, where $e \in G_o$. There is an obvious bijection between $(G \ltimes A)_o$ and G_o. It is easy to check that for $(g, a) \in G \ltimes A$, $(\mathbf{d}(g), 0_{\mathbf{d}(g)})$ and $(\mathbf{r}(g), 0_{\mathbf{r}(g)})$ are respectively left and right identities for (g, a). We therefore define

$$\mathbf{d}(g, a) = (\mathbf{d}(g), 0_{\mathbf{d}(g)}) \quad \text{and} \quad \mathbf{r}(a, g) = (\mathbf{r}(g), 0_{\mathbf{r}(g)}).$$

We show that the product in $G \ltimes A$ is associative. Let $(g_1, a_1), (g_2, a_2), (g_3, a_3) \in G \ltimes A$, and suppose that $\exists g_1 g_2 g_3$. Then

$$((g_1, a_1)(g_2, a_2))(g_3, a_3) = (g_1 g_2, a_1 \cdot g_2 + a_2)(a_3, g_3) = (g_1 g_2 g_3, (a_1 \cdot g_2 + a_2) \cdot g_3 + a_3)$$

and

$$\begin{aligned} (g_1, a_1)((g_2, a_2)(g_3, a_3)) &= (g_1, a_1)(g_2 g_3, a_2 \cdot g_3 + a_3) \\ &= (g_1 g_2 g_3, a_1 \cdot (g_2 g_3) + a_2 \cdot g_3 + a_3) \\ &= (g_1 g_2 g_3, (a_1 \cdot g_2 + a_2) \cdot g_3 + a_3). \end{aligned}$$

Hence $G \ltimes A$ is a category.

To see that $G \ltimes A$ is a groupoid, let $(g, a) \in G \ltimes A$. Then $\exists (-a) \cdot g^{-1}$, so $(g^{-1}, -(a \cdot g^{-1})) \in G \ltimes A$. Furthermore

$$\begin{aligned} (g, a)(g^{-1}, -(a \cdot g^{-1})) &= (gg^{-1}, a \cdot g^{-1} - (a \cdot g^{-1})) = (\mathbf{r}(g), 0_{\mathbf{r}(g)}) \\ \text{and} \quad (g^{-1}, -(a \cdot g^{-1}))(g, a) &= (g^{-1} g, -(a \cdot g^{-1}) \cdot g + a) = (\mathbf{d}(g), 0_{\mathbf{d}(g)}). \end{aligned}$$

Hence $G \ltimes A$ is a groupoid with $(g, a)^{-1} = (g^{-1}, -(a \cdot g^{-1}))$.

We show that $G \ltimes A$ is an ordered groupoid. Let $(g, a), (h, b) \in G \ltimes A$ with

$$(g, a) \leqslant (h, b) \quad \text{that is} \quad a \leqslant b \text{ and } g \leqslant h.$$

Then $a^{-1} \leqslant b^{-1}$ and $g^{-1} \leqslant h^{-1}$. So by (GM5), $(-a) \cdot g^{-1} \leqslant (-b) \cdot h^{-1}$. Hence $(g, a)^{-1} \leqslant (h, b)^{-1}$, and (OG1) holds. To see that (OG2) holds, let $(g, a) \leqslant (h, b)$ and $(g', a') \leqslant (h', b')$ in $G \ltimes A$, and suppose $\exists gg'$, $\exists hh'$. By Lemma 9.4, $a \cdot g' \leqslant b \cdot h'$, so $a \cdot g' + a' \leqslant b \cdot h' + b'$, thus $(g, a)(g', a') \leqslant (h, b)(h', b')$, and (OG2) holds. Now let $(g, a) \in G \ltimes A$ and $e \in G_o$ with $e \leqslant \mathbf{d}(g)$; that is $(e, 0_e) \leqslant \mathbf{d}(g, a)$. It is easy to see that (OG3) holds with

$$((g, a)|(e, 0_e)) = ((g|e), (a|0_e)).$$

■

The semidirect product, provides an example of an extension. Let A be a G-module, and let $\pi : G \ltimes A \longrightarrow G$ be the canonical projection onto G. Clearly π has kernel $G_o \rtimes A$. Define $i : A \longrightarrow G \ltimes A$ by $i(a) = (e, a)$, where e is the unique element of G_o with $\mathbf{d}(a) = 0_e$. It is immediate that

$$\begin{tikzcd} A \arrow[r, hook, "i"] & G \ltimes A \arrow[r, "\pi"] & G \end{tikzcd}$$

is an extension of A by G.

An extension $\mathcal{E} = (\iota, U, \sigma)$ of an abelian ordered groupoid A by an ordered groupoid G *splits* if σ has a right inverse; that is, if there is an ordered functor $\tau : G \longrightarrow U$ with $\sigma\tau = \mathrm{Id}_G$. Such a functor τ is called a *splitting* for the extension $\mathcal{E}$.

The semidirect product above splits with splitting

$$\tau : G \longrightarrow G \ltimes A \quad \text{given by} \quad \tau(g) = (g, 0_{\mathbf{r}(g)}).$$

The following result tells us that semidirect products are precisely split extensions.

Proposition 10.9 *Any split extension* $\mathcal{E} : \begin{tikzcd} A \arrow[r, hook, "\iota"] & U \arrow[r, shift left, "\sigma"] & G \arrow[l, shift left, "\tau"] \end{tikzcd}$ *is congruent to the semidirect product extension arising from the G-module A.*

Proof. Recall from Proposition 10.5 that the G-module structure of A is given by $0_e = \iota^{-1}(\sigma|U_o)^{-1}(e)$ and $\iota(a \cdot g) = \tau(g)^{-1}\iota(a)\tau(g)$. We need to construct an ordered functor $\mu : U \longrightarrow G \ltimes A$ making the diagram below commute.

$$\begin{tikzcd} & U \arrow[dd, dashed, "\mu"] \arrow[dr, "\sigma"] & \\ A \arrow[ur, hook, "\iota"] \arrow[dr, hook, "i"'] & & G \\ & G \ltimes A \arrow[ur, "\pi"'] & \end{tikzcd}$$

Let u be an element of U. Then $\tau\sigma(u^{-1})u \in \iota(A)$, by Lemma 10.4. Define $\mu : U \longrightarrow G \ltimes A$ by

$$\mu : u \longmapsto (\sigma(u), \iota^{-1}(\tau\sigma(u^{-1})u)).$$

Let $a \in A$ and write $e = \sigma\iota(a)$. Then

$$\mu\iota(a) = (\sigma\iota(a), \iota^{-1}(\tau\sigma\iota(a^{-1})\iota(a))) = (e, a),$$

so $\mu\iota = i$. It is clear that $\pi\mu = \sigma$. Hence the diagram above commutes. It remains to show that μ is an ordered functor. Let u and v be elements of U with $\mathbf{d}(u) = \mathbf{r}(v)$. Then

$$\begin{aligned}
\mu(u)\mu(v) &= (\sigma(u), \iota^{-1}(\tau\sigma(u^{-1})u))(\sigma(v), \iota^{-1}(\tau\sigma(v^{-1})v)) \\
&= (\sigma(u)\sigma(v), \iota^{-1}(\tau\sigma(u^{-1})u) \cdot \sigma(v) + \iota^{-1}(\tau\sigma(v^{-1})v)) \\
&= (\sigma(uv), \iota^{-1}(\tau\sigma(v)^{-1}\tau\sigma(u^{-1})u\tau\sigma(v)) + \iota^{-1}(\tau\sigma(v^{-1})v)) \\
&= (\sigma(uv), \iota^{-1}(\tau\sigma(v)^{-1}\tau\sigma(u^{-1})u\tau\sigma(v)\tau\sigma(v^{-1})v)) \\
&= (\sigma(uv), \iota^{-1}(\tau\sigma(u^{-1}v^{-1})uv)) \\
&= \mu(uv).
\end{aligned}$$

Hence μ is a functor. Since τ and σ are both order preserving, so also is μ.

■

The above result tells us that there is only one split extension of G by A (up to equivalence) associated to the given action of G on A. Nevertheless, there is a classification problem involving split extensions: given that an extension splits, classify all possible splittings.

Definition Let G be an ordered groupoid and A a G-module. A *derivation* of A is an order-preserving function $\phi : G \longrightarrow A$ such that

(D1) $\phi(g) \in A_{\mathbf{d}(g)}$, for all $g \in G$.

(D2) $\phi(gh) = \phi(g) \cdot h + \phi(h)$, for all $g, h \in G$ such that $\exists gh$.

Note that if $e \in G_o$, then by (D2)

$$\phi(e) = \phi(ee) = \phi(e) \cdot e + \phi(e).$$

So $\phi(e) = 0_e$.

We shall now show that derivations may be used to classify splittings.

Lemma 10.10 *Every splitting of a split extension determines, and is determined by, a derivation.*

PROOF. By Proposition 10.9 we may assume that the split extension in question is the semidirect product extension $(i, G \ltimes A, \pi)$. Let $\tau : G \longrightarrow G \ltimes A$ be a function such that $\pi\tau = \mathrm{Id}_G$. Then $\tau(g) = (g, \overline{\tau}(g))$, where $\overline{\tau}(g)$ is an element of A such that $\mathbf{d}(\overline{\tau}(g)) = 0_{\mathbf{d}(g)}$, thus $\overline{\tau}$ satisfies (D1). To see that (D2) holds, let $g, h \in G$ such that $\exists gh$. Then

$$\tau(g)\tau(h) = (g, \overline{\tau}(g))(h, \overline{\tau}(h)) = (gh, \overline{\tau}(g) \cdot h + \overline{\tau}(h)),$$

so τ will be an ordered functor if, and only if, $\overline{\tau}$ is order-preserving and satisfies (D2).

Conversely, if $\phi : G \longrightarrow A$ is a derivation, then $(\phi(g), g) \in A \rtimes G$ by (D1). Define

$$\phi^* : G \longrightarrow A \rtimes G \quad \text{by} \quad \phi^*(g) = (\phi(g), g),$$

it is easy to check that ϕ^* is a splitting.

■

Let $Z^1(G, A)$ denote the set of all derivations of a G-module A.

Lemma 10.11 *$Z^1(G, A)$ is an abelian group under pointwise addition.*

PROOF. Let $\phi, \psi \in Z^1(G, A)$. Then $\psi + \phi$ is defined by

$$\psi + \phi : g \longmapsto \psi(g) + \phi(g), \quad \phi, \psi \in Z^1(G, A).$$

To see that this is well-defined, suppose $\exists gh$ in G. Then

$$\begin{aligned}
(\psi + \phi)(gh) &= \psi(gh) + \phi(gh) \\
&= \psi(g) \cdot h + \psi(h) + \phi(g) \cdot h + \phi(h) \\
&= (\psi(g) + \phi(g)) \cdot h + (\psi(h) + \phi(h)) \\
&= ((\psi + \phi)(g)) \cdot h + (\psi + \phi)(h).
\end{aligned}$$

Thus $\psi + \phi$ is a derivation. The operation $+$ is clearly associative and abelian. The identity in $Z^1(G, A)$ is the function $\overline{0} : G \longrightarrow A$ defined by $\overline{0}(g) = 0_{\mathbf{d}(g)}$. As for inverses, let $\phi \in Z^1(G, A)$ and $g \in G$. Define

$$-\phi : G \longrightarrow A \quad \text{by} \quad (-\phi)(g) = -\phi(g).$$

To see that $-\phi$ is a derivation suppose that $\exists gh$ in G. Then

$$(-\phi)(gh) = -(\phi(g) \cdot h + \phi(h)) = -(\phi(g) \cdot h) - \phi(h) = (-\phi(g)) \cdot h - \phi(h).$$

It is immediate that $\phi - \phi = \overline{0} = -\phi + \phi$. Hence $Z^1(G, A)$ is an abelian group.

■

The following result provides an important class of derivations.

Lemma 10.12 *Let A be a G-module and $\delta : G_o \longrightarrow A$ an order-preserving function such that $\delta(e) \in A_e$. The function*

$$\partial\delta : G \longrightarrow A \quad \textit{defined by} \quad \partial\delta : g \longmapsto \delta(\mathbf{r}(g)) \cdot g - \delta(\mathbf{d}(g))$$

is a derivation.

PROOF. It is immediate that (D1) holds. To see that (D2) holds, suppose that $\exists gh$ in G. Then

$$\begin{aligned}
\partial\delta(g) \cdot h + \partial\delta(h) &= (\delta(\mathbf{r}(g)) \cdot g - \delta(\mathbf{d}(g))) \cdot h + (\delta(\mathbf{r}(h)) \cdot h - \delta(\mathbf{d}(h))) \\
&= \delta(\mathbf{r}(g)) \cdot (gh) - \delta(\mathbf{d}(g)) \cdot h + \delta(\mathbf{r}(h)) \cdot h - \delta(\mathbf{d}(h)) \\
&= \delta(\mathbf{r}(g)) \cdot (gh) - \delta(\mathbf{d}(h)) \\
&= \partial\delta(gh).
\end{aligned}$$

■

We call derivations of the form $\partial\delta$ *principal derivations*. Denote the set of all principal derivations of a G-module A by $B^1(G, A)$.

Lemma 10.13 *$B^1(G, A)$ is a subgroup of $Z^1(G, A)$.*

PROOF. To see that the sum of principal derivations is principal, let $\delta, \varepsilon : G_o \longrightarrow A$ be order-preserving functions such that $\delta(e), \varepsilon(e) \in A_e$. The function $\delta + \varepsilon : G_o \longrightarrow A$ defined by $(\delta + \varepsilon)(e) = \delta(e) + \varepsilon(e)$ is order-preserving and $\delta + \varepsilon \in A_e$. It is routine to check that $\partial(\delta + \varepsilon) = \partial\delta + \partial\varepsilon$. To see that the identity derivation $\overline{0}$ is principal, let $g \in G$. Consider

$$\partial 0(g) = 0_{\mathbf{r}(g)} \cdot g - 0_{\mathbf{d}(g)} = 0_{\mathbf{d}(g)} - 0_{\mathbf{d}(g)} = 0_{\mathbf{d}(g)},$$

so $\partial 0 = \overline{0}$. For any principal derivation $\partial\delta$, define $-\delta : G_o \longrightarrow A$ by $(-\delta)(e) = -\delta(e)$, it is immediate that $\partial(-\delta) = -\partial\delta$.

■

Derivations provide an easy way of characterising splittings. In order to classify splittings we introduce the notion of 'A-conjugation'. Let $\mathcal{E}$ be a split extension of A by G and let τ and λ be two splittings of $\mathcal{E}$ with corresponding derivations $\overline{\tau}$ and $\overline{\lambda}$ respectively. We say that τ and λ are *A-conjugate* if there is an order-preserving function $\delta : G_o \longrightarrow A$ such that $\delta(e) \in A_e$ and

$$\overline{\tau}(g) = \delta(\mathbf{r}(g)) \cdot g + \overline{\lambda}(g) - \delta(\mathbf{d}(g))$$

for all $g \in G$. The following result is immediate.

Lemma 10.14 *Two derivations correspond to A-conjugate splittings if, and only if, their difference is a principal derivation.*

■

Theorem 10.15 *For any ordered groupoid G and G-module A, the A-conjugacy classes of splittings of the split extension*

$$A \overset{i}{\hookrightarrow} A \rtimes G \xrightarrow{\pi} G$$

are in one-to-one correspondence with the elements of the quotient group of derivations modulo principal derivations.

Proof. By Lemma 10.10 there is a bijection between the set of splittings of a split extension and the group $Z^1(G, A)$ of derivations of A. By Lemma 10.13 the set $B^1(G, A)$ of principle derivations is a subgroup of $Z^1(G, A)$. By Lemma 10.14 two splittings are A-conjugate precisely when the corresponding derivations are in the same coset of $B^1(G, A)$ in $Z^1(G, A)$.

■

10.3 Factor sets

In this section we shall classify extensions of ordered groupoids by defining factor sets for G-modules.

Let G be an ordered groupoid, and $n \geqslant 1$. An *n-staircase* over G is an n-tuple $(g_n, \ldots, g_2, g_1)$, where $g_i \in G$ and $\mathbf{r}(g_i) \leqslant \mathbf{d}(g_{i+1})$, for $i = 1, \ldots, n-1$. We denote the set of n-staircases over G by $S_n(G)$.

Definition Let G be an ordered groupoid. A *factor set* for a G-module A is a function

$$\zeta : S_2(G) \longrightarrow A$$

satisfying the following conditions:

(FS1) $\zeta(g, h) \in A_{\mathbf{d}(h)}$, for all $(g, h) \in S_2(G)$.

(FS2) For all $(g, h, k) \in S_3(G)$, $(\zeta(g, h)|0_{\mathbf{r}(k)}) \cdot k + \zeta(g \otimes h, k) = \zeta(g, h \otimes k) + \zeta(h, k)$.

The following result will be useful.

Lemma 10.16 *Let ζ be a factor set for a G-module A. Then*

(i) $\zeta(\mathbf{d}(g), \mathbf{d}(g)) = \zeta(g, \mathbf{d}(g))$, *for all* $g \in G$.

(ii) $\zeta(\mathbf{r}(g), \mathbf{r}(g)) \cdot g = \zeta(\mathbf{r}(g), g)$, *for all* $g \in G$.

(iii) $\zeta(g, g^{-1}) \cdot g + \zeta(\mathbf{r}(g), g) = \zeta(g, \mathbf{d}(g)) + \zeta(g^{-1}, g)$, *for all* $g \in G$.

(iv) *If* $e, f \in G_o$ *with* $f \leqslant e$, *then* $(\zeta(e, e)|0_f) = \zeta(e, f)$.

(v) *If* $g, h \in G$ *with* $h \leqslant g$, *then* $(\zeta(g, \mathbf{d}(g))|0_{\mathbf{d}(h)}) = \zeta(\mathbf{d}(g), \mathbf{d}(h))$.

(vi) *If* $g, h \in G$ *with* $h \leqslant g$, *then* $\zeta(g, \mathbf{d}(h)) \cdot h^{-1} + \zeta(h, h^{-1}) = \zeta(g, h^{-1}) + \zeta(\mathbf{d}(h), h^{-1})$.

(vii) *If* $g, h \in G$ *with* $h \leqslant g$, *then* $(\zeta(g, g^{-1})|0_{\mathbf{r}(h)}) + \zeta(\mathbf{r}(g), \mathbf{r}(h)) = \zeta(g, h^{-1}) + \zeta(g^{-1}, \mathbf{r}(h))$.

