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Stacks and Formal Maps of Crossed Modules

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Contents

Abstract

If X is a topological space then there is an equivalence between the category, $\pi_1(X) - \mathbf{Set}$, of actions of the fundamental group of X on sets, and the category of covering spaces on X. Moreover the latter is also equivalent to the category of locally constant sheaves on X.

Grothendieck has conjectured that this should be the '$n = 1$' case of a result which is true for all n, and it is the '$n = 2$' case we look at in this thesis.

The desired generalisation should replace actions of the group $\pi_1(X)$ (which is an algebraic model for the 1-type of X) by actions of a crossed module (i.e., by an algebraic model for the 2-type) on groupoids; 'locally constant sheaves of sets' by 'locally constant stacks of groupoids'; and 'covering space' by a locally trivial object whose fibres are groupoids.

This last object we handle using the machinery of simplicial fibre bundles (twisted Cartesian products) and formal maps, building a simplicial object, $Z(\lambda)$, where the fibre is now a (nerve of) a groupoid. To interpret $Z(\lambda)$ as a stack, we show that just as sheaves on X are equivalent to étale spaces, we can define a notion of 2-étale space corresponding to stacks and show that from $Z(\lambda)$ we can construct a locally constant stack on X.

1 Introduction

1.1 Introduction and Motivation

If X is a topological space, then covering spaces on X correspond to actions of the fundamental group, $\pi_1(X)$, on sets (this is sometimes called the 'Galois-Poincaré correspondence'). Since 'fundamental group' can be generalised to a higher dimensional algebraic object, Grothendieck conjectured in his famous Pursuing Stacks [25] that this correspondence generalises to higher dimensions in the following (approximate) way.

First we note that the fundamental group, $\pi_1(X)$, classifies (i.e., is an algebraic version of) the 1-type of the space X, and 'action' means a functor into **Set**, the category of 0-types (sets). On the other side of the original correspondence, 'covering spaces' (i.e., locally trivial bundles $E \longrightarrow X$ whose fibre is a set) are equivalent to locally trivial sheaves of sets i.e., a '1-stack of 0-types'.

Grothendieck's conjecture is that we can generalise the above to get a correspondence between 'locally trivial' n-stacks of $(n-1)$-types and actions of an algebraic model for the n-type of X on structures that model $(n-1)$-types. Of course this has all been lacking in details: part of the problem (the hardest part?) is defining all these things correctly. For that we need higher dimensional category theory—for example 'correspondence' will presumably be some form of n-equivalence of (some form of) n-categories. In this thesis we look only at the case $n = 2$: we know from [37] that the 2-type for a space can be modelled by a crossed module (up to weak equivalence), and what Grothendieck called a '2-stack in 1-types' is usually just called a 'stack of Groupoids' (or just simply a stack in an algebraic geometry context). Thus we should be looking at actions of crossed modules on groupoids (i.e., functors from the crossed module, considered as a 2-group, to the category **Grpoids**).

Note that this is linked to the idea of categorification ([2, 1]) namely the replacing of sets by groupoids and groupoids by 2-groupoids.

There is a standard construction which, given a group G (or more precisely, given a set of generators and relations for G) gives us a space X with $\Pi_1(X) = G$. Simply take a bouquet of circles, one for each generator, and add in a disk for each relation, attaching it according to its generators. Thus we can start with the group rather than needing to first have a topological space. Similarly, any crossed module is the fundamental crossed module of some space X, so instead of starting with X and trying to classify actions of its fundamental crossed module we can instead take the (notationally less complicated) approach of starting with a crossed module $M = \left(C \xrightarrow{\partial} P\right)$ (since we can always construct an appropriate X if we need to).

However, a given X only determines M up-to weak equivalence. If X is given as a CW-complex, then the crossed module we are interested in is given by $M = \left(\pi_2(X, X^{(1)}) \xrightarrow{\partial} \pi_1(X^{(1)})\right)$, but now if X' is a second decomposition of X as a CW-complex, then the corresponding M' is weakly equivalent to M rather than isomorphic. (In the original Galois-Poincaré case the relevant groups are isomorphic: $\pi_1(X) \cong \pi_1(X')$, so we have replaced isomorphism with equivalence here). Thus we need to regard M as only being defined up-to weak equivalence. (See section 5.2.)

Finally we note that general topological spaces are difficult to use, or at least it is easier to use a more algebraic model, namely simplicial sets. (Quillen's theory of model categories shows that the homotopy theory one gets from simplicial sets is equivalent to that coming from nice topological spaces [44, 16, 29] so there should not be a problem to make the move from topology to the simplicial world here.)

Describing the Galois Poincaré correspondence in a simplicial setting is not new, but it does not appear to have been written down in one place in a modern, categorical form. So we first give an exposition of the classical correspondence, then look at some generalisations, getting some of the way to an answer.

For the analogue to 'actions of the group $\pi_1(X)$' we consider actions of our crossed module M on a groupoid Y, which we model as simplicial maps $\operatorname{Ner}\mathfrak{X}(M)_{\mathrm{v}} \longrightarrow \operatorname{aut}(\operatorname{Ner} Y)$. Here $\operatorname{Ner}\mathfrak{X}(M)_{\mathrm{v}}$ is the 'vertical nerve' of M (see section 5.1.2) and $\operatorname{aut}(\operatorname{Ner} Y)$ is the simplicial set of automorphisms of the nerve of Y (see section 2.2.2). In chapter 7 and section 8.3.5 we just consider the case where $Y = \operatorname{Ner}\mathfrak{X}(M)_{\mathrm{v}}$ and the actions comes from that of M on itself by multiplication, but for more general actions we could perform similar constructions without difficulty. For the generalisation of 'covering space' we fix a cover $\mathcal{U}$ of X: our 'higher dimensional covering space' should be isomorphic to a trivial bundle over the sets U_i in $\mathcal{U}$ (i.e., over U_i it should be a product of U_i with the nerve of the groupoid Y), and these isomorphisms should be compatible over double intersections U_{ij}. To get such an object, we use the theory of simplicial fibre bundles (TCPs, chapter 4): we apply the $\overline{W}$ construction to our simplicial group $\operatorname{Ner}\mathfrak{X}(M)_{\mathrm{v}}$, to get a simplicial set $\overline{W}(M)$ with a universal fibre bundle $W(M)$ (which actually depends on the action of M on Y). We can interpret these as simplicial étale spaces on X by taking Cartesian products $X \times \overline{W}(M)$ and putting the discrete topology on $\overline{W}(M)$ (and similarly for $W(M)$). The cover $\mathcal{U}$ also gives a simplicial étale space through the well-known Čech nerve construction (section 6.2), and so from a simplicial formal map $\lambda\colon \check{C}(\mathcal{U}) \longrightarrow \operatorname{Ner}\mathfrak{X}(M)_{\mathrm{v}}$ we can induce, by pullback, an object $Z(\lambda)$ over $\check{C}(\mathcal{U})$. In section 7.3.1 we see that this is a reasonable simplicial analogue to a covering space in the case where $M = 1 \longrightarrow P$, and in section 7.3.2 we find it has the properties we would expect for the desired 'higher dimensional covering space' where the fibre is the Y on which M is acting. To interpret this as a stack, we generalise, in section 8.2, the relationship between sheaves on X and étale spaces to a 2-equivalence between stacks and what we call 2-bundles on X, and in section 8.3 we use this to reinterpret $Z(\lambda)$ as a locally constant stack on X.

This is not quite a complete '$n = 2$' version of the Galois-Poincaré correspondence since we do not give a complete description of the 2-category structure on the $Z(\lambda)$ objects. (However, the description of morphisms of simplicial fibre bundles in Theorem 4.1.5.54

is a partial result in this direction.) We also note that since simplicial sets form a simplicially enriched category, we have a simplicial set of maps between fibre bundles and so conjecture that the most general statement for even the $n = 2$ case will need a simplicial enrichment on the 2-category of stacks.

2 Simplicial Sets

Much of this chapter is standard material which we shall need later. Mostly things are included because there is no one reference that includes them using modern (i.e., categorical) notation. We should note that we are writing composition in the 'opposite' way to normal, so when f, g are some objects that may be composed then $f \# g$ means we have composed them in the order 'f then g'; if f and g are 2-cells (or higher dimensional objects) then $f \#_i g$ means to join f and g along a common i-cell, again with f 'first', so $f \# g$ can also be written $f \#_0 g$.

2.1 Simplicial Sets

2.1.1 The topologists' Δ

We write Δ for 'the topologists' Δ', meaning the skeletal category of non-empty totally ordered finite sets and non-decreasing maps, i.e., objects are the non-empty finite totally ordered sets $[n] = \{0 < 1 < \cdots < n\}$ (one for each $n \geqslant 0$), and morphisms are order-preserving functions.

From [35] (but noting that our Δ is his Δ^+) we have the following description of morphisms in Δ. Let $\delta_i \colon [n-1] \rightarrowtail [n]$ be the injective map which 'omits i from its image', and let $\sigma_i \colon [n+1] \twoheadrightarrow [n]$ be the surjective map which 'doubles i in its image', e.g., for $n = 1$, $\sigma_1 \colon [2] \longrightarrow [1]$ sends 0 to 0 and both 1 and 2 to 1. Every map $t \colon [m] \longrightarrow [n]$ can be uniquely reduced to 'normal form'

$$t = \sigma_{j_h} \# \cdots \# \sigma_{j_1} \# \delta_{i_k} \# \ldots \# \delta_{i_1} \qquad (2.1.1.1)$$

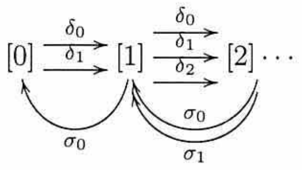
with $m - h + k = n$, $n \geqslant i_1 > \cdots > i_k \geqslant 0$, and $0 \leqslant j_1 < \cdots < j_h \leqslant m - 1$ using the following identities

$$\delta_i \# \delta_j = \delta_j \# \delta_{i+1} \qquad j \leqslant i \qquad (2.1.1.2)$$

$$\sigma_i \# \sigma_j = \sigma_{j+1} \# \sigma_i \qquad j \geqslant i \qquad (2.1.1.3)$$

$$\delta_i \# \sigma_j = \begin{cases} \sigma_{j-1} \# \delta_i & i < j \\ \mathrm{id} & i = j \\ \mathrm{id} & i = j+1 \\ \sigma_j \# \delta_{i-1} & i > j+1 \end{cases} \qquad (2.1.1.4)$$

This gives the following picture of the (first few) generating morphisms of Δ:



(2.1.1.5)

The category, **SSet**, of *simplicial sets* is the presheaf category $[\Delta^{\mathrm{op}}, \mathbf{Set}]$; we refer to morphisms of simplicial sets as *simplicial maps*—they are of course just natural transformations.

More generally for any category $\mathcal{C}$, we have a category $\mathrm{simp}(\mathcal{C})$ of functors from Δ^{op} to $\mathcal{C}$. Apart from $\mathcal{C} = \mathbf{Set}$ we will also be interested in simplicial groups ($\mathrm{simp}(\mathbf{Grp})$), and (special kinds of) simplicial groupoids.

2.1.2 Faces and Degeneracies

From (2.1.1.1) we see that $X \in \mathbf{SSet}$ is determined by the images under X of the following data from Δ.

- Objects $[n] \in \Delta$. We write $X_n := X([n])$ and call these sets the *n-simplices of X*.
- The injective maps $\delta_i \colon [n-1] \longrightarrow [n]$. We write $X(\delta_i) = d_i^{X_\bullet}$ (or just d_i if context allows) and call d_i a *face map for X*.
- The surjective maps $\sigma_i \colon [n+1] \longrightarrow [n]$. We write $X(\sigma_i) = s_i^{X_\bullet}$ (or just s_i) and call s_i a *degeneracy map for X*.

In view of the notation for the set of n-simplices, it is common to use a notation such as $X_\bullet$ (instead of just X) for a simplicial set. Of course we cannot just choose arbitrary functions for d_i and s_i: the basic simplicial morphisms δ_i and σ_i satisfy certain equations (as generated by the basic identities shown in (2.1.1.2)), and the functor X must preserve these equations; the relations satisfied by d_i and s_i are called the *simplicial identities* and are as follows [11, 40, 14, 35]

$$d_j \mathbin{\#} d_i = d_{i+1} \mathbin{\#} d_j \qquad j \leqslant i \qquad (2.1.2.1)$$

$$s_j \mathbin{\#} s_i = s_i \mathbin{\#} s_{j+1} \qquad j \geqslant i \qquad (2.1.2.2)$$

$$s_j \mathbin{\#} d_i = \begin{cases} d_i \mathbin{\#} s_{j-1} & i < j \\ \mathrm{id} & i = j \\ \mathrm{id} & i = j+1 \\ d_{i-1} \mathbin{\#} s_j & i > j+1 \end{cases}. \qquad (2.1.2.3)$$

2.1.3 The standard n-simplex $\Delta[n]$

The representable functor $\Delta[n] = \mathrm{Hom}(-, [n]) \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is called the *standard n-simplex*. We can look at these in more detail since they are important. The 0-simplex $\Delta[0]$ has $(\Delta[0])_m = \Delta(m, 0) = \{\star\}$, a 1-element set for all m. This is clearly the terminal object in **SSet** (this is also true for categorical reasons: $[0]$ is terminal in Δ, and the Yoneda embedding preserves limits).

An element $\sigma \in \Delta[n]_m = \Delta(m, n)$ can be identified with the $(m+1)$-element list $\sigma(0)\sigma(1)\ldots\sigma(m)$, where the symbols $\sigma(i)$ are increasing and each $0 \leqslant \sigma(i) \leqslant n$. If we let $m = 0$ we have that the 0-simplices of $\Delta[n]$ are just the n singletons, $0, 1, \ldots, n$.

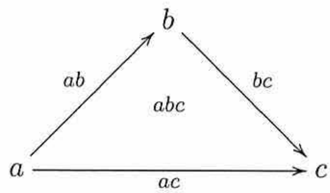
The list $d_i(\sigma)$ has the ith element removed and $s_i(\sigma)$ has the element in position i doubled (where we start counting at zero), so $s_1(012) = 0112$. Hence every simplex in $\Delta[n]$ can be obtained by applying some combination of the face and degeneracy operators to the n-simplex $\mathrm{id}_n = 0123\ldots n$; this list corresponds to the identity map $(\mathrm{id}_n \colon [n] \longrightarrow [n])$ in $\Delta[n]_n = \Delta(n, n)$.

We saw in section 2.1.2 that d_i and s_i must satisfy the simplicial identities: if we think of d_i as "delete the ith element of a list", and s_i as "double the ith element of a list" then we can work out all the identities that d_i and s_i must satisfy, for example $d_1 d_0 = d_0 d_0$ because both sides delete the first two entries in the list.

Geometrically, the identities for the d_i say how, given an n-simplex t, we can fit its faces (i.e., all the $d_i(t)$) together. A 0-simplex has no faces, it is just a vertex, and a 1-simplex ab ($0 \leqslant a \leqslant b \leqslant n$) has two vertices a and b which we call vertex number 0 and 1 respectively, numbering by the position in the list ab. The faces are $d_0(ab) = b$ and $d_1(ab) = a$. Thus $d_i(ab)$ is the vertex that does not contain the ith vertex, i.e., *d_i is the face opposite vertex i.* Thus we draw our 1-simplex as

$$a \xrightarrow{ab} b \tag{2.1.3.1}$$

with the arrow pointing from vertex 0 to vertex 1. Similarly the 2-simplex abc has faces $d_0(abc) = bc$, $d_1(abc) = ac$, and $d_2(abc) = ab$. These faces fit together to give a triangle, because, for example $d_0 d_2(abc) = b = d_1 d_0(abc)$, so we think of our 2-simplex as the "filled-in triangle"



. (2.1.3.2)

We can continue, drawing a 3-simplex as a tetrahedron, and a 4-simplex as the analogous object in 4 dimensions.

So far we have been drawing individual m-simplices from some $(\Delta[n])_m$, but the same pictures illustrate the whole simplicial set $\Delta[n]$ itself. For example, $\Delta[2]$ has a 2-simplex 012 which looks like (2.1.3.2), but every m-simplex in $\Delta[2]$ is obtained from this one by applications of d_i or s_i: we can think of (2.1.3.2) as illustrating the non-degenerate elements of $\Delta[2]$. (A simplex is *degenerate* if it is equal to $s_i(t)$ for some i and t). Similarly, $\Delta[1]$ can be adequately represented by the picture (2.1.3.1).

2.1.4 Geometric Interpretation

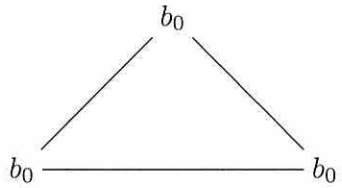
The Yoneda lemma for **SSet** says that

$$\mathbf{SSet}(\Delta[n], X_\bullet) \cong X_n \tag{2.1.4.1}$$

so we may think of X_n as 'n-simplices in X'. So for example a 2-simplex $x \in X_2$ can be drawn exactly like (2.1.3.2), but a, b, c are elements of X_0, and we replace the edges ab, bc, ac with elements of X_1.

Vertices and generalised vertices

Elements of X_0 are called *vertices* of X. Starting with a vertex $b_0 \in X_0$ we can apply degeneracy operators to get higher dimensional degenerate simplices which 'look like b_0'. Because $[0]$ is terminal in Δ, once we get above dimension 1 it does not matter which of the s_i we apply—we always get the same result, namely a simplex with all faces 'at b_0. For example $s_0^2(b_0)$ can be pictured as



(2.1.4.2)

where each edge is degenerate. We call these iterated degenerate copies of b_0 *generalised vertices*, thus the above is a generalised vertex of dimension 2.

2.1.5 Representing Maps

Definition

The Yoneda lemma says that that for every $x \in X_n$ we have a unique simplicial map $\overline{x}\colon \Delta[n] \longrightarrow X_\bullet$ with $\overline{x}(\mathrm{id}_n) = x$. ($\mathrm{id}_n \in \Delta[n]_n$ is the identity map $\mathrm{id}_n\colon [n] \longrightarrow [n]$ in Δ.)

Explicitly, naturality of $\overline{x}$ says that for any $\sigma\colon [m] \longrightarrow [n]$ in $\Delta[n]_m$ the following diagram commutes

$$\begin{array}{ccc} \Delta[n]_n & \xrightarrow{\overline{x}} & X_n \\ {\scriptstyle \sigma\#-}\downarrow & & \downarrow{\scriptstyle X(\sigma)} \\ \Delta[n]_m & \xrightarrow[\overline{x}]{} & X_m \end{array} \tag{2.1.5.1}$$

Chasing the element $\mathrm{id}_n \in \Delta[n]_n$ shows that $\overline{x}(\sigma) = X(\sigma)(x)$ where $X(\sigma)\colon X_n \longrightarrow X_m$ is the map induced by σ (regarding X as a contravariant functor); we call this $\overline{x}$ the *representing map for x*.

Calculus of representing maps

The following properties of representing maps are more or less immediate from the defining property (2.1.5.1).

Proposition 2.1.5.2. *Let $\sigma\colon [m] \longrightarrow [n]$ be an n-simplex of $(\Delta[n])_m$.*

1. *Let $b \in B_n$ be an n-simplex of the simplicial set B. Then:*

$$\overline{b}(\sigma) = B(\sigma)(b) \tag{2.1.5.3}$$
$$\overline{b}(\delta_i) = d_i b \tag{2.1.5.4}$$
$$\overline{b}(\sigma_i) = s_i b \tag{2.1.5.5}$$
$$\overline{b}([m_1] \xrightarrow{\sigma'} [m] \xrightarrow{\sigma''} [n]) = \overline{\overline{b}(\sigma'')}\big(\sigma'\big) \tag{2.1.5.6}$$
$$\overline{f(b)} = \overline{b} \# f \quad \text{for } f\colon B \longrightarrow C \text{ a simplicial map} \tag{2.1.5.7}$$

2. *Let G by a simplicial group(oid) and $g, h, \in G_n$ (composable). Write e_n for the identity of the group G_n, and the multiplication (composition) as $\#$) then*

$$\overline{e_n}(\sigma) = e_m \tag{2.1.5.8}$$
$$\overline{g \# h}(\sigma) = \overline{g}(\sigma) \# \overline{h}(\sigma) \tag{2.1.5.9}$$

2.1.6 Nerves of Categories

As a final example of a simplicial set we introduce the *nerve of a category*. Let $\mathcal{C}$ be a category and consider the poset $[n]$ as a category. A functor $[n] \longrightarrow \mathcal{C}$ is then an n-tuple of composable maps

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \longrightarrow \cdots \xrightarrow{f_n} A_n \tag{2.1.6.1}$$

in $\mathcal{C}$ (when $n = 0$ this just means an object A_0). $\operatorname{Ner}\mathcal{C}$ is the simplicial set whose n-simplices are these n-tuples:

$$(\operatorname{Ner}\mathcal{C})_n = \mathbf{Cat}([n], \mathcal{C}). \tag{2.1.6.2}$$

The face maps, d_i, takes an n-tuple to an $(n-1)$-tuple by composing f_i with f_{i+1}, and the degeneracy s_i inserts id_{A_i} after f_i.

This gives us a functor $\operatorname{Ner}\colon \mathbf{Cat} \longrightarrow \mathbf{SSet}$ which is in fact full and faithful.

2.2 Cartesian Closedness of SSet

2.2.1 The internal hom

Definition of $\underline{\mathcal{S}}(X, Y)$

As with every presheaf category, **SSet** is Cartesian closed,

$$\mathbf{SSet}(X \times Y, Z) \cong \mathbf{SSet}(X, \underline{\mathcal{S}}(Y, Z)). \tag{2.2.1.1}$$

The 'internal hom' is given by

$$\underline{S}(X,Y) = \mathcal{Y}_\Delta \mathbin{\#} (X \times -) \mathbin{\#} \mathbf{SSet}(-,Y), \tag{2.2.1.2}$$

where $\mathcal{Y} = \Delta[-]$ is the Yoneda embedding $\Delta \longrightarrow \mathbf{SSet}$. To see this, put $X = \Delta[n]$ in (2.2.1.1), and deduce that the internal hom (if it exists) must have, as n-simplices, the set $\mathbf{SSet}(X \times \Delta[n], Y)$. To finish the proof, show that (2.2.1.2) does always satisfy (2.2.1.1) (in fact there is nothing special about Δ here, and the same proof may be used to show that any presheaf category is Cartesian closed).

Faces and degeneracies in $\underline{S}(X,Y)$

We can also find the face and degeneracy maps from (2.2.1.2): for $f\colon X \times \Delta[n] \longrightarrow Y$ an n-simplex in $\underline{S}(X,Y)$,

$$(d_i f) = X \times \Delta[n-1] \xrightarrow{\mathrm{id}\times(-\#\Delta[\delta_i])} X \times \Delta[n] \xrightarrow{f} Y \tag{2.2.1.3}$$

$$(d_i f)(x, [m] \xrightarrow{\tau} [n-1]) = f(x, [m] \xrightarrow{\tau} [n-1] \xrightarrow{\delta_i} [n]), \tag{2.2.1.4}$$

and analogously for the degeneracy maps $s_i f$. Note that f itself is simplicial, so we have $d_i(f(x,\sigma)) = f(d_i x, d_i(\sigma)) = f(d_i x, [m-1] \xrightarrow{\delta_i} [m] \xrightarrow{\sigma} [n-1]$; in particular we note that $(d_i f)(x,\tau) \neq d_i(f(x,\tau))$. (Indeed both sides cannot both be defined at once, since $d_i f(x,\tau) \in Y_m$, but $d_i(f(x,\tau))$ would be in Y_{m-1}).

As an example of the face maps we offer this lemma which is useful for proving proposition 4.1.2.8

Lemma 2.2.1.5. *For* $(f \times \Delta[n]) \in \underline{S}(A, B \times \Delta[n])_n$, *i.e.*, $f \times \Delta[n]\colon A \times \Delta[n] \longrightarrow B \times \Delta[n]$ *we have* $d_i(f) = f \times \Delta[\delta_i] = (f \times \Delta[n-1]) \mathbin{\#} (B \times \Delta[\delta_i])$.

Proof. The map $d_i(f)$ is the diagonal of the commutative square

$$\begin{array}{ccc} A \times \Delta[n-1] & \xrightarrow{A \times \Delta[\delta_i]} & A \times \Delta[n] \\ {\scriptstyle f \times \Delta[n-1]}\big\downarrow & & \big\downarrow{\scriptstyle f \times \Delta[n]} \\ B \times \Delta[n-1] & \xrightarrow[B \times \Delta[\delta_i]]{} & B \times \Delta[n] \end{array} . \tag{2.2.1.6}$$

□

Similarly, we have lemma 2.2.1.12 for the degeneracies.

Representing maps in $\underline{S}(A, B)$

If $t \in \underline{S}(A,B)_n$ then the representing map is $\overline{t}\colon \Delta[n] \longrightarrow \underline{S}(A,B)$, which must map $\sigma\colon [m] \longrightarrow [n]$ to $\overline{t}(\sigma) \in \underline{S}(A,B)_m$. Using (2.1.5.3) we get

$$\overline{t}(\sigma)(y_k, \tau\colon [k] \longrightarrow [m]) = t(y_m, [k] \xrightarrow{\tau} [m] \xrightarrow{\sigma} [n]) \tag{2.2.1.7}$$

This is exactly the formula for transposing the map $t\colon A \times \Delta[n] \longrightarrow B$ across the Cartesian closed adjunction.

If $b \in B_n$ is any n-simplex, we can regard the map $\overline{b}$ as an n-simplex of $\underline{S}(\Delta[0], B)$, and in this simplicial set we get

$$\begin{aligned}(d_i\overline{b})(\sigma) &= \overline{d_i b}(\sigma) \quad (\text{in } \underline{S}(\Delta[0],B)_{n-1}) \\ &= \overline{b}([m] \xrightarrow{\sigma} [n-1] \xrightarrow{\delta_i} [n])\end{aligned} \tag{2.2.1.8}$$

$$\begin{aligned}(s_i\overline{b})(\sigma) &= \overline{s_i b}(\sigma) \quad (\text{in } \underline{S}(\Delta[0],B)_{n+1}) \\ &= \overline{b}([m] \xrightarrow{\sigma} [n+1] \xrightarrow{\sigma_i} [n]).\end{aligned} \tag{2.2.1.9}$$

Alternate description of $\underline{S}(X, Y)$

If $\mathcal{C}$ is a category and $A \in \mathcal{C}$ an object, write $\mathbf{SSet}/\Delta[n]$ for the *slice category* of objects over A.

The universal property of products tells us that we have mutually inverse bijections

$$\underline{S}(X,Y)_n \quad \cong \quad \mathbf{SSet}/\Delta[n]\left(\begin{pmatrix} X \times \Delta[n] \\ \downarrow{\scriptstyle \mathrm{pr}} \\ \Delta[n] \end{pmatrix}, \begin{pmatrix} Y \times \Delta[n] \\ \downarrow{\scriptstyle \mathrm{pr}} \\ \Delta[n] \end{pmatrix}\right) \tag{2.2.1.10}$$

$$f \longmapsto (f, \mathrm{pr}_{\Delta[n]})$$

$$g \# \mathrm{pr}_{\Delta[n]} \longleftarrow\!\shortmid\; g$$

pictorially,

$$\begin{array}{ccccc} & & X \times \Delta[n] & & \\ & {\scriptstyle f}\swarrow & \downarrow{\scriptstyle g} & \searrow{\scriptstyle \mathrm{pr}_{\Delta[n]}} & \\ & & Y \times \Delta[n] & & \\ & \swarrow{\scriptstyle \mathrm{pr}_Y} & & {\scriptstyle \mathrm{pr}_{\Delta[n]}}\searrow & \\ Y & & & & \Delta[n] \end{array} . \tag{2.2.1.11}$$

Lemma 2.2.1.12. *If* $a\colon A \longrightarrow B$ *is regarded as a* 0*-simplex of* $\underline{S}(A,B)$*, then* $s_0(a) \in \underline{S}(A,B)_1$ *is equal to the composite* $A \times \Delta[1] \xrightarrow{\mathrm{pr}_A} A \xrightarrow{a} B$ *and corresponds to* $a \times \Delta[1]\colon A \times \Delta[1] \longrightarrow B \times \Delta[1]$ *under* (2.2.1.10). □

2.2.2 The internal automorphism group

The description of (2.2.1.10) makes it clear that we have a simplicial map

$$\#\colon \underline{S}(X,Y) \times \underline{S}(Y,Z) \longrightarrow \underline{S}(X,Z) \tag{2.2.2.1}$$

(just compose maps in the slice category). In terms of the original definition (2.2.1.2),

$$(s \# t)(x,\sigma) := t(s(x,\sigma),\sigma). \tag{2.2.2.2}$$

Another way of saying this is that for $(a\colon A \times \Delta[n] \longrightarrow B) \in \underline{S}(A,B)_n$ and $(b\colon B \times \Delta[n] \longrightarrow C) \in \underline{S}(B,C)_n$, the composite $a \# b \in \underline{S}(A,C)_n$ is

$$a \# b = \left(A \times \Delta[n] \xrightarrow{\alpha} B \times \Delta[n] \xrightarrow{b} C\right) \tag{2.2.2.3}$$

where a corresponds to α under the isomorphism (2.2.1.10).

Taking $X = Y = Z$, we see that $\underline{S}(Y,Y)$ is an internal monoid in **SSet**. Pass now to the slice category description of $\underline{S}(Y,Y)$, and we have a subobject consisting of the invertible simplicial maps. This gives us a simplicial group (see section 2.5) $\mathrm{aut}(Y) \subseteq \underline{S}(Y,Y)$, with

$$\mathrm{aut}(Y)_n = \left\{ a\colon Y \times \Delta[n] \xrightarrow{\cong} Y \times \Delta[n]\colon a \text{ is an invertible simplicial map over } \Delta[n] \right\} \tag{2.2.2.4}$$

In terms of the original definition, an n-simplex $t\colon Y \times \Delta[n] \longrightarrow Y$ is in $\mathrm{aut}(Y)$ iff for every m and every $\sigma \in \Delta[n]_m$ the map $y \mapsto t(y,\sigma)$ is a bijection $Y_m \longrightarrow Y_m$ i.e., iff $t(-,\sigma)$ is a bijection for all σ.

2.2.3 The simplicial action of $\mathrm{aut}(Y)$ on Y

In any Cartesian closed category with terminal object, 1, the internal hom of maps from 1 to the object A is isomorphic to A. For **SSet** this is just the Yoneda lemma, as elements of $\underline{S}(\Delta[0],Y)_n$ are maps $t\colon \Delta[0] \times \Delta[n] \longrightarrow Y$ where the domain of such a map is isomorphic to $\Delta[n]$ (because $\Delta[0]$ is terminal), so we get a map $\Delta[n] \longrightarrow Y$ which is the representing map for $t(\star, \mathrm{id}_n) \in Y_n$.

Taking $X = \Delta[0]$ and $Z = Y$ the map (2.2.2.1) says that $\underline{S}(Y,Y)$ acts on Y (and moreover this is a simplicial action). Explicitly $t \in \mathrm{aut}(Y)_n$ acts on Y_n via

$$y^t := t(y, \mathrm{id}_n) \in Y_n. \tag{2.2.3.1}$$

(This is a right action: $(y^t)^s = y^{t\#s}$ because we wrote composition on the right as '#'. If we had written composition the other way, as '∘', we would get a left action here.)

To see the explicit form of the action, from $y \in Y_n$ find the representing map for y and consider it as $\overline{y}\colon \Delta[0] \times \Delta[n] \cong \Delta[n] \longrightarrow Y$, an element of $\underline{\mathcal{S}}(\Delta[0], Y)_n$. Then

$$y^t = \overline{y} \# t\colon \Delta[0] \times \Delta[n] \longrightarrow Y, \qquad (2.2.3.2)$$

which (using the definition of # from (2.2.2.2), and the Yoneda correspondence from section 2.1.5) corresponds to the n-simplex

$$(\overline{y} \# t)(\star, \mathrm{id}_n) = t(\overline{y}(\star, \mathrm{id}_n), \mathrm{id}_n) = t(y, \mathrm{id}_n) \in Y_n. \qquad (2.2.3.3)$$

(where $\star$ is the unique element of $\Delta[0]_n$) Setting $\tau = \mathrm{id}_m$ in (2.2.1.7) gives us

$$y_m^{\overline{t}(\sigma)} = t(y_m, \sigma). \qquad (2.2.3.4)$$

Note that this is true more generally: if we have $t \in \underline{\mathcal{S}}(A, B)_n$ and $a \in A_n$, we can write a^t for $t(a, \mathrm{id}_n)$, and (2.2.1.7) tells us that $t(a, \sigma) = a^{\overline{a}(\sigma)}$, whilst (2.2.2.2) gives us

$$a^{(s \# t)} = (a^s)^t. \qquad (2.2.3.5)$$

When we specialise to $\underline{\mathcal{S}}(Y, Y)$ acting on Y, we get the right-action property $(a^s)^t = a^{(s \# t)}$.

As an example, take $n = 0$. A 0-simplex of $\mathrm{aut}(Y)$ is just an automorphism of the simplicial set Y, and the action on Y_0 is $y^t = t(y)$.

To further understand the action of $\mathrm{aut}(Y)$ on Y we offer the following observation. Although (2.2.3.4) tells us we can recover t from the actions of all the maps $\overline{t}(\sigma)$, it is not true that if $y^t = y$ for all $y \in Y_n$ we must have $t = \mathrm{id}$: even for $n = 0$, we have that $t(y_0) = y_0$ for all $y_0 \in Y_0$, so the zeroth level of t is trivial, but t need not fix higher levels. For example, let G be any group with t a non-identity automorphism. Using section 2.1.6, there is only one vertex in $Y = \mathrm{Ner}\, G$, so t must induce something in $\mathrm{aut}(Y)_0$ acting trivially on the vertices.

We do, however, always have that $b^t = c^t$ iff $b = c$ (just act on both sides by t^{-1}).

2.3 $\mathrm{simp}(\mathcal{C})$ as an $\mathcal{S}$-Cat

We can generalise the previous section to more general simplicial objects; this section follows [33] closely.

2.3.1 The Grothendieck construction for a simplicial set

If we regard a set as a discrete category (i.e., only identity morphisms), then a simplicial set, X, can be regarded as a functor into **Cat**. Further, if we regard X as a *covariant* functor $X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ then we can apply the Grothendieck construction (see section 3.1) to get $\begin{array}{c} \Delta^{\mathrm{op}} \int X \\ \big\downarrow {\scriptstyle \mathrm{Grot}(X)} \\ \Delta^{\mathrm{op}} \end{array}$ with $\mathrm{Grot}(X)(n, x) = n$.

For example, when $X = \Delta[n]$, we have

$$\mathrm{Grot}(\Delta[n]) \cong \begin{array}{c} (\Delta/[n])^{\mathrm{op}} \\ \big\downarrow U \\ \Delta^{\mathrm{op}} \end{array} \qquad (2.3.1.1)$$

the forgetful functor.

2.3.2 Definition of SIMP($\mathcal{C}$)

For any $\mathcal{C}$ we let $\mathrm{SIMP}(\mathcal{C})(X,Y)_n$ be the set of natural transformations from $\mathrm{Grot}(\Delta[n]) \# X$ to $\mathrm{Grot}(\Delta[n]) \# Y$.

Explicitly, $\alpha \in \mathrm{SIMP}(\mathcal{C})(X,Y)_n$ is specified by a family of maps $\alpha(\sigma)\colon m \longrightarrow n$, one for each $\sigma\colon m \longrightarrow n$ in Δ and the naturality condition says that for $\tau\colon q \longrightarrow m$ in Δ (i.e., for each $\tau\colon \sigma \longrightarrow \tau \# \sigma$ in $(\Delta/[n])^{\mathrm{op}}$), the square

$$\begin{array}{ccc} X_m & \xrightarrow{\alpha(\sigma)} & Y_m \\ {\scriptstyle X(\tau)}\big\downarrow & & \big\downarrow{\scriptstyle Y(\tau)} \\ X_q & \xrightarrow[\alpha(\tau\#\sigma)]{} & Y_q \end{array} \qquad (2.3.2.1)$$

commutes in $\mathcal{C}$. Thus we get a $\mathcal{S}$-**Cat**, denoted SIMP($\mathcal{C}$) with the objects of simp(C) but using SIMP($\mathcal{C}$) as the simplicial set of morphisms.

2.3.3 $\mathcal{C}$ = Set

Note that α is a function $\alpha\colon \Delta(n,m) \longrightarrow \mathcal{C}(X_m, Y_m)$ and when $\mathcal{C} = \mathbf{Set}$ we can use the Cartesian Closed property to get a family $\overline{\alpha(m)}\colon X_m \times \Delta[n]_m \longrightarrow Y_m$; the square (2.3.2.1) corresponds exactly to the condition that $\overline{\alpha}$ be a natural transformation $X \times \Delta[n] \longrightarrow Y$, i.e., we recover the 'traditional' definition of the internal hom in **SSet**.

2.3.4 Copowered categories $\mathcal{C}$ and tensoring in simp($\mathcal{C}$)

More generally, suppose $\mathcal{C}$ is *copowered.* By definition (see e.g., [35]) this means that for $A \in \mathcal{C}$ and $F \in \mathbf{Set}$ we have an object $A \otimes F \in \mathcal{C}$ and a natural isomorphism

$$\mathcal{C}(A \otimes F, B) \cong \mathbf{Set}(F, \mathcal{C}(A,B)). \qquad (2.3.4.1)$$

Immediately we observe that this gives an adjunction $A \otimes - \dashv \mathcal{C}(A,-)$, but also $A \otimes F$ is a coproduct, $\coprod_{f\in F} A$, so we only need $\mathcal{C}$ to have small coproducts. If $\mathcal{C} = \mathbf{Set}$ then $\otimes$ is just Cartesian product and we are back in the Cartesian closed situation.

Just as in the case where $\mathcal{C} = \mathbf{Set}$, we can now transpose the map $\alpha(m)$ to get a map $\overline{\alpha(m)}\colon X_m \otimes \Delta(n,m) \longrightarrow Y_m$ in $\mathcal{C}$. The naturality condition (2.3.2.1) can be written as the equality of these two composites

$$\Delta(n,m) \xrightarrow{\alpha(m)} \mathcal{C}(X_m, Y_m) \xrightarrow{\mathcal{C}(X_m, Y\tau)} \mathcal{C}(X_m, Y_q) \qquad (2.3.4.2)$$

and

$$\Delta(n,m) \xrightarrow{\Delta[n](\tau)} \Delta(q,n) \xrightarrow{\alpha(q)} \mathcal{C}(X_q, Y_q) \xrightarrow{\mathcal{C}(X\tau, Y_q)} \mathcal{C}(X_m, Y_q). \tag{2.3.4.3}$$

The transpose of this equality across the adjunction (2.3.4.1) gives us the commuting square

$$\begin{array}{ccc} X_m \otimes \Delta[n]_m & \xrightarrow{\overline{\alpha(m)}} & Y_m \\ {\scriptstyle X\tau\otimes\Delta[n]\tau}\downarrow & & \downarrow{\scriptstyle Y\tau} \\ X_q \otimes \Delta[n]_q & \xrightarrow[\overline{\alpha(q)}]{} & Y_q \end{array} \tag{2.3.4.4}$$

hence, for copowered $\mathcal{C}$ we have $\mathrm{SIMP}(\mathcal{C})(X,Y)_n \cong \mathrm{simp}(C)(X \otimes \Delta[n], Y)$ where we have extended the $\otimes$ notation to cope with $A \in \mathrm{simp}(\mathcal{C})$ and $F \in \mathbf{SSet}$ by defining $(A \otimes F)_n = A_n \otimes F_n$. This is a direct generalisation of section 2.3.3.

2.3.5 simp(C) is a tensored $\mathcal{S}$-Cat

In this section we will see that (for copowered $\mathcal{C}$) simp($\mathcal{C}$) is a *tensored* $\mathcal{S}$-**Cat**. By definition (see e.g., [10]) this means that we have a natural isomorphism

$$\mathrm{SIMP}(\mathcal{C})(A \otimes K, B) \cong \underline{\mathcal{S}}(K, \mathrm{SIMP}(\mathcal{C})(A,B)) \tag{2.3.5.1}$$

Proof. We can reduce to the 'non-enriched version'

$$\mathrm{simp}(\mathcal{C})(A \otimes K, B) \cong \mathbf{SSet}(K, \mathrm{SIMP}(\mathcal{C})(A,B)) \tag{2.3.5.2}$$

as replacing K by $K \times \Delta[n]$ in (2.3.5.2) gives us (2.3.5.1) (this depends on section 2.3.4 and the isomorphism $(A \otimes K) \otimes L \cong A \otimes (K \times L)$ which is easily proved using the definition of $\otimes$).

To prove (2.3.5.2) we first note that if $K = \Delta[n]$ then both sides are just the set $\mathrm{SIMP}(\mathcal{C})(A,B)_n$. K is a presheaf on the category Δ so can be expressed as a colimit of representables: $K \cong \int^M \Delta[M]$. Because colimits are computed pointwise and $A \otimes -$ is a left adjoint, we have

$$A \otimes K \cong \int^M A \otimes \Delta[M], \tag{2.3.5.3}$$

and then we are left with an easy exercise using the end/coend calculus:

$$
\begin{aligned}
\mathrm{simp}(\mathcal{C})(A \otimes K, B) &\cong \mathrm{simp}(\mathcal{C})(\int^M A \otimes \Delta[M], B) \\
&\cong \int_M \mathrm{simp}(\mathcal{C})(A \otimes \Delta[M], B) \\
&\cong \int_M \mathrm{SIMP}(\mathcal{C})(A, B)_M \\
&\cong \int_M \mathbf{SSet}(\Delta[M], \mathrm{SIMP}(\mathcal{C})(A, B)) \\
&\cong \mathbf{SSet}(\int^M \Delta[M], \mathrm{SIMP}(\mathcal{C})(A, B)) \\
&\cong \mathbf{SSet}(K, \mathrm{SIMP}(\mathcal{C})(A, B)).
\end{aligned}
\tag{2.3.5.4}
$$

□

2.4 Homotopy of Simplicial Sets

We have already looked at the category of simplicial sets, but there is a 2-category structure obtained by adding homotopies of simplicial maps. We follow [33] as well as the standard references [11] and [40].

2.4.1 Definition

Let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be simplicial maps, then a *simplicial homotopy* $h\colon f \xrightarrow{\simeq} g$ is a 1-simplex $f \xrightarrow{h} g$ in $\underline{S}(A, B)$, i.e., a simplicial map $h\colon A \times \Delta[1] \longrightarrow B$ with $d_1(h) = f$ and $d_0(h) = g$.

2.4.2 Interpreting the definition

The following is adapted from [33].

We have

$$d_i(h)(a, \sigma) = h(a, [m] \xrightarrow{\sigma} [1] \xrightarrow{\delta_i} [0]) \tag{2.4.2.1}$$

$$= h(a, [m] \xrightarrow{\delta_i} [1]). \tag{2.4.2.2}$$

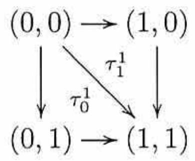
Using the notation of section 2.1.3, δ_1 is the list with every entry zero and δ_0 is the list with everything equal to 1, so we sometimes write $h(a, 0) = g(a)$ and $h(a, 1) = f(a)$ (for this reason some authors write $h\colon g \xrightarrow{\simeq} f$).

Note that this determines h on vertices of $A \times \Delta[1]$. Let $(a, \sigma) \in (A \times \Delta[1])_n$, we have a diagram

$$\Delta[n] \times \Delta[1] \xrightarrow{\overline{a} \times \Delta[1]} A \times \Delta[1] \xrightarrow{h} B \tag{2.4.2.3}$$

and $h(a,\sigma) = h(\overline{a} \times \mathrm{id})(\mathrm{id}_n, \sigma)$. The non-degenerate simplices in $(\Delta[n] \times \Delta[1])_{n+1}$ are $\tau_i^n = (s_i(\mathrm{id}_n), t_i)$ where t_i has image $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$ (the last 0 occurs in position i so that t_i is degenerate, but is not equal to $s_i(u)$ for any u, because $s_i(\mathrm{id}_n) \neq s_k(u)$ for $k \neq i$ we see that τ_i^n is non-degenerate). If (the list corresponding to) σ involves only one of the symbols 0 and 1 then $h(a, \sigma)$ is either $f(a)$ or $g(a)$, otherwise $\sigma = d_i \tau_i^n$ for some i, and hence $h(a, \sigma) = h(\overline{a} \times \mathrm{id})(d_i \tau_i)$.

Write $h_i \colon A_n \longrightarrow B_{n+1}$ for the map (of sets) sending a to $h(\overline{a} \times \mathrm{id})(\mathrm{id}_n, \tau_i^n)$, then the h_i determine h. Of course not every set of maps h_i determine a homotopy, for example in $(\Delta[1] \times \Delta[1])_2$ we have $d_1 \tau_0^1 = d_1 \tau_1^1$



(2.4.2.4)

so that we must have $d_1 h_0(a) = d_1 h_1(a)$. The complete list of conditions for a family $\{ h_i \colon A_n \longrightarrow B_{n+1} \colon 0 \leqslant i \leqslant n \}$ to determine a homotopy $h \colon f \xrightarrow{\simeq} g$ is as follows (where we omit the a's using the obvious conventions) [40]

$$d_0 h_0 = f, \qquad d_{n+1} h_n = g \tag{2.4.2.5}$$

$$d_i h_j = \begin{cases} h_{j-1} d_i & 0 \leqslant i < j \leqslant n \\ d_j h_{j-1} & 1 \leqslant i = j \leqslant n \\ d_{j+1} h_{j+1} & 1 \leqslant i = j+1 \leqslant n \\ h_j d_{i-1} & 1 \leqslant j+1 < i \leqslant n+1 \end{cases} \tag{2.4.2.6}$$

$$s_i h_j = \begin{cases} h_{j+1} s_i & 0 \leqslant i \leqslant j \leqslant n \\ h_j s_{i-1} & 0 \leqslant j < i \leqslant n \end{cases} \tag{2.4.2.7}$$

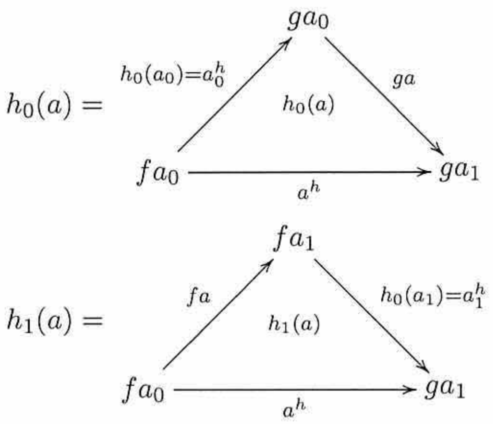
Visualising a simplicial homotopy

For $n = 0$, each vertex $a \in A_0$ is assigned a 1-simplex $f(a) \xrightarrow{h_0(a)} g(a)$ in B. Explicitly,

$$\begin{aligned} h_0(a) &= h(\overline{a} \times \mathrm{id})(\tau_0) \\ &= h(\overline{a} \times \mathrm{id})((00), (01)) \\ &= h(s_0 a, \mathrm{id}) \\ &= (s_0 a)^h \end{aligned} \tag{2.4.2.8}$$

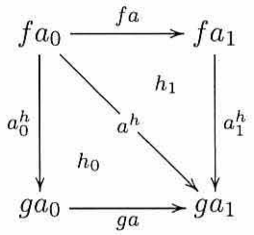
where in the last line we are extending the notation of section 2.2.3 for the map $\# \colon A \times \underline{S}(A, B) \longrightarrow B$. We will further abbreviate $(s_0 a)^h$ to just a^h.

For $a \in A_1$ the conditions on $h_0(a)$ and $h_1(a)$ give us two 2-simplices as shown here



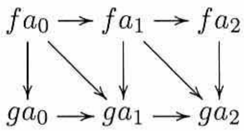
(2.4.2.9)

where the common 1-face is easily checked to be a^h, giving us the square



(2.4.2.10)

In dimension 2 the h_i give us three tetrahedra which fit into a triangular prism with fa and ga as the two end triangular faces. If we 'flatten' the 2-simplex fa and draw it as $fa_0 \longrightarrow fa_1 \longrightarrow fa_2$ (this of course makes perfect sense if A and B are nerves of categories) then the prism becomes



(2.4.2.11)

and in general we just add more 'squares with diagonals' on the right, and, in flattened notation, $h_i(a) = (fa_0 \longrightarrow \cdots \longrightarrow fa_i \xrightarrow{a_i^h} ga_i \longrightarrow \cdots \longrightarrow ga_n)$.

Contracting homotopies

For example, if g is the identity map and f maps everything to a (iterated degeneracy of a) vertex, $\star$ then all we need is to give each $a \in A_n$ an 'extra degeneracy' $s_{-1}(a)$, which looks like $(\star \longrightarrow a_0 \longrightarrow \cdots \longrightarrow a_n)$, and is called a *contracting homotopy.*

2.5 Simplicial Groups and Groupoids

We now introduce the machinery needed to understand bundles. The main reference for these results is [40], with [11] providing an overview. The only difference is we have attempted to adopt more explicitly categorical notation.

2.5.1 Definitions

A simplicial group is a simplicial object in **Grp**, thus it is a simplicial set G in which all the G_n are groups, and with all d_i and s_i group homomorphisms.

An example is the simplicial automorphism group discussed in section 2.2.2.

2.5.2 Simplicial actions

Recall from section 2.2 that **SSet** is Cartesian closed. Thus we have a bijection between the sets of simplicial maps (of simplicial *sets*) of the form

$$\star \colon Y \times G \longrightarrow Y \tag{2.5.2.1}$$

$$\beta \colon G \longrightarrow \operatorname{aut}(Y) \tag{2.5.2.2}$$

(the Cartesian property gives us β going into the internal hom, but, as we will see shortly, it has image inside the automorphism group). Explicitly, given the action as above, define β in dimension n by

$$\begin{gathered} \beta(g_n) = \left(Y \times \Delta[n] \xrightarrow{\operatorname{id} \times \overline{g_n}} Y \times G \xrightarrow{\star} Y \right) \\ \beta(g_n)(y_m, [m] \xrightarrow{\sigma} [n]) = y_m \star \overline{g_n}(\sigma) \end{gathered} \tag{2.5.2.3}$$

(here y_m is an m-simplex of Y). It is easy to see that β is a group homomorphism iff $\star$ satisfies the axioms for a right action, so in particular $\beta(g_n)$ really is in $\operatorname{aut}(Y)$. (Note however, that for β to be a homomorphism in a single dimension n requires more than just the corresponding action axioms in that dimension: for β to be a group homomorphism in dimension n we must have that $y_m \star (\overline{g_n}(\sigma) \# \overline{h_n}(\sigma)) = (y_m \star \overline{g_n}(\sigma)) \star \overline{h_n}(\sigma)$ for all m and all σ: Thus we need that the $g_n \in G_n$ act on *all* the Y_m via $\overline{g}(\sigma)$.) Conversely, from β we get an action with $y_n \star g_n := \beta(g_n)(y_n, \operatorname{id}_n)$. This is $y_n^{\beta(g_n)}$ using the notation of section 2.2.3, and indeed that section can be recovered by taking $G = \operatorname{aut}(Y)$ and $\beta = \operatorname{id}$. Diagrammatically, this says that given an action $\star$ of G on Y, the corresponding β is the unique map making the diagram

$$\begin{array}{ccc} Y \times G & \xrightarrow{\star} & Y \\ {\scriptstyle Y \times \beta} \big\downarrow & \nearrow_{\#} & \\ Y \times \operatorname{aut}(Y) & & \end{array} \tag{2.5.2.4}$$

commute, where $\#$ is the action of $\operatorname{aut}(Y)$ on Y from section 2.2.3; thus we might regard $\operatorname{aut}(Y)$ with the action $\#$ as the 'universal action of a group on Y'.

2.5.3 The regular Representation

Let G be a simplicial group, and write $G \times G \longrightarrow G$ as composition, so the result of multiplying g and h is written as $g \# h$. This gives us a (right) action of G on itself, and applying section 2.5.2 gives us a map of simplicial groups, $\rho_G \colon G \longrightarrow \mathrm{aut}(G)$, called the *(right) regular representation for G.*

Explicitly, $\rho_G(g)(h, \sigma) = h\overline{g}(\sigma)$.

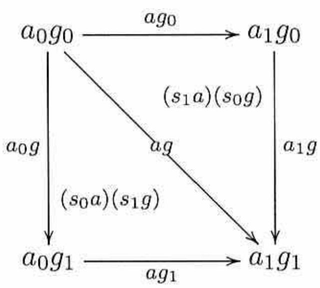
Examples of the regular representation

Some examples. We will see what $\rho_G(g)$ does to (x, σ) where $x \in G_m$ and $\sigma \in \Delta[n]_m$ are m-simplices and $g \in G_n$ is an n-simplex.

- Let $n = 0$ and $g \in G_0$ a 0-simplex. Then $\sigma \colon [m] \longrightarrow [0]$ can only be σ_0^m, and hence $\rho_G(g)(x, \sigma) = x \# \overline{g}(\sigma) = x \# s_0^m(g)$ so $\rho_G(g)$ is 'multiplication by the generalised vertex corresponding to g'.

- Let $n = 1$ and $g \in G_1$ a 1-simplex. Then $\rho_G(g) \colon G \times \Delta[1] \longrightarrow G$ is (by definition; see section 2.4.1) a homotopy from $\rho_G(d_1 g)$ to $\rho_G(d_0 g)$.

  Referring to the pictures in section 2.4.2 we can draw our homotopy (in dimension 1) as



(2.5.3.1)

- For a general $g \in G_n$, let $\sigma = \mathrm{id}_n$ and $h \in G_n$ also. Then $\rho_G(g)(h, \mathrm{id}_n) = h \# g$. Note that the left hand side is also $h^{\rho_G(g)}$, the action from section 2.2.3.

- Completely generally, $\rho_G(g)(h, \sigma) = h \# \overline{g}(\sigma)$ by (2.5.2.3).

2.5.4 From Groups to Groupoids

Classically, generalising groups to groupoids is 'a good thing' [6], and the simplicial case is no exception. However since a simplicial group is both an internal group in **SSet** and a 1-object category enriched over **SSet**, there are two "obvious" generalisations available, both useful.

'Simplicially Enriched-Groupoids' versus 'Simplicial Groupoids'

A *simplicially enriched groupoid* is a groupoid enriched in **SSet**. So we have a set[1] of objects $A, B, \ldots$; for each pair of objects, A and B we have a simplicial set $\mathrm{Hom}(A, B)_\bullet$ playing the rôle of 'Hom set'; and composition is a family of simplicial morphisms $\#\colon \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C)$ which satisfy obvious associativity and identity axioms (see [34] for enrichment in general). The category of these is denoted $\mathcal{S}$-**Gpds**.

A *simplicial groupoid* is a simplicial object in **Grpoids**, i.e., for each n we have a groupoid of n-simplices, with face and degeneracy *functors* satisfying the simplicial identities.

We picture a simplicial groupoid $H_\bullet$ as

$$\cdots \begin{pmatrix} M_2 \\ \downdownarrows \\ O_2 \end{pmatrix} \underset{d_2}{\overset{d_0}{\underset{d_1}{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}}}} \begin{pmatrix} M_1 \\ \downdownarrows \\ O_1 \end{pmatrix} \underset{d_1}{\overset{d_0}{\rightrightarrows}} \begin{pmatrix} M_0 \\ \downdownarrows \\ O_0 \end{pmatrix} \tag{2.5.4.1}$$

and notice that (2.5.4.1) gives us two simplicial sets[2], $M_\bullet$ and $O_\bullet$ which we call the "simplicial set of morphisms" and "simplicial set of objects" respectively. If all the groupoids $H_i := (M_i \rightrightarrows O_i)$ have the same objects ("have the same base"), and if the face and degeneracy functors are the identity on objects, then $O_\bullet$ is constant: a *constant simplicial object of objects.*

Proposition 2.5.4.2. *The category of simplicially enriched groupoids is equivalent to the category of simplicial groupoids with a constant simplicial object of objects.*

Proof. The relationship between a simplicially enriched groupoid X and a simplicial groupoid $M_\bullet \rightrightarrows O_\bullet$ with $O_\bullet$ constant is given by the equations

$$M_n(A, B) = X(A, B)_n \tag{2.5.4.3}$$
$$O_n = \mathrm{Ob}\, X \tag{2.5.4.4}$$

(and similarly for composition and identities) where if we are given either one of the sides we may use it to define the other. □

2.5.5 The $\mathcal{G} \dashv \overline{W}$ adjunction

From $\mathcal{S}$-Gpds to SSet: The $\overline{W}$ construction

There is a functor $\overline{W}\colon \mathcal{S}$-**Gpds** $\longrightarrow$ **SSet** which looks a lot like a nerve. It was discovered independently by Joyal–Tierney and Dwyer–Kan. Let $H \in \mathcal{S}$-**Gpds**. If H was an ordinary groupoid we would call two maps h_1 and h_2 composable if h_2 starts where

[1] Let us assume all categories are "small" for simplicity

[2] Again we need a smallness assumption here.

h_1 finishes: $\mathrm{dom}(h_2) = \mathrm{cod}(h_1)$, an equation between objects. In the **SSet**-enriched case, where H has a simplicial set of morphisms, the equation $\mathrm{dom}(h_2) = \mathrm{cod}(h_1)$ still makes sense even though we may only actually compose the two maps when their dimensions agree.

Recalling that the n-simplices of the nerve of a category are the composable n-tuples of morphisms in that category, we define $(\overline{W}H)_n$ as the set of 'composable' n-tuples of the form

$$A_0 \xrightarrow{h_1} A_1 \longrightarrow \cdots \xrightarrow{h_n} A_n \tag{2.5.5.1}$$

where each h_i is an $(n-i)$-simplex in $H(A_{i-1}, A_i)$, and 'composable' means only that $\mathrm{cod}\, h_i = \mathrm{dom}\, h_{i+1}$ (in the notation of section 2.5.4, $h_i \in M_{n-i}(A_{i-1}, A_i)$ is a map in the groupoid H_i). For $n = 0$, (2.5.5.1) is to be interpreted as the set of objects of H (and hence of each groupoid H_i).

The face and degeneracy maps are in general more complicated than those of the nerve: for the lowest dimensions, $(\overline{W}H)_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} (\overline{W}H)_0$ are the domain and codomain functions from H (and the degeneracy, s_0, is the 'identities' map), which is the same as the usual nerve, but in higher dimensions we have

$$d_i(h) = \begin{cases} (h_2, \ldots, h_n) & i = 0 \\ (d_{i-1}h_1, d_{i-2}h_2, \ldots, [d_0h_i] \# h_{i+1}, h_{i+2}, \ldots, h_n) & 0 < i < n \\ (d_{n-1}h_1, d_{n-2}h_2, \ldots, d_1h_{n-1}) & i = n \end{cases} \tag{2.5.5.2}$$

and

$$s_i(h) = \begin{cases} (\mathrm{id}_{A_0}, h_1, \ldots, h_n) & i = 0 \\ (s_{i-1}h_1, \ldots, s_0h_i, \mathrm{id}_{A_i}, h_{i+1}, \ldots, h_n) & 0 < i \leqslant n \end{cases} \tag{2.5.5.3}$$

See [17] for a detailed proof that this does give a simplicial set.

On morphisms $\overline{W}$ acts in the obvious way: $(\overline{W}f)(h_1, \ldots, h_n) = (f(h_1), \ldots, f(h_n))$.