(viii) *If* $g, h, k \in G$ *with* $k \leqslant h \leqslant g$, *then*

$$(\zeta(g, \mathbf{d}(h))|0_{\mathbf{d}(k)}) + \zeta(h, \mathbf{d}(k)) = \zeta(g, \mathbf{d}(k)) + \zeta(\mathbf{d}(h), \mathbf{d}(k)).$$

(ix) *If* $g, g', h, h' \in G$ *are such that* $\exists gg'$, $\exists hh'$, $h \leqslant g$ *and* $h' \leqslant g'$. *Then*

$$(\zeta(g, g')|0_{\mathbf{d}(h')}) + \zeta(gg', \mathbf{d}(h')) = \zeta(g, h') + \zeta(g', \mathbf{d}(h')).$$

(x) *If* $g, g', h, h' \in G$ *are such that* $\exists gg'$, $\exists hh'$, $h \leqslant g$ *and* $h' \leqslant g'$. *Then*

$$\zeta(g, \mathbf{r}(h')) \cdot h' + \zeta(h, h') = \zeta(g, h') + \zeta(\mathbf{r}(h'), h').$$

PROOF. To show that (i) holds, apply the condition (FS2) to the 3-staircase $(g, \mathbf{d}(g), \mathbf{d}(g))$. This gives

$$\zeta(g, \mathbf{d}(g)) + \zeta(g, \mathbf{d}(g)) = \zeta(g, \mathbf{d}(g)) + \zeta(\mathbf{d}(g), \mathbf{d}(g)),$$

as required. To prove (ii), apply (FS2) to the 3-staircase $(\mathbf{r}(g), \mathbf{r}(g), g)$. To prove (iii), apply (FS2) to the 3-staircase (g, g^{-1}, g). To prove (iv), apply (FS2) to the 3-staircase (e, e, f). To prove (v), apply (FS2) to the 3-staircase $(g, \mathbf{d}(g), \mathbf{d}(h))$. To prove (vi), apply (FS2) to the 3-staircase $(g, \mathbf{d}(h), h^{-1})$. To prove (vii), apply (FS2) to the 3-staircase $(g, g^{-1}, \mathbf{r}(h))$. To prove (viii), apply (FS2) to the 3-staircase $(g, \mathbf{d}(h), \mathbf{d}(k))$. To prove (ix), apply (FS2) to the 3-staircase $(g, g', \mathbf{d}(h'))$. To prove (x), apply (FS2) to the 3-staircase $(g, \mathbf{r}(h'), h')$.

■

Suppose that $\mathcal{E}$ is an extension of an abelian ordered groupoid A by an ordered groupoid G. Let l be a transversal for $\mathcal{E}$ and let $(g, h) \in S_2(G)$. By Lemma 10.3, the products $l(g) \otimes l(h)$ and $l((g \otimes h)^{-1}) \otimes l(g)$ exist. By Corollary 10.3.1, $\exists l(g \otimes h)^{-1} \otimes l(g)$.

Proposition 10.17 *Let $\mathcal{E} = (\iota, U, \sigma)$ be an extension of an abelian ordered groupoid A by an ordered groupoid G, and let $l : G \longrightarrow U$ be a transversal for $\mathcal{E}$. Define*

$$\zeta : S_2(G) \longrightarrow A \quad \textit{by} \quad \iota(\zeta(g,h)) = l(g \otimes h)^{-1} \otimes l(g) \otimes l(h).$$

Then ζ is a factor set.

PROOF. The function ζ is well-defined since, by Proposition 10.7, each $\zeta(g,h) \in A$ is unique. (FS1) is immediate. To see that (FS2) holds, let $(g,h,k) \in S_3(G)$. We compute $l(g) \otimes l(h) \otimes l(k)$ in two ways. Consider

$$\begin{aligned}
(l(g) \otimes l(h)) \otimes l(k) &= l(g \otimes h) \otimes \iota\zeta(g,h) \otimes l(k) \\
&= l(g \otimes h) \otimes \mathbf{r}(l(k)) \otimes \iota\zeta(g,h) \otimes l(k) \\
&= l(g \otimes h) \otimes l(k) \otimes l(k)^{-1} \otimes \iota\zeta(g,h) \otimes l(k) \\
&= l(g \otimes h \otimes k) \otimes \iota\zeta(g \otimes h, k) \otimes l(k)^{-1} \otimes \iota\zeta(g,h) \otimes l(k) \\
&= l(g \otimes h \otimes k)\iota\zeta(g \otimes h, k) l(k)^{-1}\iota(\zeta(g,h)|0_{\mathbf{r}(k)}) l(k).
\end{aligned}$$

On the other hand

$$\begin{aligned}
l(g) \otimes (l(h) \otimes l(k)) &= l(g) \otimes l(h \otimes k) \otimes \iota\zeta(h,k) \\
&= l(g \otimes h \otimes k) \otimes \iota\zeta(g, h \otimes k) \otimes \iota\zeta(h,k) \\
&= l(g \otimes h \otimes k)\iota\zeta(g, h \otimes k)\iota\zeta(h,k).
\end{aligned}$$

Since both factorisations of $l(g) \otimes l(h) \otimes l(k)$ are equal, we have

$$\iota\zeta(g \otimes h, k) l(k)^{-1}\iota(\zeta(g,h)|0_{\mathbf{r}(k)}) l(k) = \iota\zeta(g, h \otimes k)\iota\zeta(h,k).$$

That is $\iota(\zeta(g \otimes h, k) + (\zeta(g,h)|0_{\mathbf{r}(k)}) \cdot k) = \iota\zeta(g, h \otimes k) + \zeta(h,k))$. The result follows, since ι is injective.

■

Note that if the transversal l is an ordered functor, then $l(g \otimes h) = l(g) \otimes l(h)$. Thus we can think of the factor set constructed above as a measure of how l fails to be an ordered functor. This idea is made precise in the following Lemma.

Lemma 10.18 *Let $\mathcal{E}$ be an extension of an abelian ordered groupoid A by an ordered groupoid, let l be a transversal for $\mathcal{E}$ and let ζ be the factor set defined by $l(g) \otimes l(h) = l(g \otimes h)\iota(\zeta(g,h))$. Then l is an ordered functor if, and only if, $\zeta(g,h) = 0_{\mathbf{d}(g)}$, for all $(g,h) \in S_2(G)$.*

PROOF. Suppose that l is an ordered functor and let $(g,h) \in S_2(G)$. Then

$$\begin{aligned} l(g \otimes h) &= l((g|0_{\mathbf{r}(h)})h) \\ &= l(g|\mathbf{r}(h))l(h) \quad \text{since } l \text{ is a functor} \\ &= (l(g)|l(\mathbf{r}(h)))l(h) \quad \text{since } l \text{ is a order-preserving} \\ &= l(g) \otimes l(h). \end{aligned}$$

Hence $\zeta(g,h) = 0_{\mathbf{d}(h)}$.

Conversely, suppose that $\zeta(g,h) = 0_{\mathbf{d}(h)}$ for all $(g,h) \in S_2(G)$. Let $g,h \in G$ where $\exists gh$. Then $l(g)l(h) = l(gh)\zeta(g,h) = l(gh)$, so l is a functor. To see that l is order-preserving let $g,h \in G$ with $h \leqslant g$. Then

$$l(g|\mathbf{d}(h)) = l(g \otimes \mathbf{d}(h)) = l(g) \otimes l(\mathbf{d}(h))\iota\zeta(g,\mathbf{d}(h)) = l(g) \otimes l(\mathbf{d}(h)) = (l(g)|l(\mathbf{d}(h))$$

as required.

■

Proposition 10.17 tells us that an extension together with a transversal determines a factor set. We now consider the relationship between the factor sets given by different transversals.

Proposition 10.19 *Let $\mathcal{E} = (\iota, U, \sigma)$ be an extension of an abelian ordered groupoid A by an ordered groupoid G and let $l, l' : G \longrightarrow U$ be transversals for $\mathcal{E}$. Consider the factor sets $\zeta, \zeta' : S_2(G) \longrightarrow A$ defined by*

$$\iota\zeta(g,h) = l(g \otimes h)^{-1} \otimes l(g) \otimes l(h) \quad \text{and} \quad \iota\zeta'(g,h) = l'(g \otimes h)^{-1} \otimes l'(g) \otimes l'(h).$$

There is a function $\varepsilon : G \longrightarrow A$ defined by $\iota\varepsilon(g) = l(g)^{-1}l'(g)$, and

$$\zeta'(g,h) = \zeta(g,h) - \varepsilon(g \otimes h) + \varepsilon(h) + (\varepsilon(g)|0_{\mathbf{r}(h)}).$$

PROOF. Let $g \in G$. Since $\sigma(l(g)) = \sigma(l'(g))$, by Lemma 10.4, there is a unique element $\varepsilon(g)$ of A such that

$$l'(g) = l(g)\iota(\varepsilon(g)).$$

Let $(g,h) \in S_2(G)$. Then

$$\begin{aligned} l'(g) \otimes l'(h) &= l(g) \otimes \iota(\varepsilon(g)) \otimes l(h)\iota(\varepsilon(h)) \\ &= l(g) \otimes \iota(\varepsilon(g)) \otimes l(h) \otimes l(h)^{-1} \otimes l(h) \otimes \iota(\varepsilon(h)) \\ &= l(g) \otimes l(h) \otimes l(h)^{-1} \otimes \iota(\varepsilon(g)) \otimes l(h) \otimes \iota(\varepsilon(h)) \\ &= l(g) \otimes l(h) \otimes \iota(\varepsilon(h)) \otimes l(h)^{-1} \otimes \iota(\varepsilon(g)) \otimes l(h) \end{aligned}$$

$$
\begin{aligned}
&= l(g \otimes h) \otimes \iota\zeta(g,h) \otimes \iota(\varepsilon(h)) \otimes l(h)^{-1} \otimes \iota(\varepsilon(g)) \otimes l(h) \\
&= l(g \otimes h)\iota\zeta(g,h)\iota(\varepsilon(h))l(h)^{-1}(\iota(\varepsilon(g))|0_{\mathbf{r}(h)})l(h) \\
&= l(g \otimes h)\iota\zeta(g,h)\iota(\varepsilon(h))\iota((\varepsilon(g)|0_{\mathbf{r}(h)}) \cdot h) \\
&= l'(g \otimes h)\iota(\varepsilon(g \otimes h))^{-1})\iota\zeta(g,h)\iota(\varepsilon(h))\iota((\varepsilon(g)|0_{\mathbf{r}(h)}) \cdot h) \\
&= l'(g \otimes h)\iota(\zeta(g,h) - \varepsilon(g \otimes h) + \varepsilon(h) + (\varepsilon(g)|0_{\mathbf{r}(h)}) \cdot h)
\end{aligned}
$$

So by Lemma 10.4, $\zeta'(g,h) = \zeta(g,h) - \varepsilon(g \otimes h) + \varepsilon(h) + (\varepsilon(g)|0_{\mathbf{r}(h)}) \cdot h$.

■

The definitions in the following result are motivated by Proposition 10.7.

Proposition 10.20 *Let G be an ordered groupoid and A a G-module. Suppose that ζ is a factor set for A. Let*

$$G * A = \{(g,a) \in G \times A \mid \mathbf{d}(a) = 0_{\mathbf{d}(g)}\}.$$

If $(g,a),(b,h) \in G * A$ and $\exists gh$, then define

$$(g,a)(b,h) = (gh, \zeta(g,h) + a \cdot h + b).$$

Define an order on $G * A$ as follows:

$$(b,h) \leqslant (g,a) \quad \Longleftrightarrow \quad h \leqslant g \text{ and } b = (a|0_{\mathbf{d}(h)}) + \zeta(g,\mathbf{d}(h)) - \zeta(\mathbf{d}(h),\mathbf{d}(h)).$$

Then $G * A$ is an ordered groupoid. Furthermore, if we define

$$i : A \longrightarrow G * A \quad \text{by} \quad i : a \longmapsto (e, a - \zeta(e,e)), \quad a \in A_e,$$

and let $\pi : G * A \longrightarrow G$ be the canonical projection. Then

$$A \overset{i}{\hookrightarrow} G * A \xrightarrow{\pi} G$$

is an extension of A by G which induces the given action of G on A.

Proof. We begin by showing that $G * A$ is a groupoid.

It is clear that the partial product is well-defined. To locate the identities of $G * A$, suppose that $(g,a) \in G * A$ and $\exists (g,a)(g,a) = (g,a)$, that is

$$(gg, \zeta(g,g) + a \cdot g + a) = (g,a).$$

Clearly g is an identity, hence $a \cdot g = a$. Therefore $a = a + a + \zeta(g,g)$, that is, $a = -\zeta(g,g)$. Hence the the identities of $G * A$ are of the form $(e, -\zeta(e,e))$, where $e \in G_o$. There is an obvious bijection between $(G * A)_o$ and A_o. Now

$$\begin{aligned}(g,a)(\mathbf{d}(g), -\zeta(\mathbf{d}(g),\mathbf{d}(g))) &= (g, \zeta(g,\mathbf{d}(g)) + a\cdot\mathbf{d}(g) - \zeta(\mathbf{d}(g),\mathbf{d}(g))) \\ &= (g, \zeta(g,\mathbf{d}(g)) + a - \zeta(g,\mathbf{d}(g))) \quad \text{by Lemma 10.16(i)} \\ &= (g,a).\end{aligned}$$

Therefore we define

$$\mathbf{d}(g,a) = (\mathbf{d}(g), -\zeta(\mathbf{d}(g),\mathbf{d}(g))).$$

Similarly,

$$\mathbf{r}(g,a) = (\mathbf{r}(g), -\zeta(\mathbf{r}(g),\mathbf{r}(g))).$$

To see that the product is associative, let $(g,a),(h,b),(k,c) \in G*A$ where $\exists ghk$. Then

$$\begin{aligned}((g,a)(h,b))(k,c) &= (gh, \zeta(g,h) + a\cdot h + b)(k,c) \\ &= (ghk, \zeta(gh,k) + (\zeta(g,h) + a\cdot h + b)\cdot k + c) \\ &= (ghk, \zeta(gh,k) + \zeta(g,h)\cdot k + a\cdot(hk) + b\cdot k + c)\end{aligned}$$

and

$$\begin{aligned}(g,a)((h,b)(k,c)) &= (g,a)(hk, \zeta(h,k) + b\cdot k + c) \\ &= (ghk, \zeta(g,hk) + a\cdot(hk) + \zeta(h,k) + b\cdot k + c).\end{aligned}$$

So associativity follows from the condition (FS2): $\zeta(g,h)\cdot k + \zeta(gh,k) = \zeta(h,k) + \zeta(g,hk)$.

To find the left inverse of $(g,a) \in G * A$ solve the equation

$$(g^{-1}, b)(g,a) = (\mathbf{d}(g), -\zeta(\mathbf{d}(g),\mathbf{d}(g))).$$

That is

$$-\zeta(\mathbf{d}(g),\mathbf{d}(g)) = \zeta(g^{-1},g) + b\cdot g + a.$$

So $b = -\zeta(\mathbf{d}(g)\mathbf{d}(g))\cdot g^{-1} - \zeta(g^{-1},g)\cdot g^{-1} - a\cdot g^{-1}$. To find the right inverse of (g,a) solve $(g,a)(g^{-1},b') = (\mathbf{r}(g), -\zeta(\mathbf{r}(g),\mathbf{r}(g)))$. That is

$$-\zeta(\mathbf{r}(g),\mathbf{r}(g)) = \zeta(g,g^{-1}) + a\cdot g^{-1} + b'.$$

So $b' = -\zeta(\mathbf{r}(g),\mathbf{r}(g)) - \zeta(g,g^{-1}) - a\cdot g$. To show that (g,a) has a two sided inverse, we need to prove that $b = b'$. Now

$$\begin{aligned}b' &= -\zeta(\mathbf{d}(g),\mathbf{d}(g))\cdot g^{-1} - \zeta(g^{-1},g)\cdot g^{-1} - a\cdot g^{-1} \\ &= -(\zeta(g,\mathbf{d}(g)) + \zeta(g^{-1},g) + a)\cdot g^{-1} \quad \text{by Lemma 10.16(i)} \\ &= -(\zeta(g,g^{-1})\cdot g + \zeta(\mathbf{r}(g),g) + a)\cdot g^{-1} \quad \text{by Lemma 10.16(iii)} \\ &= -(\zeta(g,g^{-1})\cdot g + \zeta(\mathbf{r}(g),\mathbf{r}(g))\cdot g + a)\cdot g^{-1} \quad \text{by Lemma 10.16(ii)} \\ &= \zeta(g,g^{-1}) - \zeta(\mathbf{r}(g),\mathbf{r}(g)) - a\cdot g^{-1} \\ &= b.\end{aligned}$$

Hence each $(g, a) \in G * A$ has a two sided inverse

$$(g, a)^{-1} = (g^{-1}, -\zeta(g, g^{-1}) - \zeta(\mathbf{r}(g), \mathbf{r}(g)) - a \cdot g^{-1}).$$

We have therefore proved that $G * A$ is a groupoid.

We now show that $G * A$ is an ordered groupoid. Firstly we need to show that the order we have defined is in indeed a partial order.

Let $(g, a) \in G * A$. To show that the order is reflexive we need

$$a = (a|0_{\mathbf{d}(g)}) + \zeta(g, \mathbf{d}(g)) - \zeta(\mathbf{d}(g), \mathbf{d}(g)).$$

This is immediate from Lemma 10.16(i).

To show that the order is anti-symmetric, let $(g, a), (h, b) \in G * A$ and suppose that $(h, b) \leqslant (g, a)$ and $(g, a) \leqslant (h, b)$. Then $h \leqslant g$ and $g \leqslant h$, so $h = g$. Now $(g, a) \leqslant (g, b)$ implies that

$$\begin{aligned} a &= (b|0_{\mathbf{d}(g)}) + \zeta(g, \mathbf{d}(g)) - \zeta(\mathbf{d}(g), \mathbf{d}(g)) \\ &= (b|0_{\mathbf{d}(g)}) + \zeta(g, \mathbf{d}(g)) - \zeta(g, \mathbf{d}(g)) \\ &= (b|0_{\mathbf{d}(g)}), \end{aligned}$$

so $a \leqslant b$. Similarly $(g, b) \leqslant (g, a)$ implies that $b \leqslant a$. Hence $a = b$ as required.

We now prove that the order is transitive. Let $(k, c), (h, b), (g, a) \in G * A$, and suppose that $(k, c) \leqslant (h, b)$ and $(h, b) \leqslant (g, a)$. It is immediate that $k \leqslant h \leqslant g$. From $(k, c) \leqslant (h, b)$, we have that

$$c = (b|0_{\mathbf{d}(h)}) + \zeta(h, \mathbf{d}(k)) - \zeta(\mathbf{d}(k), \mathbf{d}(k)),$$

and since $(h, b) \leqslant (g, a)$, we have

$$b = (a|0_{\mathbf{d}(h)}) + \zeta(g, \mathbf{d}(h)) - \zeta(\mathbf{d}(h), \mathbf{d}(h)).$$

It is required to show that

$$c = (a|0_{\mathbf{d}(k)}) + \zeta(g, \mathbf{d}(k)) - \zeta(\mathbf{d}(k), \mathbf{d}(k)).$$

Now

$$\begin{aligned} c &= (b|0_{\mathbf{d}(k)}) + \zeta(h, \mathbf{d}(k)) - \zeta(\mathbf{d}(k), \mathbf{d}(k)) \\ &= (((a|0_{\mathbf{d}(h)}) + \zeta(g, \mathbf{d}(h)) - \zeta(\mathbf{d}(h), \mathbf{d}(h)))|0_{\mathbf{d}(k)}) + \zeta(h, \mathbf{d}(k)) - \zeta(\mathbf{d}(k), \mathbf{d}(k)) \\ &= (a|0_{\mathbf{d}(k)}) + (\zeta(g, \mathbf{d}(h))|0_{\mathbf{d}(k)}) - (\zeta(\mathbf{d}(h), \mathbf{d}(h))|0_{\mathbf{d}(k)}) + \zeta(h, \mathbf{d}(k)) - \zeta(\mathbf{d}(k), \mathbf{d}(k)) \\ &= (a|0_{\mathbf{d}(k)}) + \zeta(g, \mathbf{d}(k)) + \zeta(\mathbf{d}(h), \mathbf{d}(k)) \\ &\qquad - (\zeta(\mathbf{d}(h), \mathbf{d}(h))|0_{\mathbf{d}(k)}) - \zeta(\mathbf{d}(k), \mathbf{d}(k)) \quad \text{by Lemma 10.16(vii)} \\ &= (a|0_{\mathbf{d}(k)}) + \zeta(g, \mathbf{d}(k)) - \zeta(\mathbf{d}(k), \mathbf{d}(k)) \quad \text{by Lemma 10.16(iv)} \end{aligned}$$

as required.