From SSet back to $\mathcal{S}$-Gpds: The $\mathcal{G}$ construction

The left adjoint to $\overline{W}$ is the *loop groupoid functor* $\mathcal{G}\colon \mathbf{SSet} \longrightarrow \mathcal{S}\text{-}\mathbf{Gpds}$, again it was first studied by both Joyal–Tierney and Dwyer–Kan. If $X \in \mathbf{SSet}$ a simplicial set, then $\mathcal{G}X$ is the simplicially enriched groupoid with:

- Objects $\{\, \overline{x} \colon x \in X_0 \,\}$
- n-arrows the free groupoid on X_{n+1} modulo $s_0(X_n)$, i.e., arrows are words in $\overline{y}$, with $y \in X_{n+1}$, but $s_0(x) \sim ()$, the empty word, for each $x \in X_n$. (Since we are just killing some generators, the set of n-arrows is still a free groupoid)
- The source map is $\mathrm{dom}\,\overline{y} := \overline{d_1 d_2 \ldots d_{n+1} y}$, and the target map is $\mathrm{cod}\,\overline{y} := \overline{d_0 d_2 \ldots d_{n+1} y}$. (If $y_0, y_1, \cdots, y_n$ are the vertices of y then $\overline{y}\colon \overline{y_0} \longrightarrow \overline{y_1}$)

- The simplicial structure on the arrows is given by

$$d_0\overline{y} = (\overline{d_1 y}) \# (\overline{d_0 y})^{-1} \tag{2.5.5.4}$$

$$d_i\overline{y} = \overline{d_{i+1} y} \qquad i > 0 \tag{2.5.5.5}$$

$$s_i\overline{y} = \overline{s_{i+1} y} \qquad i \geqslant 0 \tag{2.5.5.6}$$

On morphisms $f \colon X \longrightarrow Y$, $\mathcal{G}f$ is defined on the generators in the obvious way: $(\mathcal{G}f)(\overline{y}) := \overline{f(y)}$.

The adjunction $\mathcal{G} \dashv \overline{W}$

Proposition 2.5.5.7. *We have an adjunction*

$$\mathbf{SSet} \underset{\overline{W}}{\overset{\mathcal{G}}{\underset{\perp}{\rightleftarrows}}} \mathcal{S}\text{-}\mathbf{Gpds} \tag{2.5.5.8}$$

Proof. The unit is given (in dimension n) by

$$\begin{aligned} \eta_X \colon X &\longrightarrow \overline{W}\mathcal{G}X \\ x &\longmapsto (\overline{x}, \overline{d_0 x}, \overline{d_0^2 x}, \ldots, \overline{d_0^{n-1} x}) \end{aligned} \tag{2.5.5.9}$$

and the counit is defined on generators by

$$\begin{aligned} \varepsilon_H \colon \mathcal{G}\overline{W}H &\longrightarrow H \\ \overline{(g_1, \ldots, g_n)} &\longmapsto g_1 \end{aligned} \tag{2.5.5.10}$$

We shall omit the rest of the proof, which is just verifying that the triangle identities hold. □

3 Homotopy Colimits and the Grothendieck construction

This chapter is to introduce the notion of homotopy colimits [4, 47] which will be needed later. The theorem that homotopy colimits are related to the Grothendieck construction is proved (in more generality than we give here) in [47] and there is a different proof in [9]; our proof here is slightly different, using elements from each.

3.1 The Grothendieck construction

From any strict functor[1] $F\colon \mathcal{C} \longrightarrow \mathbf{Cat}$ there is standard construction called the *Grothendieck construction*, $\mathcal{C} \int (F)$, which is a generalisation of the semidirect product of groups.

3.1.1 Functors as actions

If $b\colon B \longrightarrow C$ is a map in $\mathcal{C}$ then $F(b)\colon FB \longrightarrow FC$ is a functor. So if f is a map in the category FB, we get a map $F(b)(f)$ in the category FC. Denote this map by f^b, then the condition that $F(b)$ be a functor is just the familiar rules

$$\begin{aligned} (f \# g)^b &= f^b \# g^b \\ \mathrm{id}^b &= \mathrm{id} \end{aligned} \qquad (3.1.1.1)$$

In this notation, the coherence maps are isomorphisms (if F is strict then these are also equalities) $x^{(ab)} \longrightarrow (x^a)^b$ and $x^{\mathrm{id}} \longrightarrow x$.

As an example of this 'action notation', suppose G is a group, then to regard G as a category, $\mathcal{G}$, we take a single object $*$ and one map $* \xrightarrow{g} *$ for each $g \in G$. We want the composition in $\mathcal{G}$ to reflect the multiplication in G, but we could equally well define the composite $* \xrightarrow{g} * \xrightarrow{h} *$ to be $* \xrightarrow{gh} *$ or $* \xrightarrow{hg} *$. Of course the resulting categories are both dual and isomorphic, but to be consistent we shall always choose the first way. This means that a right action of G on a set X is a precisely a functor $\mathcal{G} \longrightarrow \mathbf{Set}$ sending $*$ to X; a (right) action of G on a group H is precisely a functor $\mathcal{G} \longrightarrow \mathbf{Grp}$ sending $*$ to $\mathcal{H}$ (where $\mathcal{H}$ is obtained from the group H in the same way we got $\mathcal{G}$ from G); and the rules (3.1.1.1) are the usual definition of a (right) action. Since $\mathbf{Grp} \subseteq \mathbf{Cat}$, from every G-action we get a (strict) functor from $\mathcal{G}$ to $\mathbf{Cat}$. (In later sections we will use the same symbol, G for the group and the category.)

[1]When F is merely op-lax, with $F(a \# b) \xrightarrow{\cong} (Fa) \# (Fb)$ and $F(\mathrm{id}_A) \xrightarrow{\cong} \mathrm{id}_{FA}$ just natural maps, a slight modification lets the following construction still works.

3.1.2 The Grothendieck construction for F a (strict) functor

Given a functor $F\colon \mathcal{C} \longrightarrow \mathbf{Cat}$, we can construct a functor $\mathrm{Grot}(F)\colon \mathcal{C} \int F \longrightarrow \mathcal{C}$.

The category $\mathcal{C} \int F$ is constructed as follows: for objects take all pairs (a, x) where $A \in \mathcal{C}$ and $x \in FA$; for morphisms take a map $(a, f)\colon (A, x) \longrightarrow (B, y)$ for every $a\colon A \longrightarrow B$ in $\mathcal{C}$ and $f\colon Fa(x) \longrightarrow y$ in FB. The composition

$$(A, x) \xrightarrow{(a,f)} (B, y) \xrightarrow{(b,g)} (C, z) \tag{3.1.2.1}$$

is given by the pair

$$\begin{gathered} A \xrightarrow{a\#b} C \\ Fab(x) = Fb(Fa(x)) \xrightarrow{Fb(f)} Fb(y) \xrightarrow{g} z \end{gathered} \tag{3.1.2.2}$$

(where we have used that F was a strict functor for the first equality). Using our notation (3.1.1.1), the above just says that

$$(a, f) \# (b, g) = (a \# b, f^b \# g), \tag{3.1.2.3}$$

which is the usual group law in a semidirect product. So when $\mathcal{C} = \mathcal{G}$ (the category corresponding to a group G), and F represents an action of G on a group H, then $\mathcal{C} \int F$ corresponds to the group $G \ltimes H$.

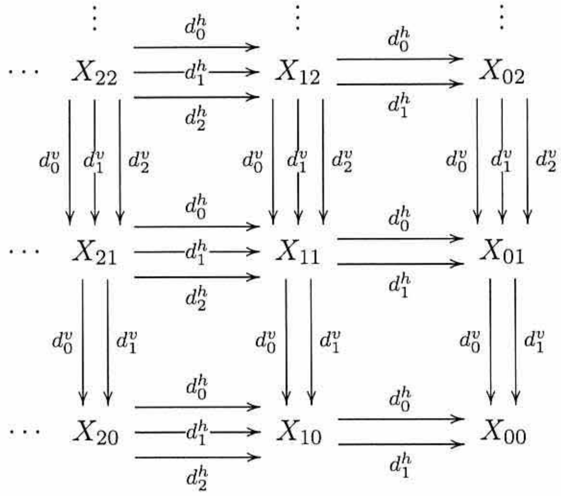
3.2 Bisimplicial Sets

3.2.1 Definition

A *bisimplicial set*, X, is a simplicial object in $\mathbf{SSet}$, i.e., a functor $X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{SSet}$, which looks like

$$\cdots X_2 \underset{d_2}{\overset{d_0}{\underset{d_1}{\rightrightarrows}}} X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \tag{3.2.1.1}$$

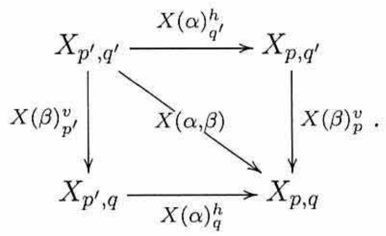
with each X_i a simplicial set, and each d_i and s_i a simplicial map. Now X may be considered as a functor $X\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ sending $([p], [q])$ to $(X_p)_q$. In other words we are writing the simplicial sets X_p vertically, and redrawing (3.2.1.1) as a 2-dimensional (infinite in both directions) square array,



(3.2.1.2)

Here, the d_i^h are horizontal face maps—these are $X(\delta_i) = d_i$ in the notation of (3.2.1.1), which is $X(\delta_i, \mathrm{id})$ in the bifunctor notation above (to get the component of the simplicial map d_i at level n, replace the id by id_n). The vertical maps d_i^v are the face maps inside the X_n, which are $X(\mathrm{id}_n, \delta_i)$ in the bifunctor notation. Of course we also have horizontal and vertical degeneracy maps, denoted s_i^h and s_i^v, which we omit from diagrams for purely typographical reasons.

We can extend the 'horizontal–vertical' notation to any map $\alpha\colon [p] \longrightarrow [p']$ in Δ, writing $X(\alpha)^v := X(\mathrm{id}, \alpha)$ and $X(\alpha)^h := X(\alpha, \mathrm{id})$. If $\beta\colon [q] \longrightarrow [q']$ is another map in Δ, then we have the following square



(3.2.1.3)

The resulting category of bisimplicial sets is denoted **BiSSet**.

3.2.2 Example: The Double Nerve

Let $\mathcal{D}$ be a double category, then its *nerve*, $\mathrm{Ner}\,\mathcal{D}$, is the bisimplicial set with $\mathrm{Ner}\,\mathcal{D}_{p,q}$ the grids of $p \times q$ squares from $\mathcal{D}$ (thus we may also call it the *double nerve*). If $q = 0$ we have p horizontal arrows, if $p = 0$ we have q vertical arrows, and if $p = q = 0$ we have the points of $\mathcal{D}$.

The face maps are induced by composition of squares, exactly analogous to the usual nerve of categories.

3.2.3 Some functors involving Bisimplicial sets

The Diagonal

The diagonal functor, $\Delta\colon \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ induces a functor

$$\mathrm{diag} = \Delta \,\#_0\, - \colon \mathbf{BiSSet} \longrightarrow \mathbf{SSet}, \tag{3.2.3.1}$$

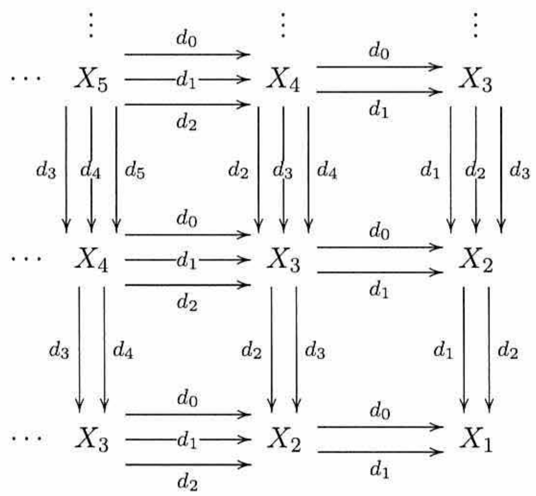
with $(\mathrm{diag}\, X)_n = X_{nn}$ (referring to picture (3.2.1.2), the diagonal is on the main-diagonal, going from bottom-right upwards and leftwards). We have $d_i(x) = d_i^h d_i^v(x) = d_i^v d_i^h(x)$ and similarly for the degeneracies.

The 'Total Dec'

Ordinal sum $+\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$ is defined by $[n]+[m] = [n+m+1]$. For morphisms $f\colon n \longrightarrow n'$ and $g\colon m \longrightarrow m'$ we have

$$(f+g)(i) = \begin{cases} f(i) & i = 0,\dots,n \\ n'+1+g(i-n-1) & i = n+1,\dots,n+m+1. \end{cases} \tag{3.2.3.2}$$

This induces a functor $\mathrm{Dec} = + \,\#\, (-)\colon \mathbf{SSet} \longrightarrow \mathbf{BiSSet}$ called *total dec*, which has $(\mathrm{Dec}\, X)_{p,q} = X_{p+q+1}$. The picture is



(3.2.3.3)

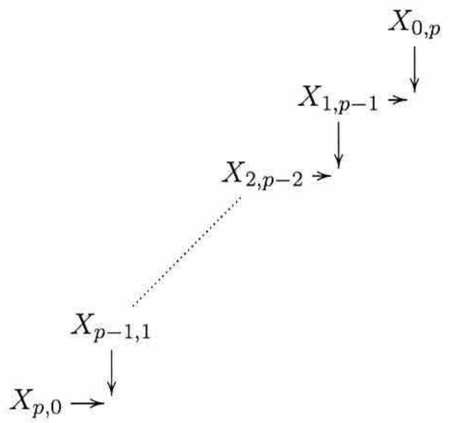
(for example, from $\delta_i\colon [p-1] \longrightarrow [p]$ and $\mathrm{id}_q\colon [q] \longrightarrow [q]$ we get $\delta_i + \mathrm{id}_n$ is the map $\delta_i\colon [(p+q+1)-1] \longrightarrow [p+q+1]$, from which d_i^h of $\mathrm{Dec}\, X$ is $d_i\colon X_{p+q+1} \longrightarrow X_{p+q}$, and similarly, $d_i^v = d_{p+i+1}$).

3.2.4 The ∇ construction

The functor Dec has a right adjoint, $\nabla\colon \mathbf{BiSSet} \longrightarrow \mathbf{SSet}$. From the desired adjunction isomorphism it is possible to construct ∇ directly, and this is done in [17], where the following definition is derived (and the adjunction proved in detail): Let X be a bisimplicial set, then $\nabla(X)$ is the simplicial set with

$$(\nabla X)_p = \left\{ (x_0, \dots, x_p) \colon x_i \in X_{i,p-i}, \text{ and } d_0^v(x_i) = d_{i+1}^h(x_{i+1}) \text{ for } 0 \leqslant i < p \right\}. \tag{3.2.4.1}$$

We can picture $\mathbf{x} = (x_0, \dots, x_p) \in \nabla(X)_p$ as lying on the pth 'codiagonal': (in the diagram below, vertical arrows are d_0^v, and horizontal arrows are d_{i+1}^h, so where two arrows meet is where the '$d_0^v(x_i) = d_{i+1}^h(x_{i+1})$' equality takes place).



(3.2.4.2)

The degeneracies are given by

$$d_0\mathbf{x} = (d_0^h(x_1), \dots, d_0^h(x_p)) \tag{3.2.4.3}$$

$$d_i\mathbf{x} = (d_i^v(x_0), d_{i-1}^v(x_1), \dots, d_1^v(x_{i-1}), d_i^h(x_{i+1}), d_i^h(x_{i+2}), \dots, d_i^h(x_p)) \tag{3.2.4.4}$$

$$s_i(\mathbf{x}) = (s_i^v(x_0), s_{i-1}^v(x_1), \dots, s_0^v(x_i), s_i^h(x_i), s_i^h(x_{i+1}), s_i^h(x_{i+2}), \dots, s_i^h(x_p)) \tag{3.2.4.5}$$

(note that the formula for d_0 is actually the same as the formula for the other faces). These are similar to the classical $\overline{W}$ construction defined in section 2.5.5, and indeed $\nabla(\operatorname{Ner}\mathcal{G})$ is the same as $\overline{W}\mathcal{G}$.

3.2.5 Example of ∇

To further understand (3.2.4.1), let us consider the example of $\nabla \operatorname{Ner}\mathcal{D}$, where $\mathcal{D}$ is a double category, and $\operatorname{Ner}\mathcal{D}$ is its double nerve as defined in section 3.2.2. Let us examine the 3-simplices, $\mathbf{x} = (x_0, x_1, x_2, x_3) \in (\nabla \operatorname{Ner}\mathcal{D})_3$, which are as follows: start with $x_0 \in \operatorname{Ner}\mathcal{D}_{0,3}$: it is 3 composable vertical maps from $\mathcal{D}$ (i.e., a 3-simplex of the nerve of the vertical structure of $\mathcal{D}$). We must have $d_0^v(x_0) = d_1^h(x_1)$, which determines one half of the objects in x_1; using $d_0^v(x_1) = d_2^h(x_2)$ then tells us something about x_2, and we can proceed from top-right to bottom-left as indicated in (3.2.5.1).

$$
\begin{array}{ccccccc}
 & & & & x_0 = & \begin{pmatrix} a \\ \downarrow \\ b \\ \downarrow \\ c \\ \downarrow \\ d \end{pmatrix} \in X_{0,3} \\
 & & & & & \downarrow \\
 & & x_1 = & \begin{pmatrix} b \to b' \\ \downarrow \quad \downarrow \\ c \to c' \\ \downarrow \quad \downarrow \\ d \to d' \end{pmatrix} & \longrightarrow & \begin{pmatrix} b \\ \downarrow \\ c \\ \downarrow \\ d \end{pmatrix} \\
 & & & \downarrow \\
x_2 = & \begin{pmatrix} c \to c' \to c'' \\ \downarrow \quad \downarrow \quad \downarrow \\ d \to d' \to d'' \end{pmatrix} & \longrightarrow & \begin{pmatrix} c \to c' \\ \downarrow \quad \downarrow \\ d \to d' \end{pmatrix} \\
 & \downarrow \\
\end{array}
$$

$$
x_3 = \begin{pmatrix} d \to d' \to d'' \to d''' \end{pmatrix} \longrightarrow \begin{pmatrix} d \to d' \to d'' \end{pmatrix} \tag{3.2.5.1}
$$

thus $\mathbf{x}$ is the '*staircase*'

$$
\begin{array}{ccccccc}
a \\
\downarrow \\
b & \longrightarrow & b' \\
\downarrow & & \downarrow \\
c & \longrightarrow & c' & \longrightarrow & c'' \\
\downarrow & & \downarrow & & \downarrow \\
d & \longrightarrow & d' & \longrightarrow & d'' & \longrightarrow & d'''
\end{array} \quad . \tag{3.2.5.2}
$$

The staircase above has 3 steps, (starting at a, we can take 0, 1, 2 or 3 steps down the staircase to get to d''' at the bottom; another way to think of this is that we can move from x_0 to x_i where i goes from 0 to 3). The face and degeneracies then act on the steps of this staircase picture: d_0 removes the zeroth step, (and with it the first column); d_1 removes the first step, (so the new staircase will have steps from a to c'' to d''', thus we must remove b', and compose squares and arrows so that we get a two-step staircase,

$$d_1(\mathbf{x}) = \left(\begin{array}{ccc} a & & \\ \downarrow & & \\ c & \longrightarrow & c'' & \\ \downarrow & & \downarrow & \\ d & \longrightarrow & d'' & \longrightarrow & d''' \end{array} \right) \tag{3.2.5.3}$$

i.e., something with a single square in the bottom-left corner); d_2 removes the second step, which takes out c'', giving

$$d_2(\mathbf{x}) = \left(\begin{array}{ccccc} a & & & & \\ \downarrow & & & & \\ b & \longrightarrow & b' & & \\ \downarrow & & \downarrow & & \\ d & \longrightarrow & d' & \longrightarrow & d''' \end{array} \right) \tag{3.2.5.4}$$

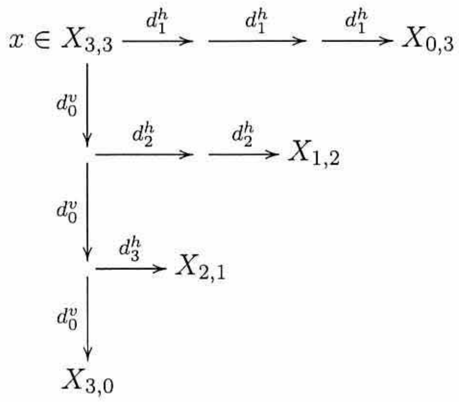
and finally d_3 removes the bottom row completely. Similarly, the face map s_i doubles the ith step in the appropriate way.

The map ϕ: diag $\longrightarrow \nabla$

There is a natural map from the diagonal to the ∇ construction, given (at level p) by

$$\phi(x) = \Big((d_1^h)^p(x), (d_2^h)^{p-1}(d_0^v)(x), \ldots, (d_{j+1}^h)^{p-j}(d_0^v)^j(x), \ldots, (d_0^v)^p(x) \Big) . \tag{3.2.5.5}$$

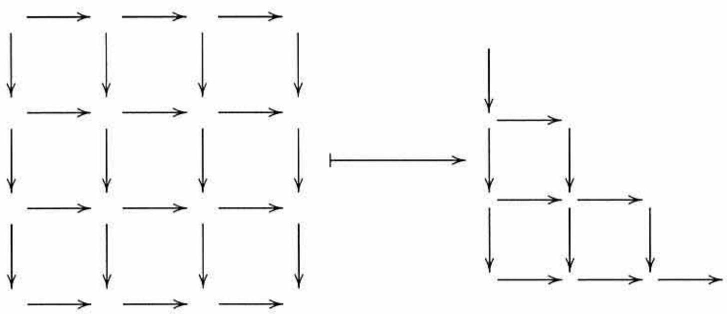
In general, remembering that elements of ∇X lie on the anti-diagonals (see (3.2.4.2)), ϕ can be pictured as follows (for $p = 3$ for brevity)



(3.2.5.6)

Example of ϕ

Returning to the example of section 3.2.5, the map $\phi\colon \operatorname{diag} \operatorname{Ner} \mathcal{D} \longrightarrow \nabla \operatorname{Ner} \mathcal{D}$ takes square grids to their 'sub-diagonal', so for $p = 3$ we have



(3.2.5.7)

ϕ is a weak equivalence

Theorem 3.2.5.8. *ϕ is always a weak equivalence*

Proof. See [9] for a proof. □

Combining this with proposition 3.3.2.10 gives us

Corollary 3.2.5.9. *For any $F\colon \mathbb{I} \longrightarrow \mathbf{SSet}$, we have a weak equivalence*

$$\operatorname{hocolim} F \xrightarrow{\sim} \nabla \coprod\nolimits_{\star} F \tag{3.2.5.10}$$

3.3 Homotopy Colimits

3.3.1 Definition

Slice Categories

Let $\mathbb{I}$ be a category, and $I \in \mathbb{I}$ an object. We get a functor $-/\mathbb{I} \colon \mathbb{I}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ sending I to the slice category $I/\mathbb{I}$. On maps we send $u \colon I \longrightarrow J$ to the functor $u/\mathbb{I} \colon J/\mathbb{I} \longrightarrow I/\mathbb{I}$ where

$$\begin{array}{ccc} & u/\mathbb{I} \colon J/\mathbb{I} \longrightarrow I/\mathbb{I} & \\ & \begin{pmatrix} J \\ \downarrow \alpha \\ K \end{pmatrix} \longmapsto \begin{pmatrix} I \\ \downarrow u\#\alpha \\ K \end{pmatrix} & \\ v \colon \begin{pmatrix} J \\ \downarrow \alpha \\ K \end{pmatrix} \longrightarrow \begin{pmatrix} J \\ \downarrow \beta \\ K' \end{pmatrix} & \longmapsto & v \colon \begin{pmatrix} J \\ \downarrow u\alpha \\ K \end{pmatrix} \longrightarrow \begin{pmatrix} J \\ \downarrow u\beta \\ K' \end{pmatrix} \end{array} \tag{3.3.1.1}$$

(i.e., $u/\mathbb{I}$ acts as the 'identity' on maps)

The bifunctor for homotopy colimit and homotopy limit

We get a functor $\mathrm{Ner} -/\mathbb{I} \longrightarrow \mathbf{SSet}$ sending I to $\mathrm{Ner}\, I/\mathbb{I}$, where

$$(\mathrm{Ner}\, I/\mathbb{I})_n = \left\{ \begin{array}{c} I \overset{f_0}{\swarrow} \cdots \\ I_0 \xrightarrow[f_1]{} I_1 \xrightarrow[f_2]{} \cdots \xrightarrow[f_n]{} I_n \end{array} \right\}, \tag{3.3.1.2}$$

the set of strings of n composable maps under I—the dotted arrows may be inferred from the arrows $f_0, f_1, \ldots, f_n$, so we may omit them, giving the description

$$(\mathrm{Ner}\, I/\mathbb{I})_n = \{ (f_0 | f_1, \ldots, f_n) \colon \operatorname{dom} f_0 = I, (f_0, \ldots, f_n) \in (\mathrm{Ner}\, \mathbb{I})_{n+1} \}. \tag{3.3.1.3}$$

Then, from our given $F \colon \mathbb{I} \longrightarrow \mathbf{SSet}$, we can form $\mathrm{Ner} -/\mathbb{I} \times F$ which is a bifunctor $\mathbb{I}^{\mathrm{op}} \times \mathbb{I} \longrightarrow \mathbf{SSet}$, and this will have a coend, $\int^I \mathrm{Ner}\, I/\mathbb{I} \times FI$ which is the *homotopy colimit* of F.

3.3.2 Constructing the homotopy colimit

Coend construction

The standard construction of a coend $\int^I f(I, I)$ for a functor $f \colon \mathbb{I} \longrightarrow \mathcal{C}$ is

$$\int^I f(I, I) = \mathrm{CoEqualiser} \left(\coprod_{u \colon I \longrightarrow J} f(J, I) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{I \in \mathbb{I}} f(I, I) \right), \tag{3.3.2.1}$$

where the maps α and β are defined to make the following squares commute.

$$\begin{array}{ccc} f(J,I) & \xrightarrow{f(\mathrm{id},u)} & f(J,J) \\ {\scriptstyle i_u}\downarrow & & \downarrow{\scriptstyle i'_J} \\ \coprod_u f(J,I) & \xrightarrow[\alpha]{} & \coprod_I f(I,I) \end{array} \tag{3.3.2.2a}$$

$$\begin{array}{ccc} f(J,I) & \xrightarrow{f(u,\mathrm{id})} & f(I,I) \\ {\scriptstyle i_u}\downarrow & & \downarrow{\scriptstyle i'_I} \\ \coprod_u fI & \xrightarrow[\beta]{} & \coprod_I fI \end{array} \tag{3.3.2.2b}$$

First we need to understand the maps from (3.3.2.2): here the 'f' now becomes $\mathrm{Ner}\,-/\mathbb{I} \times F$. First consider the domain of α and β, $\coprod_{u\colon I \longrightarrow J} \mathrm{Ner}\, J/\mathbb{I} \times FI$. We will frequently represent a coproduct $\coprod_{a \in A} B(a)$ as the set $\{\, a, B(a) \,\}$, then the domain of α and β has, as n-simplices, the set

$$\{\, (u, (f_0|f_1, \dots, f_n), x) \colon x \in (FI)_n \,\} \tag{3.3.2.3}$$

where $(f_0|f_1, \dots, f_n)$ is the notation we used in (3.3.1.3) for an n-simplex in the nerve of the slice category.

Then from (3.3.2.2) we see that α sends the triple $(u, (f_0|f_1, \dots, f_n), x)$ to $(J, (f_0|f_1, \dots, f_n), x^u)$ (which is in $\coprod_I \mathrm{Ner}\, I/\mathbb{I} \times FI$ and, as usual, x^u is $Fu(x) \in (FJ)_n$). Similarly, β takes that same triple to $(I, (u \# f_0|f_1, \dots, f_n), x)$, and hence the construction (3.3.2.1) says that

$$(\mathrm{hocolim}\, F)_n \cong \frac{\{\, (I, (f_0|f_1, \dots, f_n), x \in (FI)_n) \,\}}{(I, (I \xrightarrow{u} J \xrightarrow{f_0} I_0|f_1, \dots, f_n), x) \sim (J, (f_0|f_1, \dots, f_n), x^u)} \tag{3.3.2.4}$$

If we write elements of the above as $x \otimes (f_0|f_1, \dots, f_n)$, then the relations say that

$$x \otimes (u \# f_0|f_1, \dots, f_n) = x^u \otimes (f_0|f_1, \dots, f_n) \tag{3.3.2.5}$$

so we have a 'tensor product'.

Simplicial replacement

Working from (3.3.2.4), the relations say that

$$\begin{aligned} (I, (f_0|f_1, \dots, f_n), x \in FI_n) &= (I, (I \xrightarrow{f_0} I_0 \xrightarrow{\mathrm{id}} I_0|f_1, \dots, f_n), x \in FI_n) \\ &\sim (I_0, (\mathrm{id}_{I_0}|f_1, \dots, f_n), x^{f_0} \in (FI_0)_n) \end{aligned} \tag{3.3.2.6}$$

Now $(\mathrm{id}\,|\,f_1,\ldots,f_n)$ gives us the string $(f_1,\ldots,f_n) \in \operatorname{Ner}\mathbb{I}$, and we get a simplicial isomorphism

$$\begin{aligned}\operatorname{hocolim} F &\cong \{\,(I_0,(f_1,\ldots,f_n),y)\colon I_0 \in \mathbb{I}, (f_1,\ldots,f_n) \in \operatorname{Ner}\mathbb{I}, \operatorname{dom} f_1 = I_0, y \in (FI_0)_n\,\} \\ &\cong \coprod_{\left(I_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} I_n\right) \in (\operatorname{Ner}\mathbb{I})_n} (FI_0)_n\end{aligned} \tag{3.3.2.7}$$

Define the *simplicial replacement,* $\coprod_\star F$ for F to be the bisimplicial set with

$$(\textstyle\coprod_\star F)_{p,q} := \coprod_{\left(I_0 \xrightarrow{f_1} \cdots \xrightarrow{f_p} I_p\right) \in (\operatorname{Ner}\mathbb{I})_p} (FI_0)_q \tag{3.3.2.8}$$

This is a bisimplicial set as follows: the definition above says that (p,q)-simplices are of the form $(x,\mathbf{a})$ with $x \in F(A_0)_p$ and $\mathbf{a} = \left(A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_q} A_q\right) \in (\operatorname{Ner}\mathcal{C})_q$. The horizontal simplicial structure is that of $F(A_0)$, meaning

$$d_i^h(x,\mathbf{a}) = (d_i x, \mathbf{a}) \tag{3.3.2.9a}$$

$$s_i^h(x,\mathbf{a}) = (s_i x, \mathbf{a}) \tag{3.3.2.9b}$$

while the vertical structure comes from $\operatorname{Ner}\mathcal{C}$:

$$d_i^v(x,\mathbf{a}) = \begin{cases} (x, d_i\mathbf{a}) & i \geqslant 1 \\ (x^{a_1}, d_0\mathbf{a}) & i = 0 \end{cases} \tag{3.3.2.9c}$$

$$s_i^v(x,\mathbf{a}) = (x, s_i\mathbf{a}) \tag{3.3.2.9d}$$

the 'twist' in d_0^v is because when we apply d_0 to $\mathbf{a}$, we remove A_0, leaving a string of arrows starting from A_1. Thus for the result to be a simplex of $\coprod_\star F$, we need to move the x from $F(A_0)$ to $F(A_1)$.