To prove that $G*A$ is an ordered groupoid, we need show that it satisfies the conditions (OG1)–(OG3) given in Section 3. To see that (OG1) holds, let $(g,a),(h,b) \in G*A$ with $(h,b) \leqslant (g,a)$. To show that $(h,b)^{-1} \leqslant (g,a)^{-1}$, we need

$$\begin{aligned} -\zeta(h,h^{-1}) - \zeta(\mathbf{r}(h),\mathbf{r}(h)) - b \cdot h^{-1} &= ((\zeta(g,g^{-1}) - \zeta(\mathbf{r}(g),\mathbf{r}(g)) - a \cdot g^{-1})|0_{\mathbf{r}(h)}) \\ &\quad + \zeta(g^{-1},\mathbf{r}(h)) - \zeta(\mathbf{r}(h),\mathbf{r}(h)). \end{aligned} \tag{10.1}$$

Since $(h,b) \leqslant (g,a)$, we have $b = (a|0_{\mathbf{d}(h)}) + \zeta(g,\mathbf{d}(h)) - \zeta(\mathbf{d}(h),\mathbf{d}(h))$. So the left-hand side of (10.1) is equal to

$$\begin{aligned} &-\zeta(h,h^{-1}) - \zeta(\mathbf{r}(h),\mathbf{r}(h)) - ((a|0_{\mathbf{d}(h)}) + \zeta(g,\mathbf{d}(h)) - \zeta(\mathbf{d}(h),\mathbf{d}(h))) \cdot h^{-1} \\ &= -\zeta(\mathbf{r}(h),\mathbf{r}(h)) - (a \cdot g^{-1}|0_{\mathbf{r}(h)}) - \zeta(h,h^{-1}) - \zeta(g,\mathbf{d}(h)) \cdot h^{-1} + \zeta(\mathbf{d}(h),\mathbf{d}(h)) \cdot h^{-1} \\ &= -\zeta(\mathbf{r}(h),\mathbf{r}(h)) - (a \cdot g^{-1}|0_{\mathbf{r}(h)}) - \zeta(g,h^{-1}) \\ &\qquad - \zeta(\mathbf{d}(h),h^{-1}) + \zeta(\mathbf{d}(h),\mathbf{d}(h)) \cdot h^{-1} \quad \text{by Lemma 10.16(vi)} \\ &= -\zeta(\mathbf{r}(h),\mathbf{r}(h)) - (a \cdot g^{-1}|0_{\mathbf{r}(h)}) - \zeta(g,h^{-1}) \\ &\qquad - \zeta(\mathbf{d}(h),h^{-1}) + \zeta(\mathbf{d}(h),h^{-1}) \quad \text{by Lemma 10.16(ii)} \\ &= -\zeta(\mathbf{r}(h),\mathbf{r}(h)) - (a \cdot g^{-1}|0_{\mathbf{r}(h)}) - \zeta(g,h^{-1}). \end{aligned}$$

The right-hand side of (10.1) is equal to

$$\begin{aligned} &-(a \cdot g^{-1}|0_{\mathbf{r}(h)}) - \zeta(\mathbf{r}(h),\mathbf{r}(h)) - (\zeta(g,g^{-1})|0_{\mathbf{r}(h)}) - (\zeta(\mathbf{r}(g),\mathbf{r}(g))|0_{\mathbf{r}(h)}) + \zeta(g^{-1},\mathbf{r}(h)) \\ &= -(a \cdot g^{-1}|0_{\mathbf{r}(h)}) - \zeta(\mathbf{r}(h),\mathbf{r}(h)) + \zeta(\mathbf{r}(g),\mathbf{r}(h)) - \zeta(g,h^{-1}) \\ &\qquad -\zeta(g^{-1},\mathbf{r}(h)) - (\zeta(\mathbf{r}(g),\mathbf{r}(g))|0_{\mathbf{r}(h)}) + \zeta(g^{-1},\mathbf{r}(h)) \quad \text{by Lemma 10.16(vii)} \\ &= -(a \cdot g^{-1}|0_{\mathbf{r}(h)}) - \zeta(\mathbf{r}(h),\mathbf{r}(h)) - \zeta(g,h^{-1}) + \zeta(\mathbf{r}(g),\mathbf{r}(h)) - (\zeta(\mathbf{r}(g),\mathbf{r}(g))|0_{\mathbf{r}(h)}) \\ &= -(a \cdot g^{-1}|0_{\mathbf{r}(h)}) - \zeta(\mathbf{r}(h),\mathbf{r}(h)) - \zeta(g,h^{-1}) \quad \text{by Lemma 10.16(ix).} \end{aligned}$$

So (10.1) holds. Hence (OG1) is satisfied.

We now show that $G*A$ satisfies the condition (OG2). Let $(g,a),(g',a'),(h,b),(h',b') \in G*A$, where $\exists gg', hh'$ and $(h,b) \leqslant (g,a)$, $(h',b') \leqslant (g',a')$. We need to prove that $(h,b)(h',b') \leqslant (g,a)(g',a')$. Now

$$(g,a)(g',a') = (gg', \zeta(g,g') + a \cdot g' + a')$$

and

$$(h,b)(h',b') = (hh', \zeta(h,h') + b \cdot h' + b').$$

It is immediate that $hh' \leqslant gg'$. We need to show that

$$\zeta(h,h') + b \cdot h' + b' = ((\zeta(g,g') + a \cdot g' + a')|0_{\mathbf{d}(h')}) + \zeta(gg',\mathbf{d}(h')) - \zeta(\mathbf{d}(h'),\mathbf{d}(h')). \tag{10.2}$$

Since $(h,b) \leqslant (g,a)$ and $(h',b') \leqslant (g',a')$ we have

$b = (a|0_{\mathbf{d}(h)}) + \zeta(g,\mathbf{d}(h)) - \zeta(\mathbf{d}(h),\mathbf{d}(h))$ and $b' = (a'|0_{\mathbf{d}(h')}) + \zeta(g',\mathbf{d}(h')) - \zeta(\mathbf{d}(h'),\mathbf{d}(h'))$

respectively. Substituting these into the left-hand side of (10.2) gives

$$\begin{aligned}
&\zeta(h,h') + ((a|0_{\mathbf{d}(h)}) + \zeta(g,\mathbf{d}(h)) - \zeta(\mathbf{d}(h),\mathbf{d}(h))) \cdot h' \\
&\qquad + (a'|0_{\mathbf{d}(h')}) + \zeta(g'\mathbf{d}(h')) - \zeta(\mathbf{d}(h'),\mathbf{d}(h')) \\
&= \quad (a \cdot g'|0_{\mathbf{d}(h')}) + (a'|0_{\mathbf{d}(h')}) - \zeta(\mathbf{d}(h'),\mathbf{d}(h')) \\
&\qquad + \zeta(h,h') + \zeta(g,\mathbf{d}(h)) \cdot h' - \zeta(\mathbf{d}(h),\mathbf{d}(h)) \cdot h' + \zeta(g'\mathbf{d}(h')) \\
&= \quad (a \cdot g'|0_{\mathbf{d}(h')}) + (a'|0_{\mathbf{d}(h')}) - \zeta(\mathbf{d}(h'),\mathbf{d}(h')) + \zeta(h,h') + \zeta(g,\mathbf{d}(h)) \cdot h' \\
&\qquad - \zeta(\mathbf{d}(h),\mathbf{d}(h)) \cdot h' + (\zeta(g,g')|0_{\mathbf{d}(h')}) + \zeta(gg',\mathbf{d}(h')) - \zeta(g,h') \quad \text{by Lemma 10.16(ix)} \\
&= \quad ((a \cdot g' + a' - \zeta(g,g'))|0_{\mathbf{d}(h')}) + \zeta(gg',\mathbf{d}(h')) - \zeta(\mathbf{d}(h'),\mathbf{d}(h')) \\
&\qquad + \zeta(h,h') + \zeta(g,\mathbf{d}(h)) \cdot h' - \zeta(\mathbf{d}(h),\mathbf{d}(h)) \cdot h' - \zeta(g,h') \\
&= \quad ((a \cdot g' + a' - \zeta(g,g'))|0_{\mathbf{d}(h')}) + \zeta(gg',\mathbf{d}(h')) - \zeta(\mathbf{d}(h'),\mathbf{d}(h')) \\
&\qquad + \zeta(h,h') + \zeta(g,\mathbf{d}(h)) \cdot h' - \zeta(\mathbf{r}(h'),h') - \zeta(g,h') \quad \text{by Lemma 10.16(ii)} \\
&= \quad ((a \cdot g' + a' - \zeta(g,g'))|0_{\mathbf{d}(h')}) + \zeta(gg',\mathbf{d}(h')) - \zeta(\mathbf{d}(h'),\mathbf{d}(h')) \quad \text{by Lemma 10.16(x)}
\end{aligned}$$

which is the right-hand side of (10.2). Hence (OG2) holds.

Now let $(g,a) \in G * A$ and $e \leqslant \mathbf{d}(g)$. To show that (OG3) holds, we need to define the restriction of (g,a) to $(e,-\zeta(e,e))$. Define

$$((g,a)|(e,-\zeta(e,e))) = ((g|e),(a|0_e) + \zeta(g,e) - \zeta(e,e)).$$

It is clear that $\mathbf{d}((g,a)|(e,-\zeta(e,e))) = (e,-\zeta(e,e))$ and $((g,a)|(e,-\zeta(e,e))) \leqslant (g,a)$. Hence (OG3) holds.

We now show that

$$A \overset{i}{\hookrightarrow} G * A \xrightarrow{\pi} G$$

is an extension of A by G. Where i is defined by $i(a) = (e, a - \zeta(e,e))$ for $a \in A_e$, and π is the canonical projection. It is immediate that i is an injective functor and that $i\pi = 0$. It is clear that π is surjective and an order isomorphism on identities.

It remains to show that the extension above gives rise to the given action of G on A. Let $l : G \longrightarrow G * A$ be a transversal for the extension, let $g \in G$ and let $a \in A_{\mathbf{r}(g)}$. We need to show that $i(a \cdot g) = l(g)^{-1} i(a) l(g)$. Since $\pi l = \mathrm{Id}_G$, $l(g)$ is of the form $l(g) = (g, \overline{a})$, where $\overline{a} \in A_{\mathbf{d}(g)}$. Now

$$\begin{aligned}
&l(g)^{-1} i(a) l(g) \\
&\quad = \quad (g,\overline{(}a))^{-1}(\mathbf{r}(g), a - \zeta(\mathbf{r}(g),\mathbf{r}(g)))(g,\overline{(}a))
\end{aligned}$$

$$
\begin{aligned}
&= (g^{-1}, -\zeta(g, g^{-1}) - \zeta(\mathbf{r}(g), \mathbf{r}(g)) - \overline{a} \cdot g^{-1})(\mathbf{r}(g), a - \zeta(\mathbf{r}(g), \mathbf{r}(g)))(g, \overline{(a)}) \\
&= (g^{-1}, \zeta(g^{-1}, \mathbf{r}(g)) - \zeta(g, g^{-1}) - \zeta(\mathbf{r}(g), \mathbf{r}(g)) - \overline{a} \cdot g^{-1} + a - \zeta(\mathbf{r}(g), \mathbf{r}(g)))(g, \overline{a}) \\
&= (g^{-1}, \zeta(g^{-1}\mathbf{r}(g)) - \zeta(g, g^{-1}) - \zeta(g^{-1}, \mathbf{r}(g)) \\
&\qquad - \overline{a} \cdot g^{-1} + a - \zeta(\mathbf{r}(g), \mathbf{r}(g)))(g, \overline{a}) \quad \text{by Lemma 10.16(i)} \\
&= (g^{-1}, -\zeta(g, g^{-1}) - \overline{a} \cdot g^{-1} + a - \zeta(\mathbf{r}(g), \mathbf{r}(g)))(g, \overline{a}) \\
&= (\mathbf{d}(g), \zeta(g^{-1}, g) - \zeta(g, g^{-1}) \cdot g - \zeta(\mathbf{r}(g), \mathbf{r}(g)) \cdot g - \overline{a} + a \cdot g + \overline{a}) \\
&= (\mathbf{d}(g), \zeta(g^{-1}, g) - \zeta(g, g^{-1}) \cdot g - \zeta(\mathbf{r}(g), \mathbf{r}(g)) \cdot g + a \cdot g) \\
&= (\mathbf{d}(g), -\zeta(g, \mathbf{d}(g)) + a \cdot g) \quad \text{by Lemma 10.16(iii)} \\
&= (\mathbf{d}(g), a \cdot g - \zeta(\mathbf{d}(g), \mathbf{d}(g))) \quad \text{by Lemma 10.16(i)} \\
&= i(a \cdot g)
\end{aligned}
$$

as required.

■

Theorem 10.21 *Let $\mathcal{E} = (\iota, U, \sigma)$ be an extension of an abelian ordered groupoid A by an ordered groupoid G. Let l be a transversal for $\mathcal{E}$ and let ζ be the factor set constructed in Proposition 10.17. The extension of A by G constructed from ζ as in Proposition 10.20 is congruent to $\mathcal{E}$.*

PROOF. The factor set $\zeta : S_2(G) \longrightarrow A$ is defined by $l(g) \otimes l(h) = l(g \otimes h)\iota(\zeta(g,h))$. We shall construct an ordered functor $\mu : U \longrightarrow G * A$. By Lemma 10.4, each $u \in U$ can be written uniquely as $u = l(g)\iota(a)$ where $a \in A$ and $g = \sigma(u)$, so the function

$$\mu : U \longrightarrow G * A \quad \text{with} \quad \mu : l(g)\iota(a) \longmapsto (g, a)$$

is bijective. To see that μ is a functor let $u = l(g)\iota(a)$ and $v = l(h)\iota(b)$ be elements of U. The product uv exists if, and only if, $\mathbf{d}(g) = \mathbf{r}(h)$, in which case

$$
\begin{aligned}
uv &= l(g)\iota(a)l(h)\iota(b) \\
&= l(g)\iota(a)l(h)l(h)^{-1}l(h)\iota(b) \\
&= l(g)l(h)l(h)^{-1}\iota(a)l(h)\iota(b) \\
&= l(gh)\iota(\zeta(g,h)))\iota(a \cdot h)\iota(b),
\end{aligned}
$$

so

$$\mu(uv) = \mu(l(gh)\iota(\zeta(g,h) + a \cdot h + b)) = (gh, \zeta(g,h) + a \cdot h + b)$$

Now $\mu(u)\mu(v) = (g,a)(h,b) = (gh, \zeta(g,h) + a \cdot h + b)$. Hence $\mu(uv) = \mu(u)\mu(v)$ as required.

To see that μ is order-preserving, let $u = l(g)\iota(a)$ and $v = l(h)\iota(b)$ be elements of U and suppose that $u \leqslant v$. We need to show that $(g, a) \leqslant (h, b)$. Now $\sigma : U \longrightarrow G$ is order-preserving so $\sigma(u) \leqslant \sigma(v)$; that is $\sigma(l(g)\iota(a)) \leqslant \sigma(l(h)\iota(b))$, so $g \leqslant h$. By Lemma 10.3, $\mathbf{d}(l(g)) \leqslant \mathbf{d}(l(h))$. To prove that $(g, a) \leqslant (h, b)$, we need

$$a = (b|0_{\mathbf{d}(g)}) + \zeta(h, \mathbf{d}(g)) - \zeta(\mathbf{d}(g), \mathbf{d}(g)).$$

Now

$$\begin{aligned}
\iota(a) &= l(g)^{-1}u \\
&= l(g)^{-1}(l(h)\iota(b)|\mathbf{d}(l(g))) \\
&= l(g)^{-1}(l(h)|\mathbf{d}(l(g)))(\iota(b)|\mathbf{d}(l(g))) \\
&= l(g)^{-1}(l(h) \otimes (\iota(b)|\mathbf{d}(l(g)))) \\
&= l(g)^{-1} \otimes l(h) \otimes (\iota(b)|\mathbf{d}(l(g))),
\end{aligned}$$

and

$$\begin{aligned}
&\iota(\zeta(h, \mathbf{d}(g)) - \zeta(\mathbf{d}(g), \mathbf{d}(g)) + (b|0_{\mathbf{d}(g)})) \\
&= \iota\zeta(h, \mathbf{d}(g))(\iota\zeta(\mathbf{d}(g), \mathbf{d}(g)))^{-1}\iota(b|0_{\mathbf{d}(g)}) \\
&= (l(h \otimes \mathbf{d}(g))^{-1} \otimes l(h) \otimes l(\mathbf{d}(g)))(l(\mathbf{d}(g))^{-1}l(\mathbf{d}(g))l(\mathbf{d}(g)))^{-1}(\iota(b)|\mathbf{d}(l(g))) \\
&= l(g)^{-1} \otimes l(h) \otimes l(\mathbf{d}(g)) \otimes l(\mathbf{d}(g))^{-1} \otimes (\iota(b)|\mathbf{d}(l(g))) \\
&= l(g)^{-1} \otimes l(h) \otimes (\iota(b)|\mathbf{d}(l(g))) \\
&= \iota(a).
\end{aligned}$$

Hence $\mu(u) \leqslant \mu(v)$ as required.

It remains to show that the diagram below commutes.

$$\begin{tikzcd}
 & U \arrow[dd, "\mu"] \arrow[dr, "\sigma"] & \\
A \arrow[ur, hook, "\iota"] \arrow[dr, "i"'] & & G \\
 & A * G \arrow[ur, "\pi"'] &
\end{tikzcd}$$

Let $b \in A$, and write $\mathbf{d}(b) = 0_e$. Then there is a unique $a \in A$ such that $\iota(b) = l(e)\iota(a)$. So

$$\mu\iota(b) = (e, \iota^{-1}(l(e)^{-1}\iota(b))) = (e, -\iota^{-1}l(e) + b).$$

But $\iota\zeta(e, e) = l(e)^{-1}l(e)l(e) = l(e)$, so $\mu\iota(b) = (e, -\zeta(e, e) + b)$. Which is equal to $i(b)$. Therefore $\mu\iota = i$. To see that $\sigma = \pi\mu$, let $u = l(g)\iota(a)$ be an element of U. Then

$$\pi\mu(u) = \pi(g, a) = g \quad \text{and} \quad \sigma(u) = \sigma(l(g)\iota(a)) = \sigma(l(g))\sigma(\iota(a)) = g.$$

Hence result. ∎

Let G be an ordered groupoid and A a G-module. Denote the set of factor sets by $Z^2(G, A)$.

Define addition on $Z^2(G, A)$ pointwise. That is for $\zeta, \eta \in Z^2(G, A)$

$$\zeta + \eta : S_2(G) \longrightarrow A \quad \text{is defined by} \quad (\zeta + \eta)(g, h) = \zeta(g, h) + \eta(g, h)$$

Lemma 10.22 *With the above addition $Z^2(G, A)$ is an abelian group.*

PROOF. We begin by showing that $\zeta + \eta$ is a factor set. It immediate that (FS1) holds. To see that (FS2) holds, let $g, h, k \in S_2(G)$. Then

$$\begin{aligned}
&((\zeta + \eta)(g, h)|0_{\mathbf{r}(k)}) \cdot k + (\zeta + \eta)(g \otimes h, k) \\
&\quad = \quad (((\zeta(g, h) + \eta(g, h))|0_{\mathbf{r}(k)}) \cdot k + \zeta(g \otimes h, k) + \eta(g \otimes h, k) \\
&\quad = \quad (\zeta(g, h)|0_{\mathbf{r}(k)}) \cdot k + \zeta(g \otimes h, k) + (\eta(g, h)|0_{\mathbf{r}(k)}) \cdot k + \eta(g \otimes h, k) \\
&\quad = \quad \zeta(g, h \otimes k) + \zeta(g, h) + \eta(g, h \otimes k) + \eta(g, h) \\
&\quad = \quad (\zeta + \eta)(g, h \otimes k) + (\zeta + \eta)(g, h).
\end{aligned}$$

As required. Hence the addition is well-defined. It is clear that the addition is associative and commutative.