Then we have shown

Proposition 3.3.2.10. *For any $F\colon \mathbb{I} \longrightarrow$ **SSet**, its homotopy colimit is*

$$\operatorname{hocolim} F \cong \operatorname{diag} \textstyle\coprod_\star F \tag{3.3.2.11}$$

3.3.3 An Example of a homotopy colimit

Consider the category

$$\mathbb{I} = \left(\begin{array}{ccc} 0 & \xrightarrow{01} & 1 \\ {\scriptstyle 02}\downarrow & & \\ 2 & & \end{array}\right) \tag{3.3.3.1}$$

whose limit is a pushout. For a functor $F\colon \mathbb{I} \longrightarrow \mathbf{SSet}$ the homotopy colimit is a *homotopy pushout*, which we will construct using the coend construction of section 3.3.2.

A functor $F\colon \mathbb{I} \longrightarrow \mathbf{SSet}$ is a 'corner'

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\big\downarrow & & \\ C & & \end{array} \tag{3.3.3.2}$$

The first step is to understand $\operatorname{Ner} I/\mathbb{I} \times FI$ for $I \in \mathbb{I}$.

- When $I = 1$, $1/\mathbb{I}$ has just one object, $\left(\begin{array}{c} 1 \\ \big\downarrow{\scriptstyle \mathrm{id}} \\ 1 \end{array}\right)$, and just one (identity) morphism: $1/\mathbb{I} \cong [0]$, therefore $\operatorname{Ner} 1/\mathbb{I} \cong \Delta[0]$, the terminal object of $\mathbf{SSet}$, and $\operatorname{Ner} 1/\mathbb{I} \times F1 \cong F1 = B$
- When $I = 2$, a similar argument gives $\operatorname{Ner} 2/\mathbb{I} \times F2 \cong C$
- $I = 0$ is the initial object of $\mathbb{I}$, so $0/\mathbb{I} \cong \mathbb{I}$, which we can see explicitly, as $0/\mathbb{I}$ looks like this

$$\begin{array}{ccc} \left(\begin{array}{c} 0 \\ \big\downarrow{\scriptstyle \mathrm{id}} \\ 0 \end{array}\right) & \longrightarrow & \left(\begin{array}{c} 0 \\ \big\downarrow{\scriptstyle 01} \\ 1 \end{array}\right) \\ \big\downarrow & & \\ \left(\begin{array}{c} 0 \\ \big\downarrow{\scriptstyle 02} \\ 2 \end{array}\right) & & \end{array} \tag{3.3.3.3}$$

  So $\operatorname{Ner} 0/\mathbb{I} \cong \operatorname{Ner} \mathbb{I}$, so we must understand $\operatorname{Ner} \mathbb{I}$.

n	$(\operatorname{Ner} \mathbb{I})_n$
0	$\operatorname{Ob} \mathbb{I} = \{0, 1, 2\}$
1	$0 \longrightarrow 1$, $0 \longrightarrow 2$ plus degenerate arrows (identities)
2	composable pairs, (e.g., $0 \longrightarrow 1 \longrightarrow 1$), all of which are degenerate

  so in higher dimensions everything is degenerate, and we see that $\operatorname{Ner} \mathbb{I}$ is generated by $0 \longrightarrow 1$ and $0 \longrightarrow 2$ in dimension 2.[2] Moreover, since the non-degenerate

[2] *Generates* means that every element (in every dimension) may be obtained from these two by repeatedly applying face and degeneracy maps—See [17].

1-simplices have equal d_1 faces we see that $\operatorname{Ner}\mathbb{I}$ is a pushout in **SSet**,

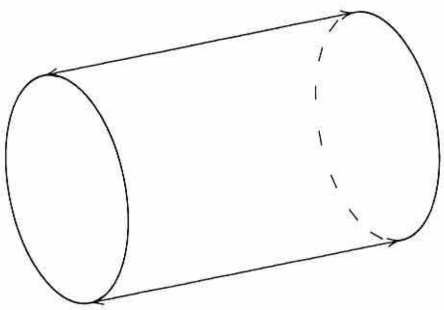
$$\begin{array}{ccc} \Delta[0] & \xrightarrow{\Delta[\delta_1]} & \Delta[1] \\ {\scriptstyle \Delta[\delta_0]}\downarrow & & \downarrow \\ \Delta[1] & \longrightarrow & \operatorname{Ner}\mathbb{I} \end{array} \tag{3.3.3.4}$$

($\Delta[\delta_i]$ is the image of the basic face map defining the ith face; alternatively it is the representing map for vertex i in $\Delta[1]$: $\Delta[1](\mathrm{id}_0) = i$, and by the Yoneda embedding this defines the whole map. The unnamed maps pick out the non-degenerate simplices of $\operatorname{Ner}\mathbb{I}$),

The above shows that $\operatorname{Ner}\mathbb{I}$ is two copies of $\Delta[1]$ glued at the ends—a subdivided interval

(3.3.3.5)

and hence $A \times \operatorname{Ner}\mathbb{I}$ is a subdivided cylinder with cross-section A



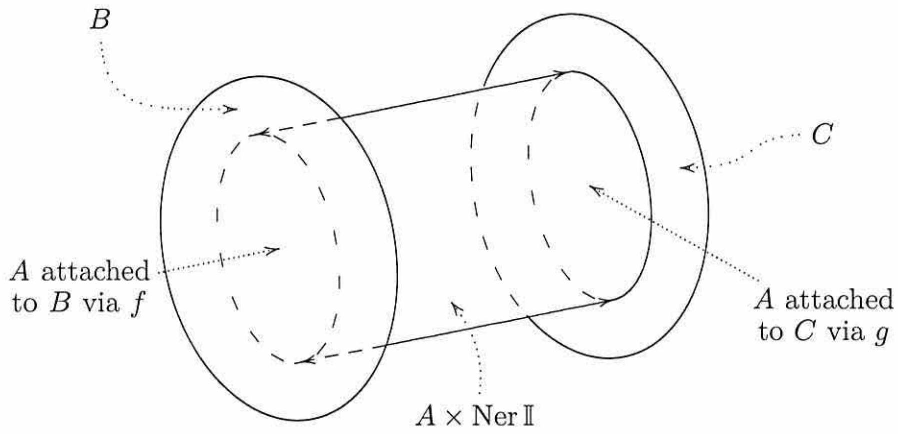
(3.3.3.6)

To finish the construction of the homotopy pushout, we now glue B and C to the above cylinder by the relations (3.3.2.4). For example, (using the same notation as (3.3.2.4), when $u = 0 \longrightarrow 1$, we have that the f_i must be identities id_1, and we get the relation

$$(0, (0 \xrightarrow{u} 1 \xrightarrow{\mathrm{id}} 1 |\, \mathrm{id}, \ldots, \mathrm{id}), a \in A_n) \sim (1, (\mathrm{id} \,|\, \mathrm{id}, \ldots, \mathrm{id}), f(a)) \tag{3.3.3.7}$$

the left hand side is the copy of A at the right-hand end of the cylinder (3.3.3.6), and right hand side is just $f(a) \in B$, so we are saying that B is glued to the end of the cylinder via f. Taking $u = 0 \longrightarrow 2$ tells us to glue C to the other end of the cylinder, and the only other possibilities are $u = \mathrm{id}$, but these give us nothing new as they just say things like $(0, (0 \xrightarrow{\mathrm{id}} 0 \xrightarrow{\mathrm{id}} 0 |\, \mathrm{id}, \ldots, \mathrm{id}), a \in A_n) \sim (0, (\mathrm{id} \,|\, \mathrm{id}, \ldots, \mathrm{id}), a^{\mathrm{id}} = a)$ i.e., they identify already equal elements.

Thus the homotopy pushout is a *double mapping cylinder*, which we picture as follows.



(3.3.3.8)

3.3.4 Thomason's theorem on homotopy colimits in Cat

We now turn to Thomason's theorem, which tells us how to compute homotopy colimits of functors into **Cat**: if $F\colon \mathcal{C} \longrightarrow \mathbf{Cat}$ is a functor from a small index category $\mathcal{C}$, then we can compose F with the nerve functor to get $\operatorname{Ner} F\colon \mathcal{C} \longrightarrow \mathbf{SSet}$, which will have a homotopy colimit, $\operatorname{hocolim} \operatorname{Ner} F$: Thomason's theorem says that this simplicial set is $\operatorname{Ner}(\mathcal{C} \int F)$, the nerve of the category $\mathcal{C} \int F$ from section 3.1.

The double category $\mathbb{D}(F)$

Let $F\colon \mathcal{C} \longrightarrow \mathbf{Cat}$ be a functor, and write x^a for $Fa(x) \in FB$ whenever $x \in FA$ and $a\colon A \longrightarrow B$. Define the *Thomason double category of* F, $\mathbb{D}(F)$, by taking squares of the form

$$\begin{array}{ccc} (A,x) & \xrightarrow{(A,f)} & (A,y) \\ {\scriptstyle (a,x)}\downarrow & (a,f) & \downarrow{\scriptstyle (a,y)} \\ (B,x^a) & \xrightarrow[(B,f^a)]{} & (B,y^a) \end{array} \tag{3.3.4.1}$$

So: squares are (a, f) with $a\colon A \longrightarrow B$ in $\mathcal{C}$ and $f\colon x \longrightarrow y$ a map in FA; horizontal arrows are (A, f) with $A \in \mathcal{C}$ and $f\colon x \longrightarrow y$ in FA. vertical arrows are (a, x) with $a\colon A \longrightarrow B$ in $\mathcal{C}$ and x an object of FA. Thus a square is actually determined by its domain arrows (top and left sides).

Proof of Thomason's theorem

Lemma 3.3.4.2.

$$\operatorname{Ner}(\mathbb{D}(F)) \cong \coprod\nolimits_{\star}(\operatorname{Ner} F) \tag{3.3.4.3}$$

Proof. A typical element of $\operatorname{Ner}\mathbb{D}(F)_{p,q}$ is exactly determined by its top and left-hand edge:; the top is an element of $(\operatorname{Ner}\mathcal{C}_p)$ and the edge is in $(\operatorname{Ner} FA_0)_q$.

$$\begin{array}{ccccccc}
(A_0, x_0) & \xrightarrow{(A_0,f_1)} & (A_0, x_1) & \xrightarrow{(A_0,f_2)} & \cdots & \xrightarrow{(A_0,f_q)} & (A_0, x_p) \\
{\scriptstyle (a_1,x_0)}\downarrow & & \vdots & & & & \vdots \\
(A_1, {x_0}^{a_0}) & \dashrightarrow & \cdot & \dashrightarrow & \cdots & \dashrightarrow & \cdot \\
{\scriptstyle (a_2,{x_0}^{a_0})}\downarrow & & \vdots & & & & \vdots \\
\vdots & & \vdots & & \ddots & & \vdots \\
{\scriptstyle (a_q,x_0^{a_0a_1\cdots a_{q-1}})}\downarrow & & \vdots & & & & \vdots \\
(A_q, x_0^{a_0\cdots a_{q-1}a_q}) & \dashrightarrow & \cdot & \dashrightarrow & \cdots & \dashrightarrow & \cdot
\end{array} \qquad (3.3.4.4)$$

□

Lemma 3.3.4.5. *For any functor* $F\colon \mathcal{C} \longrightarrow \mathbf{Cat}$*,*

$$\nabla \operatorname{Ner}\mathbb{D}(F)\mathcal{C} \int F \qquad (3.3.4.6)$$

Proof. We saw how to calculate ∇ of a nerve of a double category in section 3.2.5, so taking $\mathcal{D} = \operatorname{Ner}\mathbb{D}(F)$ we have that elements of $(\nabla \operatorname{Ner}\mathbb{D}(F))_3$ are of the form

$$\begin{array}{ccccccc}
(A, x) & & & & & & \\
\downarrow{\scriptstyle (a,x)} & & & & & & \\
(B, x^a) & \xrightarrow{(B,f)} & (B, y) & & & & \\
\downarrow{\scriptstyle (b,x^a)} & & \downarrow{\scriptstyle (b,y)} & & & & \\
(C, x^{ab}) & \xrightarrow{(C,f^b)} & (C, y^b) & \xrightarrow{(C,g)} & (D, z) & & \\
\downarrow{\scriptstyle (c,x^{ab})} & & \downarrow{\scriptstyle (c,y^b)} & & \downarrow{\scriptstyle (c,z)} & & \\
(D, x^{abc}) & \xrightarrow{(D,f^{bc})} & (D, y^{bc}) & \xrightarrow{(D,g^c)} & (D, z^c) & \xrightarrow{(D,h)} & (D, w)
\end{array} \qquad (3.3.4.7)$$

which corresponds to (and is determined by)

$$(A, x) \xrightarrow{(a,f)} (B, y) \xrightarrow{(b,g)} (C, z) \xrightarrow{(c,h)} (D, w) \qquad (3.3.4.8)$$

giving us a simplicial isomorphism as required. □

To finish the proof we note that we have a weak equivalence built as follows:

$$\begin{aligned}
\operatorname{hocolim} \operatorname{Ner} F &\cong \operatorname{diag} \coprod\nolimits_{\star} (\operatorname{Ner} F) \\
&\cong \operatorname{diag} \operatorname{Ner} \mathbb{D}(F) \\
&\xrightarrow{\phi} \nabla \operatorname{Ner} \mathbb{D}(F) \\
&\cong \operatorname{Ner} \mathcal{C} \int F.
\end{aligned} \tag{3.3.4.9}$$

This completes the proof of Thomason's theorem.

4 Bundles and Twisted Cartesian Products

We recall the notion of twisted Cartesian products and recall how they are classified by homotopy classes of maps into the classifying space of a simplicial group: in the next chapter we generalise this result, replacing 'group' with 'crossed module' and look at a generalisation of 'classifying space' and what that might classify.

4.1 Fibrations and Fibre Bundles

We follow both [40, 11] in this section. We wish to consider 'locally trivial' structures over some base simplicial set, and the notion of fibre bundle, and its representation as a 'twisted Cartesian product' provide the structure we need.

4.1.1 Bundles

A bundle over a topological space B is an element of the slice category $\mathbf{Top}/B$.[1] We often require extra structure on such an object (i.e., choose a suitable subcategory), but in a general $\mathcal{C}$ it is not clear a priori which subcategory to choose, so introduce the following general concept. (This viewpoint is taken from [3].)

In any category $\mathcal{C}$, the category of *bundles over* B is the slice category $\mathcal{C}/B$. The *total space* of a bundle $\left(\begin{array}{c} E \\ \downarrow p \\ B \end{array}\right)$ is just the object E. The *trivial bundles with fibre* $Y \in \mathcal{C}$ are those (isomorphic to) the projections from a product with Y, $Y \times B \longrightarrow B$, and a *locally trivial* bundle is a p for which there is some pullback diagram

$$\begin{array}{ccc} Y \times A & \longrightarrow & E \\ {\scriptstyle \mathrm{pr}}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array} \tag{4.1.1.1}$$

Here we say f *trivialises* the bundle p, and instead of a single map f, it can also be useful to require a family of trivialisations (often this is equivalent since we can define A to be the coproduct of these and get a single trivialisation). For our purposes in simplicial categories we just want our trivialisations to be representing maps: $f = \overline{b}$ to be a representing map, for some b from B. If B was a topological space, we would look

[1] Note that the slice categories in this chapter are dual to those used in section 3.3.1.

at inclusions of open sets covering B (so we could have one map for each open set in the cover, or form the coproduct of all such).

4.1.2 Kan Fibrations

We will be using some of the well-known theory of Kan fibrations from [11] and other places.

Definition 4.1.2.1. A bundle $\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ in **SSet** is a *Kan fibration* if every commuting square

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{h} & E \\ {\scriptstyle \text{inclusion}} \downarrow & & \downarrow {\scriptstyle p} \\ \Delta[n] & \xrightarrow[w]{} & B \end{array} \tag{4.1.2.2}$$

induces a map $\Delta[n] \longrightarrow E$ making the resulting triangles commute.

Interpreting definition 4.1.2.1 we see that p is a Kan fibration iff the following occurs: whenever we have a horn, h, in E whose image under p has a filler w in B, then we can lift w to a filler for the original horn in E.

A simplicial set E is called *Kan* if the unique map to the terminal $\Delta[0]$ is a Kan fibration.

Here is an application of the diagram-based definition.

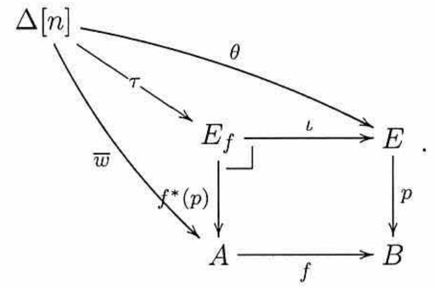
Proposition 4.1.2.3. *The property "being a Kan fibration" is pullback-stable*

Proof. Let p be a Kan fibration and f a simplicial map. We must show that the pullback bundle $f^*(p)$ is a Kan fibration, so consider a square of the form (4.1.2.2), and adjoin the pullback square defining $f^*(p)$

$$\begin{array}{ccccc} \Lambda^k[n] & \xrightarrow{h} & E_f & \xrightarrow{\iota} & E \\ {\scriptstyle \text{inclusion}} \downarrow & & \downarrow {\scriptstyle f^*(p)} & \lrcorner & \downarrow {\scriptstyle p} \\ \Delta[n] & \xrightarrow[w]{} & A & \xrightarrow[f]{} & B \end{array} . \tag{4.1.2.4}$$

Because p is a fibration, the combined square has a diagonal filling map $\theta \colon \Delta[n] \longrightarrow E$.

We have $\theta \# p = \overline{w} \# f$ and, since E is a pullback, we get a map $\tau \colon \Delta[n] \longrightarrow E_f$



(4.1.2.5)

We just need to show that τ is a filler for the left-hand square of (4.1.2.4). The bottom triangle of (4.1.2.5) is the same as the bottom triangle of (4.1.2.4), so we just need to show that (inclusion) $\# \, \tau = h$. To see this we use the fact that ι and $f^*(p)$ are jointly monic: we have (inclusion) $\# \, \tau \# \iota =$ (inclusion) $\# \, \theta = h \# \iota$ and (inclusion) $\# \, \tau \# f^*(p) =$ (inclusion) $\# \, \overline{w} = h \# f^*(p)$. $\square$

Proposition 4.1.2.6. *$\underline{S}(A, -)$ preserves Kan fibrations, i.e., if* $\left(\begin{array}{c} E \\ \downarrow p \\ B \end{array}\right)$ *is a Kan fibration then so is* $\left(\begin{array}{c} \underline{S}(A,E) \\ \downarrow \underline{S}(A,p) \\ \underline{S}(A,B) \end{array}\right)$.

Proof. There is a direct proof in [11], but we can (almost) see this directly as follows. Given a square of the form (4.1.2.2) (with p replaced by $\underline{S}(A,p)$), transpose it across the adjunction $A \times - \dashv \underline{S}(A,-)$ and observe that the outside of the diagram

$$\begin{array}{ccccc} A \times \Lambda^k[n] & \xrightarrow{A \times h} & A \times \underline{S}(A,E) & \xrightarrow{\#} & E \\ {\scriptstyle A \times \text{inclusion}} \downarrow & & \downarrow {\scriptstyle A \times \underline{S}(A,p)} & & \downarrow {\scriptstyle p} \\ A \times \Delta[n] & \xrightarrow[A \times \overline{w}]{} & A \times \underline{S}(A,B) & \xrightarrow[\#]{} & B \end{array} \qquad (4.1.2.7)$$

commutes. Then, since p is a fibration, we can (for example by constructing the map at each dimension by induction, see also [22]) fill the rectangle by $\theta \colon A \times \Delta[n] \longrightarrow E$, whose transpose fills the original square for $\underline{S}(A,p)$. $\square$

In the next proposition we use the more explicit 'horn lifting' property.

Proposition 4.1.2.8. *Let* $\left(\begin{array}{c} E \\ \big\downarrow p \\ B \end{array}\right)$ *be a Kan fibration and let* $A \overset{f}{\underset{g}{\rightrightarrows}} B$ *be homotopic via* $F\colon f \xrightarrow{\simeq} g$. *Then the bundles induced by pulling-back over* A *are homotopy equivalent:* $f^*(p) \simeq g^*(p)$.

Proof. This proof is taken from [40], but changed to use a more modern, categorical, notation. First we construct the pullback $f^*(p)$ by writing $f = d_1(F)$ as the composite $(A \times \Delta[\delta_1]) \# F$,

$$\begin{array}{ccccc} E_f & \xrightarrow{\iota_f} & E^* & \longrightarrow & E \\ {\scriptstyle f^*(p)}\big\downarrow & & \big\downarrow{\scriptstyle p^*} & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow[A\times\Delta[\delta_1]]{} & A \times \Delta[1] & \xrightarrow[F]{} & B \end{array} \tag{4.1.2.9}$$

where we have identified A and $A \times \Delta[0]$. Similarly, for $g = d_0(F)$, we get

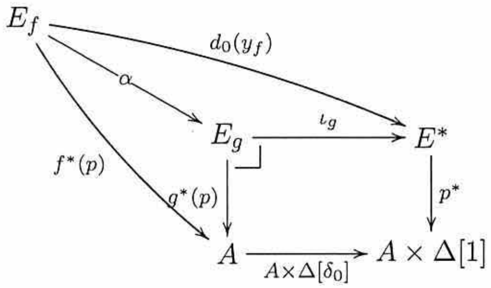
$$\begin{array}{ccccc} E_g & \xrightarrow{\iota_g} & E^* & \longrightarrow & E \\ {\scriptstyle g^*(p)}\big\downarrow & & \big\downarrow{\scriptstyle p^*} & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow[A\times\Delta[\delta_0]]{} & A \times \Delta[1] & \xrightarrow[F]{} & B \end{array}. \tag{4.1.2.10}$$

Now $q_f := \underline{S}(E_f, -)(p^*)\colon \underline{S}(E_f, E^*) \longrightarrow \underline{S}(E_f, A \times \Delta[1])$ is a Kan fibration by propositions 4.1.2.3 and 4.1.2.6, and similarly we have $q_g := \underline{S}(E_g, -)(p^*)$. Recall that these maps are simply "compose with p^*", for example $q_f(\iota_f)$ is the diagonal of the right-hand pullback square in (4.1.2.9). By lemma 2.2.1.5 we have $q_f(\iota_f) = f^*(p) \times \Delta[\delta_1] = d_1(f^*(p) \times \Delta[1])$. But now we have a horn $(-, \iota_f)$ in $\underline{S}(E_f, E^*)_0$ whose image, $(-, q_f(\iota_f))$, under q_f has a filler $f^*(p) \times \Delta[1] \in \underline{S}(E_f, A \times \Delta[1])_1$; therefore, since q_f is a Kan fibration, we can lift the filler $f^*(p) \times \Delta[1]$ to some 1-simplex $y_f \in \underline{S}(E_f, E^*)_1$ with $d_1(y_f) = \iota_f$ (and $q_f(y_f) = f^*(p) \times \Delta[1]$).

We now claim that the other end, $d_0(y_f) \in \underline{S}(E_f, A \times \Delta[1])_0$, induces a map, α between our induced bundles. To see this, calculate

$$d_0(y_f) \# p^* = q_f(d_0(y_f)) \qquad \text{(definition of } q_f\text{)} \tag{4.1.2.11}$$
$$= d_0(q_f(y_f)) \qquad (q_f \text{ is a simplicial map}) \tag{4.1.2.12}$$
$$= d_0(f^*(p) \times \Delta[1]) \qquad \text{(defining property of } y_f\text{)} \tag{4.1.2.13}$$
$$= f^*(p) \# (A \times \Delta[\delta_0]) \qquad \text{(by lemma 2.2.1.5)} \tag{4.1.2.14}$$

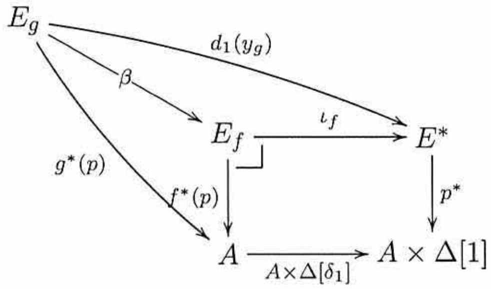
and then α is the factorisation through the right-hand pullback square in (4.1.2.10)



(4.1.2.15)

(the lower triangle says that α is a map $f^*(p) \longrightarrow g^*(p)$ of bundles. ι_g is monic as it is the pullback of the monic $A \times \Delta[\delta_0]$, so the upper triangle commuting says that α is $d_0(y_f)$ 'corestricted to its image').

A similar argument produces first $y_g \in \underline{S}(E_g, E^*)_1$ with $d_0(y_g) = \iota_g$ and then $q_g(y_g) = g^*(p) \times \Delta[1]$, and finally β making



(4.1.2.16)

commute.

We now have maps between the two bundles over A, and we must show that the composites $\alpha \# \beta$ and $\beta \# \alpha$ are homotopic to identities. To this end, regard α as a 0-simplex of $\underline{S}(E_f, E_g)$. Then, using the composition defined on the internal hom (2.2.2.2), we have an element $\xi := s_0(\alpha) \# y_g \in \underline{S}(E_f, E^*)_1$. We can write ξ as $E_f \times \Delta[1] \xrightarrow{\alpha \times \Delta[1]} E_g \times \Delta[1] \xrightarrow{y_g} E^*$ using (2.2.2.3) and the construction of y_g. We also have

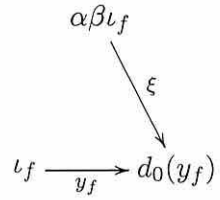
$$\begin{aligned} d_0(\xi) &= d_0 s_0(\alpha) \# d_0(y_g) \\ &= \alpha \# \iota_g \\ &= d_0(y_f) \end{aligned} \tag{4.1.2.17}$$

and

$$\begin{aligned} d_1(\xi) &= d_1 s_0(\alpha) \# d_1(y_g) \\ &= \alpha \# \beta \# \iota_f, \end{aligned} \tag{4.1.2.18}$$

which gives us a horn, h say, in $\underline{S}(E_f, E^*)_2$.

$h =$

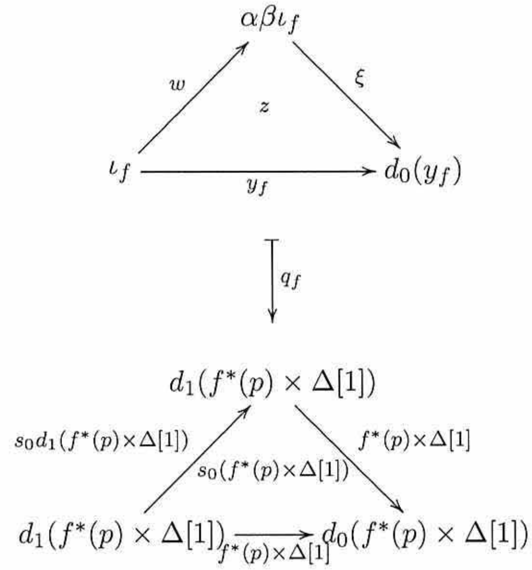


(4.1.2.19)

Now $q_f(y_f) = f^*(p) \times \Delta[1]$ by the definition of y_f, and

$$\begin{aligned}
q_f(\xi) &= \xi \# p^* && && (4.1.2.20)\\
&= (\alpha \times \Delta[1]) \# y_g \# p^* && \text{(definition of } \xi) && (4.1.2.21)\\
&= (\alpha \times \Delta[1]) \# (g^*(p) \times \Delta[1]) && \text{(definition of } y_g) && (4.1.2.22)\\
&= (\alpha \# g^*(p)) \times \Delta[1] && && (4.1.2.23)\\
&= f^*(p) \times \Delta[1] && \text{(definition of } \alpha) && (4.1.2.24)
\end{aligned}$$

which means that the image of the horn h under q_f is filled by $s_0(f^*(p) \times \Delta[1])$ in $\underline{S}(E_f, A \times \Delta[1])_2$. Since q_f is Kan we can lift to get a filler, $z \in \underline{S}(E_f, E^*)_2$, for h. Let $w := d_2(z) \in \underline{S}(E_f, E^*)_1$.