Define a function

$$0 : S_2(G) \longrightarrow A \quad \text{by} \quad 0 : (g, h) \longmapsto 0_{\mathbf{d}(h)}.$$

It is easy to show that 0 is a factor set satisfying

$$0 + \zeta = \zeta = \zeta + 0,$$

for all $\zeta \in Z^2(G, A)$.

For each factor set ζ define

$$-\zeta : S_2(G) \longrightarrow A \quad \text{by} \quad (-\zeta)(g, h) = -(\zeta_0(g, h)).$$

It is straightforward to show that $-\zeta$ is a factor set satisfying

$$-\zeta + \zeta = 0 = \zeta - \zeta.$$

Hence $Z^2(G, A)$ is an abelian group.

∎

By Proposition 10.17, every extention $\mathcal{E}$ of of an abelian ordered groupoid A by an ordered groupoid G, together with a transversal l determines a factor set for the G-module A. Let l and l' be two transversals for $\mathcal{E}$, and let ζ and ζ' denote the resulting factor sets. By Proposition 10.19, there is a function $\varepsilon : G \longrightarrow A$ defined by $\iota\varepsilon(g) = l(g)^{-1}l'(g)$, and

$$\zeta'(g,h) = \zeta(g,h) - \varepsilon(g \otimes h) + \varepsilon(h) + (\varepsilon(g)|0_{\mathbf{r}(h)}).$$

Generally, for any G-module A and function $\varepsilon : G \longrightarrow A$ such that $\varepsilon(g) = 0_{\mathbf{d}(g)}$, for all $g \in G$. Define

$$\partial\varepsilon : S_2(G) \longrightarrow A \quad \text{by} \quad \partial\varepsilon(g,h) = (\varepsilon(g)|0_{\mathbf{r}(h)}) \cdot h + \varepsilon(h) - \varepsilon(g \otimes h).$$

We call the function $\partial\varepsilon$ a *principal factor set* for A. Denote the set of principal factor sets by by $B^2(G,A)$. The following is immediate.

Proposition 10.23 *Let $\mathcal{E} = (\iota, U, \sigma)$ be an extension of an abelian ordered groupoid A by an ordered groupoid G and let $l, l' : G \longrightarrow U$ be transversals for $\mathcal{E}$. Consider the factor sets $\zeta, \zeta' : S_2(G) \longrightarrow A$ defined by*

$$\iota\zeta(g,h) = l(g \otimes h)^{-1} \otimes l(g) \otimes l(h) \quad \text{and} \quad \iota\zeta'(g,h) = l'(g \otimes h)^{-1} \otimes l'(g) \otimes l'(h).$$

Then there is a principal factor set $\partial\varepsilon$ such that

$$\zeta'(g,h) = \zeta(g,h) + \partial\varepsilon(g,h).$$

■

Lemma 10.24 *$B^2(G,A)$ is a subgroup of $Z^2(G,A)$.*

PROOF. We begin by showing that principal factor sets are factor set. Let $\varepsilon : G \longrightarrow A$ be a function from such that $\varepsilon(g) \in A_{\mathbf{d}(g)}$, for all $g \in G$. Let $\partial\varepsilon$ be the corresponding principal factor set. It is immediate that $\partial\varepsilon$ satisfies (FS1). To see that (FS2) holds, let $(g,h,k) \in S_3(G)$. Then

$$\begin{aligned}
&(\partial\varepsilon(g,h)|0_{\mathbf{r}(k)}) \cdot k + \partial\varepsilon(g \otimes h, k) \\
&= (((\varepsilon(g)|0_{\mathbf{r}(h)}) \cdot h + \varepsilon(h) - \varepsilon(g \otimes h))|0_{\mathbf{r}(k)}) \cdot k \\
&\qquad + (\varepsilon(g \otimes h)|0_{\mathbf{r}(k)}) \cdot k + \varepsilon(k) + \varepsilon(g \otimes h \otimes k) \\
&= (\varepsilon(g)|0_{\mathbf{r}(h\otimes k)}) \cdot (h \otimes k) + (\varepsilon(h)|0_{\mathbf{r}(k)}) \cdot k - (\varepsilon(g \otimes h)|0_{\mathbf{r}(k)}) \cdot k \\
&\qquad + (\varepsilon(g \otimes h)|0_{\mathbf{r}(k)}) \cdot k + \varepsilon(k) - \varepsilon(g \otimes h \otimes k) \\
&= (\varepsilon(g)|0_{\mathbf{r}(h\otimes k)}) \cdot (h \otimes k) + (\varepsilon(h)|0_{\mathbf{r}(k)}) \cdot k + \varepsilon(k) - \varepsilon(g \otimes h \otimes k),
\end{aligned}$$

and

$$\begin{aligned}
&\partial\varepsilon(g, h\otimes k) + \partial\varepsilon(h,k) \\
&= (\varepsilon(g)|0_{\mathbf{r}(h\otimes k)})\cdot(h\otimes k) + \varepsilon(h\otimes k) - \varepsilon(g\otimes h\otimes k) + (\varepsilon(h)|0_{\mathbf{r}(k)})\cdot k + \varepsilon(k) - \varepsilon(h\otimes k) \\
&= (\varepsilon(g)|0_{\mathbf{r}(h\otimes k)})\cdot(h\otimes k) + (\varepsilon(h)|0_{\mathbf{r}(k)})\cdot k + \varepsilon(k) - \varepsilon(g\otimes h\otimes k).
\end{aligned}$$

Therefore (FS2) holds. Hence $B^2(G,A)$ is a subset of $Z^2(G,A)$.

The function $\overline{0} : G \longrightarrow A$ is defined by $\overline{0}(g) = 0_{\mathbf{d}(g)}$. Let $(g,h) \in \mathrm{Ner}_2(G)$. Then

$$\partial\overline{0}(g,h) = (0_{\mathbf{d}(g)}|0_{\mathbf{r}(h)})\cdot h + 0_{\mathbf{d}(h)} - 0_{\mathbf{d}(g\otimes h)} = 0_{\mathbf{d}(h)} = 0(g,h).$$

So $0 \in B^2(G,A)$. It remains to show that $B^2(G,A)$ is closed under addition. Let $\varepsilon : G \longrightarrow A$ and $\delta : G \longrightarrow A$ be functions satisfying $\varepsilon(g), \delta(g) \in A_{\mathbf{d}(g)}$. The function $\varepsilon+\delta : G \longrightarrow A$ is defined by $(\varepsilon+\delta)(g) = \varepsilon(g)+\delta(g)$. We show that $\partial(\varepsilon+\delta)$ is a principal factor set. Let $(g,h) \in S_2(G)$. Then

$$\begin{aligned}
(\partial\varepsilon + \partial\delta)(g,h) &= (\varepsilon(g)|0_{\mathbf{r}(h)})\cdot h + \varepsilon(h) - \varepsilon(g\otimes h) + (\delta(g)|0_{\mathbf{r}(h)})\cdot h + \delta(h) - \delta(g\otimes h) \\
&= ((\varepsilon(g)|0_{\mathbf{r}(h)}) + (\delta(g)|0_{\mathbf{r}(h)}))\cdot h + \varepsilon(h) + \delta(h) - \varepsilon(g\otimes h) - \delta(g\otimes h) \\
&= ((\varepsilon+\delta)(g)|0_{\mathbf{r}(h)})\cdot h + (\varepsilon+\delta)(h) - (\varepsilon+\delta)(g\otimes h) \\
&= \partial(\varepsilon+\delta).
\end{aligned}$$

Thus $B^2(G,A)$ is a subgroup of $Z^2(G,A)$.

■

The following result shows that extensions are classified by factor sets and principal factor sets.

Theorem 10.25 *Let G be an ordered groupoid and A a G-module. There is a bijection between the set of all congruence classes of extensions of A by G and the quotient group $Z^2(G,A)/B^2(G,A)$. Under this correspondence the class of the semidirect product extension corresponds to the identity.*

Proof. Let $\mathrm{Ext}(G,A)$ denote the set of congruence classes of extensions of factor sets.

Let ζ be an element of $Z^2(G,A)$ and let

$$\mathcal{E}_\zeta = A \overset{i_\zeta}{\hookrightarrow} (G * A)_\zeta \xrightarrow{\pi_\zeta} G$$

be the extension of A by G constructed in Proposition 10.20. Define

$$\omega : Z^2(G,A)/B^2(G,A) \longrightarrow \mathrm{Ext}(G,A) \quad \text{by} \quad \omega(\zeta + B^2(G,A)) \longmapsto [\mathcal{E}_\zeta].$$

To see that ω is well-defined, let η be in the same coset as ζ. That is $\zeta = \eta + \partial\varepsilon$, for some $\varepsilon : G \longrightarrow A$ satisfying $\varepsilon(g) \in A_{\mathbf{d}(g)}$. Define

$$\mu : (G * A)_\zeta \longrightarrow (G * A)_\eta \quad \text{by} \quad \mu : (g, a) \longmapsto (g, a + \varepsilon(g)).$$

We shall prove that μ is a congruence of extensions.

We show that μ is a functor. Let $(g, a) \in (G * A)_\zeta$. Then

$$\mathbf{d}(\mu(g, a)) = \mathbf{d}(g, a + \varepsilon(g)) = (\mathbf{d}(g), -\eta(\mathbf{d}(g), \mathbf{d}(g))),$$

and

$$\mu(\mathbf{d}(g, a)) = \mu(\mathbf{d}(g), -\zeta(\mathbf{d}(g), \mathbf{d}(g))) = (\mathbf{d}(g), -\zeta(\mathbf{d}(g), \mathbf{d}(g)) + \varepsilon(\mathbf{d}(g))).$$

But

$$\begin{aligned}
\eta(\mathbf{d}(g), \mathbf{d}(g)) &= \zeta(\mathbf{d}(g), \mathbf{d}(g)) - \partial\varepsilon(\mathbf{d}(g), \mathbf{d}(g)) \\
&= \zeta(\mathbf{d}(g), \mathbf{d}(g)) - \varepsilon(\mathbf{d}(g)) - \varepsilon(\mathbf{d}(g)) + \varepsilon(\mathbf{d}(g)) \\
&= \zeta(\mathbf{d}(g), \mathbf{d}(g)) - \varepsilon(\mathbf{d}(g).
\end{aligned}$$

Hence $\mathbf{d}(\mu(g, a)) = \mu(\mathbf{d}(g, a))$. Similarly $\mathbf{r}(\mu(g, a)) = \mu(\mathbf{r}(g, a))$ Now let $(g, a), (h, b) \in (G * A)_\zeta$ with $\mathbf{d}(g) = \mathbf{r}(h)$. Then

$$\begin{aligned}
\mu((g, a)(h, b)) &= \mu(gh, \zeta(g, h) + a \cdot h + b) \\
&= (gh, \zeta(g, h) + a \cdot h + b + \varepsilon(gh))
\end{aligned}$$

and

$$\begin{aligned}
\mu(g, a)\mu(h, b) &= (g, a + \varepsilon(g))(h, b + \varepsilon(h)) \\
&= (gh, \eta(g, h) + a \cdot h + \varepsilon(g) \cdot h + b + \varepsilon(h))
\end{aligned}$$

but

$$\begin{aligned}
\zeta(g, h) + \varepsilon(gh) &= \eta(g, h) + \partial\varepsilon(g, h) + \varepsilon(gh) \\
&= \eta(g, h) + \varepsilon(g) \cdot h + \varepsilon(h) - \varepsilon(gh) + \varepsilon(gh) \\
&= \eta(g, h) + \varepsilon(g) \cdot h + \varepsilon(h).
\end{aligned}$$

Hence μ is a functor.

To see that μ is order-preserving, let $(g, a), (h, b) \in (G * A)_\zeta$ with $(h, b) \leqslant (g, a)$, that is $g \leqslant h$ and $b = (a|0_{\mathbf{d}(h)}) + \zeta(g, \mathbf{d}(h)) - \zeta(\mathbf{d}(h), \mathbf{d}(h))$. Now

$$\begin{aligned}
h + \varepsilon(h) &= (a|0_{\mathbf{d}(h)}) + \zeta(g, \mathbf{d}(h)) - \zeta(\mathbf{d}(h), \mathbf{d}(h)) + \varepsilon(h) \\
&= (a|0_{\mathbf{d}(g)}) + \eta(g, \mathbf{d}(h)) + \partial\varepsilon(g, \mathbf{d}(h)) - \eta(\mathbf{d}(h), \mathbf{d}(h)) - \partial\varepsilon(\mathbf{d}(h), \mathbf{d}(h)) + \varepsilon(h)
\end{aligned}$$

$$
\begin{aligned}
&= (a|0_{\mathbf{d}(h)}) + \eta(g, \mathbf{d}(h)) + (\varepsilon(g)|\mathbf{d}(h)) + \varepsilon(\mathbf{d}(h)) - \varepsilon(h) \\
&\quad - \eta(\mathbf{d}(h), \mathbf{d}(h)) - \varepsilon(\mathbf{d}(h)) - \varepsilon(\mathbf{d}(h)) + \varepsilon(\mathbf{d}(h)) + \varepsilon(h) \\
&= (a|0_{\mathbf{d}(h)}) + \eta(g, \mathbf{d}(h)) - \eta(\mathbf{d}(h), \mathbf{d}(h)) + (\varepsilon(g)|0_{\mathbf{d}(h)}) \\
&= ((a + \varepsilon(g))|0_{\mathbf{d}(h)}) + \eta(g, \mathbf{d}(h)) - \eta(\mathbf{d}(h), \mathbf{d}(h))
\end{aligned}
$$

so $(g, a + \varepsilon(g)) \leqslant (h, b + \varepsilon(h))$, thus μ is order-preserving.

It is immediate that the diagram below commutes.

$$
\begin{tikzcd}
 & (A * G)_\zeta \arrow[dd, "\mu"] \arrow[dr, "\pi_\zeta"] & \\
A \arrow[ur, hook, "i_\zeta"] \arrow[dr, hook, "i_\eta"'] & & G \\
 & (A * G)_\eta \arrow[ur, "\pi_\eta"'] &
\end{tikzcd}
$$

Therefore μ is an congruence of extensions. Hence ω is well-defined.

To see that ω is injective, let ζ and η be factor sets and let $\mu : \mathcal{E}_\zeta \cong \mathcal{E}_\eta$ be an equivalence of the resulting extensions. We show that ζ and η lie in the same coset. Let $g \in G$, then $(g, 0_{\mathbf{d}(g)}) \in (G * A)_\zeta$. Since $\pi_\zeta = \pi_\eta \mu$, we have

$$
\mu(g, 0_{\mathbf{d}(g)}) = (g, \varepsilon(g))
$$

for some $\varepsilon(g) \in A_{\mathbf{r}(g)}$. It is clear that $\varepsilon : G \longrightarrow A$ is a well defined function. Let $(a, g) \in (A * G)_\zeta$. Observe that

$$
(g, 0_{\mathbf{d}(g)})(\mathbf{d}(g), a - \zeta(\mathbf{d}(g), \mathbf{d}(g))) = (g, \zeta(g, \mathbf{d}(g)) + a - \zeta(\mathbf{d}(g), \mathbf{d}(g)) = (g, a),
$$

by Lemma 10.16(i). Hence

$$
\begin{aligned}
\mu(g, a) &= \mu(g, 0_{\mathbf{d}(g)})\mu(\mathbf{d}(g), a - \zeta(\mathbf{d}(g), \mathbf{d}(g))) \\
&= (g, \varepsilon(g))\mu i_\zeta(a) \\
&= (g, \varepsilon(g)) i_\eta(a) \\
&= (g, \varepsilon(g))(\mathbf{d}(g), a - \eta(\mathbf{d}(g), \mathbf{d}(g))) \\
&= (g, \eta(g, \mathbf{d}(g)) + \varepsilon(g) + a - \eta(\mathbf{d}(g), \mathbf{d}(g))) \\
&= (a, g)
\end{aligned}
$$

by Lemma 10.16(i). Let $(g, a), (h, b) \in (G * A)_\zeta$, were $\exists gh$. Then

$$
\begin{aligned}
\mu(g, a)\mu(h, b) &= (g, a + \varepsilon(g))(h, b + \varepsilon(h)) \\
&= (gh, \eta(g, h) + (a + \varepsilon(g)) \cdot h + b + \varepsilon(h)).
\end{aligned}
$$

But

$$\begin{aligned}\mu((g,a)(h,b)) &= \mu(gh, \zeta(g,h) + a\cdot h + b)\\ &= (gh, \zeta(g,h) + a\cdot h + b + \varepsilon(gh)).\end{aligned}$$

Hence $\eta(g,h) + a\cdot h + \varepsilon(g)\cdot h + b + \varepsilon(h) = \zeta(g,h) + a\cdot h + b + \varepsilon(gh)$, that is

$$\zeta(g,h) = \eta(g,h) + \varepsilon(g)\cdot h + \varepsilon(h) + \varepsilon(gh).$$

So $\zeta = \eta + \partial\varepsilon$, as required. Hence ω is injective.

Let $\mathcal{E} = (\iota, U, \sigma)$ be an extension of A by G. We can choose a transversal l for $\mathcal{E}$. By Proposition 10.17, $\mathcal{E}$ and l give rise to a factor set ζ and by Proposition 10.21 $\mathcal{E}$ is congruent to $\mathcal{E}_\zeta$. By Proposition 10.23, a different choice of transversal gives rise to a factor set in the same coset as ζ. That is $\omega(\zeta + B^2(G,A)) = [\mathcal{E}]$. Hence ω is surjective.

We have thus constructed a bijection ω from the group $Z^2(G,A)/B^2(G,A)$ to the set $\mathrm{Ext}(G,A)$. The semidirect product extension $(i, A \rtimes G, \pi)$ is split and so has the identity 0 as one of its factor sets by Lemma 10.18, conversely, any extension which has 0 as a factor set is congruent to the semidirect product by Proposition 10.9. Thus $\omega(B^2(G,A)) = \omega(0 + B^2(G,A)) = [\mathcal{E}_0] = [(i, A \rtimes G, \pi)]$.

■

10.4 Cohomology and extensions

In this section we show that the second cohomology group of an ordered groupoid with a maximal identity, can be characterised by means of extensions.

Theorem 10.26 *Let G be an ordered groupoid which has a maximal identity 1, and let A be a G-module. Then*

(i) *The first cohomology group $H^1(G,A)$ is in bijective correspondence with the set of A-conjugacy classes of split extensions of A by G.*

(ii) *The second cohomology group $H^2(G,A)$ is in bijective correspondence with the set of congruence classes of extensions of A by G.*

PROOF. We will work in the category $\mathbf{Ab}^{C(G)^{\mathrm{op}}}$. Recall that the cohomology of the G-module A is defined as the cohomology of the corresponding right $C(G)$-module, which we denote by $\mathcal{A}$. We shall calculate the cohomology groups by constructing a projective resolution of the free right $C(G)$-module $\Delta\mathbb{Z}$.

1. We construct some free $C(G)$-modules .

Consider the $C(G)_o$ set defined by

$$K(e) = \begin{cases} \emptyset & \text{if } e \neq 1 \\ \{*\} & \text{if } e = 1. \end{cases}$$

We shall construct the free $C(G)$-module over K, using the construction of Section 8.2.2. Consider the functor $\mathbb{F}(K) : C(G)^{\text{op}} \longrightarrow \mathbf{Set}$ constructed in Proposition 8.6. Since $(1, g) \in C(G)$ for all $g \in G$, we have that

$$\mathbb{F}(K)(e) = \{(*, (1, g)) \mid \mathbf{d}(g) = e\}.$$

We write elements of $\mathbb{F}(K)(e)$ as (g) rather than $(*, (1, g))$. Thus $\mathbb{F}(K)(e)$ is the $\mathcal{L}$-class of G at e. For each element (e, g) of $C(G)$, the function

$$\mathbb{F}(K)(e,g) : \mathbb{F}(K)(e) \longrightarrow (K)(\mathbf{d}(g)) \quad \text{is given by} \quad \mathbb{F}(K)(e,g) : (h) \longmapsto (h \otimes g).$$

We let $\mathbb{Z}\mathcal{L} = \mathbb{Z}(K)$ denote the free $C(G)$-module over K. That is, $\mathbb{Z}\mathcal{L}$ is the free abelian group generated by the $\mathcal{L}$-class

$$L_e = \{g \in G \mid \mathbf{d}(g) = e\},$$

and for each $(e, g) \in C(G)$, $\mathbb{Z}\mathcal{L}(e, g)$ is the homomorphism induced by the function $\mathbb{F}(K)(e, g)$. The $C(G)_o$-morphism

$$\eta : K \longrightarrow \mathbb{Z}\mathcal{L}$$

is given by $\eta(*) = 1$, and is the empty function otherwise.

To construct some more free $C(G)$-modules, let $n \geqslant 1$, and let $S_n(e)$ denote the n-staircase over G starting at the identity e. That is

$$S_n(e) = \{(g_n, \ldots, g_1) \in G \times \cdots \times G \mid \mathbf{d}(g_0) = e \text{ and } \mathbf{r}(g_i) \leqslant \mathbf{d}(g_{i+1}),\ i = 1, \ldots, n-1\}.$$

Clearly S_n is a $C(G)_o$-set. We shall construct the free $C(G)$-module over S_n. Let $\mathcal{S}_n = \mathbb{F}(S_n) : C(G)^{\text{op}} \longrightarrow \mathbf{Set}$ denote the functor constructed in Proposition 8.6. Then

$$\mathcal{S}_n(e) = \{((g_n, \ldots, g_1), h) \in S_n \times G \mid \mathbf{d}(h) = e,\ \mathbf{r}(h) \leqslant \mathbf{d}(g_1)\}.$$

It is clear that $\mathcal{S}_n(e) = S_{n+1}(e)$. We therefore write elements of $\mathcal{S}_n$ as $(g_n, \ldots, g_1, g_0)$. Let $(e, h) \in C(G)$. The function

$$\mathcal{S}_n(e,h) : \mathcal{S}_n(e) \longrightarrow \mathcal{S}_n(\mathbf{d}(h)) \quad \text{is given by} \quad \mathcal{S}_n(e,g) : (g_n, \ldots, g_0) \longmapsto (g_n, \ldots, g_0 \otimes h).$$

Let $\mathbb{Z}\mathcal{S}_n$ denote the free $C(G)$-module over S_n. The abelian group $\mathbb{Z}\mathcal{S}_n(e)$ is generated by the set $\mathcal{S}_n(e)$. For each $(e, g) \in C(G)$, the homomorphism $\mathbb{Z}\mathcal{S}_n(e, g) : \mathbb{Z}\mathcal{S}_n(e) \longrightarrow \mathbb{Z}\mathcal{S}_n(\mathbf{d}(g))$, is induced by the function $\mathcal{S}_n(e, h)$. The inclusion $C(G)_o$-morphism

$$\eta_n : S_n \longrightarrow \mathbb{Z}\mathcal{S}_n \quad \text{is given by} \quad \eta_n : (g_n, \ldots, g_1) \longmapsto (g_n, \ldots, g_1, \mathbf{d}(g_1)).$$

We have therefore constructed free $C(G)$-modules $\mathbb{Z}\mathcal{L}$ and $\mathbb{Z}\mathcal{S}_n$, for $n \geqslant 1$.

2. We construct a chain complex. We shall define $C(G)$-morphisms between the $C(G)$-modules constructed above, to obtain a chain complex.

Define

$$\overline{\varepsilon} : K \longrightarrow \Delta\mathbb{Z} \quad \text{by} \quad \overline{\varepsilon}(*) = 1_1,$$

and the empty function otherwise. Since $\mathbb{Z}\mathcal{L}$ is free over K, the $C(G)_o$-morphism $\overline{\varepsilon}$ extends uniquely to a $C(G)$-morphism $\varepsilon : \mathbb{Z}\mathcal{L} \longmapsto \Delta\mathbb{Z}$ such that $\varepsilon\eta = \overline{\varepsilon}$. By Corollary 8.8.1, ε is defined on the generators of $\mathbb{Z}\mathcal{L}(G)$ by

$$\varepsilon(g) = \mathbb{Z}\mathcal{L}(g)(\overline{\varepsilon}(*)) = \Delta\mathbb{Z}(1,g)(1_1) = 1_{\mathbf{d}(g)}.$$

Now define a $C(G)_o$-morphism

$$\overline{d}_0 : S_1 \longrightarrow \mathbb{Z}\mathcal{L}(G) \quad \text{by} \quad \overline{d}_0 : (g) \longmapsto (g) - (\mathbf{d}(g)).$$

Since $\mathbb{Z}\mathcal{S}_1$ is free over S_1, there is a unique $C(G)$-morphism d_0 from $\mathbb{Z}\mathcal{S}_1$ to $\mathbb{Z}\mathcal{L}$. By 8.8.1 d_0 is given by

$$d_0(g,h) = (g \otimes h) - (\mathbf{d}(g) \otimes h) = (g \otimes h) - (h).$$

For each $n \geqslant 1$ define a $C(G)_o$-function

$$\overline{d}_n : S_{n+1} \longrightarrow \mathbb{Z}\mathcal{S}_n$$

by

$$\begin{aligned} \overline{d}_n(g_{n+1}, \ldots, g_1) &= (g_{n+1}, \ldots, g_2, g_1) \\ &\quad + \sum_{j=1}^{n} (-1)^j (g_{n+1}, \ldots, g_{j+1} \otimes g_j, \ldots, g_1, \mathbf{d}(g_1)) \\ &\quad + (-1)^{n+1}(g_n, \ldots, g_1, \mathbf{d}(g_1)) \end{aligned}$$

Since $\mathbb{Z}\mathcal{S}_{n+1}$ is free over S_{n+1}, there is a unique $C(G)$-morphism $d_n : \mathbb{Z}\mathcal{S}_{n+1} \longrightarrow \mathbb{Z}\mathcal{S}_n$ making the diagram below commute.

$$\begin{tikzcd} \mathbb{Z}\mathcal{S}_{n+1} \arrow[r, "d_n"] & \mathbb{Z}\mathcal{S}_n \\ \\ \mathcal{S}_n \arrow[uu, "\eta_{n+1}"] \arrow[uur, "\overline{d}_n"'] & \end{tikzcd}$$

By Corollary 8.8.1, d_n is defined on the generators of the free abelian group $\mathbb{Z}(\mathrm{Ner}_n(C))$ by

$$\begin{aligned} d_n(g_{n+1}, \ldots, g_1, g_0) &= \mathbb{Z}\mathcal{S}_{n+1}(\mathbf{d}(g_1), g_0)(\overline{d}(g_{n+1}, \ldots, g_1)) \\ &= (g_{n+1}, \ldots, g_1 \otimes g_0) + \sum_{j=1}^{n} (-1)^j (g_{n+1}, \ldots, g_{j+1} \otimes g_j, \ldots, g_1, g_0) \\ &\quad + (-1)^{n+1}(g_n, \ldots, g_1, g_0). \end{aligned}$$

That is

$$d_n(g_{n+1},\ldots,g_0)=\sum_{j=0}^{n}(-1)^j(g_{n+1},\ldots,g_{j+1}\otimes g_j,\ldots,g_0)+(-1)^{n+1}(g_n,\ldots,g_0).$$

We have therefore constructed the following diagram of $C(G)$-modules and $C(G)$-morphisms

$$\cdots \longrightarrow \mathbb{Z}\mathcal{S}_2 \xrightarrow{d_1} \mathbb{Z}\mathcal{S}_1 \xrightarrow{d_0} \mathbb{Z}\mathcal{L} \xrightarrow{\epsilon} \Delta\mathbb{Z} \longrightarrow 0$$

We shall prove that this is a chain complex. To see that $\varepsilon d_0=0$, let (g,h) be a generator for $\mathbb{Z}\mathcal{S}_1$. Then

$$\varepsilon d_0(g,h)=\varepsilon((g\otimes h)-(h))=1_{\mathbf{d}(h)}-1_{\mathbf{d}(h)}=0_{\mathbf{d}(h)}.$$

To see that $d_0d_1=0$, let (g,h,k) be a generator for $\mathbb{Z}\mathcal{S}_2$. Then

$$\begin{aligned} d_0d_1(g,h,k) &= d_0((g,h\otimes k)-(g\otimes h,k)-(h,k)) \\ &= ((g\otimes h\otimes k)-(h\otimes k))-((g\otimes h\otimes k)-(k))+((h\otimes k)-(k)) \\ &= 0_{\mathbf{d}(k)}. \end{aligned}$$

We omit the proof that $d_{n+1}d_n=0$ in general.

3 We show that the sequence is a projective resolution The $C(G)$-modules $\mathbb{Z}\mathcal{L}$ and $\mathbb{Z}\mathcal{S}_n$ are free by construction. Hence by Proposition 8.9, each $C(G)$-module in the sequence

$$\cdots \longrightarrow \mathbb{Z}\mathcal{S}_2 \xrightarrow{d_1} \mathbb{Z}\mathcal{S}_1 \xrightarrow{d_0} \mathbb{Z}\mathcal{L} \xrightarrow{\epsilon} \Delta\mathbb{Z} \longrightarrow 0$$

is a projective object in $\mathbf{Ab}^{C^{\mathrm{op}}}$. To prove that the sequence is a projective resolution, it remains to show that the sequence is exact. The sequence is a chain complex, so by Proposition 7.8, to show that the sequence is exact, it is enough to construct a contracting homotopy.

Define a $C(G)_o$-morphism

$$\tau:\Delta\mathbb{Z}\longrightarrow\mathbb{Z}\mathcal{L}\quad\text{by}\quad\tau(n_e)=n(e).$$

Clearly $\tau|\Delta\mathbb{Z}(e)$ are group homomorphisms. Now define a $C(G)_o$-morphism $\sigma_0:\mathbb{Z}\mathcal{L}\longrightarrow\mathbb{Z}\mathcal{S}_1$ on the generators of $\mathbb{Z}\mathcal{L}$ by

$$\sigma_0:(g)\longmapsto(g,\mathbf{d}(g)).$$

It is straightforward to show that $\sigma_0|\mathbb{Z}\mathcal{L}(e)$ is a group homomorphism for all $e\in G_o$.

Let $n\geq 1$. Define a $C(G)_o$-morphism

$$\overline{\sigma}_n:\mathcal{S}_n\longrightarrow\mathbb{Z}\mathcal{S}_{n+1}\quad\text{by}\quad\sigma_n:(g_n,\ldots,g_0)\longmapsto(g_n,\ldots,g_0,\mathbf{d}(g_0)).$$

Since $\mathbb{Z}\mathcal{S}_n$ is freely generated by $\mathcal{S}_n$, $\overline{\sigma}_n$ extends to a unique $C(G)_o$-morphism σ_n : $\mathbb{Z}\mathcal{S}_n(G) \longrightarrow \mathbb{Z}\mathcal{S}_{n+1}(G)$, such that each $\sigma_n|\mathbb{Z}\mathcal{S}_n(e)$ is a group homomorphism.

To show that σ and τ define a contracting homotopy, it is required to show that

$$\varepsilon\tau = \mathrm{Id}_{\Delta\mathbb{Z}}, \quad \tau\varepsilon + d_0\sigma_0 = \mathrm{Id}_{\mathbb{Z}\mathcal{S}_0} \quad \text{and} \quad d_n\sigma_n + \sigma_{n-1}d_{n-1} = \mathrm{Id}_{\mathbb{Z}\mathcal{S}_n}.$$

The first condition is immediate. We show that $\tau\varepsilon + d_0\sigma_0 = \mathrm{Id}_{\mathbb{Z}\mathcal{S}_0}$. Let $(g) \in \mathbb{Z}\mathcal{S}_0$. Then

$$(\tau\varepsilon + d_0\sigma_0)(g) = \tau\varepsilon(g) + d_1\sigma_0(g) = \tau(1_{\mathbf{d}(g)}) + d_0(g, \mathbf{d}(g)) = (\mathbf{d}(g)) + (g) - (\mathbf{d}(g)) = (g).$$

To show that $d_n\sigma_n + \sigma_{n-1}d_{n-1} = \mathrm{Id}_{\mathrm{Id}_{\mathbb{Z}\mathcal{S}_n}}$, let $(g_n, \ldots, g_0)$ be a generator for $\mathbb{Z}\mathcal{S}_n$. Then

$$\begin{aligned}
\sigma_{n-1}d_{n-1}(g_n, \ldots, g_0) &= \sigma\Big(\sum_{i=0}^{n-1}(-1)^i(g_n, \ldots, g_{i+1} \otimes g_i, \ldots, g_0) \\
&\quad +(-1)^n(\mathbf{d}(g_n), g_{n-1}, \ldots, g_0)\Big) \\
&= \sum_{i=0}^{n-1}(-1)^i(g_n, \ldots, g_{i+1} \otimes g_i, \ldots, g_0, \mathbf{d}(g_0)) \\
&\quad +(-1)^n(\mathbf{d}(g_n), g_{n-1}, \ldots, g_0, \mathbf{d}(g_0))
\end{aligned}$$

and

$$\begin{aligned}
d_n\sigma_n(g_n, \ldots, g_0) &= d_n(g_n, \ldots, g_0, \mathbf{d}(g_0)) \\
&= (g_n, \ldots, x_1, g_0\mathbf{d}(g_0)) \\
&\quad +\sum_{i=0}^{n-1}(-1)^{j+1}(g_n, \ldots, g_{i+1} \otimes g_i, \ldots, g_0, \mathbf{d}(g_0)) \\
&\quad +(-1)^{n+1}(\mathbf{d}(g_n), g_{n-1}, \ldots, g_0, \mathbf{d}(g_0)) \\
&= (g_n, \ldots, g_0) \\
&\quad -\sum_{i=0}^{n-1}(-1)^j(g_n, \ldots, g_{i+1} \otimes g_i, \ldots, g_0, \mathbf{d}(g_0)) \\
&\quad -(-1)^n(\mathbf{d}(g_{n-1}, \ldots, g_0, \mathbf{d}(g_0)) \\
&= (g_n, \ldots, g_0) - \sigma_{n-1}d_n(g_n, \ldots, g_0).
\end{aligned}$$

Hence $d_n\sigma_n + \sigma_{n-1}d_{n-1} = \mathrm{Id}_{\mathbb{Z}(\mathrm{Ner}_n(C))}$.

Therefore $\mathbb{Z}\mathcal{S}_\bullet$ is a projective resolution.

4 The corresponding cochain complex We apply the left-exact functor $\hom(-, \mathcal{A})$ to the projective resolution constructed above. Thus we obtain a cochain complex

$$\hom(\mathbb{Z}\mathcal{L}(G), \mathcal{A}) \xrightarrow{d_*^0} \hom(\mathbb{Z}\mathcal{S}_1, \mathcal{A}) \xrightarrow{d_*^1} \hom(\mathbb{Z}\mathcal{S}_2, \mathcal{A}) \xrightarrow{d_*^2} \hom(\mathbb{Z}\mathcal{S}_3, \mathcal{A}) \longrightarrow \cdots.$$

where $d_*^n(\alpha) = \alpha d_n$.

This complex has a simpler description, which we now give. Define

$$X^0(G, \mathcal{A}) = \{\varepsilon : G_o \longrightarrow \mathcal{A} \mid \varepsilon(e) \in \mathcal{A}_e \text{ and if } e \leqslant f, \text{ then } \varepsilon(e) = \mathcal{A}(e, f)(\varepsilon(f))\}.$$

We will prove that $X^0(G, \mathcal{A})$ in bijective correspondence with $\hom(\mathbb{Z}\mathcal{L}(G), \mathcal{A})$. Let $\alpha : \mathbb{Z}\mathcal{L}(G) \longrightarrow \mathcal{A}$ be a $C(G)$-morphism. We shall construct an element of $X^0(G, \mathcal{A})$. Now $\mathbb{Z}\mathcal{L}(G)$ is the free $C(G)$-module over the $C(G)_o$-set K. So by Corollary 8.8.1, α defines and is defined by a $C(G)_o$-morphism

$$\overline{\alpha} : K \longrightarrow \mathcal{A} \quad \text{where} \quad \alpha(g) = \mathcal{A}(1, g)(\overline{\alpha}(*)).$$

Define

$$\hat{\alpha} : G_o \longrightarrow \mathcal{A} \quad \text{by} \quad \hat{\alpha} : e \longmapsto \mathcal{A}(1, e)(\overline{\alpha}(*)).$$

This is clearly a $C(G)_o$-morphism. Furthermore, if $e, f \in G_o$ and $e \leqslant f$, then

$$\hat{\alpha}(e) = \mathcal{A}(1, e)(\overline{\alpha}(*)) = \mathcal{A}((1, f)(f, e))(\overline{\alpha}(*)) = (f, e)(1, f)(\overline{\alpha}(*)) = \mathcal{A}(f, e)\hat{\alpha}(f).$$

So $\hat{\alpha} \in X^0(G, \mathcal{A})$. Conversely, let $\varepsilon \in X^0(G, \mathcal{A})$. We shall construct an element of $\hom(\mathbb{Z}\mathcal{L}(G), \mathcal{A})$. Define

$$\varepsilon' : K \longrightarrow \mathcal{A} \quad \text{by} \quad \varepsilon'(*) = \varepsilon(1).$$

Since $\mathbb{Z}\mathcal{L}$ is free over K, there is a $C(G)$-morphism

$$\varepsilon^* : \mathbb{Z}\mathcal{L} \longrightarrow \mathcal{A} \quad \text{with} \quad \varepsilon^*(1) = \varepsilon'(*) = \varepsilon(1).$$

By Corollary 8.8.1, $\varepsilon^*(g) = \mathcal{A}(1, g)\varepsilon'(*)$; that is

$$\varepsilon^*(g) = \mathcal{A}(\mathbf{r}(g), g)\mathcal{A}(1, \mathbf{r}(g))(\varepsilon(1)) = \mathcal{A}(\mathbf{r}(g), g)(\varepsilon(\mathbf{r}(g))).$$

We shall now show that $(\hat{\alpha})^* = \alpha$ and $\widehat{(\varepsilon^*)} = \varepsilon$, for all $\alpha \in \hom(\mathbb{Z}\mathcal{L}(G), \mathcal{A})$ and $\varepsilon \in X^0(G, \mathcal{A})$. Let $g \in G$. Then

$$\begin{aligned} (\hat{\alpha})^*(g) &= \mathcal{A}(\mathbf{r}(g), g)(\hat{\alpha}(\mathbf{r}(g))) \\ &= \mathcal{A}(\mathbf{r}(g), g)\mathcal{A}(1, \mathbf{r}(g))(\overline{\alpha}(*)) \\ &= \mathcal{A}(1, g)(\overline{\alpha}(*)) \\ &= \alpha(g). \end{aligned}$$

Now let $e \in G_o$. Then

$$\begin{aligned} \widehat{(\varepsilon^*)}(e) &= \mathcal{A}(1, e)(\overline{\varepsilon^*}(*)) \\ &= \mathcal{A}(1, e)\varepsilon(1) \\ &= \varepsilon(e). \end{aligned}$$

Hence there is a bijection between $X^0(G, \mathcal{A})$ and $\hom(\mathbb{Z}\mathcal{L}(G), \mathcal{A})$.