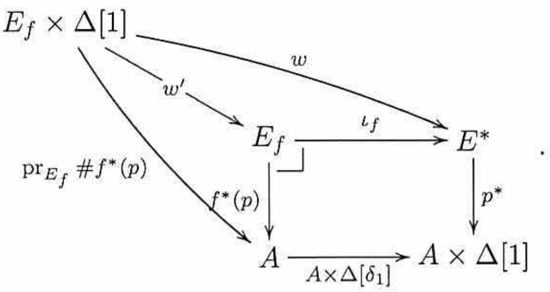


(4.1.2.25)

This w is almost the required homotopy $\alpha \# \beta \simeq \mathrm{id}$, but it has the wrong codomain, so we calculate

$$\begin{aligned}
w \#_0 p^* &= q_f(w) && \text{(definition of } q_f) && (4.1.2.26)\\
&= s_0 d_1(f^*(p) \times \Delta[1]) && \text{(definition of } w) && (4.1.2.27)\\
&= s_0(f^*(p) \# (A \times \Delta[\delta_1])) && \text{(by lemma 2.2.1.5 with } n = 1) && (4.1.2.28)\\
&= \mathrm{pr}_{E_f} \# f^*(p) \# (A \times \Delta[\delta_1]) && \text{(by lemma 2.2.1.12)} && (4.1.2.29)
\end{aligned}$$

which tells us that w factors through the pullback to give us $w' \colon E_f \times \Delta[1] \longrightarrow E_f$



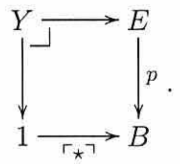
(4.1.2.30)

This says that w' is part of a strong homotopy (i.e., has the identity on the base) on the bundle $f^*(p)$, and further $d_i(w') \# \iota_f = (E_f \times \Delta[\delta_i]) \# w' \# \iota_f = (E_f \times \Delta[\delta_i]) \# w = d_i(w)$ hence $d_0(w') \# \iota_f = \alpha\beta\iota_f$ giving $d_0(w') = \alpha \# \beta$ because ι_f is monic. Similarly $d_1(w') = \mathrm{id}$ so w' is the required homotopy from the identity to $\alpha\beta$.

A similar construction gives a homotopy between $\beta\alpha$ and the identity and we are done. □

4.1.3 Fibres

The *fibre*, Y, of a bundle $\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ over a point $\star$ is the inverse image $Y = p^{-1}(\star)$. In a general category $\mathcal{C}$ (with finite limits), interpret a point, $\star$, of B as a map, $\ulcorner \star \urcorner$, from the terminal object to B, and the inverse image as the pullback



(4.1.3.1)

From the above diagram we can also see that the sequence $Y \longrightarrow E \longrightarrow B$ is 'exact' in the sense that the composite collapses all of Y to a point, and the 'kernel' of p (i.e., the inverse image over $\star$) is the 'image' of $Y \longrightarrow E$. This 'exact sequences' framework can be made rigorous, for example, if $\mathcal{C}$ is an Abelian category.

Let $\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ be any bundle (in **SSet**), and let $b_0 \in B_0$ any vertex. Then the fibre over

b_0 is given by the pullback

$$\begin{array}{ccc} Y & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta[0] & \xrightarrow[\overline{b_0}]{} & B \end{array} \tag{4.1.3.2}$$

and "$Y = p^{-1}(b_0)$", means that $Y_0 = p^{-1}(b_0)$, $Y_1 = p^{-1}(s_0 b_0)$, $Y_2 = p^{-1}(s_0 s_0 b_0)$, etc.

If we pull back (4.1.3.2) against the unique map $\Delta[n] \longrightarrow \Delta[0]$, we get

$$\begin{array}{ccccc} Y \times \Delta[n] & \longrightarrow & Y & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta[n] & \longrightarrow & \Delta[0] & \longrightarrow & B \end{array} \tag{4.1.3.3}$$

where the bottom map is the representing map for the generalised vertex, $s_0^n(b_0)$, of dimension n corresponding to b_0. So knowing the fibre above b_0 determines the fibre above the generalised vertices corresponding to b_0. But of course we cannot say anything about other n-simplices of B, nor can we even say that all vertices have the same fibre. This brings us to the notion of fibre bundle.

4.1.4 Fibre Bundles

A *fibre bundle of fibre* Y over a simplicial set B is a bundle $\left(\begin{array}{c} E \\ \downarrow{\scriptstyle f} \\ B \end{array}\right)$ such that for any n-simplex, $b \in B_n$, the pullback against the representing map is a trivial bundle.

$$\begin{array}{ccc} Y \times \Delta[n] & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta[n] & \xrightarrow[\overline{b}]{} & B \end{array} \tag{4.1.4.1}$$

Lemma 4.1.4.2. *The trivial bundle over B with fibre Y is a fibre bundle*

Proof. Trivial bundles are stable under pullback. Explicitly, we have a commuting diagram

$$\begin{array}{ccccc} Y \times \Delta[n] & \xrightarrow{(\mathrm{pr}_1 \# \overline{b}) \times \mathrm{id}} & Y \times B & \longrightarrow & Y \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} & \lrcorner & \downarrow \\ \Delta[n] & \xrightarrow[\overline{b}]{} & B & \longrightarrow & \Delta[0] \end{array} \tag{4.1.4.3}$$

where the top composes to give the projection from the product $Y \times \Delta[n]$ to Y. Hence the outer rectangle is a pullback, and hence the left square is also a pullback, showing that $Y \times B$ is a fibre bundle. $\square$

4.1.5 Twisted Cartesian Products (TCPs)

Motivation

We shall use the discussion of atlases in [40] to motivate the idea of a TCP. The definition can also be found in [11].

Atlases of a fibre bundle

Let $\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ be a fibre bundle as above. Looking more closely at the definition in section 4.1.4 we see that for $b \in B_n$ we have an isomorphism $\alpha(b)$ from $Y \times \Delta[n]$ to the standard pullback $E \times_B \Delta[n]$.

$$\begin{array}{ccccc} Y \times \Delta[n] & \xrightarrow{\alpha(b)} & E \times_B \Delta[n] & \xrightarrow{\pi_1} & E \\ & \searrow^{\pi_2} & \downarrow \pi_2 & & \downarrow p \\ & & \Delta[n] & \xrightarrow[\bar{b}]{} & B \end{array} \tag{4.1.5.1}$$

Let $a(b) := \alpha(b) \# \pi_1$. Since $\alpha(b)(y, \sigma) = (a(b)(y, \sigma), \sigma)$, $a(b)$ determines α and we call either of the two families $\{ \alpha(b) \colon b \in B \}$ or $\{ a(b) \colon b \in B \}$ an *atlas for p*. We usually regard $\alpha(b)$ as an element of $\underline{S}(Y, E \times_B \Delta[n])_n$ and $a(b) \in \underline{S}(Y, E)_n$. Let G be a subgroup of $\operatorname{aut}(Y)$, then any family of elements $g(b) \in G$ defines a new atlas $\{ g(b) \# \alpha(b) \colon b \in B \}$ (or $\{ g(b) \# a(b) \colon b \in B \}$). We call two atlases *$G$-equivalent* if they differ by such a family of $g(b)$'s.

Atlas Normalisation

We do not necessarily have $a(s_i b) = s_i a(b)$ in $\underline{S}(Y, E)$, but we can define a new atlas $\{ a'(b) \colon b \in B \}$ by

$$a'(b) = \begin{cases} a(b) & b \text{ non-degenerate} \\ s_i(a'(b)) & b = s_i c \text{ is degenerate} \end{cases} \tag{4.1.5.2}$$

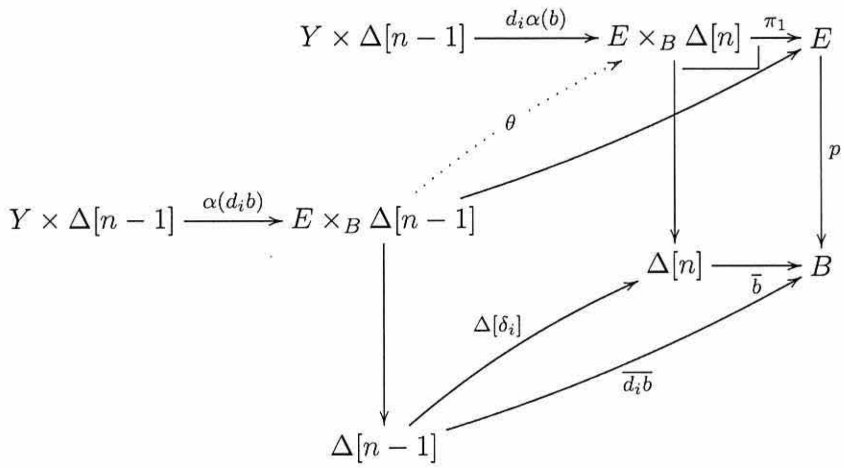
(by induction this is well-defined. We would get the same formula if we used the maps $\{ \alpha(b) \colon b \in B \}$ instead of $\{ a(b) \colon b \in B \}$). We call the new atlas $\{ a'(b) \colon b \in B \}$ *normalised.*

Transition elements

Let $\{ \alpha(b) \colon b \in B \}$ be an atlas. We do not necessarily have $a(d_i b) = d_i a(b)$, but

$$d_i(\alpha(b)) = \left(Y \times \Delta[n-1] \rightarrowtail^{Y \times \Delta[\sigma_i]} Y \times \Delta[n] \xrightarrow[\alpha(b)]{\cong} E \times_B \Delta[n] \right) \tag{4.1.5.3}$$

is an isomorphism onto its image. We get a θ making



(4.1.5.4)

commute. We have $\theta(e, \sigma) = (e, \sigma \# \delta_i)$, and the images of the maps $d_i(\alpha(b))$ and $\alpha(d_ib) \# \theta$ are the same, both being the elements of the form $(e, \sigma \# \delta_i)$. Thus there is a (unique) isomorphism $t_i(b)\colon Y \times \Delta[n-1] \longrightarrow Y \times \Delta[n-1]$ with

$$t_i(b) \# a(d_i(b)) = d_i(a(b)). \tag{4.1.5.5}$$

We call the $t_i(b)$ the *transition elements of the atlas.* If the transition elements all lie in G then we say $\{ a(b) \colon b \in B \}$ is a *G-atlas* (we omit the G when possible). An atlas is *regular* if its transition elements are all identities for $i > 0$.

Lemma 4.1.5.6. *Every (normalised) G-atlas is G-equivalent to a (normalised) regular G-atlas.*

Proof. Let $\{ a(b) \colon b \in B \}$ be a (normalised) regular atlas with transition elements $t_i(b)$ as above. We define a new G-atlas $\{ a'(b) \colon b \in B \}$ which is (normalised and) regular and G-equivalent to $\{ a(b) \colon b \in B \}$ by induction. On dimension 0 we let $a'(b) = a(n)$, so assume we have defined $\{ a'(b) \colon b \in B \}$ on dimensions at most $n-1$, and let $b \in B_n$ (non-degenerate, as the normalisation process takes care of the degenerate elements). Find elements $g_i(b) \in G_{n-1}$ with

$$a'(d_ib) = g_i(b) \# a(d_i(b)) \tag{4.1.5.7}$$
$$= g_i(b) \# t_i(b)^{-1} \# d_i(a(b)) \tag{4.1.5.8}$$
$$= h_i \# d_i(a(b)) \tag{4.1.5.9}$$

where $h_i := g_i(b) \# t_i(b)^{-1}$. Since $\{ a'(b) \colon b \in B \}$ is regular we have, for $0 < i \leqslant j$,

$$d_ih_j \# d_id_ja(b) = d_i(a'(d_jb)) \tag{4.1.5.10}$$
$$= a'(d_id_jb) = a'(d_{j-1}d_ib) \tag{4.1.5.11}$$
$$= d_{j-1}a'(d_ib) \tag{4.1.5.12}$$
$$= d_{j-1}h_i \# d_{j-1}d_ia(b) \tag{4.1.5.13}$$
$$= d_{j-1}h_i \# d_id_ja(b) \tag{4.1.5.14}$$

which implies $d_i h_j = d_{j-1} h_i$. Now use the Kan property of G to get $h \in G_n$ with $d_i h = h_i$ for $i > 0$, and define $a'(b) := h \# a(b)$. We have

$$d_i(a'(b)) = d_i h \# d_i a(b) \tag{4.1.5.15}$$
$$= h_i \# d_i a(b) \tag{4.1.5.16}$$
$$= a'(d_i b) \tag{4.1.5.17}$$

for $i > 0$ and so a' is still (normalised and) regular and G-equivalent to a. □

Fibre Bundles are TCPs

Let $\{ a(b) \colon b \in B \}$ be a normalised regular G-atlas and let $t(b) = t_0(b)$ which means that $d_0 a(b) = t(b) \# a(d_0 b)$. Note that the uniqueness condition in (4.1.5.5) implies that t satisfies some properties which are listed in (4.1.5.30); these are verified by applying a d_i or s_i to the equation $d_0 a(b) = t(b) \# a(d_0 b)$, and noting that $t(b)$ is the unique solution to that equation. Deviating from the standard references, we write $\varphi(b, \sigma)$ for the unique element of G for which

$$\sigma \cdot a(b) = \varphi(b, \sigma) \# a(\sigma \cdot b). \tag{4.1.5.18}$$

Here $b \in B_n$ and $\sigma \in \Delta[n]$, so the defining property of $t(b)$ says that $\varphi(b, \delta_0) = t(b)$. The calculation

$$(\tau \# \sigma) \cdot a(b) = \tau \cdot (\sigma \cdot a(b)) \tag{4.1.5.19}$$
$$= \tau \cdot (\varphi(b, \sigma) a(\sigma \cdot b)) \tag{4.1.5.20}$$
$$= (\tau \cdot \varphi(b, \sigma)) \# (\tau \cdot a(\sigma \cdot b)) \tag{4.1.5.21}$$
$$= (\tau \cdot \varphi(b, \sigma)) \# (\varphi(\sigma \cdot b, \tau) a(\tau \cdot \sigma \cdot b)) \tag{4.1.5.22}$$

shows that

$$\varphi(b, (\tau \# \sigma)) = \tau \cdot \varphi(b, \sigma) \# \varphi(\sigma \cdot b, \tau). \tag{4.1.5.23}$$

We also note that if σ can be written (for example using the normal form (2.1.1.1)) as a product of σ_j ($j \geqslant 0$) and δ_i ($i > 0$) only (i.e., no δ_0), then $\varphi(b, \sigma) = \mathrm{id}$. Thus taking τ to be δ_i and σ_i in (4.1.5.23) gives us the following rules

$$\varphi(b, d_0(\sigma)) = d_0(\varphi(b, \sigma)) \# t(\sigma \cdot b) \tag{4.1.5.24a}$$
$$\varphi(b, d_i(\sigma)) = d_i(\varphi(b, \sigma)) \qquad i > 0 \tag{4.1.5.24b}$$
$$\varphi(b, s_i(\sigma)) = s_i(\varphi(b, \sigma)) \qquad i \geqslant 0. \tag{4.1.5.24c}$$

We now note that our atlas *almost* gives a simplicial map $a \colon B \longrightarrow \underline{S}(Y, E)$, except that a does not commute with d_0 (and φ is measuring the degree to which a is not simplicial; the normalisation process just says that we can move all the failure to the d_0-face). Nevertheless, we can 'transpose a across the Cartesian-closed adjunction' to get $\xi \colon Y \times B \longrightarrow E$:

$$\xi = \left(Y \times B \xrightarrow{Y \times a} Y \times \underline{S}(Y, E) \xrightarrow{\#} E \right) \tag{4.1.5.25}$$

which sends (y, b) to $y^{a(b)}$. ξ inherits all the 'simplicialness' of a, i.e., it commutes with all s_j and all d_i with $i \geqslant 0$, and

$$d_0(\xi(y,b)) = (d_0 y)^{d_0(a(b))} \tag{4.1.5.26}$$
$$= (d_0 y)^{t(b) \# a(d_0 b)} \tag{4.1.5.27}$$
$$= \xi((d_0 y)^{t(b)}, d_0 b) \tag{4.1.5.28}$$

So we can make ξ simplicial by defining a new simplicial set, which we call $Y \times_t B$, which is the same as $Y \times B$ but we redefine the 0-face to be $d_0(y, b) := ((d_0 y)^{t(b)}, d_0 b)$. Of course to be a simplicial set we need to check the simplicial identities, but these follow from the aforementioned properties of t (i.e., (4.1.5.5)).

Proposition 4.1.5.29. *The map ξ is an isomorphism $Y \times_t B \cong E$.*

Proof. Using the notation of the diagram (4.1.5.1), we have $\alpha(y, b) = (a(y, b), b)$, the inverse is given by $\alpha(e, \sigma) = (\nu(e, \sigma), \sigma)$ where $\nu = \alpha(b)^{-1} \# \mathrm{pr}_Y$ is a simplicial map $\nu(b) \colon E(b) \longrightarrow Y$.

When $e \in E$, we get $(e, \mathrm{id}_n) \in E(p(e))$, so it makes sense to write $e^{\nu(pe)}$ for $\nu(b)(e, \mathrm{id}_n)$, and it is easy to check that the map sending e to this element $e^{\nu(pe)}$ is an inverse for ξ. □

We can consider $Y \times_\tau B$ as a bundle $\left(\begin{array}{c} Y \times_t B \\ \big\downarrow p_t \\ B \end{array}\right)$ where $p_t(y, b) = b$ and then the above shows that $p_t \cong p$ as bundles.

Definition

Abstracting from the previous section, let G be a simplicial group acting principally on the simplicial set Y. A *twisted Cartesian product (*TCP*)* with base B, fibre Y, group G and twisting function t is the bundle $\left(\begin{array}{c} Y \times_t B \\ \big\downarrow p_t \\ B \end{array}\right)$ where

- As a set, $(Y \times_t B)_n = (Y \times B)_n = Y_n \times B_n$
- The *twisting function* $t \colon B_{n+1} \longrightarrow G_n$ (one t for each n) satisfies

$$t(d_1 b) = d_0(t(b)) \# t(d_0 b) \tag{4.1.5.30a}$$
$$t(d_i b) = d_{i-1} t(b) \qquad i \geqslant 2 \tag{4.1.5.30b}$$
$$t(s_i b) = s_{i-1} t(b) \qquad i \geqslant 1 \tag{4.1.5.30c}$$
$$t(s_0 b) = \mathrm{id} \tag{4.1.5.30d}$$

- The faces and degeneracies are given by

$$d_0(b,y) = \left(d_0 b, (d_0 y)^{t(b)}\right) \tag{4.1.5.31a}$$

$$d_i(b,y) = (d_i b, d_i y) \qquad i \geqslant 1 \tag{4.1.5.31b}$$

$$s_i(b,y) = (s_i b, s_i y) \qquad i \geqslant 0 \tag{4.1.5.31c}$$

- The map to B (making the TCP a bundle) is projection, $p_t(y,b) = b$.

If Y is the underlying simplicial set of G and the action is the regular representation (section 2.5.3) then call $Y \times_t B$ a *principal* TCP *(*PTCP*)*.

Lemma 4.1.5.32. *The class of* TCP*s (with fibre Y) is stable under pullback, with $f^*(p_t) \cong p_{f\#t}$,*

$$\begin{array}{ccc} Y \times_{f\#t} A & \longrightarrow & Y \times_t B \\ {\scriptstyle p_{f\#t}} \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle p_t} \\ A & \xrightarrow[f]{} & B \end{array} . \tag{4.1.5.33}$$

Proof. Using the standard description of pullbacks in **SSet**, we have that the set of n-simplices in the pullback is $\{ ((y,b),a) \colon (y,b) \in Y \times_t B, \quad a \in A, \quad f(a) = b \}$ and the faces are just like a product except for the zero face, which is given by $d_0((y,b),a) = (((d_0 y)^{t(b)}, d_0 b), a) = (((d_0 y)^{t(f(b))}, f(d_0 a)), d_0 a)$. Thus we have a simplicial isomorphism from 'the standard pullback' to $Y \times_{f\#t} A$ sending $((y,b),a)$ to (y,a). □

We saw above that every fibre bundle gives rise to a TCP, we will show in proposition 4.1.5.36 that every possible TCP (i.e., every choice of t satisfying (4.1.5.30)) gives a fibre bundle. Since we are starting only with t, we do not, a priori, have the map φ from (4.1.5.18), however we can define φ as follows. If $b \in B_n$ and $\sigma \in \Delta[n]_m$ then write σ in normal form using (2.1.1.1); if this form does not involve δ_0 then set $\varphi(b,\sigma) = \mathrm{id}$, otherwise we have $\sigma = \sigma' \# \delta_0 \# \delta'$ where σ' is a product of σ_i $(i \geqslant 0)$ and δ' is a product of δ_j $(j > 0)$, and we define

$$\varphi(b, \sigma' \# \delta_0 \# \delta') = \sigma' \cdot t(\delta' b) \tag{4.1.5.34}$$

Lemma 4.1.5.35. *Let t be a twisting function and φ defined as in* (4.1.5.34)*, then φ satisfies the properties of* (4.1.5.24).

Proof. This is an easy exercise using the simplicial identities. □

Proposition 4.1.5.36. *Any* TCP*, $Y \times_t B$ (as defined in section 4.1.5) is a fibre bundle*

Proof. Let $b \in B_n$; using lemma 4.1.5.32 we just need to find an isomorphism $\Delta[n] \times_{\overline{b}\#t} Y \cong \Delta[n] \times Y$. This can be done by the following map

$$\begin{array}{rcl} \eta \colon Y \times \Delta[n] & \longrightarrow & \Delta[n] \times_{\overline{b}\#t} Y \\ (y,\sigma) & \longmapsto & (y^{\varphi(b,\sigma)}, \sigma) \end{array} \tag{4.1.5.37}$$

where $\varphi(b, \sigma)$ is defined from t as in (4.1.5.34). To show η is simplicial is exactly the properties (4.1.5.24); because $\varphi(b, \sigma) \in G$, η is a bijection in each dimension, with inverse sending (y, σ) to $(y^{\varphi(b,\sigma)^{-1}}, \sigma)$. □

Corollary 4.1.5.38. *The* TCP $Y \times_t B$ *has a normalised atlas given by*

$$a_t(b)(y, \sigma) = (y^{\varphi(b,\sigma)}, \bar{b}(\sigma)) \tag{4.1.5.39}$$

i.e., we have a pullback diagram

$$\begin{array}{ccc} Y \times \Delta[n] & \xrightarrow{a_t(b)} & Y \times_t B \\ \downarrow & & \downarrow{\scriptstyle p_t} \\ \Delta[n] & \xrightarrow[\bar{b}]{} & B \end{array} \tag{4.1.5.40}$$

Proof. $a_t(b)$ is just η from the proposition composed with the top map in the pullback square (4.1.5.33) from (4.1.5.37) □

We note in particular that putting $\sigma = \mathrm{id}_n$ in (4.1.5.39) tells us that $y^{a_t(b)} = (y, b)$.

Twisting functions as simplicial maps

Lemma 4.1.5.41. *Twisting functions* $t \colon B_{n+1} \longrightarrow G_n$ *correspond naturally to maps of* **$\mathcal{S}$-Gpds**, $t' \colon \mathcal{G}B \longrightarrow G$, *and hence to simplicial maps* $B \longrightarrow \overline{W}G$

Proof. By definition of $\mathcal{G}B$ (section 2.5.5) $t' \colon \mathcal{G}B \longrightarrow G$ in level n is exactly specified by its action on the generators, i.e., by a function $t \colon B_{n+1} \longrightarrow G_n$ with $t'(s_0 x) = \mathrm{id}$: $t(b) = t'(\bar{b})$.

The condition that t' should commute with the simplicial structure on the arrows of $\mathcal{G}B$ is exactly the rules (4.1.5.30) applied to t. For example, $t'(d_0^{\mathcal{G}B}\bar{b}) = d_0^G(t'(\bar{b}))$ says exactly that $d_0 t(b) = t(d_1 b) \# (t(d_0 b))^{-1}$ because $d_0(t'(\bar{b})) = d_0(t(b))$ and

$$\begin{aligned} t'(d_0\bar{b}) &= t'\left(\overline{d_0 b} \# \left(\overline{d_1 b}\right)^{-1}\right) \\ &= t'(\overline{d_0 b}) \# \left(t'(\overline{d_1 b})\right)^{-1} \\ &= t(d_0 b) \# (t(d_1 b))^{-1} \end{aligned} \tag{4.1.5.42}$$

The final statement follows by transposing across the $\mathcal{G} \dashv \overline{W}$ adjunction. □

Note that a given twisting function alone does not determine the TCP, as different actions of G on Y will give different values for d_0—it is both the twisting function and the action that are required.

Classification of TCPs by homotopy classes

In this section we will see how TCPs with base B and fixed fibre are classified by homotopy classes of maps $B \longrightarrow \overline{W}G$. By *fixed fibre* we mean that we choose a group G, fibre Y and a *fixed* action of G on Y—what we really classify is twisting functions from B into G, and then to get a TCP you just need to choose an appropriate action.

Furthermore all TCPs arise as pullbacks against a 'universal TCP'. The obvious candidate for a universal twisting function is given by the counit of $\mathcal{G} \dashv \overline{W}$: we have $\varepsilon_G \colon \mathcal{G}\overline{W}G \longrightarrow G$ which gives us a twisting function which we will still call ε_G. The corresponding TCP, $Y \times_{\varepsilon_G} \overline{W}G$, is denoted by $W(G)_Y$. If the fibre Y is the underlying simplicial set of G itself, and the action is multiplication on the right (i.e., the regular representation from section 2.5.3), then we denote the TCP by $W(G)$.

We have the following corollary to lemma 4.1.5.32.

Corollary 4.1.5.43. *Any* TCP p_t *(with fibre* Y*) arises as a pullback against the* TCP *with twisting function* ε_G *(and the same fibre as* p_t*):*

$$p_t \cong \overline{t'}^*(p_{\varepsilon_G}) \tag{4.1.5.44}$$

where $\overline{t'} \colon B \longrightarrow \overline{W}G$ *is the transpose of the simplicial map* $t' \colon \mathcal{G}B \longrightarrow G$ *corresponding to the twisting function* t.

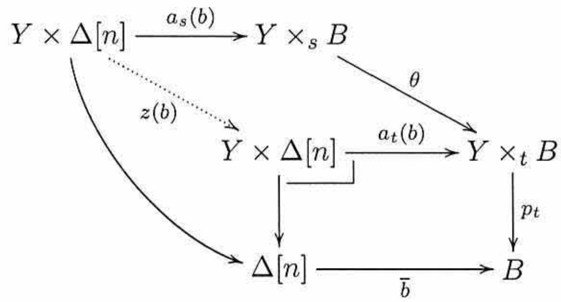
Proof. $t' = \mathcal{G}(\overline{t'}) \# \varepsilon_G$, so the result follows from lemma 4.1.5.32. □

In [11, 40] the following approach is now used: First show that TCPs are Kan fibrations and then use proposition 4.1.2.8 to show that homotopic maps induce the same TCP. To complete the classification, they then show that two maps inducing the same TCP over A are homotopic.

However we can take a slightly different path to this result. Suppose we have a map

$$\theta \colon \begin{pmatrix} Y \times_s B \\ \big\downarrow p_s \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} Y \times_t B \\ \big\downarrow p_t \\ B \end{pmatrix} \tag{4.1.5.45}$$

If $b \in B_n$ we have a pullback diagram



(4.1.5.46)

where the dotted line is an element of $\underline{S}(Y,Y)_n$ using the description in section 2.2.1; it is convenient to write this map $z(b)(y,\sigma) = (z(b)(y,\sigma),\sigma)$ where now $z(b)(y,\sigma) \in Y_n$.

We now calculate

$$\begin{aligned}
\theta(y,b) &= \theta(a_s(b)(y,\mathrm{id}_n)) && (4.1.5.47)\\
&= a_t(b)(z(b)(y,\mathrm{id}_n),\mathrm{id}_n) && (4.1.5.48)\\
&= (z(b)(y,\mathrm{id}_n),b) && (4.1.5.49)\\
&= (y^{z(b)},b) && (4.1.5.50)
\end{aligned}$$

It is easy to check that the following equations will ensure that θ be simplicial

$$\begin{aligned}
z(s_i b) &= s_i z(b) && i \geqslant 0 && (4.1.5.51)\\
z(d_i b) &= d_i z(b) && i > 0 && (4.1.5.52)\\
s(b) \# z(d_0 b) &= (d_0 z(b)) \# t(b) && && (4.1.5.53)
\end{aligned}$$

Conversely, if θ is simplicial, we would get, for each of the equations a statement that each side acts on all y in the same way, (so for example $y^{s_i z(b)} = y^{z(s_i b)}$ for all y), and it should in fact be possible to prove that the equations hold without the $y^{(-)}$ part from this. In any case, we will always be constructing our θ from a z which definitely satisfies the above equations.

We will say that θ is a *morphism of* TCP*s of group G* if those equations hold and $z(b) \in G_n$. As an example we note that the equations (4.1.5.24) say exactly that the map η from the proof of proposition 4.1.5.36 is a morphism of TCPs.

The requirement that $z(b) \in G_n$ implies that θ must be an isomorphism, for it has an inverse given $\theta^{-1}(y,b) := (y^{z(b)^{-1}},b)$.

We saw earlier that TCPs are classified by maps $B \longrightarrow \overline{W}G$, i.e., by elements of $\underline{S}(B,\overline{W}G)_0$. In our standard references [11, 40] it is shown that isomorphism classes of TCPs are classified by the homotopy classes in $\underline{S}(B,\overline{W}G)_0$ based on the fact that a TCP is a Kan fibration and using proposition 4.1.2.8; in fact we can follow a more modern spirit of categorification (in the sense of, for example [1, 2]) and instead of classifying the homotopy classes we will instead look at the actual homotopies themselves.

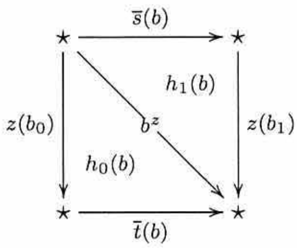
We first note that elements of $(\overline{W}H)_0$ are the objects of the simplicially enriched groupoid H, so in dimension 0, $\overline{s}$ maps every object to $\star \in (\overline{W}G)_0$ where $\star$ is the single object we have when we consider the simplicial group G as a simplicially enriched groupoid. In higher dimensions, $\overline{s}(b)$ is the list $(s(b), s(d_0 b), s(d_0^2 b), \ldots, s(d_0^{n-1}(b)))$ (which is an element of $(\overline{W}G)_n$.

Theorem 4.1.5.54. *Morphisms $\theta\colon Y \times_s B \longrightarrow Y \times_t B$ of* TCP*s induce homotopies $\overline{z}\colon \overline{s} \xrightarrow{\simeq} \overline{t}$ between the elements of $\underline{S}(B,\overline{W}G)_0$ corresponding to the* TCP*s we started with.*

Proof. Referring to section 2.4.2, we will just draw the lowest parts of the homotopy—it is clear how the construction can be continued for the higher levels, but we will not give all the details here.

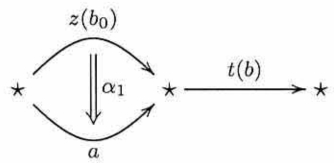
In dimension 0 we assign to each $b \in B_0$ the 1-simplex $\overline{s}(b) \xrightarrow{z(b)} \overline{t}(b)$.

In dimension 1, we must assign to $b_0 \xrightarrow{b} b_1$ a pair of 2-simplices which fit together to form the square (which lives in $\overline{W}G$)



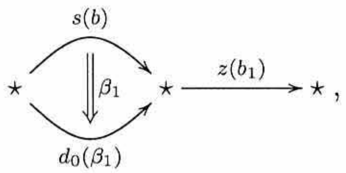
(4.1.5.55)

here the common face b^z will be determined when we define $h_i(b)$. Now $\overline{t}(b) \in \overline{W}(G)_1$ is just the element $t(b) \in G_0$, and $h_0(b) \in \overline{W}(G)_2$ consists some $\alpha_1 \in G_1$ and $\alpha_2 \in G_0$. As we will find in later sections, it is helpful to take a categorical view and draw α_1 as a 2-cell, so that $h_0(b)$ is drawn as the following diagram in G.



(4.1.5.56)

The formula for d_0 and d_2 in $\overline{W}(G)$ tell us everything apart from $d_0(\alpha_1))$. The d_1 part tells us only that $d_0(\alpha) \# t(b) = b^z$, but once we choose $\alpha \in G_1$ we will have specified $h_0(b)$ completely. Similarly, specifying $h_1(b)$, which is drawn as



(4.1.5.57)

is equivalent to giving $\beta_1 \in G_1$ with

$$d_0(\beta_1) \# z(b_1) = b^z = d_0(\alpha) \# t(b). \tag{4.1.5.58}$$

Comparing this requirement to the last equation (4.1.5.53) suggests we take $\alpha_1 = z(b)$ and $\beta_1 = s_0(s(b))$. This specifies the homotopy in dimension 1, and clearly the process can be continued for dimensions 2 and above: the h_i (as described in section 2.4.2) are built from $z(b)$ and various degeneracies. □

Corollary 4.1.5.59. *If we fix an action of G on Y, then isomorphism classes of* TCP*s, $Y \times_t B$, are classified by homotopy classes of maps $B \longrightarrow \overline{W}(G)$.*

Proof. We saw in the theorem that we can construct a homotopy from an isomorphism, for the converse we can either do a similar construction in reverse, building z (and hence θ) from the homotopy, or note that in the proof of proposition 4.1.2.8 we constructed, from a homotopy, maps, α and β, between the induced bundles E_f and E_g, and these will automatically be isomorphisms of TCPs. □

In fact one should be able to do the construction in further dimensions, i.e., if given a 'triangle' of maps of TCPs,

$$\begin{array}{ccccc} & & Y \times_t B & & \\ & \overset{\theta_1}{\nearrow} & & \overset{\theta_2}{\searrow} & \\ Y \times_s B & & \xrightarrow[\theta_3]{\qquad\qquad} & & Y \times_u B \end{array} \tag{4.1.5.60}$$

a homotopy $X\colon \theta_1 \xrightarrow{\simeq} \theta_3\theta_2^{-1}$ will, given some conditions on X (i.e., the analogues of the conditions on z) correspond to an element of $\underline{\mathcal{S}}(B, \overline{W}(G))_2$. Indeed we should expect that higher n-homotopies of TCPs should be the same, in some sense, as elements of $\underline{\mathcal{S}}(B, \overline{W}(G))_n$ for all n. However even drawing the pictures as in the proof of Theorem 4.1.5.54 becomes rather involved.

Returning to the case of a single map $\theta\colon Y \times_s B \longrightarrow Y \times_t B$, we can instead interpret s and t as maps $\mathcal{G}B \longrightarrow G$, and we can interpret our homotopies in a similar manner: s and t are morphisms of $\mathcal{S}$-**Gpds**, so we expect our θ to be some kind of natural transformation between them; a brief sketch of this idea follows.

First the objects: both maps s and t can only map the object of $\mathcal{G}B$ corresponding to $b \in B_0$ to the single object $\star \in G$. We have a $z(b) \in G_0$ between these objects as we might expect for a natural transformation.

On the morphisms it gets more complicated. Dimension zero of $\mathcal{G}B$ is generated by the $b_0 \xrightarrow{b} b_1$ from B_1, and this is mapped to $s(b)$ and $t(b)$ in G_0. Our map θ provides a 1-simplex, $z(b)$, of the form

$$z(b_0) \xrightarrow{z(b)} s(b)z(b_1)t(b)^{-1} \tag{4.1.5.61}$$

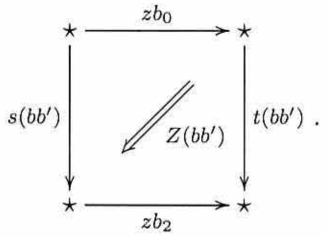
or equivalently we can look at $Z(b) := z(b) \# s_0(t(b))$, which may be pictured as a 2-cell filling the square

$$\begin{array}{ccc} \star & \xrightarrow{zb_0} & \star \\ {\scriptstyle sb}\downarrow & \swarrow\!\!\!\!\swarrow Z(b) & \downarrow{\scriptstyle tb} \\ \star & \xrightarrow[zb_1]{} & \star \end{array} . \tag{4.1.5.62}$$

The other morphisms are products of the generators, and can deal with them by combining those by multiplying the basic squares of the form (4.1.5.62). For example, if we have a second 1-simplex $b_1 \xrightarrow{b'} b_2$ in B_1 (which corresponds a 0-morphism b' which we can compose with the one corresponding to b) we can form $Z(bb')$ as

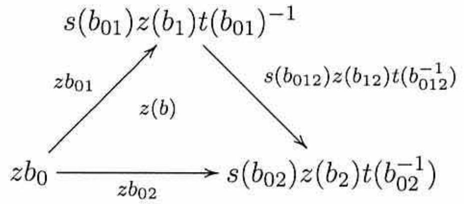
$$z(b_0)t(bb') \xrightarrow{Z(b)\#s_0(z(b_1)^{-1})\#Z(b')} s(bb')z(b_2) \tag{4.1.5.63}$$

which fills the square



(4.1.5.64)

In the next dimension (when b is in B_2, corresponding to a generator for $\mathcal{G}(B)_1$) we get a $z(b)$,



(4.1.5.65)

which, if we multiply it by $s_0(t(b))$, can be interpreted as a 3-cell filling in a prism in which the three square sides are essentially $Z(d_ib)$ $(i = 0, 1, 2)$. In this way our homotopy corresponds to a sort of 'infinitely lax natural transformation' between the functors s and t.

Relationship with homotopy colimits

Let G be an ordinary (not simplicial) group acting on the simplicial set Y. This means that, considering G as a category with one object, $\star$, we have a functor $\rho\colon G \longrightarrow \mathbf{SSet}$ sending $\star$ to Y. Write $K(G, 0)$ for the simplicial group with $K(G, 0)_n = G$ for all n (and all face and degeneracy maps are the identity).

Theorem 4.1.5.66. *For G an ordinary group acting on the simplicial set Y via a functor ρ, the universal* TCP $W(G)_Y$ *is isomorphic to the homotopy colimit of ρ.*

Proof. Because all the faces and degeneracy maps are constant, $\overline{W}(K(G, 0)) \cong \operatorname{Ner} G$ and the universal twisting just picks out the first g_0 from $(g_0, g_1, \dots, g_n) \in \operatorname{Ner} G$.

Then level n of $W(K(G, 0))_Y$ consists of elements of the form $(y, (g_0, g_1, \dots, g_n))$, with $y \in Y_n$, and

$$d_0(y, (g_0, g_1, \dots, g_n)) = ((d_0 y)^{g_0}, d_0(g_0, g_1, \dots, g_n)) \tag{4.1.5.67}$$

is exactly the formula from the simplicial replacement formulation of the homotopy colimit (see (3.3.2.11)). □

In the light of this theorem, we can regard a TCP as a kind of generalisation of a homotopy colimit: the simplicial set $Y \times_t B$ is a sort of 'formal homotopy colimit' where we have replaced the nerve of the category G with the simplicial set B and have replaced ε with t in some sense.

We will generalise this result in section 7.4.

5 The Classifying space of a crossed module

We now apply the 'classifying space' machinery of the previous section to crossed modules.

5.1 2-groups, crossed modules

Let $M = \left(C \xrightarrow{\partial} P\right)$ be a crossed module.

One of the difficulties is that we may view M as a crossed module, cat^1-group, simplicially-enriched group(oid) or as a special kind of 2- or double category. All of these views will be useful, so let us set out some notation for the various (equivalent) views of M.

Above we said $C \xrightarrow{\partial} P$ was a crossed module. We must clarify whether we are using right or left actions here; for action now, we will use right actions, thus the group P acts on the group C on the right and ∂ is a group homomorphism which satisfies the following rules:

$$\partial(c^p) = p^{-1}\partial c p \quad \text{in } P, \tag{CM1}$$
$$c^{\partial c_1} = c_1^{-1} c c_1 \quad \text{in } C. \tag{CM2}$$

5.1.1 'Algebraic' views of M

cat^1-groups

It is well-known ([37], and see also the exposition [21]), that the category of crossed modules is equivalent to the category of cat^1-groups, the latter being (equivalent to) internal categories in **Grp** (the category of groups). Here is the cat^1-group corresponding to our crossed module M:

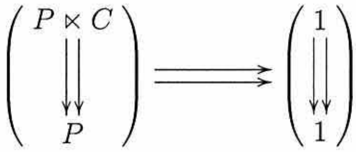
$$\begin{array}{c} P \ltimes C \\ \downarrow\downarrow \\ P \end{array} \tag{5.1.1.1}$$

i.e., P is the group of objects, $P \ltimes C$ the group of arrows. The source and target maps are such that we have $(p, c)\colon p \longrightarrow p\partial c$, with composition given by $(p, c) \# (p', c') = (p, cc')$. Note that this is automatically an internal groupoid in **Grp**, simply because all internal categories in **Grp** are in fact internal groupoids.

Let us write $\mathfrak{X}(M)_\mathrm{v}$ for the cat^1-group arising from M; all such cat^1-groups arise (up-to isomorphism) from some M.

Internal Categories, Internal Group(oids)

An internal category in groups is also an internal group in categories, and since **Grp** is a subcategory of **Grpoids**, we can replace 'group' by 'groupoid with one object' giving the 'horizontal' picture



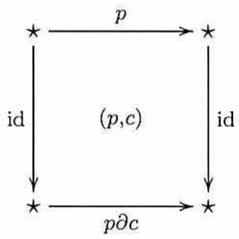
(5.1.1.2)

i.e., the category of objects is the terminal category and the category of arrows is the category described in the previous section. The composition is given by multiplication in $P \ltimes C$: $(p, c) \# (p', c') = (pp', c^{p'} c')$.

Although we do not need it for our present purposes, for completeness we should write $\mathfrak{X}(M)_{\mathrm{h}}$ for this internal group in **Grpoids**; all such objects arise (again up-to isomorphism) from some M.

Double categories, Double groupoids

A category object in **Cat** is a double category, and the previous two sections can be explained as showing the vertical and horizontal structures of a double category; the squares look like



(5.1.1.3)

with vertical and horizontal composition of squares exactly as in the previous two sections.

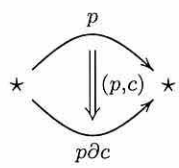
Because both vertical and horizontal structures are groupoids, we actually have a double groupoid, and since there is just one object, $\star$, we have a double group.

Let us call this double group $\mathfrak{X}(M)_{\mathrm{d}}$; all double groups with trivial vertical structure arise in this way.

2-groups

Because the category of vertical maps is trivial, we actually have a 2-groupoid; and even a 2-group: a single 0-cell and all higher cells invertible.

Here is a typical 2-cell in our 2-group.



(5.1.1.4)

We will use the '$\#_i$' notation (which seems to be due to Crans) for composition, so we write horizontal composition as $(p,c)\,\#_0\,(p',c') = (pp', c^{p'}c')$, and vertical composition is as $(p,c)\,\#_1\,(p',c') = (p, cc')$.

Let us write $\mathfrak{X}(M)$ for the 2-group arising from M. All 2-groups arise (up-to isomorphism) as $\mathfrak{X}(M)$ for some M.

5.1.2 'Simplicial' views of M

From any category object (in a category with pullbacks) we may produce a simplicial object by taking the (internal) nerve: the n-simplices are the strings of n composable arrows in the internal category (in general these are constructed using pullbacks, but we only need this for categories in which we have actual elements so will omit the details here); because the nerve functor is full and faithful this loses no information and offers a "simplicial view" of M. From our crossed module we thus have several simplicial views available, one for each of the more "algebraic" views above. Let us quickly list these.

Vertical nerve

Taking the nerve (internal to **Grp**) of $\mathfrak{X}(M)_\mathrm{v}$ gives a simplicial group, $\mathrm{Ner}\,\mathfrak{X}(M)_\mathrm{v}$ which looks like

$$\cdots P \ltimes C \ltimes C \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} P \ltimes C \substack{\longrightarrow\\ \longrightarrow} P\,. \tag{5.1.2.1}$$

Horizontal nerve

Taking the nerve (internal to **Grpoids**) of $\mathfrak{X}(M)_\mathrm{h}$ gives a simplicial groupoid $\mathrm{Ner}\,\mathfrak{X}(M)_\mathrm{h}$, which looks like this:

$$\cdots \begin{pmatrix} (P\ltimes C)^2 \\ \downdownarrows \\ P^2 \end{pmatrix} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \begin{pmatrix} P\ltimes C \\ \downdownarrows \\ P \end{pmatrix} \substack{\longrightarrow\\ \longrightarrow} \begin{pmatrix} 1 \\ \downdownarrows \\ 1 \end{pmatrix}. \tag{5.1.2.2}$$

The 'double nerve'

The double category, $\mathfrak{X}(M)_\mathrm{d}$ has a *double nerve* as in section 3.2.2 which we denote $\mathrm{Ner}\,\mathfrak{X}(M)_\mathrm{d}$.

Getting a simplicial set

Because $\mathfrak{X}(M)_\mathrm{v}$ is a simplicial *group*, it can be considered as a simplicially-enriched groupoid (with one object), and we can apply $\overline{W}$ to get a simplicial set.

Alternatively we can apply the Artin–Mazur codiagonal to the double nerve of $\mathfrak{X}(M)_\mathrm{d}$ (the other obvious approach is to apply the diagonal, but this gives the same simplicial set up-to weak equivalence [9]).

The above constructions give the same simplicial set and a final way to arrive at this set is that M is a 2-category, i.e., a category enriched over **Cat**. Replace each hom-category with its nerve, and we have a **SSet**-enriched category (in fact a **SSet**-enriched group). The obvious nerve construction of a **SSet**-enriched category is a bisimplicial set, and taking the codiagonal gives exactly the description above.

In summary, this simplicial set, which we will call $\mathrm{Ner}\, M$, is

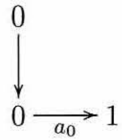
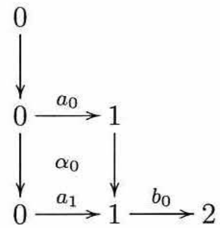
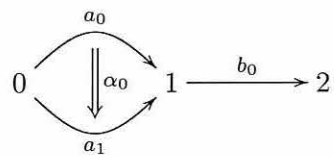
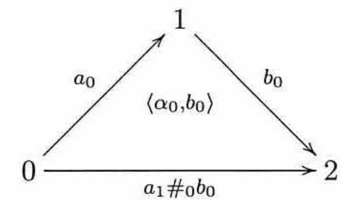
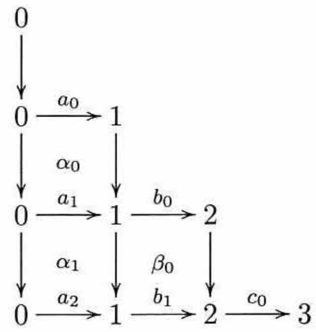
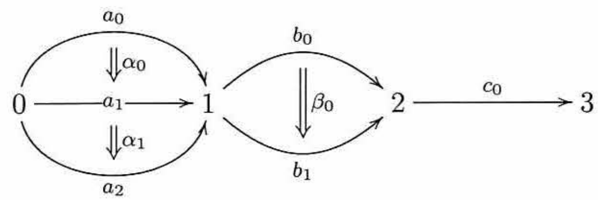
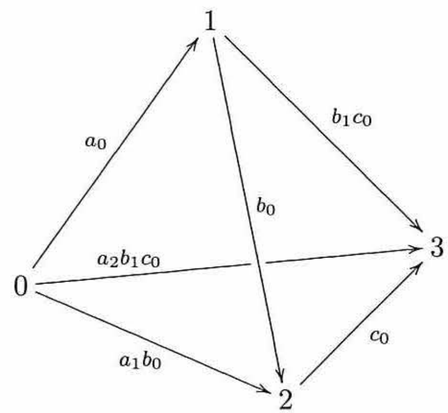
$$\mathrm{Ner}\, M = \overline{W}(M) = \overline{W}\mathfrak{X}(M)_{\mathrm{v}} \cong \nabla\, \mathrm{Ner}\, \mathfrak{X}(M)_{\mathrm{d}} \tag{5.1.2.3}$$

There are further generalisations to get nerves of bicategories, for example [14], but we do not need these here.

5.1.3 Explicit description of the nerve

Using section 5.1.2, we can give an explicit description of the nerve of a crossed module. It is probably just as easy to give the case of a general 2-category, so let us do that.

In general, $(\mathrm{Ner}\, \mathcal{C})_n$ involves 'generalised whiskering'. Here are the first few levels, drawn both as diagrams in $\mathcal{C}$ and as simplices. Unnamed arrows are identities and $0, 1, \ldots$ are arbitrary objects of $\mathcal{C}$ (so all are equal to $\star$ in the crossed module case).

n	staircase	2-category picture	simplex picture
0	A_0	A_0	A_0
1		$0 \xrightarrow{a_0} 1$	$0 \xrightarrow{a_0} 1$
2			
3			

The faces of the above 3-simplex (which we may denote by $\langle \alpha_0, \alpha_1, \beta_0, c_0 \rangle$) are: $d_0 = \langle \beta_0, c_0 \rangle$, $d_1 = \langle \alpha_1 \#_0 \beta_0, c_0 \rangle$, $d_2 = \langle \alpha_0 \#_1 \alpha_1, b_0 \#_0 c_0 \rangle$ and $d_3 = \langle \alpha_0, b_0 \rangle$.

5.2 Weak Equivalences and the Vertical Nerve of a crossed module

5.2.1 Homotopy groups of a crossed module

Let $M = (C \xrightarrow{\partial} P)$ be our crossed module. The *homotopy groups of M* are defined to be the homotopy groups of the Moore complex of M. This means that M has just two non-zero groups $\pi_1(M) = \operatorname{coker} \partial = P/_{\operatorname{im} \partial}$ and $\pi_2(M) = \ker \partial$.

5.2.2 Weak Equivalences of Crossed Modules

Let M_1 and M_2 be crossed modules with $M_i = C_i \xrightarrow{\partial_i} P_i$. A map $\varphi \colon M_1 \longrightarrow M_2$ is (as usual) a *weak equivalence* if it induces isomorphisms on the homotopy groups, i.e., on the kernel and cokernel:

$$\begin{array}{ccc}
\ker \partial_1 & \xrightarrow[\cong]{\varphi} & \ker \partial_2 \\
\downarrow & & \downarrow \\
C_1 & \xrightarrow{\varphi} & C_2 \\
{\scriptstyle \partial_1}\downarrow & & \downarrow{\scriptstyle \partial_2} \\
P_1 & \xrightarrow{\varphi} & P_2 \\
\downarrow & & \downarrow \\
\operatorname{coker} \partial_1 & \xrightarrow[\cong]{\varphi} & \operatorname{coker} \partial_2
\end{array} \qquad (5.2.2.1)$$

Write π_1 and π_2 for the common homotopy groups. We can subsume the isomorphisms (given by φ) into maps into C_i (or out of P_i), and rewrite the above diagram as

$$\begin{array}{ccccc}
 & \swarrow & \pi_2 & \searrow & \\
C_1 & & \xrightarrow{\varphi} & & C_2 \\
{\scriptstyle \partial_1}\downarrow & & & & \downarrow{\scriptstyle \partial_2} \\
P_1 & & \xrightarrow{\varphi} & & P_2 \\
 & {\scriptstyle q_1}\searrow & \pi_1 & \swarrow{\scriptstyle q_2} &
\end{array} . \qquad (5.2.2.2)$$

5.2.3 Example of a weak equivalence

Let $M_1 = N \xrightarrow{\partial} P$ be the crossed module corresponding to the inclusion of a normal subgroup $N \lhd P$, and let $M_2 = 1 \longrightarrow G$ a 1-type with $G = {}^{P}/_{N}$ the quotient group. Then the quotient map, $q\colon P \longrightarrow G$, induces a weak equivalence $M_1 \xrightarrow{\simeq} M_2$.

5.2.4 Weak equivalences induce homotopy equivalences on the vertical nerve

Proposition 5.2.4.1. *Let φ be a weak equivalence $\varphi\colon M_1 \xrightarrow{\simeq} M_2$. Then φ induces a homotopy equivalence on the vertical nerves.*

Proof. **Case 1: φ comes from a quotient map**
It is instructive to first consider the case where φ comes from a quotient map, q, as in section 5.2.3.

We have to prove that the simplicial set $\mathrm{Ner}_v[\mathfrak{X}(M_1)_{\mathrm{v}}]$ is homotopy equivalent to $\mathrm{Ner}_v[\mathfrak{X}(M_2)_{\mathrm{v}}] = K(G,0)$. We produce a contraction by specifying an extra degeneracy s_{-1}

$$\cdots P \ltimes N \ltimes N \underset{d_1}{\overset{d_0}{\rightrightarrows}} P \ltimes N \xrightarrow{q} G \qquad (P \ltimes N \ltimes N \xleftarrow{s_{-1}} P \ltimes N \xleftarrow{s_{-1}} G) \tag{5.2.4.2}$$

Note that, since q is a quotient map, $q(a) = q(b) \iff a = bn$ for some $n \in N$.

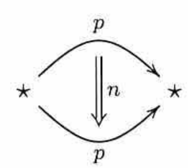
Let s be a splitting in **Set** for q (of course no splitting exists in the category of groups unless $N = 1$), i.e., $s\colon UG \longrightarrow UP$ is a map of the underlying sets such that $q(s(g)) = 1$ for all $g \in G$. This means that $s(g)$ is any choice of element from the coset g, i.e., s is a *transversal.*

We can now form a map

$$\begin{array}{ccc} U(G \times N) & \longrightarrow & UP \\ (g,n) & \longmapsto & s(g)n \end{array} \tag{5.2.4.3}$$

this map is a bijection. (For any $p \in P$ we have $q(p) = qsq(p)$, hence $p = sq(p)n$ for some n; the inverse sends p to (sqp, n).) We use this bijection to define our extra degeneracy s_{-1}. Namely, given $(p, \mathbf{n})$ write $p = s(g)n$ (where $g = q(p) \in G$) and set $s_{-1}(p, \mathbf{n}) = (sg, (n, \mathbf{n}))$.

Note that the equation $p = s(g)n$ says that we have a 2-cell



(5.2.4.4)

in M_1 (equivalently: a 1-simplex in the its nerve) and s_{-1} just prepends this 2-cell to the n-simplex $(p, \mathbf{n})$.

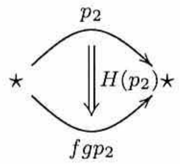
The General Case

We now turn to a general weak equivalence $\varphi\colon M_1 \xrightarrow{\simeq} M_2$ (using the notation of section 5.2.2; in particular see (5.2.2.2)).

As before, pick a section s_1 for the quotient map q_1. This gives us a section s_2 for q_2 with $s_2(g) = \varphi s_1(g)$.

We require maps $f\colon \operatorname{Ner}\mathfrak{X}(M_1)_{\mathrm{v}} \longrightarrow \operatorname{Ner}\mathfrak{X}(M_2)_{\mathrm{v}}$, $g\colon \operatorname{Ner}\mathfrak{X}(M_2)_{\mathrm{v}} \longrightarrow \operatorname{Ner}\mathfrak{X}(M_1)_{\mathrm{v}}$ and homotopies $H\colon \mathrm{id} \xrightarrow{\simeq} fg$ and $L\colon gf \xrightarrow{\simeq} \mathrm{id}$. We set $f = \operatorname{Ner}\varphi$ and define g and H together.

In dimension 0 we need g to map P_2 to P_1, so define $g(p_2) = s_1(q_2(p_2))$. H must assign to p_2 a 1-simplex $p_2 \Longrightarrow fgp_2$ in $\mathfrak{X}(M_2)_{\mathrm{v}}$. We have $q_2(fgp_2) = q_2\varphi s_1 q_2 p_2 = q_2 s_2 q_2 p_2 = q_2(p_2)$ so there is $H(p_2) \in C_2$ with $fgp_2 = p_2\partial_2 H(p_2)$, i.e., we have a 1-simplex/2-cell

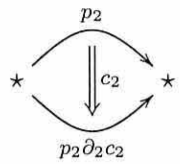


(5.2.4.5)

as needed for H.

Note that the defining property $fgp_2 = p_2\partial_2 H(p_2)$ does not uniquely determine $H(p_2)$; in fact it determines it uniquely only modulo $\ker \partial_2 = \pi_2$

Now consider a 2-cell

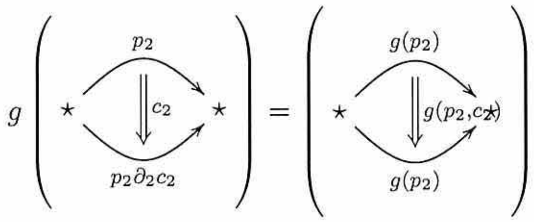


(5.2.4.6)

We have $g(p_2\partial_2 c_2) = s_1 q_2(p_2\partial_2 c_2) = s_1 q_2(p_2) = g(p_2)$; hence $fg(p_2) = fg(p_2\partial_2 c_2)$, which means that $\partial_2 c_2 \partial_2 H(p_2\partial_2 c_2) = \partial_2 H(p_2)$, or equivalently, $H(p_2)^{-1}c_2 H(p_2\partial_2 c_2) \in \ker\partial_2 = \pi_2$. So we may define a map (of sets)

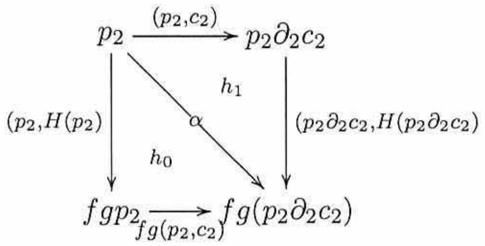
$$
\begin{aligned}
g\colon P_2 \ltimes C_2 &\longrightarrow \pi_2 \\
(p_2, c_2) &\longmapsto H(p_2)^{-1}c_2 H(p_2\partial_2 c_2)
\end{aligned}
\tag{5.2.4.7}
$$

using the map $\pi_2 \rightarrowtail C_1$ we can consider g to map into C_1, (i.e., $\varphi g(p_2, c_2) = H(p_2)^{-1}c_2 H(p_2\partial_2 c_2) \in C_2$ (here we are using that φ gives an isomorphism on π_2), and this defines g in dimension 1, i.e.,



(5.2.4.8)

For our homotopy, H, we need 2-simplices h_0 and h_1 with



(5.2.4.9)

since the vertical nerve is determined by dimensions 0 and 1 only, the above diagram determines h_0 and h_1 provided the common d_1-face is well-defined, i.e., we must have $c_2 H(p_2\partial_2 c_2) = H(p_2) fg(p_2, c_2)$ which is true by construction of g. Again using the coskeletal property of the vertical nerve this is all we need to construct the map g and homotopy $H\colon \mathrm{id} \xrightarrow{\simeq} fg$.

A similar construction gives us g' and $L\colon \mathrm{id} \xrightarrow{\simeq} gf$. By lemma 5.2.4.10 this shows that the two vertical nerves are homotopy equivalent. □

Lemma 5.2.4.10. *Suppose we have* $A \xrightarrow{h} B \xrightarrow{f} A \xrightarrow{g} B$ *composable maps and homotopies* $h \# f \xrightarrow{\simeq} \mathrm{id}_A$ *and* $f \# g \xrightarrow{\simeq} \mathrm{id}_B$. *Then* f *is a homotopy equivalence* $B \xrightarrow{\simeq} A$.

Proof. $g \# f \simeq (h \# f) \# g \# f = h \# (f \# g) \# f \simeq h \# f \simeq \mathrm{id}_A$. □

6 Sheaves and Covering spaces

In this chapter we recall some well-known tools from topology (sheaves, covering spaces and the nerve of a cover) that will be needed in the next chapters. The main reference is [36].

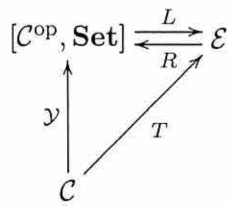
6.1 Sheaves, Étale spaces, and Covering Spaces

To make a connection with locally trivial bundles in topology, and to start moving towards stacks, we recall the proof that locally constant sheaves on a space are equivalent to covering spaces. The proof comes from [36] and [3] but the result is rather older. We will attempt to generalise some of this section in section 8.2.