Now define

$$X^n(G, \mathcal{A}) = \{\phi : S_n \longrightarrow \mathcal{A} \mid \phi(g_n, \ldots, g_1) \in \mathcal{A}_{\mathbf{d}(g_1)}\},$$

that is $X^n(G, \mathcal{A})$ is the set of G_o-morphisms from the set of n-staircases over G to $\mathcal{A}$.

By Corollaries 8.8.1 and 8.8.2, there is a bijective correspondence between the group $\hom(\mathbb{Z}\mathcal{S}_n, \mathcal{A})$ and the set $X^n(G, \mathcal{A})$, in which a $C(G)$-morphisms $\alpha : \mathbb{Z}\mathcal{S}_n \longrightarrow \mathcal{A}$ corresponds to the $C(G)_o$-morphism $\overline{\alpha} : S_n \longrightarrow \mathcal{A}$ given by

$$\overline{\alpha}(g_n, \ldots, g_1) = \alpha(g_n, \ldots, g_1, \mathbf{d}(g_1)).$$

The $C(G)$-morphism α can be recaptured from $\overline{\alpha}$ and $\mathcal{A}$ as

$$\alpha(g_n, \ldots, g_0) = \mathcal{A}(\mathbf{d}(g_1), g_0)(\overline{\alpha}(g_n, \ldots, g_1)).$$

Now consider the homomorphism

$$d_*^0 : \hom(\mathbb{Z}\mathcal{L}, \mathcal{A}) \longrightarrow \hom(\mathbb{Z}\mathcal{S}_1, \mathcal{A}) \quad \text{given by} \quad d_*^0(\alpha) = \alpha d_0.$$

We shall construct a function $\partial^0 : X^0(G, \mathcal{A}) \longrightarrow X^1(G, \mathcal{A})$.

Let $\varepsilon : G_o \longrightarrow \mathcal{A}$ be an element of $X^0(G, \mathcal{A})$. Recall that ε is determined by a $C(G)_o$-morphism $\varepsilon' : K \longrightarrow \mathcal{A}$ where $\varepsilon(e) = \mathcal{A}(1, e)(\varepsilon(*))$. Also recall that ε corresponds to the $C(G)$-morphism $\varepsilon^* : \mathbb{Z}\mathcal{L} \longrightarrow \mathcal{A}$ given by $\varepsilon^*(g) = \mathcal{A}(1, g)\varepsilon'(*)$. By Corollary 8.8.2, the $C(G)$-morphism $d_*^0(\varepsilon^*) : \mathbb{Z}\mathcal{S}_1 \longrightarrow \mathcal{A}$ determines, and is determined by, a $C(G)_o$-morphism

$$\partial^0(\varepsilon) : \mathcal{S}_1 \longrightarrow \mathcal{A} \quad \text{where} \quad \eta_1 \partial^0(\varepsilon) = d_*^0(\varepsilon^*).$$

Now

$$\begin{aligned}
\partial^0(\varepsilon)(g) &= \varepsilon^* d_0(g, \mathbf{d}(g)) \\
&= \varepsilon^*((g) - (\mathbf{d}(g))) \\
&= \varepsilon^*(g) - \varepsilon^*(\mathbf{d}(g)) \\
&= \mathcal{A}(1, g)(\varepsilon'(*)) - \mathcal{A}(1, \mathbf{d}(g))(\varepsilon'(*)) \\
&= \mathcal{A}(\mathbf{r}(g), g)\mathcal{A}(1, \mathbf{r}(g))(\varepsilon'(*)) - \mathcal{A}(1, \mathbf{d}(g))(\varepsilon'(*)) \\
&= \mathcal{A}(\mathbf{r}(g), g)\varepsilon(\mathbf{r}(g)) - \varepsilon(\mathbf{d}(g)).
\end{aligned}$$

We have therefore defined a function

$$\partial^0 : X^0(G, \mathcal{A}) \longrightarrow X^1(G, \mathcal{A}),$$

arising from d_*^0.

Now let $n \geqslant 1$. We shall construct a function $\partial^n : X^n(G,A) \longrightarrow X^{n+1}(G,A)$, arising from the homomorphism

$$d_*^n : \hom(\mathbb{Z}\mathcal{S}_n, \mathcal{A}) \longrightarrow \hom(\mathbb{Z}\mathcal{S}_{n+1}, \mathcal{A}).$$

Let $\alpha : S_n \longrightarrow \mathcal{A}$ be a $C(G)_o$-morphism. Since $\mathbb{Z}\mathcal{S}_n$ is free over S_n, there is a unique $C(G)$-morphism $\alpha^* : \mathbb{Z}\mathcal{S}_n \longrightarrow \mathcal{A}$ such that $\alpha = \alpha^*\eta_n$; that is

$$\alpha(g_n, \ldots, g_1) = \alpha^*(g_n, \ldots, g_1, \mathbf{d}(g_1)).$$

Furthermore

$$\alpha^*(g_n, \ldots, g_1, g_0) = \mathcal{A}(\alpha(g_n, \ldots, g_1)),$$

by Corollary 8.8.1.

Consider the $C(G)$-morphism

$$d_*^n(\alpha^*) = \alpha^* d_n : \mathbb{Z}\mathcal{S}_{n+1} \longrightarrow \mathcal{A}.$$

Define

$$\partial^n(\alpha) : S_{n+1} \longrightarrow \mathcal{A} \quad \text{by} \quad \partial^n(\alpha) = \alpha^* d_n \eta_{n+1}.$$

The situation is illustrated below.

$$\begin{tikzcd}
\mathbb{Z}\mathcal{S}_{n+1} \arrow[r, "d_n"] & \mathbb{Z}\mathcal{S}_n \arrow[r, "\alpha^*"] & \mathcal{A} \\
\mathcal{S}_{n+1} \arrow[u, hook, "\eta_{n+1}"] \arrow[ur, "\overline{d}_n"'] & \mathcal{S}_n \arrow[u, hook, "\eta_n"] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

Since $\mathbb{Z}\mathcal{S}_{n+1}$ is free over S_n, $d_*^n(\alpha^*)$ is the unique morphism extending $\partial^n(\alpha)$. We have therefore defined a $C(G)_o$-function

$$\partial^n : X^n(G,A) \longrightarrow X^{n+1}(G,A)$$

corresponding to d_*^n. Let $(g_{n+1}, \ldots, g_1) \in S_{n+1}$. Then

$$\begin{aligned}
\partial^n(\alpha)(g_{n+1}, \ldots, g_1) &= \alpha^* d_n \eta_{n+1}(g_{n+1}, \ldots, g_1) \\
&= \alpha^* \overline{d}_n(g_{n+1}, \ldots, g_1) \\
&= \alpha^*\Big((g_{n+1}, \ldots, g_2, g_1) \\
&\quad + \sum_{j=1}^{n} (-1)^j (g_{n+1}, \ldots, g_{j+1} \otimes g_j, \ldots, g_1, \mathbf{d}(g_1)) \\
&\quad + (-1)^{n+1}(g_n, \ldots, g_1, \mathbf{d}(g_1))\Big) \\
&= \alpha^*(g_{n+1}, \ldots, g_2, g_1)
\end{aligned}$$

$$
\begin{aligned}
& +\sum_{j=1}^{n}(-1)^j \alpha^*(g_{n+1},\dots,g_{j+1}\otimes g_j,\dots,g_1,\mathbf{d}(g_1)) \\
& +(-1)^{n+1}\alpha^*(g_n,\dots,g_1,\mathbf{d}(g_1)) \\
= \quad & \mathcal{A}(\mathbf{d}(g_2),g_1)(\alpha(g_{n+1},\dots,g_2)) \\
& +\sum_{j=1}^{n}(-1)^j \alpha(g_{n+1},\dots,g_{j+1}\otimes g_j,\dots,g_1) \\
& +(-1)^{n+1}\alpha^*(g_n,\dots,g_1).
\end{aligned}
$$

We have therefore shown that the cochain complex

$$X^0(G,A) \xrightarrow{\partial^0} X^1(G,A) \xrightarrow{\partial^1} X^2(G,A) \xrightarrow{\partial^2} X^3(G,A) \xrightarrow{\partial^3} \cdots$$

is a simpler representation of the complex $\hom(\mathbb{Z}\mathcal{S}_\bullet,\mathcal{A})$.

5 The first cohomology group We now calculate the first cohomology groups arising from the cochain complex $X^\bullet(G,\mathcal{A})$. Let $(g,h)\in S_2$ and $\alpha\in X^1(G,\mathcal{A})$. The function $\partial^1 : X^1(G,\mathcal{A})\longrightarrow X^2(G,\mathcal{A})$ is defined by

$$\partial^1(\alpha)(g,h)=\mathcal{A}(\mathbf{d}(g),h)(\alpha(h))-\alpha(g\otimes h)+\alpha(g).$$

Therefore α is an element of $\mathrm{Ker}(\partial^1)$ if

$$\alpha(g\otimes h)=\mathcal{A}(\mathbf{d}(g),h)(\alpha(g))+\alpha(h).$$

In G-module notation, the 1-cocycles are therefore those functions $\alpha : G\longrightarrow A$ such that $\alpha(g)\in A_{\mathbf{d}(g)}$ and

$$\alpha(g\otimes h)=(\alpha(g)|0_{\mathbf{r}(h)})\cdot h+\alpha(h).$$

Note that, if $e\in G_o$ then

$$\alpha(e)=\alpha(e\otimes e)=(\alpha(e)|0_e)\cdot e+\alpha(e);$$

that is $\alpha(e)=0_e$. To prove that 1-cocycles are the derivations of the G-module A, it remains to show that α is order-preserving. Let $g,h\in G$ with $h\leqslant g$. Then $(g,\mathbf{d}(h))\in S_2$, so

$$\alpha(g\otimes \mathbf{d}(h))=(\alpha(g)|0_{\mathbf{d}(h)})\cdot \mathbf{d}(h)+\alpha(\mathbf{d}(h)).$$

Hence $\alpha(h)=(\alpha(g)|0_{\mathbf{d}(h)})$. We have therefore shown that the 1-cocycles of our complex are precisely the derivations of the G-module A.

We now calculate the 1-coboundaries of the complex. Let ε be an element of $X^0(G,A)$. Then $\varepsilon : G_o\longrightarrow\mathcal{A}$ is a $C(G)_o$-morphism such that $\varepsilon(e)=\mathcal{A}(f,e)(\varepsilon(f))$ for all $e,f\in G_o$

such that $e \leqslant f$. In the G-module corresponding to $\mathcal{A}$, this last condition becomes $\varepsilon(e) = (\varepsilon(f)|0_e)$; that is ε is order-preserving. Now let $g \in G$, then

$$\partial^0(\varepsilon)(g) = \mathcal{A}(\mathbf{r}(g), g)(\varepsilon(\mathbf{r}(g))) - \varepsilon(\mathbf{d}(g)).$$

Thus in G-module notation, the 1-coboundaries are functions $\partial^0\varepsilon : G \longrightarrow A$ such that

$$\partial^0\varepsilon(g) = \varepsilon(\mathbf{r}(g)) \cdot g - \varepsilon(\mathbf{d}(g)).$$

Where $\varepsilon : G_o \longrightarrow A$ is an order-preserving function such that $\varepsilon(e) \in A_e$. Hence the 1-coboundaries of the complex are precisely the principal derivations of the G-module A.

We have therefore shown that the group $H^2(G, A) = \mathrm{Ker}(\partial^1)/\mathrm{Im}(\partial^0)$ is precisely the quotient group of derivations modulo principal derivations. By Theorem 10.15, this group is in one-to-one correspondence with the set of A-conjugacy classes of splittings of the canonical split extension $(i, G \ltimes A, \pi)$.

6 The second cohomology group We now calculate the second cohomology group of the cochain complex $X^\bullet(G, A)$.

To calculate the 2-cocycles, consider the function

$$\partial^2 : X^2(G, A) \longrightarrow X^3(G, A).$$

Let $\alpha \in X^2(G, A)$, and $(g, h, k) \in S_3$. Then

$$\partial^2(\alpha)(g, h, k) = \mathcal{A}(\mathbf{d}(h), k)(\alpha(g, h)) - \alpha(g, h \otimes k) + \alpha(g \otimes h, k) - \alpha(h, k).$$

If α is a 2-cocycle (an element of $\mathrm{Ker}(\partial^2)$), then, in G-module notation,

$$(\alpha(g, h)|0_{\mathbf{r}(k)}) \cdot k + \alpha(g \otimes h, k) = \alpha(h, k) + \alpha(g, h \otimes k).$$

Hence the 2-cocycles are precisely the factor sets of the G-module A.

Let $\varepsilon \in X^1(G, A)$ and $(g, h) \in S_2$. Then

$$\partial^1(\varepsilon)(g, h) = \mathcal{A}(\mathbf{d}(g), h)(\varepsilon(g)) - \varepsilon(g \otimes h) + \varepsilon(h).$$

So in the G-module A, the 2-coboundaries are functions

$$\partial^1\varepsilon(g, h) = (\varepsilon(g)|0_{\mathbf{r}(h)}) \cdot h - \varepsilon(g \otimes h) + \varepsilon(h)$$

where $\varepsilon : G \longrightarrow A$ is such that $\varepsilon(g) \in A_{\mathbf{d}(g)}$. Hence the 2-coboundaries are precisely the principal factor sets of A.

We have therefore proved that the second cohomology group $H^2(G, A)$ is the quotient group of factor sets modulo principal factor sets. By Theorem 10.25, there is a bijection between this group and the set of all equivalence classes of extensions of A by G. Hence result. ■

Now suppose that G is an ordered groupoid (which may or may not possess a maximal identity). In order to apply the previous result to this situation we adjoin a maximal identity as follows. Let $1 \notin G$ be a symbol, and let $\{1\}$ be the singleton group consisting only of the identity 1. Define

$$G^I = G \sqcup \{1\},$$

the disjoint union of the groupoids G and $\{1\}$. The order on G is extended to an order on G^1 by defining

$$e \leqslant 1, \quad \text{for all } e \in G_o.$$

Note that $g \otimes 1 = g = 1 \otimes g$, for all $g \in G$.

Let A be a G-module. We define a G^1-module A^0. Let

$$A^0 = A \sqcup \{0_1\},$$

where $0_1 \notin A$ is a symbol and $\{0_1\}$ is a singleton group. Now $A = \bigsqcup_{e \in G_o} A_e$ is a presheaf of groups over G_o. If we write $A_1 = \{0_1\}$, then it is clear that $A^0 = \bigsqcup_{e \in G_o^1} A_e$, is a presheaf of groups over G_o^1. We extend the order on A to an order on A^0 by defining

$$0_e \leqslant 0_1, \quad \text{for all } e \in G_o.$$

That is, 0_1 is a maximal identity in A^0. The action of G on A is extended to an action of G^1 on A^0 as

$$0_1 \cdot 1 = 0_1.$$

It is clear that A^0 is indeed a G^I-module

Theorem 10.27 *Let G be an ordered groupoid and A a G-module. The second cohomology group $H^2(G^I, A^0)$ is in bijective correspondence with the set of congruence classes of extensions of A by G.*

PROOF. Since G^1 has a maximal identity, we can apply Theorem 10.26, to obtain the cohomology group $H^2(G^1, A^0)$. The 2-cocycles of the cochain complex constructed in Theorem 10.26 are precisely the factor sets of A^0. We shall prove that the factor sets of A^0 are in bijective correspondence with the factor sets of A.

Let ζ be a factor set for the G-module A. That is a function $\zeta : S_2(G) \longrightarrow A$ such that $\zeta(g,h) \in A_{\mathbf{d}(h)}$ and

$$(\zeta(g,h)|0_{\mathbf{r}(k)}) \cdot k + \zeta(g \otimes h, k) = \zeta(g, h \otimes k) + \zeta(h,k),$$

for all $(g, h, k) \in S_3(G)$.

We shall extend ζ to a factor set for the G^1-module A^0. Clearly the only 2-staircases over G^1 which involve 1 are $(1, 1)$ and $(1, g)$, for any $g \in G$. Define

$$\zeta^1 : S_2(G^1) \longrightarrow A^0 \quad \text{by} \quad \begin{cases} \zeta^1(g, h) = \zeta(g, h) & \text{for all } (g, h) \in S_2(G) \\ \zeta^1(1, h) = 0_{\mathbf{d}(h)} & \text{for all } h \in G \\ \zeta^1(1, 1) = 0_1 & \end{cases}$$

It is straightforward to show that ζ^1 is a factor set for A^0.

Conversely, every factor set for A^0 has the form of the factor set ζ^1 above. To see why, let η be a factor set for the G^1-module A^0. Consider the 3-staircase $(1, 1, g)$. By (FS4), we have

$$(\eta(1, 1)|0_{\mathbf{r}(g)}) \cdot g + \eta(I \otimes 1, g) = \eta(1, 1 \otimes g) + \eta(1, g),$$

that is $\eta(\eta(1, 1)|0_{\mathbf{r}(g)}) \cdot g = \eta(1, g)$. But $\eta(1, 1) \in A^0_I$, so $\eta(1, 1) = 0_1$. Therefore $\eta(1, g) = 0_{\mathbf{d}(g)}$. It is now clear that the factor sets for A are in bijective correspondence with the factor sets for A^0.

We now show that the principal factor sets of A are exactly the principal factor sets of A^0. Let $\partial\varepsilon$ be a principal factor set for A. That is

$$\partial\varepsilon(g, h) = (\varepsilon(g)|0_{\mathbf{r}(h)}) \cdot h + \varepsilon(h) - \varepsilon(g \otimes h),$$

for all $(g, h) \in S_2(G)$, where $\varepsilon : G \longrightarrow A$ is a function satisfying $\varepsilon(g) \in A_{\mathbf{r}(g)}$. We extend $\partial\varepsilon$ to a principal factor set for A^0 by defining

$$\varepsilon^1 : G^1 \longrightarrow A^0 \quad \text{by} \begin{cases} \varepsilon^1(g) = \varepsilon(g), & \text{for all } g \in G \\ \varepsilon^1(1) = 0_1. & \end{cases}$$

Then $\partial\varepsilon^1$ is a principal factor set for A^0.

Conversely, if $\partial\delta$ is a principal factor set for A^0. Then $\delta(1) \in A_1$, which is a singleton group. Thus $\delta(1) = 0_1$. Hence $\partial\delta$ has the same form as $\partial\varepsilon^1$ above. It is now clear that the principal factor sets of A^0 are in bijective correspondence with the principal factor sets of A.

By Theorem 10.25, the set of congruence classes of extensions of A by G is in bijective correspondence with the quotient group of factor sets of A modulo principal factor sets. But we have just seen that this group is the group of factor sets of A^0 modulo principal factor sets of A^0. By Theorem 10.26, this group is precisely the second cohomology group $H^2(G^1, A^0)$. ■

Chapter 11

Renault's cohomology

In [25], Renault adapts the cohomology of groupoids to inverse semigroups. The resulting cohomology theory differs from the approach taken by Lausch in several important ways. In this section we describe Renault's construction of extensions and factor sets for inverse semigroups.