6.1.1 Preliminaries

For a functor $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$ we can form the Grothendieck construction (section 3.1) which comes with a projection functor $\mathrm{Grot}(F)\colon \mathcal{C} \int F \longrightarrow \mathcal{C}$ back to $\mathcal{C}$.

Given any $T\colon \mathcal{C} \longrightarrow \mathcal{E}$ with $\mathcal{E}$ cocomplete, write $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ for the category of presheaves on $\mathcal{C}$, and $\mathcal{Y}$ for the Yoneda embedding. We have a diagram



(6.1.1.1)

with $L \dashv R$ and $\mathcal{Y} \mathbin{\#} L \cong T$. Explicitly,

$$L(F) = \operatorname{colim}\left(\mathcal{C} \int F \xrightarrow{\mathrm{Grot}(F)} \mathcal{C} \xrightarrow{T} \mathcal{E}\right) \tag{6.1.1.2}$$

$$R(B) = \mathcal{E}(T(-), B) \tag{6.1.1.3}$$

Proof.

$$\begin{aligned}
\mathcal{E}(LF, B) &= \mathcal{E}(\int^{(A,x)\in\mathcal{C}\int F} TA, B) \\
&\cong \int_{(A,x)} \mathcal{E}(TA, B) \\
&\cong \int_{(A,x)} R(B)(A) \\
&\cong \int_{(A,x)} [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]\,(\mathcal{Y}(A), R(B)) \\
&\cong [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]\,(\int^{(A,x)} \mathcal{Y}(A), R(B)) \\
&\cong [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]\,(F, R(B))
\end{aligned} \tag{6.1.1.4}$$

where the last line uses density[1]: $\int^{(A,x)} \mathcal{Y}(A) \cong F$. $L(\mathcal{Y}(A)) \cong TA$ now follows from putting $F = \mathcal{Y}(A)$ in the adjunction isomorphism (6.1.1.4). □

6.1.2 Germs and sections

Take $\mathcal{E} = \mathbf{Top}/X$ and T the following functor

$$\begin{aligned}
T\colon \mathcal{O}(X) &\longrightarrow \mathbf{Top}/X \\
U &\longmapsto U \hookrightarrow X \\
U \subseteq V &\longmapsto U \subseteq V.
\end{aligned} \tag{6.1.2.1}$$

From (6.1.1.2) we get $L(F) = \left(\begin{array}{c} \mathrm{Stalk}(F) \\ \downarrow \pi \\ X \end{array} \right)$ (the *stalk* of F) with

$$\mathrm{Stalk}(F) = \{\, [p,(U,x)] : p\in \mathrm{Grot}(F)(U,x)=U, (U,x)\in\mathcal{O}(X)\int(F) \,\}/_{[p,(U,x)]\sim[q,(V,y)]} \tag{6.1.2.2}$$

and where the equivalence relation defining the colimit is generated by

$$[p, (U, x)] \sim [q, (V, y)] \text{ if } (\mathrm{Grot}(F) \# T)\,((V, y) \longrightarrow (U, x))\,(q) = p. \tag{6.1.2.3}$$

Now a map $(V, y) \longrightarrow (U, x)$ exists iff $V \subseteq U$ and $x|_V = y$ (here $x|_V$ is the image of $x \in FU$ under the 'restriction' map $FU \longrightarrow FV$, which is itself the image under F of the inclusion $V \longrightarrow U$ in $\mathcal{O}(X)$), and when such a map exists (6.1.2.3) just says $q = p$. So we see that the equivalence relation is just '$[p, (U, x)] \sim [q, (V, y)]$ iff $p = q$ and there is $W \in \mathcal{O}(X)$ with $W \subseteq U, V$ and $x|_W = y|_W$'. The equivalence class of a triple (p, U, x)

[1] Density actually follows from this adjunction result: taking $T = \mathcal{Y}$, we see $R = \mathrm{id}$, whence $F = \mathrm{id}$ also.

under this equivalence relation is called the *germ of x at p* and is written $\mathrm{germ}_p(x)$; thus $\mathrm{Stalk}(F)$ is just the set of germs for F. The map π is just projection: $\pi(\mathrm{germ}_p(x)) = p$.

From (6.1.1.3) we get that R (which we will write as Γ) takes a space $\left(\begin{smallmatrix} A \\ \downarrow \alpha \\ X \end{smallmatrix}\right)$ to the presheaf $\Gamma(\alpha)$ with

$$\begin{aligned} \Gamma(\alpha)(U) &= \mathrm{Hom}_{\mathbf{Top}/X}\left(\begin{pmatrix} U \\ \downarrow \\ X \end{pmatrix}, \begin{pmatrix} A \\ \downarrow \alpha \\ X \end{pmatrix} \right) \\ &= \{\, \text{sections of } \alpha \text{ over } U \,\}, \end{aligned} \tag{6.1.2.4}$$

where a *section of α over U* is a map $s\colon U \longrightarrow A$ with $s \,\#\, \alpha = \mathrm{id}_U$.

6.1.3 The Sh(X)-Etale/X equivalence

The above gives us an adjunction between the presheaf category on $\mathcal{C}$ and spaces over X. Restricting attention to those presheaves for which the unit of $L \dashv \Gamma$ is an isomorphism gives us the full subcategory $\mathbf{Sh}(X)$ of *sheaves on X*; restricting attention to those spaces over X where the counit of $L \dashv \Gamma$ is an isomorphism gives us the full subcategory $\mathbf{Etale}/X$ of *étale spaces over X*. It then follows that that the adjunction of section 6.1.2 restricts to an equivalence $\mathbf{Sh}(X) \simeq \mathbf{Etale}/X$: the sheaves are exactly the presheaves at which the unit is invertible and the étale spaces are exactly the spaces over X where the counit is.

Finally, the sheafification functor from presheaves to sheaves is obtained as $\mathrm{Stalk} \,\#\Gamma$, and we have $\mathrm{Stalk} \,\#\Gamma \dashv$ inclusion.

6.1.4 Locally constant sheaves and covering spaces

From now on we assume our space X is *locally connected.* We recall two classical local definitions.

Definition 6.1.4.1. Call a sheaf $F \in \mathbf{Sh}(X)$ *locally constant* if each $p \in X$ has a basis of neighbourhoods, $\mathcal{N}_p$, such that whenever $U, V \in \mathcal{N}_p$ with $V \subseteq U$, the restriction $FU \longrightarrow FV$ is isomorphism.

We get a full subcategory $\mathbf{Sh}(X)_{\mathrm{lc}} \subseteq \mathbf{Sh}(X)$ of locally constant sheaves on X.

Definition 6.1.4.2. Call a space $\left(\begin{smallmatrix} E \\ \downarrow \pi \\ X \end{smallmatrix}\right) \in \mathbf{Etale}/X$ over X a *covering space* (or sometimes just a *cover* on X) if every $x \in X$ has a neighbourhood U (called *fundamental*) with $\pi^{-1}(U) \cong U \times F$ for some discrete space F (called the *fibre*).

We get a full subcategory $\mathbf{Cov}(X) \subseteq \mathbf{Etale}/X$ of covers of X.

Theorem 6.1.4.3. *The equivalence of section 6.1.3 restricts to an equivalence between locally constant sheaves and covering spaces on X: $\mathbf{Sh}(X)_{lc} \simeq \mathbf{Cov}(X)$.*

Proof. Let $\left(\begin{array}{c} E \\ \downarrow \pi \\ X \end{array}\right)$ be étale corresponding to $F \in \mathbf{Sh}(X)$. We must show that π is a cover iff F is locally constant.

($\Rightarrow$): If π is a cover, let U be a fundamental neighbourhood for $x \in X$. Since X was locally connected we can take U to be connected and it will still be open. Let $\mathcal{N}_p$ be the connected open sets in U (this is a basis since if $p \in W \in \mathcal{O}(X)$ then W contains a connected component of $W \cap U$ which is in $\mathcal{N}_p$). Then for $V \in \mathcal{N}_p$, sections, s, of π over V are of the form $s(e) = (e, \overline{s}(f))$ where $\overline{s}\colon V \longrightarrow F$ is continuous from the connected V to the discrete F—all such $\overline{s}$ are constant, so $s(e) = (e, f)$ for some fixed $f \in F$, and $\Gamma(\pi)(V) \cong F$ for all $V \in \mathcal{N}_p$.

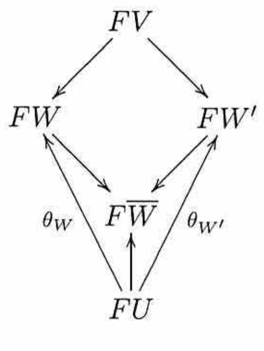
($\Leftarrow$): Let F be locally constant with $U \in \mathcal{N}_p$. Then

$$\pi^{-1}(U) = \{\, \mathrm{germ}_p(x)\colon p \in U, x \in FV \text{ for some } V \ni p \text{ open in } X. \} \tag{6.1.4.4}$$

Given U and V as in (6.1.4.4), find $W \in \mathcal{N}_p$ with $W \subseteq U \cap V$. Write θ_W for the restriction $\theta_W\colon FU \longrightarrow FW$.

Claim: $\theta_W^{-1}(x|_W) \in FU$ does not depend on the choice of $W \in \mathcal{N}_p$.

Proof of claim: If W' is another choice, we find $\overline{W} \subseteq W \cap W'$ with $\overline{W} \in \mathcal{N}_p$. We have the following commuting diagram (in **Set**, with the 'upwards pointing arrows iso)



(6.1.4.5)

since $x \in FV$, we get $\theta_W^{-1}(x|_W) = \theta_{W'}^{-1}(x|_{W'})$ as required. A similar argument shows that for $V \in \mathcal{N}_p$, $\theta_W^{-1}(x|_W)$ does not depend on the choice of $x \in \mathrm{germ}_p(x)$. Finally, if $\mathrm{germ}_p(x) = \mathrm{germ}_Q(y)$ then applying π gives us $p = q$, so we have shown that the map $\pi^{-1}(U) \longrightarrow FU \times U$ which sends $\mathrm{germ}_p(x)$ to $(\theta_W^{-1}(x|_W), p)$ is well defined. If we give FU the discrete topology then the map is continuous. It is a homeomorphism with inverse $(y, Q) \mapsto \mathrm{germ}_Q(y)$ thus we have shown that U is a fundamental neighbourhood, and π has fibre FU. □

6.1.5 Constant Sheaves

We can embed **Set** into the category of presheaves via the constant functor, $\overline{(-)}$, which is defined on objects $A \in \mathbf{Set}$ by

$$\begin{aligned} \overline{A}\colon \mathcal{O}(X)^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ U &\longmapsto A \quad . \\ U \subseteq V &\longmapsto \mathrm{id}_A \end{aligned} \tag{6.1.5.1}$$

Now $\overline{A}$ is not generally a sheaf because a sheaf must take a singleton value at the empty set.[2] So to define the notion of 'constant sheaf with value A' we must apply the sheafification functor to $\overline{A}$. By the remark at the end of section 6.1.3 we see that the sheafification is defined as follows. First we apply the stalk functor, Stalk, to $\overline{A}$ and then the sections functor Γ to the result. The resulting functor is const: $\mathbf{Set} \longrightarrow \mathbf{Sh}(X)$.

Lemma 6.1.5.2.

$$\mathrm{Stalk}(\overline{A}) \cong \begin{pmatrix} X \times A \\ \big\downarrow \mathrm{pr} \\ A \end{pmatrix} \tag{6.1.5.3}$$

Proof. The total space is

$$\left\{ (x,U,a) \colon x \in X, U \in \mathcal{O}(X), a \in \overline{A}(U) = A \right\} /_{\sim} \tag{6.1.5.4}$$

where the equivalence relation is $(x, U, a) \sim (y, V, b)$ iff $x = y$ and the restrictions of a and b to some common open set W agree. But since the restriction maps are the identity, we have $(x, U, a) \sim (x, X, a)$ for all open sets U (and all $x \in X$ and $a \in A$), thus do not need to keep U in the notation, and the total space is just $X \times A$ where A gets the discrete topology. □

Lemma 6.1.5.5.

$$\mathrm{const}(A) = \Gamma\,\mathrm{Stalk}(\overline{A}) \cong \mathbf{Set}(\Pi_0(-), A), \tag{6.1.5.6}$$

the set of locally constant functions from U to A.

Proof. $\Gamma(\mathrm{Stalk}(\overline{A})) = \{ s\colon U \longrightarrow X \times A \colon \mathrm{pr}_X(s(u)) = u \text{ all } u \in U \}$, so the elements, s are of the form $s(u) = (u, s'(u))$ where $s' = s \,\#\, \mathrm{pr}_A$ is a continuous map into the discrete space A. By definition Π_0 is the left adjoint to the "discrete space" functor $\mathbf{Set} \longrightarrow \mathbf{Top}$, so s' corresponds to a map $\Pi_0(U) \longrightarrow A$ of sets, i.e., is locally constant. □

Lemma 6.1.5.7. $\Pi_0 \dashv \mathrm{const} \dashv \Gamma(-, X)$.

[2]The empty set is open and is covered by the empty cover. A product over this cover is a product over the empty set, which is the terminal object, so the sheaf condition for F with respect to the empty cover of $\emptyset$ says that $F(\emptyset)$ is the equaliser of the parallel pair $1 \rightrightarrows 1$, which is 1 as required.

Proof.

$$\begin{aligned}
\mathbf{Sh}(X)(\mathrm{const}(A), F) &\cong \mathbf{Sh}(X)(\Gamma\,\mathrm{Stalk}\,\overline{A}, F) && (6.1.5.8)\\
&\cong [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]\,(\overline{A}, F) \quad \text{using Stalk} \#\Gamma \dashv \text{inclusions} && (6.1.5.9)\\
&\cong \mathbf{Set}(A, \lim_{U\in\mathcal{O}(X)^{\mathrm{op}}} FU) \quad \text{using } \overline{(-)} \dashv \lim && (6.1.5.10)\\
&\cong \mathbf{Set}(A, FX) \quad \text{since } X \text{ is initial in } \mathcal{O}(X)^{\mathrm{op}} && (6.1.5.11)\\
&\cong \mathbf{Set}(A, \Gamma(F, X)). && (6.1.5.12)
\end{aligned}$$

For the other side,

$$\begin{aligned}
\mathbf{Sh}(X)(F, \mathrm{const}(A)) &\cong \mathbf{Etale}/X\left(\begin{pmatrix} E \\ \downarrow p \\ X \end{pmatrix}, \begin{pmatrix} X \times A \\ \downarrow \mathrm{pr}_X \\ X \end{pmatrix}\right) && (6.1.5.13)\\
&\cong \mathbf{Etale}(E, A) \quad \text{(same argument as in (6.1.5.6))} && (6.1.5.14)\\
&\cong \mathbf{Set}(\Pi_0(E), A) && (6.1.5.15)
\end{aligned}$$

so we define $\Pi_0(F) = \Pi_0(E)$ and we are done. □

6.2 Covers and the Čech nerve

6.2.1 Open covers as simplicial sheaves

Let $\{ U_i : i \in I \}$ be an (open) cover of X. Taking coproducts gives us a space, $\mathcal{U} = \coprod_{i\in I} U_i$, and a map $p: \mathcal{U} \longrightarrow X$ which is an étale space over X. Now the terminal étale space is id_X, and (because $\mathbf{Etale}/X$ is a slice category), p also serves as the unique map

$$p: \begin{pmatrix} \mathcal{U} \\ \downarrow p \\ X \end{pmatrix} \longrightarrow \begin{pmatrix} X \\ \downarrow \mathrm{id} \\ X \end{pmatrix}. \tag{6.2.1.1}$$

Using section 6.1 we can replace p with a map $U \longrightarrow 1$ of sheaves where 1 is the terminal sheaf (whose value on every open set is a singleton—this is the sheaf-theoretic version of our space X)

6.2.2 The Čech nerve of a cover

Taking the kernel pair of our map $U \longrightarrow 1$ gives us a groupoid

$$U \times U \rightrightarrows U \tag{6.2.2.1}$$

internal to $\mathbf{Sh}(X)$ (in fact it us an equivalence relation on U). The *Čech nerve* of our cover is defined as the internal nerve of (6.2.2.1)

$$\check{C}(U) = \left(\cdots U \times U \times U \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} U \times U \rightrightarrows U \right). \qquad (6.2.2.2)$$

Note that to get the simplicial étale space, $\check{C}(\mathcal{U})$, corresponding to $\check{C}(U)$, we just replace the products by 'pullback over X'.

Explicit Description

The standard description of the coproduct $\mathcal{U} = \coprod_{i \in I} U_i$ is $\mathcal{U} \cong \{ (x, i) \colon x \in U_i \}$ with the map $p \colon \mathcal{U} \longrightarrow X$ then given by $p(x, i) = x$. We note also that there is a functor $I \longrightarrow \mathbf{Etale}/X$ where the indexing set I is regarded as a discrete category: we send $i \in I$ to the inclusion $U_i \hookrightarrow X$. The colimit $\mathcal{U}$ with the map p down to X is the colimit of this functor, but because I is discrete, this is also the *homotopy colimit* as well.

$\check{C}(\mathcal{U})$ is defined by (multiple) pullback of p against itself. We have $(\check{C}(\mathcal{U}))_0 = \mathcal{U} \times_X \mathcal{U} \cong \{ (x, i, j) \colon x \in U_{ij} \}$, and in general $(\check{C}(\mathcal{U}))_n \cong \{ (x, i_0, \ldots, i_n) \colon x \in U_{i_0, \ldots, i_n} \}$ with the obvious map down to X. The face maps act on the i_α's in the obvious way, for example $d_1(x, i, j, k) = (x, i, k)$.

7 Simplicial Formal Maps

7.1 Formal maps

We are interested in *simplicial formal maps* which are just maps of simplicial sheaves $\check{C}(U) \longrightarrow \operatorname{const} \operatorname{Ner} M$ with U coming from a cover, M a crossed module as above and $\operatorname{const}\colon \mathbf{Sh}(X) \longrightarrow \mathbf{SimpSh}(X)$ is the 'constant simplicial sheaf' functor. They have been introduced in [43, 42].

To get an explicit description it is easiest to work in the category of étale spaces over X, where the constant sheaves become trivial bundles. This is because a constant sheaf does not have $F(U) = A$ for all U (e.g., $F(\emptyset) = 1$). In fact $F(U)$ would be the set of locally constant functions $U \longrightarrow A$.

Let $\lambda\colon \check{C}(\mathcal{U}) \longrightarrow \overline{W}(M)$ be a map of simplicial **Etale** spaces, where the codomain is a constant bundle, i.e., it is really $X \times \overline{W}(M)$ with the projection map to X.

We can use the TCP formula to define $W(M) = M \times_\varepsilon \overline{W}(M)$, again a trivial bundle. By analogy with lemma 4.1.5.32 we want to study the pullback bundle $Z(\lambda)$ defined by

$$\begin{array}{ccc} Z(\lambda) & \longrightarrow & W(M) \\ \downarrow & \lrcorner & \downarrow \\ \check{C}(\mathcal{U}) & \xrightarrow[\lambda]{} & \overline{W}(M) \end{array} \quad . \qquad (7.1.0.1)$$

and from this we get something like a locally constant stack on X.

7.2 $\mathcal{G} \dashv \overline{W}$ for simplicial sheaves

We wish to extend $\mathcal{G} \dashv \overline{W}$ to an adjunction between $\operatorname{simp}(\mathbf{Etale}/X)$ and $\mathcal{S}$-$\mathbf{Gpds}$. Because the definitions are 'constructive' in the sense of topos theory, it would be possible to define the adjunction if we replace the étale category with the category of simplicial sheaves. However we can do it in the étale category as follows.

There is an obvious way to turn $\overline{W}(G)$ into a simplicial étale space: we just use the tensor of section 2.3.5 and tensor with the terminal object, i.e., the new functor $\overline{W}\colon \mathcal{S}\text{-}\mathbf{Gpds} \longrightarrow \operatorname{simp}(\mathbf{Etale}/X)$ sends G to $\left(\begin{array}{c} X \times \overline{W}(G) \\ \downarrow \mathrm{pr} \\ X \end{array}\right)$, where we give the simplicial set $\overline{W}(G)$ the discrete topology. Categorically we are composing the classical $\overline{W}\colon \mathcal{S}\text{-}\mathbf{Gpds} \longrightarrow \mathbf{SSet}$ with $1 \otimes -\colon \mathbf{SSet} \longrightarrow \operatorname{simp}(\mathbf{Etale}/X)$. We have an adjunction

$\Pi_0 \dashv 1\otimes -$, which lets us define $\mathcal{G}$ of $\left(\begin{array}{c} X_\bullet \\ \downarrow \text{pr} \\ X \end{array}\right) \in \text{simp}(\mathbf{Etale}/X)$ to be $\mathcal{G}(\Pi_0(X_\bullet))$. Because we are composing adjoint functors, the new $\mathcal{G}$ and $\overline{W}$ are still adjoint.

$$\text{simp}(\mathbf{Etale}/X) \underset{1\otimes(-)}{\overset{\Pi_0}{\rightleftarrows}} \mathbf{SSet} \underset{\overline{W}}{\overset{\mathcal{G}}{\rightleftarrows}} \mathcal{S}\text{-}\mathbf{Gpds} \qquad (7.2.0.2)$$

Moreover the counit is given by the same formula as that from the original $\mathcal{G} \dashv \overline{W}$ adjunction.

7.3 Explicit Description of $Z(\lambda)$

7.3.1 Example: $M = 1 \longrightarrow P$

First we consider the case where the crossed module M is just $1 \longrightarrow P$.

Description of $\overline{W}M$ for $M = 1 \longrightarrow P$

It is easy to see that $\text{Ner}\, M = \overline{W}(M) = \text{Ner}\, P$.

Description of a formal maps, λ, for $M = 1 \longrightarrow P$

With our $M = (1 \longrightarrow P)$ our map $\lambda\colon \check{C}(\mathcal{U}) \longrightarrow X \times \text{Ner}\, P$ is particularly simple, being determined by its effect in dimension 1: for every $i, j \in I$ we get $\lambda_{ij} = \lambda(x, i, j) \in P$ and in dimension 2 we get the requirement that $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$ (and this automatically implies $\lambda_{ii} = 1$ as required for compatibility with the degeneracies). In higher dimensions we get, for example, $\lambda(x, i, j, k) = (\lambda_{ij}, \lambda_{jk}) \in P^2$, and in general $\lambda(x, i_0, \ldots, i_n) = (\lambda_{i_0 i_1}, \ldots, \lambda_{i_{n-1} i_n})$.

Description of $W(M)$ for $M = 1 \longrightarrow P$

$W(M)$ is the TCP $M \times_\varepsilon \text{Ner}\, P$, where ε is the counit of $\mathcal{G} \dashv \overline{W}$ (section 7.2), so we have $\varepsilon(p_1, \ldots, p_n) = p_1$.

Thus $W(M)_0 = P$ and $W(M)_n = \{ (p, \mathbf{p})\colon p, p_i \in P \}$, where $\mathbf{p}$ is 'vector notation' for $(p_1, \ldots, p_n) \in (\text{Ner}\, P)_n = P^n$. The face maps are given by

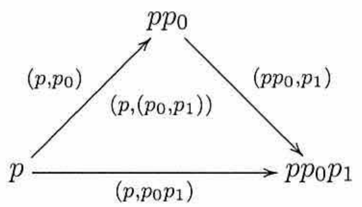
$$d_i(p, (p_1, \ldots, p_n)) = \begin{cases} (p, d_i(p_1, \ldots, p_n)) & i > 0 \\ (pp_1, d_0(p_1, \ldots, p_n)) & i = 0 \end{cases} \qquad (7.3.1.1)$$

where the d_i in the second factor comes from the nerve.

For example, a 1-simplex $(p, p_0) \in W(M)_1$ has $d_0(p, p_0) = (pp_0)$ and $d_1(p, p_0) = (p)$, which gives the picture

$$p \xrightarrow{(p,p_0)} pp_0 \tag{7.3.1.2}$$

and similarly, the picture for a typical 2-simplex is



(7.3.1.3)

$W(M)$ has the 'full equivalence relation' property that there is always exactly one 1-simplex, $(a, a^{-1}b)$ from a to b.

As étale spaces we have trivial bundles with the right-hand map, $\left(\begin{array}{c} X \times W(M) \\ \downarrow \pi \\ X \times \overline{W}(M) \end{array}\right)$, of the pullback (7.1.0.1) being just $\pi(x, p, \mathbf{p}) = (x, \mathbf{p})$.

Description of $Z(\lambda)$ for $M = 1 \longrightarrow P$

We now have enough information to describe the pullback $Z(\lambda)$. We will use the vector notation $\mathbf{p} \in (\mathrm{Ner}\, P)_n$ as above, and also $\mathbf{i} = (i_0, \ldots, i_n)$ for a $(n+1)$-tuple of indices from our open cover. We have

$$(Z(\lambda))_n \cong \{ (x, \mathbf{i}), (y, p, \mathbf{p}) \colon x \in U_{\mathbf{i}}, y \in X, p \in P, \mathbf{p} \in \mathrm{Ner}\, P_n, \lambda_{\mathbf{i}} = \mathbf{p}, x = y \} \tag{7.3.1.4}$$

$$\cong \{ (x, p, \mathbf{i}) \colon x \in U_{\mathbf{i}}, p \in P \} \tag{7.3.1.5}$$

where in the second line we have removed all redundant information.

The face maps look as follows (using the description in (7.3.1.6)).

$$d_i(x, p, \mathbf{i}) = (d_i(x, \mathbf{i}), d_i(x, p, \lambda_{\mathbf{i}})) \tag{7.3.1.6}$$

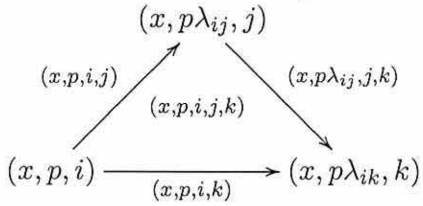
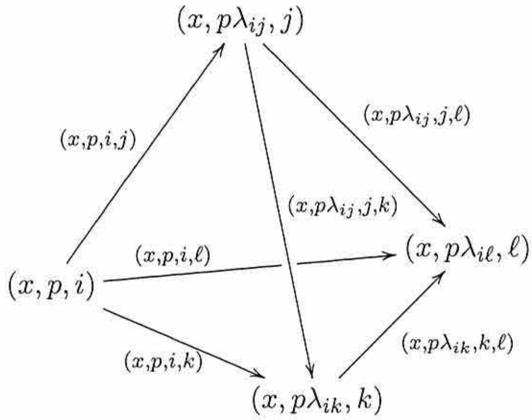
$$= \begin{cases} ((x, d_i(\mathbf{i})), (x, p, d_i(\lambda_{\mathbf{i}}))) & i > 0 \\ ((x, d_0(\mathbf{i})), (x, p\lambda_{i_0 i_1}, d_0(\lambda_{\mathbf{i}}))) & i = 0 \end{cases} \tag{7.3.1.7}$$

where we have used $(\lambda_{i_0,i_1}, \ldots, \lambda_{i_{n-1},i_n}) = \lambda_{\mathbf{i}}$ in the $i = 0$ case.

Passing to the description of $Z(\lambda)$ in the second line of (7.3.1.4) we get,

$$d_i(x, \mathbf{i}, p) = \begin{cases} (x, p, d_i(\mathbf{i})) & i > 0 \\ (x, p\lambda_{i_0 i_1}, d_0(\mathbf{i})) & i = 0 \end{cases} \tag{7.3.1.8}$$

We draw the low dimensional simplices in the following table

n	Element of $(Z(\lambda))_n$	simplex picture
0	(x,i)	(x,i)
1	(x,p,i,j)	$(x,p,i) \xrightarrow{(x,p,i,j)} (x,p\lambda_{ij},j)$
2	(x,p,i,j,k)	
3	(x,p,i,j,k,ℓ)	

We can interpret $Z(\lambda)$ as saying that over U_i we have a copy of P, thus over x we have (x,p,i), one 0-simplex for each $p \in P$. If $x \in U_{ij}$ then we have a gluing which identifies (x,p,i) with $(x,p,j\lambda_{ij})$, i.e., we are using λ to identify over double intersections. Because $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$ this is all we need to do, and we get a model for a covering space whose fibre is the underlying set of P.

Actions of P

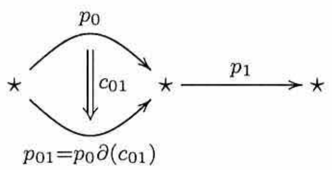
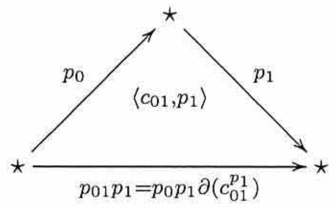
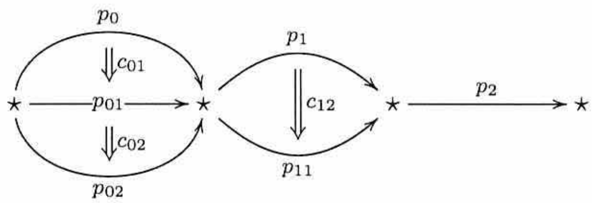
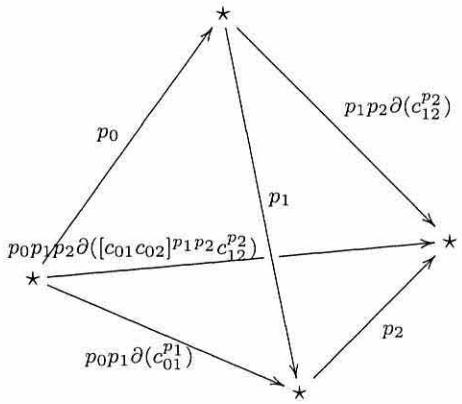
If P acts on a set Y, consider the action as a map $K(P,0) \longrightarrow \mathrm{aut}(K(Y,0))$, this gives us a different universal TCP, $W(M)_Y$, and replacing $W(M)$ above by $W(M)_Y$ gives a cover with fibre Y.