11.1 An alternative approach to factor sets

In this section we shall obtain an alternative characterisation of factor sets, which will be useful when we examine Renault's cohomology.

Let G be an ordered groupoid. Recall that $\mathrm{Ner}_2(G)$ is the set of composable pairs of elements of G and define $\mathrm{PNer}_2(G)$ to be the set of pairs of elements (h, g) of G, where $h \leqslant g$. The following result provides an alternative characterisation of factor sets.

Proposition 11.1 *The group of factor sets of a G-module A is in bijective correspondence with the set of pairs of functions*

$$\zeta_0 : \mathrm{Ner}_2(G) \longrightarrow A \quad \textit{and} \quad \zeta_1 : \mathrm{PNer}_2(G) \longrightarrow A$$

satisfying the following conditions:

(i) $\zeta_0(g, h) \in A_{\mathbf{d}(h)}$, *for all* $(g, h) \in \mathrm{Ner}_2(G)$.

(ii) $\zeta_1(h, g) \in A_{\mathbf{d}(h)}$, *for all* $h \leqslant g$.

(iii) *If* $(g, h, k) \in \mathrm{Ner}_3(G)$, *then* $\zeta_0(g, h) \cdot k + \zeta_0(gh, k) = \zeta_0(g, hk) + \zeta_0(h, k)$.

(iv) *If* $k \leqslant h \leqslant g$ *in* G, *then* $(\zeta_1(h, g)|0_{\mathbf{d}(k)}) = \zeta_1(k, g) - \zeta_1(k, h)$.

(v) *If* $g_1, g_2, h_1, h_2 \in G$ *with* $(h_2, h_1), (g_2, g_1) \in \mathrm{Ner}_2(G)$ *and* $h_2 \leqslant g_2, h_1 \leqslant g_1$, *then*

$$\begin{aligned}
&(\zeta_0(g_2, g_1)|0_{\mathbf{d}(h_1)}) - \zeta_0(h_2, h_1) \\
&\qquad = \zeta_1(h_2, g_2) \cdot h_1 + \zeta_1(h_1, g_1) - \zeta_1(h_2h_1, g_2g_1) - \zeta_1(\mathbf{r}(h_1), \mathbf{r}(g_1)) \cdot h_1.
\end{aligned}$$

PROOF. Let $\zeta : S_2(G) \longrightarrow A$ be a factor set. Recall that ζ satisfies the conditions

(FS1) $\zeta(g, h) \in A_{\mathbf{d}(h)}$, for all $(g, h) \in S_2(G)$.

(FS2) $(\zeta(g, h)|0_{\mathbf{r}(k)}) \cdot k + \zeta(g \otimes h, k) = \zeta(g, h \otimes k) + \zeta(h, k)$, for all $(g, h, k) \in S_3(G)$.

Define functions

$$\zeta_0 : \mathrm{Ner}_2(G) \longrightarrow A \quad \text{by} \quad \zeta_0 : (g_2, g_1) \longmapsto \zeta(g_2, g_1),$$

and

$$\zeta_1 : \mathrm{PNer}_2(G) \longrightarrow A \quad \text{by} \quad \zeta_1 : (h, g) \longmapsto \zeta(g, \mathbf{d}(h)) - \zeta(\mathbf{d}(h), \mathbf{d}(h)).$$

We show that (ζ_0, ζ_1) satisfy the conditions (i)–(vi) above.

It is immediate that (i) and (ii) hold, since $\zeta(g, h) \in A_{\mathbf{d}(h)}$, for all $(g, h) \in S_2(G)$.

Let $(g, h, k) \in \mathrm{Ner}_3(G)$. Applying (FS2) to the 3-staircase (g, h, k) gives

$$\zeta(g, h) \cdot k + \zeta(gh, k) = \zeta(g, hk) + \zeta(h, k).$$

So (iii) holds.

To see that (iv) holds, let $k \leqslant h \leqslant g$ in G. Then

$$\begin{aligned}
&(\zeta_1(h, g)|0_{\mathbf{d}(k)}) - \zeta_1(k, g) + \zeta_1(k, h) \\
&\quad = ((\zeta(g, \mathbf{d}(h)) - \zeta(\mathbf{d}(h), \mathbf{d}(h)))|0_{\mathbf{d}(k)}) - \zeta(g, \mathbf{d}(k)) \\
&\qquad\quad + \zeta(\mathbf{d}(k), \mathbf{d}(k)) + \zeta(h, \mathbf{d}(k)) - \zeta(\mathbf{d}(k), \mathbf{d}(k)) \\
&\quad = (\zeta(g, \mathbf{d}(h))|0_{\mathbf{d}(k)}) - (\zeta(\mathbf{d}(h), \mathbf{d}(h))|0_{\mathbf{d}(k)}) - \zeta(g, \mathbf{d}(k)) + \zeta(h, \mathbf{d}(k)) \\
&\quad = \zeta(\mathbf{d}(h), \mathbf{d}(k)) - (\zeta(\mathbf{d}(h), \mathbf{d}(h))|0_{\mathbf{d}(k)}) \quad \text{by Lemma 10.16(viii)} \\
&\quad = \zeta(\mathbf{d}(h), \mathbf{d}(k)) - \zeta(\mathbf{d}(h), \mathbf{d}(k)) \quad \text{by Lemma 10.16(iv)} \\
&\quad = 0_{\mathbf{d}(k)},
\end{aligned}$$

as required.

To see that (v) holds, consider the elements of G pictured below

$$\begin{array}{cc}
\xleftarrow{g_2} & \xleftarrow{g_1} \\
\Uparrow & \Uparrow \\
\xleftarrow[h_2]{} & \xleftarrow[h_1]{}
\end{array}$$

Consider

$$\begin{aligned}
&\zeta_1(h_2, g_2)\cdot h_1 + \zeta_1(h_1, g_1) - \zeta_1(h_2h_1, g_2g_1)\\
&\quad = \quad (\zeta(g_2, \mathbf{d}(h_2)) - \zeta(\mathbf{d}(h_2), \mathbf{d}(h_2)))\cdot h_1 + \zeta(g_1, \mathbf{d}(h_1))\\
&\qquad\qquad - \zeta(\mathbf{d}(h_1), \mathbf{d}(h_1)) - \zeta(g_2g_1, \mathbf{d}(h_1)) + \zeta(\mathbf{d}(h_1), \mathbf{d}(h_1))\\
&\quad = \quad \zeta(g_2, \mathbf{d}(h_2))\cdot h_1 - \zeta(\mathbf{d}(h_2), \mathbf{d}(h_2))\cdot h_1 + \zeta(g_1, \mathbf{d}(h_1)) - \zeta(g_2g_1, \mathbf{d}(h_1))\\
&\quad = \quad \zeta(g_2, \mathbf{d}(h_2))\cdot h_1 - \zeta(\mathbf{r}(h_1), h_1) + \zeta(g_1, \mathbf{d}(h_1)) - \zeta(g_2g_1, \mathbf{d}(h_1)) \quad \text{by Lemma 10.16(ii)}\\
&\quad = \quad \zeta(g_2, \mathbf{d}(h_2))\cdot h_1 - \zeta(\mathbf{r}(h_1), h_1) + (\zeta(g_2, g_1)|0_{\mathbf{d}(h_1)}) - \zeta(g_2, h_1) \quad \text{by Lemma 10.16(ix)}\\
&\quad = \quad \zeta(g_2, \mathbf{d}(h_2))\cdot h_1 - \zeta(\mathbf{r}(h_1), h_1) + (\zeta(g_2, g_1)|0_{\mathbf{d}(h_1)})\\
&\qquad\qquad - \zeta(g_2, \mathbf{r}(h_1))\cdot h_1 - \zeta(h_2, h_1) + \zeta(\mathbf{r}(h_1), h_1) \quad \text{by Lemma 10.16(x)}\\
&\quad = \quad (\zeta(g_2, g_1)|0_{\mathbf{d}(h_1)}) - \zeta(h_2, h_1)
\end{aligned}$$

Hence (v) holds.

Conversely, let

$$\zeta_0 : \mathrm{Ner}_2(G) \longrightarrow A, \quad \zeta_1 : \mathrm{PNer}_2(G) \longrightarrow A$$

be functions satisfying the conditions (i)–(v) above. We shall construct a factor set. Define

$$\zeta : S_2(G) \longrightarrow A \quad \text{by} \quad \zeta(g, h) = \zeta_1((g|\mathbf{r}(h)), g)\cdot h + \zeta_0((g|\mathbf{r}(h)), h).$$

It is clear that $\zeta(g, h) \in A_{\mathbf{d}(h)}$. We shall show that the condition (FS2):

$$(\zeta(g, h)|0_{\mathbf{r}(k)})\cdot k + \zeta(g\otimes h, k) = \zeta(g, h\otimes k) + \zeta(h, k),$$

holds for all $(g, h, k) \in S_3(G)$. We write $x = (g|\mathbf{r}(h))$, $y = (h|\mathbf{r}(k))$ and $z = (x|\mathbf{r}(y))$, as illustrated below

$$\begin{array}{ccc}
\xleftarrow{\;g\;} & & \\
\Uparrow & & \\
\xleftarrow[\;x\;]{} & \xleftarrow[\;h\;]{} & \\
\Uparrow & \Uparrow & \\
\xleftarrow[\;z\;]{} & \xleftarrow[\;y\;]{} & \xleftarrow[\;k\;]{}
\end{array}$$

The right-hand side of (FS2) is equal to

$$\zeta_1(z, g)\cdot yk + \zeta_0(z, yk) + \zeta_1(y, h)\cdot k + \zeta_0(y, k). \tag{11.1}$$

The left-hand side of (FS2) is

$$\begin{aligned}
&((\zeta_1(x, g)\cdot h + \zeta_0(x, h))|0_{\mathbf{r}(k)})\cdot k + \zeta_1(zy, xh)\cdot k + \zeta_0(zy, k)\\
&\quad = (\zeta_1(x, g)|0_{\mathbf{r}(y)})\cdot yk + (\zeta_0(x, h)|0_{\mathbf{r}(k)})\cdot k - +\zeta_1(zy, xh)\cdot k + \zeta_0(zy, k)\\
&\quad = (\zeta_1(z, g) - \zeta_1(z, x))\cdot yk + (\zeta_0(x, h)|0_{\mathbf{r}(k)})\cdot k + \zeta_1(zy, xh)\cdot k + \zeta_0(zy, k) \quad \text{by (iv)}\\
&\quad = (\zeta_1(zy, xh) + (\zeta_0(x, h)|0_{\mathbf{r}(k)}) - \zeta_1(z, x)\cdot y)\cdot k + \zeta_1(z, g)\cdot yk + \zeta_0(zy, k)\\
&\quad = (\zeta_0(z, y) + \zeta_1(y, h))\cdot k + \zeta_1(z, g)\cdot yk + \zeta_0(zy, k) \quad \text{by (v)}\\
&\quad = \zeta_0(z, yk) + \zeta_0(y, k) + \zeta_1(y, h)\cdot k + \zeta_1(z, g)\cdot yk \quad \text{by (iii)}.
\end{aligned}$$

Which is (11.1). Hence (FS2) holds, so ζ is a factor set.

We have therefore shown that every factor set ζ determines, and is determined by a pair of functions (ζ_0, ζ_1) satisfying (i)–(v). We now prove that this correspondence is bijective.

Let ζ be a factor set, and (ζ_0, ζ_1) be the pair of functions constructed from ζ. Thus

$$\zeta_0(g,h) = \zeta(g,h) \quad \text{and} \quad \zeta_1(h \leqslant g) = \zeta(g, \mathbf{d}(h)) - \zeta(\mathbf{d}(h), \mathbf{d}(h)).$$

Let $\overline{\zeta}$ be the factor set constructed from (ζ_0, ζ_1). Then

$$\begin{aligned}
\overline{\zeta}(g,h) &= \zeta_1((g|\mathbf{r}(h)), g) \cdot h + \zeta_0((g|\mathbf{r}(h)), h) \\
&= \zeta(g, \mathbf{r}(h)) \cdot h - \zeta(\mathbf{r}(h), \mathbf{r}(h)) \cdot h + \zeta((g|\mathbf{r}(h)), h) \\
&= \zeta(g, \mathbf{r}(h)) \cdot h - \zeta(\mathbf{r}(h), h) + \zeta((g|\mathbf{r}(h)), h) \quad \text{by Lemma 10.16(ii)} \\
&= \zeta(g,h) \quad \text{by Lemma 10.16(x).}
\end{aligned}$$

Conversely, given a pair of functions (ζ_0, ζ_1). Let ζ be the factor set constructed from (ζ_0, ζ_1), and let $(\overline{\zeta}_0, \overline{\zeta}_1)$ be the pair obtained from ζ. Then

$$\overline{\zeta}_0(g,h) = \zeta(g,h) = \zeta_1(g,g) \cdot h + \zeta_0(g,h).$$

But if we apply condition (iv) to $g \leqslant g \leqslant g$, we get

$$\zeta_1(g,g) = \zeta_1(g,g) - \zeta_1(g,g) = 0_{\mathbf{d}(g)}.$$

So $\overline{\zeta}_0(g,h) = \zeta_0(g,h)$. It remains to show that $\overline{\zeta}_1 = \zeta_1$. Now

$$\begin{aligned}
\overline{\zeta}_1(h \leqslant g) &= \zeta(g, \mathbf{d}(h)) - \zeta(\mathbf{d}(h), \mathbf{d}(h)) \\
&= \zeta_1(h,g) + \zeta_0(h, \mathbf{d}(h)) - \zeta_1(\mathbf{d}(h), \mathbf{d}(h)) + \zeta_0(\mathbf{d}(h), \mathbf{d}(h)) \\
&= \zeta_1(h,g) + \zeta_0(h, \mathbf{d}(h)) + \zeta_0(\mathbf{d}(h), \mathbf{d}(h)).
\end{aligned}$$

But if we apply the condition (iii), to $(h, \mathbf{d}(h), \mathbf{d}(h))$, we obtain

$$\zeta_0(h, \mathbf{d}(h)) = \zeta(\mathbf{d}(h), \mathbf{d}(h)).$$

Hence $\overline{\zeta}_1(h \leqslant g) = \zeta_1(h \leqslant g)$, as required.

■

The above result means that we can think of a factor set ζ as a pair of functions. A 'groupoid' function ζ_0 and an 'order' function ζ_1. This is the approach we shall take for the remainder of this chapter.

We saw in Proposition 10.17, each extension of an abelian ordered groupoid A by an ordered groupoid G, together with a transversal, determines a factor set of a G-module A. We now examine the relationship with our new characterisation of factor sets.

Let (ι, U, σ) be an extension of A by G, and let $l : G \longrightarrow U$ be a transversal for $\mathcal{E}$. By Proposition 10.17, there is a factor set ζ defined by

$$\iota\zeta(g, h) = l(g \otimes h)^{-1} \otimes l(g) \otimes l(h).$$

Consider the corresponding functions

$$\zeta_0 : \mathrm{Ner}_2(G) \longrightarrow A \quad \text{and} \quad \zeta_1 : \mathrm{PNer}_2(G) \longrightarrow A$$

constructed in Proposition 11.1. Now

$$\iota\zeta_0(g, h) = \iota\zeta(g, h) = l(gh)^{-1}l(g)l(h)$$

for all $(g, h) \in \mathrm{Ner}_2(G)$, and

$$\begin{aligned}
\iota\zeta_1(h, g) &= \iota(\zeta(g, \mathbf{d}(h)) - \zeta(\mathbf{d}(h), \mathbf{d}(h))) \\
&= l(g \otimes \mathbf{d}(h))^{-1} \otimes l(g) \otimes l(\mathbf{d}(h)) \otimes (l(\mathbf{d}(h))^{-1} \otimes l(\mathbf{d}(h)) \otimes l(\mathbf{d}(h)))^{-1} \\
&= l(h)^{-1} \otimes l(g) \\
&= l(h)^{-1}(l(g)|0_{\mathbf{d}(h)}).
\end{aligned}$$

Proposition 11.2 *Let $\mathcal{E}$ be an extension of an abelian ordered groupoid A by an ordered groupoid, let l be a transversal for $\mathcal{E}$ and let (ζ_0, ζ_1) be the factor set defined by*

$$\iota(\zeta_0(g, h)) = l(gh)^{-1}l(g)l(h) \quad \textit{and} \quad \iota(\zeta_1(h, g)) = l(h)^{-1}(l(g)|0_{\mathbf{d}(h)}).$$

Then

(i) *l is a functor if, and only if $\zeta_0(g, h) = 0_{\mathbf{d}(h)}$, for all $(g, h) \in \mathrm{Ner}_2(G)$.*

(ii) *l is order-preserving if, and only if $\zeta_1(h, g) = 0_{\mathbf{d}(h)}$, for all $h \leqslant g$ in G.*

Proof. To prove (i), suppose that every $\zeta_0(g, h) = 0_{\mathbf{d}(h)}$. Then, for all $e \in G_o$,

$$\zeta(e, e) = 0_e = l(e)^{-1}l(e)l(e) = l(e).$$

If $\exists gh$ in G, then $l(g)l(h) = l(gh)\zeta_0(g, h) = l(gh)$. Hence l is a functor. Conversely, if l is a functor, then $l(g)l(h) = l(gh)$. But $l(g)l(h) = l(gh)\iota(\zeta_0(g, h))$. So $\iota(\zeta_0(g, h)) = l(gh)^{-1}l(gh) = \iota(0_{\mathbf{d}(h)})$.

It is immediate that (ii) holds. ■

Now let (ζ_0, ζ_1) be a factor set for a G-module A. Consider the extension of A by G constructed in Proposition 10.20.

$$A \xhookrightarrow{i} A * G \xrightarrow{\pi} G$$

Recall that

$$G * A = \{(g, a) \in G \times A \mid a \in A_{\mathbf{r}(g)}\}.$$

If $(g, a), (b, h) \in G * A$ and $\exists gh$, then

$$(g, a)(b, h) = (gh, \zeta(g, h) + a \cdot h + b).$$

The order on $G * A$ is defined by

$$(b, h) \leqslant (a, g) \iff h \leqslant g \quad \text{and} \quad b = (a|0_{\mathbf{d}(h)}) + \zeta_1(h, g).$$

Factor sets form an abelian group under pointwise addition. The following result gives the addition for our new characterisation of factor sets.

Lemma 11.3 *Let (ζ_0, ζ_1) and (η_0, η_1) be factor sets for a G-module A. Then*

$$(\zeta_0, \zeta_1) + (\eta_0, \eta_1) = (\zeta_0 + \eta_0, \zeta_1 + \eta_1),$$

where the sums $\zeta_0 + \eta_0$ and $\zeta_1 + \eta_1$ are defined pointwise.

PROOF. Let $\zeta, \eta : S_2(G) \longrightarrow A$ be defined by

$$\zeta(g, h) = \zeta_1((g|\mathbf{r}(h)), g) \cdot h + \zeta_0((g|\mathbf{r}(h), h))$$

and

$$\eta(g, h) = \eta_1((g|\mathbf{r}(h)), g) \cdot h + \eta_0((g|\mathbf{r}(h), h)).$$

Then

$$\begin{aligned}
(\zeta + \eta)(g, h) &= \zeta(g, h) + \eta(g, h) \\
&= \zeta_1((g|\mathbf{r}(h)), g) \cdot h + \zeta_0((g|\mathbf{r}(h), h)) + \eta_1((g|\mathbf{r}(h)), g) \cdot h + \eta_0((g|\mathbf{r}(h), h)) \\
&= (\zeta_1((g|\mathbf{r}(h)), g) + \eta_1((g|\mathbf{r}(h)), g)) \cdot h + \zeta_0((g|\mathbf{r}(h), h)) + \eta_0((g|\mathbf{r}(h), h)) \\
&= ((\zeta_1 + \eta_1)((g|\mathbf{r}(h)), g)) \cdot h + (\zeta_0 + \eta_0)((g|\mathbf{r}(h), h)).
\end{aligned}$$

The result follows.

■

We conclude this section by examining principal factor sets. Let $\partial\varepsilon : S_2(G) \longrightarrow A$ be a principal factor set for A. That is

$$\partial\varepsilon(g,h) = (\varepsilon(g)|\mathbf{r}(h)) \cdot h + \varepsilon(h) - \varepsilon(g \otimes h)$$

for some function $\varepsilon : G \longrightarrow A$ such that $\varepsilon(g) \in A_{\mathbf{d}(g)}$. Let $(\partial\varepsilon_0, \partial\varepsilon_1)$ be the corresponding pair of functions given by Proposition 11.1. Then

$$\partial\varepsilon_0 : \mathrm{Ner}_2(G) \longrightarrow A$$

is given by

$$\partial\varepsilon_0(g,h) = \partial\varepsilon(g,h) = \varepsilon(g) \cdot h + \varepsilon(h) - \varepsilon(gh),$$

and the function $\partial\varepsilon_1 : \mathrm{PNer}_2(G) \longrightarrow A$ is given by

$$\begin{aligned} \partial\varepsilon_1(h \leqslant g) &= \partial\varepsilon(g, \mathbf{d}(h)) - \partial(\mathbf{d}(h), \mathbf{d}(h)) \\ &= (\varepsilon(g)|0_{\mathbf{d}(h)}) + \varepsilon(\mathbf{d}(h)) - \varepsilon(h) - \varepsilon(\mathbf{d}(h)) + \varepsilon(\mathbf{d}(h)) - \varepsilon(\mathbf{d}(h)) \\ &= (\varepsilon(g)|0_{\mathbf{d}(h)}) - \varepsilon(h). \end{aligned}$$

11.2 G-sheaves

For an inverse semigroup S, Renault defines S-sheaves of abelian groups. The following is our generalisation of Renault's definition to ordered groupoids.

Definition Let G be an ordered groupoid. A *G-sheaf* of abelian groups is a pair of functors

$$\rho : G_o \longrightarrow \mathbf{Ab} \quad \text{and} \quad \phi : G^{\mathrm{op}} \longrightarrow \mathbf{Ab}.$$

Were G_o is regarded G_o as a category in which there is a morphism $f \longrightarrow e$ if $e \leqslant f$. We write $\rho(e) = A_e$, and let 0_e denote the identity in A_e. These functors must satisfy the following conditions:

- ρ is injective on identities.
- $\rho(e) = \phi(e)$, for all $e \in G_o$.
- If $g \leqslant h$ in G, then the diagram below commutes

$$\begin{tikzcd} A_{\mathbf{r}(g)} \arrow[r, "\phi(g)"] \arrow[d, "\rho^{\mathbf{r}(g)}_{\mathbf{r}(h)}"'] & A_{\mathbf{d}(g)} \arrow[d, "\rho^{\mathbf{d}(g)}_{\mathbf{d}(h)}"] \\ A_{\mathbf{r}(h)} \arrow[r, "\phi(h)"'] & A_{\mathbf{d}(h)} \end{tikzcd}$$

Where we use G-modules, Renault uses G-sheaves. The following result tells us that we two approaches are equivalent.

Proposition 11.4 *Every G-sheaf determines and is determined by a G-module.*

PROOF. Let (ρ, ϕ) be a G-sheaf. Now ρ is a presheaf of abelian groups, so

$$A = \bigsqcup_{e \in G_o} A_e$$

is an abelian ordered groupoid, by Proposition 9.3. Where $\rho^e_f(a) = (a|0_f)$, for $f \leqslant e$ in G_o and $a \in A_e$. Since ρ is injective on identities, the assignment $e \longmapsto 0_e$ is a bijection from G_o to A_o. Let $e \xrightarrow{g} f$ be an element of G and let $a \in A_{\mathbf{r}(g)}$. We write

$$\phi(g)(a) = a \cdot g.$$

To prove that A is a G-module we need to show that the conditions (GM1)–(GM5) of Section 9.4 hold.

Since ϕ is a functor, the condition $a \cdot (gh) = (a \cdot g) \cdot h$ holds whenever $\exists gh$. Also the condition $a \cdot e$ holds for all $a \in A_e$. Hence (GM1) and (GM3) hold.

For all $g \in G$, $\phi(g)$ is a homomorphism. Therefore $\phi(g)(0_{\mathbf{r}(g)}) = 0_{\mathbf{d}(g)}$; that is $0_{\mathbf{r}(g)} \cdot g = 0_{\mathbf{d}(g)}$. So (GM4) holds. Also

$$\phi(g)(a + b) = \phi(g)(a) + \phi(g)(b)$$

for all $a, b \in A_{\mathbf{r}(g)}$. That is $(a + b) \cdot g = a \cdot g + b \cdot g$. So (GM2) holds.

Suppose that $g \leqslant h$ and let $a \in A_{\mathbf{r}(h)}$. Then

$$\phi(h)\rho^{\mathbf{r}(g)}_{\mathbf{r}(h)}(a) = \rho^{\mathbf{d}(g)}_{\mathbf{d}(h)}\phi(g)(a),$$

that is $(a|0_{\mathbf{r}(h)}) \cdot h = (a \cdot g|0_{\mathbf{d}(h)})$. So (GM5) holds.

Therefore every G-sheaf determines a G-module. Conversely, let A be a G-module. Define $\rho : G_o \longrightarrow \mathbf{Ab}$ by

$$\rho(e) = A_e \quad \text{and} \quad \rho^e_f(a) = (a|0_f),$$

for $f \leqslant e$ in G_o. Define $\phi : G^{\mathrm{op}} \longrightarrow A$ by $\phi(e) = A_e$ for all $e \in G_o$, and

$$\phi(g) : A_{\mathbf{r}(g)} \longrightarrow A_{\mathbf{d}(g)} \quad \text{by} \quad \phi(g)(a) = a \cdot g.$$

It is straightforward to show that (ρ, ϕ) is a G-sheaf.

■

11.3 Rigid extensions of ordered groupoids

Renault's definition of extensions of inverse semigroups (and hence ordered groupoids) differs from that of Section 10.1 in that he insists that every extension must have an order-preserving transversal. We now define this formally.

Let $A \overset{\iota}{\hookrightarrow} U \overset{\sigma}{\twoheadrightarrow} G$ be an extension of an abelian ordered groupoid A by an ordered groupoid G. The extension is said to be *rigid* if there is a transversal $k : G \longrightarrow U$ which is order-preserving.

Lemma 11.5 *If $\mathcal{E} = (\iota, U, \sigma)$ and $\mathcal{E}' = (\iota', U', \sigma')$ are congruent extensions of an abelian ordered groupoid A by an ordered groupoid G and $\mathcal{E}$ is rigid, then $\mathcal{E}'$ is rigid.*

PROOF. Let $\mu : \mathcal{E} \cong \mathcal{E}'$ be a congruence and let $k : G \longrightarrow U$ be an order-preserving transversal for $\mathcal{E}$. The situation is pictured below.

$$\begin{tikzcd}
& U \arrow[dd, "\mu"] \arrow[dr, two heads, "\sigma"] & \\
A \arrow[ur, hook, "\iota"] \arrow[dr, hook, "\iota'"'] & & G \arrow[ul, bend left, "k"] \\
& U' \arrow[ur, "\sigma'"'] &
\end{tikzcd}$$

Put $k' = \mu k$, then

$$\sigma' k'(g) = \sigma' \mu k(g) = \sigma k(g) = g \quad \text{for all } g \in G$$

and

$$k'(e) = \mu k(e) = \mu(\sigma|_{U_o})^{-1}(e) = (\sigma'|_{G_o})^{-1}(e) \quad \text{for all } e \in G_o$$

So k' is a transversal for $\mathcal{E}'$. It is clear that k' is order-preserving.

■

The following is now immediate.

Proposition 11.6 *Congruence of rigid extensions is an equivalence relation. For each abelian ordered groupoid A and ordered groupoid G, the set of congruence classes of rigid extensions of A by G is a subset of the set of congruence classes of extensions of A by G.*

■

11.4 Rigid factor sets

We have seen that factor sets correspond to extensions. In this section, we examine the factor sets which correspond to rigid extensions.

Definition Let G be an ordered groupoid. A *rigid factor set* for a G-module A is a function

$$\zeta : \mathrm{Ner}_2(G) \longrightarrow A$$

satisfying the following conditions:

(RFS1) $\zeta(g,h) \in A_{\mathbf{d}(h)}$, for all $(g,h) \in \mathrm{Ner}_2(G)$.

(RFS2) If $(g,h,k) \in \mathrm{Ner}_3(G)$, then $\zeta(g,h) \cdot k + \zeta(gh,k) = \zeta(g,hk) + \zeta(h,k)$.

(RFS3) If $(h_1,h_2),(g_1,g_2) \in \mathrm{Ner}_2(G)$ with $h_1 \leqslant g_1$ and $h_2 \leqslant g_2$. Then $\zeta(h_1,h_2) \leqslant \zeta(g_1,g_2)$.

We shall show that rigid extensions correspond to rigid factor sets, but first we show that rigid factor sets are indeed factor sets.

Proposition 11.7 *Let $\zeta : \mathrm{Ner}_2(G) \longrightarrow A$ be a rigid factor set for a G-module A. Define*

$$0_1 : \mathrm{PNer}_2(G) \longrightarrow A \quad by \quad 0_1(g \leqslant h) \longmapsto 0_{\mathbf{r}(g)}.$$

Then $(\zeta, 0_1)$ is a factor set. Conversely, if (ζ_0, ζ_1) is a factor set, and $\zeta_1 = 0_1$. Then ζ_0 is a rigid factor set.

Proof. To prove that $(\zeta, 0_1)$ is a factor set, we show that it satisfies conditions (i)–(v) of Proposition 11.1. By (RFS1) $\zeta(g,h) \in A_{\mathbf{d}(g)}$, so (i) holds.

It is immediate that (ii) holds.

The condition (RFS2) is the same as condition (iii) of Proposition 11.1.

It is immediate that 0_1 satisfies (iv).

To see that (v) holds, let $g_1, g_2, h_1, h_2 \in G$ with $\exists h_2h_1$, $\exists g_2g_1$, $h_2 \leqslant g_2$ and $h_1 \leqslant g_1$. Then

$$0_1(h_2,g_2) \cdot h_1 + 0_1(g_1,h_1) - 0_1(g_2g_1,h_2h_1) = 0_{\mathbf{d}(h_2)} \cdot h_1 + 0_{\mathbf{d}(h_1)} - 0_{\mathbf{d}(h_1)} = 0_{\mathbf{d}(h_1)}$$

and

$$(\zeta(h_2,h_1)|0_{\mathbf{r}(g_2)}) - \zeta(g_2,g_1) = \zeta(g_2,g_1) - \zeta(g_2,g_1) = 0_{\mathbf{d}(h_1)}.$$

Now let $(\zeta_0, 0_1)$ be a factor set. We show that ζ_0 is a rigid factor set. It is immediate that (RFS1) and (RFS2) hold. Let $g_1, g_2, h_1, h_2 \in G$, with $\exists g_2g_1, \exists h_2h_1, h_1 \leqslant g_1$ and $h_2 \leqslant g_2$. The condition (v) of Proposition 11.1 becomes

$$0_{\mathbf{d}(h_2)} \cdot h_1 + 0_{\mathbf{d}(h_1)} - 0_{\mathbf{d}(h_1)} = (\zeta_0(g_2,g_1)|0_{\mathbf{d}(h_1)}) - \zeta(h_2,h_1),$$

that is

$$(\zeta_0(h_2,h_1)|0_{\mathbf{r}(g_2)} = \zeta_0(g_2,g_1).$$

Hence (RFS3) holds. ■

From now on we shall identify each rigid factor set ζ with its corresponding factor set $(\zeta, 0_1)$. Let $RZ^2(G, A)$ denote the set of rigid factor sets of a G-module A. It is straightforward to show that, under pointwise addition, $RZ^2(G, A)$ is a subgroup of $Z^2(G, A)$.

Proposition 11.8 *Let $\mathcal{E}$: $A \overset{\iota}{\hookrightarrow} U \overset{\sigma}{\twoheadrightarrow} G$ be a rigid extension of ordered groupoids and let $k : G \longrightarrow U$ be an order-preserving transversal for $\mathcal{E}$. The factor set constructed from $\mathcal{E}$ and k according to Proposition 10.17 is a rigid factor set.*

Conversely, let ζ be a rigid factor set for a G-module A. The extension of A by G constructed from ζ according to Proposition 10.20 is a rigid extension.

Proof. Let (ζ_0, ζ_1) be the factor set for $\mathcal{E}$ and k constructed in Proposition 10.17. Then $\zeta_1 = 0_1$ by Lemma 10.18 (ii), and so the factor set is rigid by Proposition 11.1.

Now suppose that ζ is a rigid factor set. The extension constructed from ζ using Proposition 10.20 is $A \overset{i}{\hookrightarrow} G * A \overset{\pi}{\twoheadrightarrow} G$. Where

$$G * A = \{(g, a) \in G \times A \mid \mathbf{d}(a) = 0_{\mathbf{d}(g)}\},$$

with product

$$(g, a) + (h, b) = (gh, \zeta(g, h) + a \cdot h + b)$$

and order $(a, g) \leqslant (b, h)$ if $g \leqslant h$ and $a = (b|0_{\mathbf{r}(g)}) + 0_1(g, h) = (b|0_{\mathbf{r}(g)})$, that is $a \leqslant b$. The ordered functor π is the canonical projection map. Define $k : G \longrightarrow A * G$ by $k(g) = (g, 0_{\mathbf{d}(g)})$, then k is clearly order-preserving and $\pi k = \mathrm{Id}_G$. Hence the extension is rigid.

■

We have therefore shown that each rigid factor set determines, and is determined by an extension together with an order-preserving transversal. We now consider the effect of changing the choice of transversal.

Let $\mathcal{E}$ be a rigid extension of an abelian ordered groupoid A by an ordered groupoid G. Let k and k' be order-preserving transversals for $\mathcal{E}$. Denote the corresponding rigid factor sets by ζ and ζ' respectively. Define

$$\varepsilon : G \longrightarrow A \quad \text{by} \quad \iota\varepsilon(g) = k(g)^{-1}k'(g),$$

so ε is order-preserving and $\varepsilon(g) \in A_{\mathbf{d}(g)}$. We saw in Section 10.3 that the factor sets $(\zeta, 0_1)$ and $(\zeta', 0_1)$ differ by the principal factor set $(\partial\varepsilon_0, \partial\varepsilon_1)$ where

$$\partial\varepsilon_0(g, h) = \varepsilon(g) \cdot h + \varepsilon(h) - \varepsilon(gh)$$

and

$$\partial\varepsilon_1(h \leqslant g) = \varepsilon(g|\mathbf{d}(h)) - \varepsilon(h) = 0_{\mathbf{d}(h)}.$$

By Proposition 11.1, $\partial\varepsilon$ is a rigid factor set. Generally, we define a *rigid principal factor set* to be a function

$$\partial\varepsilon : \mathrm{Ner}_2(G) \longrightarrow A \quad \text{given by} \quad \partial\varepsilon(g,h) = \varepsilon(g)\cdot h + \varepsilon(h) - \varepsilon(gh),$$

where $\varepsilon : G \longrightarrow A$ is order-preserving and $\varepsilon(g) \in A_{\mathbf{d}(g)}$.

We have therefore shown that the effect of changing the choice of order-preserving transversal has the effect of modifying the corresponding rigid factor set by a rigid principal factor set. Hence we arrive at the following result.

Theorem 11.9 *The set of congruence classes of rigid extensions of an abelian ordered groupoid A by an ordered groupoid G is in bijective correspondence with the quotient group of rigid factor sets modulo rigid principal factor sets.*

■

11.5 The cochain complex

We conclude this section by looking at the cochain complex Renault uses for his cohomology. In Proposition 7.14 we saw that any simplicial set gives rise to a cochain complex. So for any ordered groupoid G and G-module A, we can use the simplicial set $\mathrm{Ner}(G)$ to obtain a cochain complex, the n-cochains of which are G_o-functions from $\mathrm{Ner}_n(G)$ to A. This cochain will give the cohomology of the groupoid G with the order playing no part in the construction. To compensate for this Renault demands that each n-cochain $\phi : \mathrm{Ner}_n(G) \longrightarrow A$ has the following order preserving property:

$$\phi(g_n, g_{n-1}, \ldots, g_1) \leqslant \phi(h_n, h_{n-1}, \ldots, h_1) \quad \text{if each } g_i \leqslant h_i.$$

Let $C^n(G,A)$ denote the group of such n-cochains under pointwise addition. Renault's complex is

$$C^0(G,A) \xrightarrow{\partial^0} C^1(G,A) \xrightarrow{\partial^1} C^2(G,A) \xrightarrow{\partial^2} C^3(G,A) \longrightarrow \cdots \qquad (11.2)$$

where

$$\partial^0\phi(g) = \phi(\mathbf{d}(g)) - \phi(\mathbf{r}(g))\cdot g$$

and

$$\begin{aligned}\partial^n(\phi)(g_n,\ldots,g_1) = \phi(g_n,\ldots,g_2)\cdot g_1 \quad &+ \quad \sum_{i=1}^{n-1}\phi(g_n,\ldots,g_{i+1}g_i,\ldots,g_1)\\ &+ \quad (-1)^n\phi(g_{n-1},\ldots,g_1).\end{aligned}$$

The n^{th} cohomology group of the complex is $\text{Ker}(\partial^n)/\text{Im}(\partial^{n-1})$.

Let $\varepsilon \in C^1(G, A)$. Thus $\varepsilon : G \longrightarrow A$ is an order-preserving function such that $\varepsilon \in A_{\mathbf{d}(g)}$. Then

$$\partial^1 \varepsilon(g, h) = \varepsilon(g) \cdot h - \varepsilon(gh) + \varepsilon(h).$$

So the 2-coboundaries of the complex (11.2) are precisely the rigid coboundaries of the G-module A.

To see that the 2-cocycles of the complex are the rigid factor sets of A consider the function ∂^2 given by

$$\partial^2 \phi(g, h, k) = \phi(g, h) \cdot k - \phi(g, hk) + \phi(gh, k) - \phi(h, k).$$

A function $\zeta : \text{Ner}_2(G) \longrightarrow A$ is an element of $\text{Ker}(\partial^2)$ if it satisfies the following conditions:

(i) If $\exists gh$ in G, then $\zeta(g, h) \in A_{\mathbf{d}(g)}$.

(ii) If $(g, h, k) \in \text{GNer}_3(\text{G})$, then $\zeta(g, h) \cdot k + \zeta(gh, k) = \zeta(g, hk) + \phi(h, k)$.

(iii) If $(h_2, h_1), (g_2, g_1) \in \text{GNer}_2(\text{G})$ with $h_1 \leqslant g_1$ and $h_2 \leqslant g_2$ then $\zeta(h_2, h_1) \leqslant \zeta(g_2, g_1)$.

Thus the 2-cocycles of the complex (11.2) are precisely the rigid factor sets of the G-module A.

Combining the above argument with Theorem 11.9 yields the following.

Theorem 11.10 *The set congruence classes of rigid extensions of an abelian ordered groupoid A by an ordered groupoid G is in bijective correspondence with the second cohomology group of the cochain complex (11.2).*

■

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