7.3.2 Description on $Z(\lambda)$ in the general case

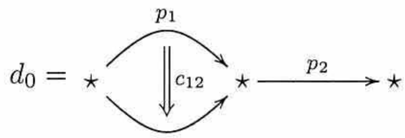
Now we can describe $Z(\lambda)$ when $M = (C \xrightarrow{\partial} P)$ is a general crossed module.

Description of $\overline{W}(M)$ for general M

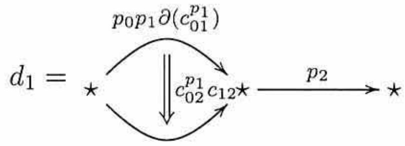
We use the following notation for elements of $\overline{W}(M)$. (Here the 2-cells of the form (p,c) are denoted merely by c, since p can be inferred from the domain.)

<table>
<tr><th>n</th><th>2-category picture</th><th>simplex picture</th></tr>
<tr><td>0</td><td>$\star$</td><td>$\star$</td></tr>
<tr><td>1</td><td>$\star \xrightarrow{p_0} \star$</td><td>$\star \xrightarrow{p_0} \star$</td></tr>
<tr><td>2</td><td>
</td><td>
</td></tr>
<tr><td>3</td><td>
</td><td>
</td></tr>
</table>

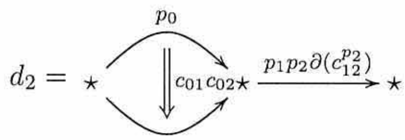
The faces of the 3-simplex are



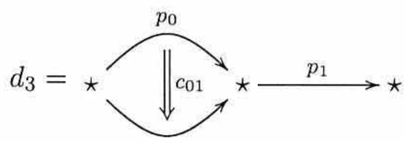
(7.3.2.1)



(7.3.2.2)



(7.3.2.3)



(7.3.2.4)

and the étale space we need is the trivial bundle $\left(\begin{array}{c} X \times \overline{W}(M) \\ \downarrow \pi \\ X \end{array}\right)$.

Description of λ for general M

The map λ has the form $\lambda(x, \mathbf{i}) = (x, \lambda_{\mathbf{i}})$ where $\lambda_{\mathbf{i}} \in \overline{W}(M)$.

Dimension 0

λ maps all 0-simplices $(x, i) \in \check{C}(\mathcal{U})$ to a 0-cell, which means we must have

$$\lambda_i = \star \qquad (7.3.2.5)$$

Dimension 1

$\lambda_{ij} \in W(M)_1$ has the form

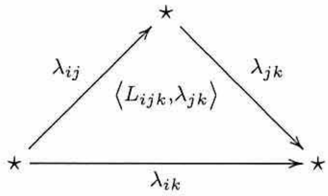
$$\star = \lambda_i \xrightarrow{\lambda_{ij}} \lambda_j = \star \qquad (7.3.2.6)$$

with $\lambda_{ij} \in P$.

Dimension 2

Using our description of $\overline{W}(M)_2$, we have

$\lambda_{ijk} =$



(7.3.2.7)

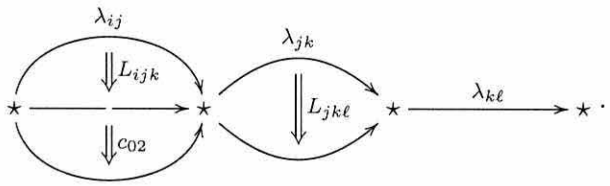
where the d_1 face gives us the equation

$$\lambda_{ik} = \lambda_{ij}\lambda_{jk}\partial(L_{ijk}^{\lambda_{jk}}) \tag{7.3.2.8}$$

in P.

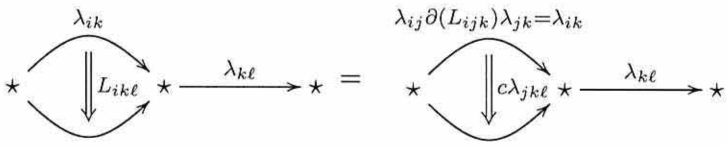
Dimension 3

The equations $d_0(\lambda_{ijk\ell}) = \lambda_{jk\ell}$ and $d_3(\lambda_{ijk\ell}) = \lambda_{ijk}$ tell us that that $\lambda_{ijk\ell}$ is the diagram



(7.3.2.9)

Now we use $d_1(\lambda_{ijk\ell}) = \lambda_{ik\ell}$ which gives



(7.3.2.10)

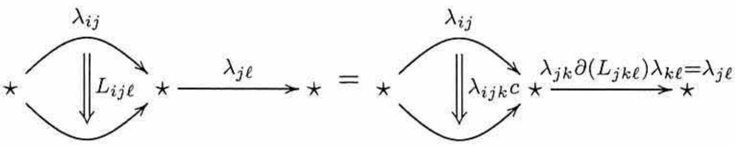
where $c\lambda_{jk\ell}$ is an abbreviation for the 2-cell

$$(\lambda_{ij}\partial(L_{ijk}), c_{02}) \#_0 (\lambda_{jk}, L_{jk\ell}) = (\lambda_{ik}, c_{02}^{\lambda_{jk}} L_{jk\ell}) \tag{7.3.2.11}$$

and we used (7.3.2.8) for its domain. Thus we must have

$$c_{02}^{\lambda_{jk}} L_{jk\ell} = L_{ik\ell}. \tag{7.3.2.12}$$

From $d_2(\lambda_{ijk\ell}) = \lambda_{ij\ell}$ we have



(7.3.2.13)

where $\lambda_{ijk}c$ stands for the 2-cell

$$(\lambda_{ij}, L_{ijk}) \#_1 (\lambda_{ij}\partial(L_{ijk}), c_{02}) = (\lambda_{ij}, L_{ijk}c_{02}). \tag{7.3.2.14}$$

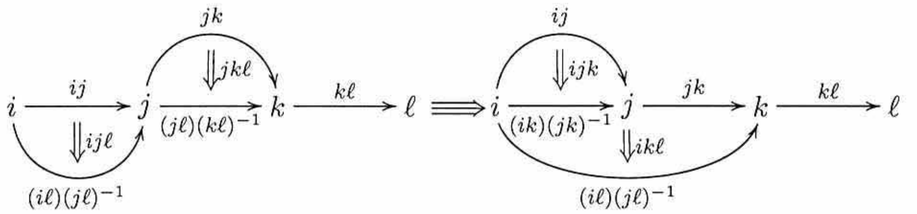
This gives us

$$L_{ijk}c_{02} = L_{ij\ell}. \tag{7.3.2.15}$$

Eliminating c_{02} from (7.3.2.15) and (7.3.2.12) we get the equation (in C)

$$L_{ij\ell}^{\lambda_{jk}} L_{jk\ell} = L_{ijk}^{\lambda_{jk}} L_{ik\ell}. \tag{7.3.2.16}$$

One way of looking at the condition (7.3.2.16) is that it says that the tetrahedron $\lambda_{ijk\ell}$ is commutative, i.e., the composition of the even faces is equal to the composition of the odd faces: $(d_2d_2 \#_0 d_0) \#_1 d_2 = (d_3 \#_0 d_0d_0) \#_1 d_1$. With this point of view we regard $\lambda_{ijk\ell}$ as being the (necessarily identity) three-cell



(7.3.2.17)

where we have omitted the λ symbols for brevity. The equation (7.3.2.16) actually corresponds to the picture (7.3.2.17) without the final λ_{kl}, but as we are working in 2-groupoids the two forms are equivalent.

Dimensions 4 and above

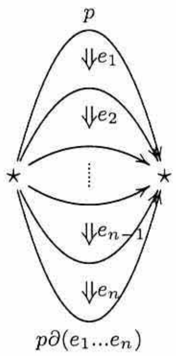
The action of λ in higher dimensions is determined by that in dimensions 0–3. (From d_0 and d_n we get all but the 2-cell c_{nn}, and any other 'interior' face allows us to determine c_{nn}. It remains to show that the resulting 2-cell does not depend on the choice of interior face, which is possible using (7.3.2.16).)

Description of $W(M)$ for general M

Now we describe $W(M) = (\mathrm{Ner}\, \mathfrak{X}(M)_{\mathrm{v}})_n \times_\varepsilon \overline{W}(M)$. As a set, the n-simplices are given by

$$W(M) = \{ (t, s) \colon t \in (\mathrm{Ner}\, \mathfrak{X}(M)_{\mathrm{v}})_n, s \in (\overline{W}(M))_n \} \tag{7.3.2.18}$$

For s we use the notation as in section 7.3.2, and

$t =$ 

$\in (\mathrm{Ner}\,\mathfrak{X}(M)_{\mathrm{v}})_n$ (7.3.2.19)

which we abbreviate to $t = (p, e_1, \ldots, e_n)$. Here we can see the twist in the d_0 face, as for $i > 0$, $d_i(p, \mathbf{e}) = (p, d_i\mathbf{e})$ but $d_0(p, \mathbf{e}) = (p\partial e_1, d_0(\mathbf{e}))$; thus we might think of the nerve as being

$$\mathrm{Ner}\,\mathfrak{X}(M)_{\mathrm{v}} = K(P, 0) \times_{\partial} \mathrm{Ner}\,C \tag{7.3.2.20}$$

The face maps are what we expect from a TCP, namely

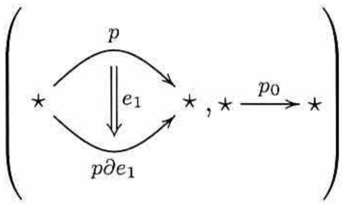
$$d_i(t, s) = \begin{cases} (d_i t, d_i s) & i > 0 \\ (d_0 t \#_0 (p, c_{01}, \ldots, c_{0,n}), d_0 s) & i = 0 \end{cases} \tag{7.3.2.21}$$

some pictures will be useful.

$(W(M))_0$

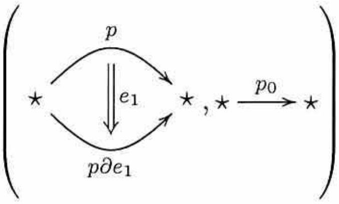
$$(W(M))_0 = \{ (p, \star) \colon p \in P \} \tag{7.3.2.22}$$

$(W(M))_1$
A 1-simplex has the form

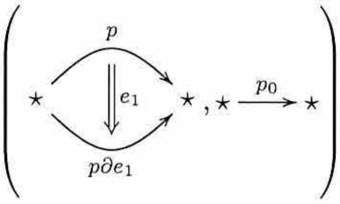


(7.3.2.23)

with

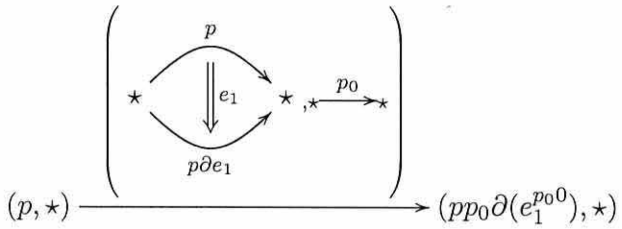
d_0 

$= ((p\partial e_1 \#_0\, p_0), \star)$ (7.3.2.24)

d_1 

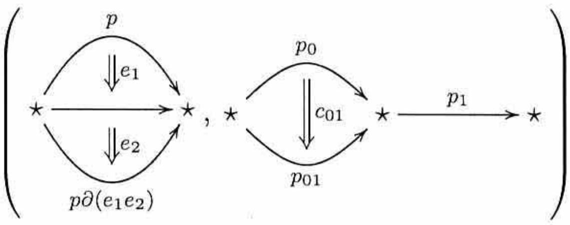
$= (p, \star)$ (7.3.2.25)

so, as a simplex, we have



(7.3.2.26)

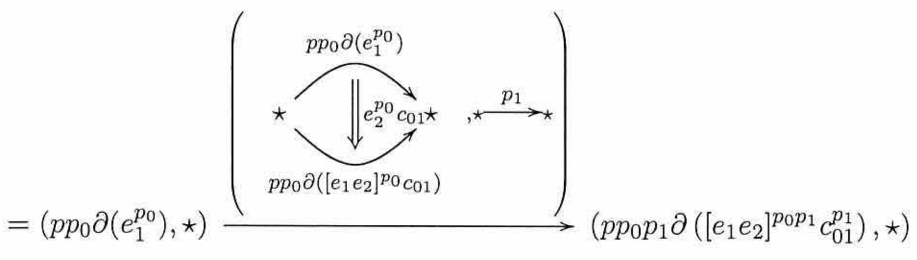
$(W(M))_2$
Going to level 2, we have elements



(7.3.2.27)

where the d_0 face is

$$d_0((p,(e_1,e_2)),(p_0,c_{01},p_1)) \quad (7.3.2.28)$$
$$= ((p\partial(e_1),e_2) \#_0 (p_0,c_{01}),p_1) \quad (7.3.2.29)$$
$$= ((p\partial(e_1)p_0, e_2^{p_0}c_{01}),p_1) \quad (7.3.2.30)$$



(7.3.2.31)

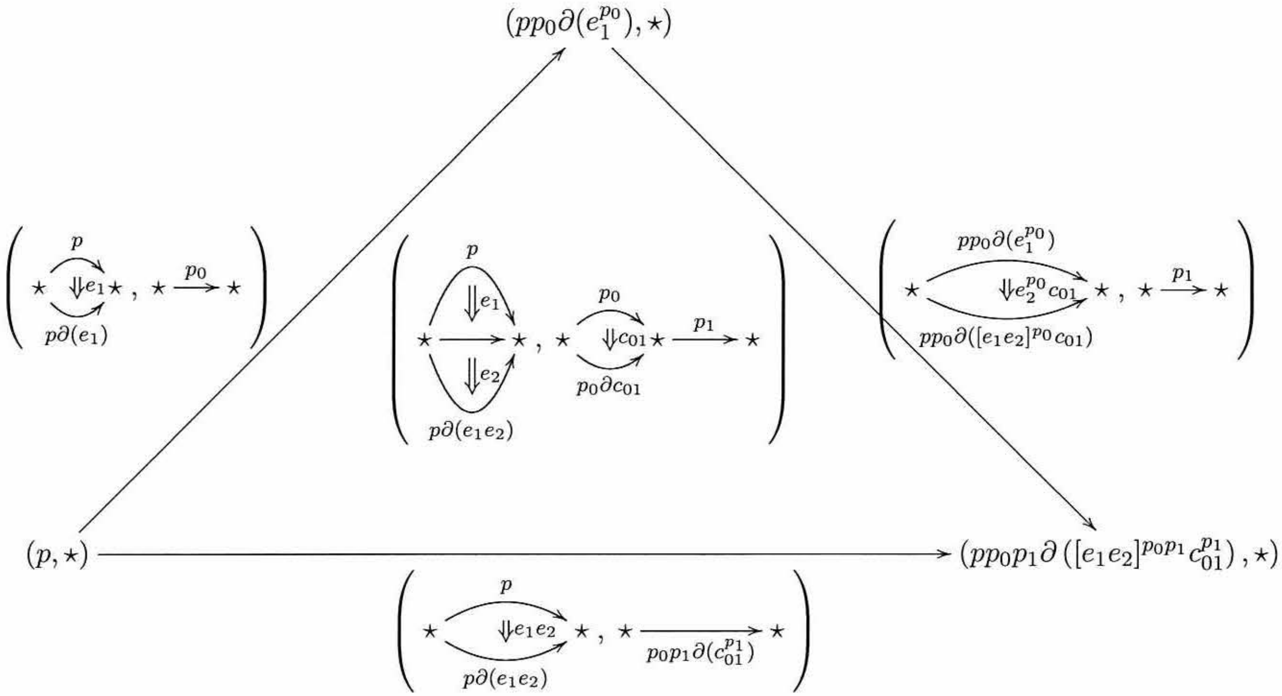
and the other faces are

$$d_1 = \left(\begin{tikzcd} \star \arrow[r, bend left, "p"{name=U}] \arrow[r, bend right, "p\partial(e_1 e_2)"'{name=D}] & \star \arrow[Rightarrow, from=U, to=D, "e_1 e_2"] \end{tikzcd}, \star \xrightarrow{p_0 p_1 \partial(c_{01}^{p_1})} \star \right) \tag{7.3.2.32}$$

$$d_2 = \left(\begin{tikzcd} \star \arrow[r, bend left, "p"{name=U}] \arrow[r, bend right, "p\partial(e_1)"'{name=D}] & \star \arrow[Rightarrow, from=U, to=D, "e_1"] \end{tikzcd}, \star \xrightarrow{p_0} \star \right) \tag{7.3.2.33}$$

Figure 7.1 shows how these fit together to give a 2-simplex.

Figure 7.1 2-simplex in $W(M)$



The pictures show that $W(M) \cong \operatorname{Dec} \overline{W}(M)$, where the isomorphism just joins the two parts along the middle $\star$ to get a 'staircase' in $\overline{W}(M)$, but of one higher dimension.

7.3.3 Description of π and p_ε in the general case

As an étale space we have $\left(\begin{array}{c} X \times W(M) \\ \downarrow \pi \\ X \end{array}\right)$ with $\pi(x,t,s) = x$, and the right-hand map of (7.1.0.1) is just

$$\begin{aligned} p_\varepsilon \colon X \times W(M) &\longrightarrow X \times \overline{W}(M) \\ (x,t,s) &\longmapsto (x,t) \end{aligned} \tag{7.3.3.1}$$

7.3.4 Description of $Z(\lambda)$ in the general case

We now describe $Z(\lambda)$ for our general crossed module M. We start with the standard description of pullbacks in $\mathcal{S}\mathbf{Etale}(X)$

$$\begin{aligned} (Z(\lambda))_n &\cong \{\, (x,\mathbf{i}),(y,t,s) \colon p_\varepsilon(y,t,s) = \lambda(x,\mathbf{i}) \,\} && (7.3.4.1)\\ &\cong \{\, (x,\mathbf{i},t,s) \colon s = \lambda_\mathbf{i} \,\} && (7.3.4.2)\\ &\cong \{\, (x,\mathbf{i},t) \colon x \in U_\mathbf{i} = U_{i_0} \cap \cdots \cap U_{i_n}, t \in (\operatorname{Ner} \mathfrak{X}(M)_\mathrm{v})_n \,\} && (7.3.4.3)\\ &= \{\, (x,\mathbf{i},(p,\mathbf{e})) \colon x \in U_\mathbf{i} = U_{i_0} \cap \cdots \cap U_{i_n}, (p,\mathbf{e}) \in (\operatorname{Ner} \mathfrak{X}(M)_\mathrm{v})_n \,\}. && (7.3.4.4) \end{aligned}$$

the faces are

$$\begin{aligned} d_i(x,\mathbf{i},(p,\mathbf{e})) &= (d_i^{\check{C}(\mathcal{U})}(x,\mathbf{i}), d_i^{X \times W(M)}(x,(p,\mathbf{e}),\lambda_\mathbf{i})) && (7.3.4.5)\\ &= \begin{cases} (x, d_i(\mathbf{i}), (p, d_i(\mathbf{e}))) & i > 0 \\ (x, d_0(\mathbf{i}), (p\partial(e_1), d_0(\mathbf{e})) \#_0 \lambda_\mathbf{i}) & i = 0 \end{cases} && (7.3.4.6) \end{aligned}$$

From this we can describe the lower dimensions explicitly.

Dimension 0
In dimension 0 we have

$$(Z(\lambda))_0 = \{\, (x,i,p) \colon x \in U_i, p \in P \,\} \tag{7.3.4.7}$$

Dimension 1
In dimension 1 we have

$$(Z(\lambda))_1 = \left\{ \left(x,i,j,\ \star \underset{p\partial(e)}{\overset{p}{\Downarrow (p,e_1)}} \star \right) \colon x \in U_{ij}, p \in P \right\} \tag{7.3.4.8}$$

with

$$d_0(x,i,j,(p,e_1)) = (x,j,p\partial(e_1)\lambda_{ij}) = (x,j,p\lambda_{ij}\partial(e_1^{\lambda_{ij}})) \tag{7.3.4.9}$$
$$d_1(x,i,j,(p,e_1)) = (x,i,p) \tag{7.3.4.10}$$

giving us the picture

$$(x,i,p) \xrightarrow{\left(x,i,j,\ \star \underset{p\partial(e)}{\overset{p}{\Downarrow (p,e_1)}} \star \right)} (x,j,p\lambda_{ij}\partial(e_1^{\lambda_{ij}})) \tag{7.3.4.11}$$

Dimension 2
In dimension 2 we have

$$(Z(\lambda))_2 = \left\{ \left(x,i,j,\ \star \underset{p\partial(e_1e_2)}{\overset{p}{\overset{\Downarrow e_1}{\underset{\Downarrow e_2}{\longrightarrow}}}} \star \right) : x \in U_{ijk}, p \in P, e_1, e_2 \in C \right\} \tag{7.3.4.12}$$

by now it is clear how to find the faces, so we will just draw the simplex in figure 7.2.

7.4 $W(M)$ as homotopy colimit

7.4.1 Simplicially enriched homotopy colimit

Nerve of a $\mathcal{S}$-Cat
The nerve of a **SSet**-enriched category, $\mathcal{C}$, is the bisimplicial set, $\operatorname{Ner}\mathcal{C}$, with

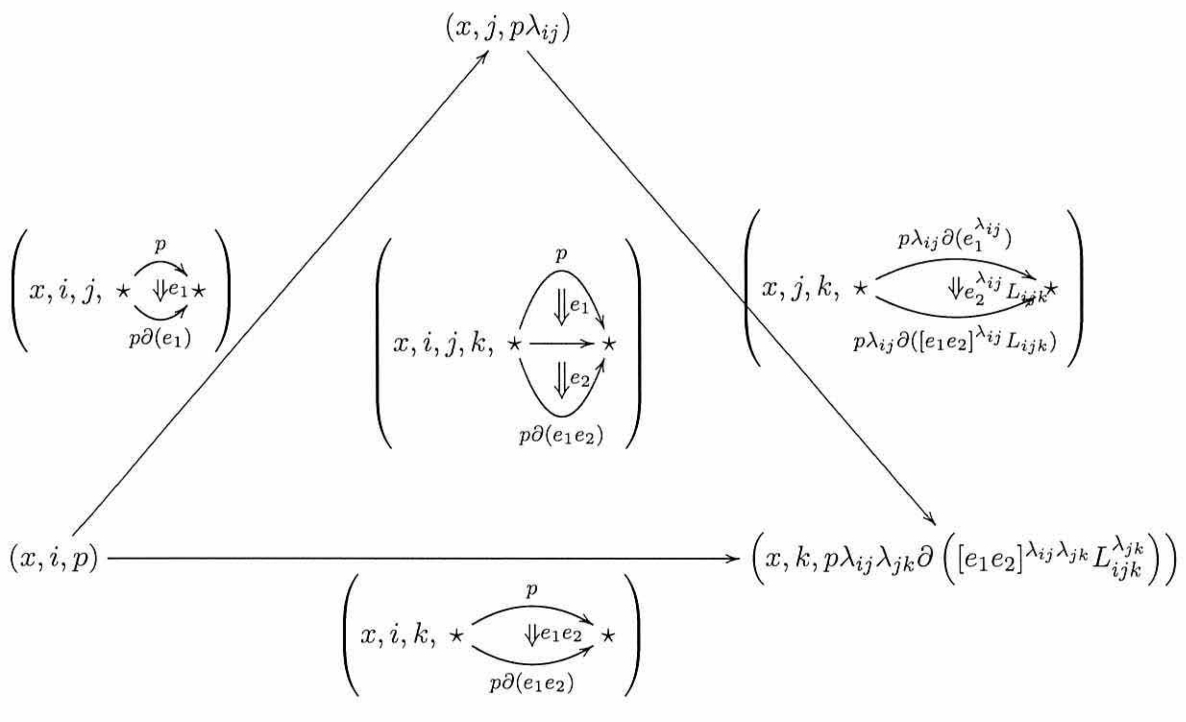
$$(\operatorname{Ner}\mathcal{C})_{pq} = (\operatorname{Ner}\mathcal{C}_q)_p = \left\{ A_0 \xrightarrow{a_1} A_1 \longrightarrow \ldots \xrightarrow{a_p} A_p \colon \dim a_i = q \right\} \tag{7.4.1.1}$$

A relevant example is when $\mathcal{C}$ is obtained from a 2-category: by replacing the hom-categories with their nerves we get an $\mathcal{S}$-**Cat** whose nerve is exactly the double nerve of the associated double category.

We can pass from **BiSSet** to **SSet** using either diag or ∇. We might write $\nabla \operatorname{Ner}\mathcal{C}$ as $\overline{W}(\mathcal{C})$ to agree with the case where $\mathcal{C}$ is in $\mathcal{S}$-**Gpds**. The argument (3.2.5.1) can be used to describe $(\nabla \operatorname{Ner}\mathcal{C})_n$ as the set of diagrams of the form

$$(\nabla \operatorname{Ner}\mathcal{C})_p = \left\{ A_0 \xrightarrow{a_1} A_1 \longrightarrow \ldots \xrightarrow{a_p} A_p \colon \dim a_i = p - i \right\} \tag{7.4.1.2}$$

Figure 7.2 2-simplex in $Z(\lambda)$



Simplicially enriched homotopy colimit

The *homotopy colimit* of a simplicially enriched functor $F\colon \mathcal{C} \longrightarrow \mathbf{SSet}$ is defined via simplicially enriched coends in [10]. As might be expected the simplicial replacement generalises as well, and we have

$$(\operatorname{hocolim} F)_n \cong \coprod_{(A_0 \xrightarrow{a_1} \dots \xrightarrow{a_n} A_n) \in (\operatorname{diag} \operatorname{Ner} \mathcal{C}_n)} (FA_0)_n \tag{7.4.1.3}$$

$$\cong \left\{ (x, (A_0 \xrightarrow{a_1} \dots \xrightarrow{a_n} A_n)) \colon x \in (FA_0)_n, \dim a_i = n - i \right\} \tag{7.4.1.4}$$

Another version of hocolim**:** $\operatorname{hocolim}^{\nabla}$

If we replace diag with ∇ in (7.4.1.3) we get a new simplicial set $\operatorname{hocolim}^{\nabla}$. Because ∇ is weakly equivalent to diag, $\operatorname{hocolim}^{\nabla}$ is a good replacement for hocolim.

7.4.2 SSet-functors

If $\mathcal{C}$ is an $\mathcal{S}$-**Cat**, then a simplicially enriched functor $F\colon \mathcal{C} \longrightarrow \mathbf{SSet}$ is specified by a map $\operatorname{Ob}(\mathcal{C}) \longrightarrow \mathbf{SSet}$ from the objects of $\mathcal{C}$ to **SSet** and either

1. simplicial maps

$$\begin{aligned} \mathcal{C}(A, B) &\longrightarrow \underline{S}(FA, FB) \\ f &\longmapsto Ff \end{aligned} \tag{7.4.2.1}$$

   with the usual functor conditions, or

2. simplicial maps

$$\begin{aligned} FA \times \mathcal{C}(A, B) &\longrightarrow FB \\ (x, f) &\longmapsto x \otimes f \end{aligned} \tag{7.4.2.2}$$

   with the usual "action" conditions $(x \otimes f) \otimes g = x \otimes (f \# g)$ and $x \otimes \mathrm{id} = x$.

(the two are equivalent by the Cartesian closed adjunction).

7.4.3 The regular representation for a crossed module

Consider a simplicially enriched category, $\mathcal{M}$, with one object $\star$. There is a **SSet**-functor $\rho_{\mathcal{M}}\colon \mathcal{M} \longrightarrow \mathbf{SSet}$ with $\rho_{\mathcal{M}}(\star) = \mathcal{M}(\star, \star)$ and whose action is the composition $\#\colon \mathcal{M}(\star, \star) \times \mathcal{M}(\star, \star) \longrightarrow \mathcal{M}(\star, \star)$ from $\mathcal{M}$. This is just like the regular representation from section 2.5.3.

Theorem 7.4.3.1. *Let M be our crossed module $C \xrightarrow{\partial} P$ and $\mathcal{M}$ the corresponding* $\mathcal{S}$-**Gpds**. *Then $W(M) \cong \operatorname{hocolim}^{\nabla} \rho_{\mathcal{M}}$.*

Proof. This is basically just the argument used in Theorem 4.1.5.66. □

8 Stacks

We now introduce stacks and attempt to reinterpret the formal maps of the last chapter in terms of stacks.

8.1 Stacks and Descent

8.1.1 Definitions

Definition 8.1.1.1. Let $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ be a functor, and

$$X_\bullet = \cdots X_2 \underset{d_2}{\overset{d_0}{\underset{d_1}{\rightrightarrows}}} X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \xrightarrow{p} X \qquad (s_0\colon X_1 \to X_2,\ s_1\colon X_1 \to X_2,\ s_0\colon X_0 \to X_1) \tag{8.1.1.2}$$

an augmented simplicial object in $\mathcal{C}$ (thus $F(X_\bullet)$ is an augmented cosimplicial category). Then the category $\mathrm{Des}(X_\bullet, F)$ of *descent data on $X_\bullet$ with coefficients in F* is the following category:

1. Objects are pairs (x, f) with $x \in FX_0$ and $f\colon Fd_1(x) \xrightarrow{\cong} Fd_0(x)$ an isomorphism satisfying the *cocycle condition*, which says that the following triangle commutes:

$$\begin{array}{ccc}
\boxed{\begin{array}{c} Fd_2(Fd_1(x)) = \\ F(d_1 \# d_2)(x) = \\ F(d_1 \# d_1)(x) = \\ Fd_1(Fd_1(x)) \end{array}} & \xrightarrow{Fd_2(f)} & \boxed{\begin{array}{c} Fd_2(Fd_0(x)) = \\ F(d_2 \# d_0)(x) = \\ F(d_0 \# d_1)(x) = \\ Fd_0(Fd_1(x)) \end{array}} \\
 & \searrow^{Fd_1(f)} & \downarrow{\scriptstyle Fd_0(f)} \\
 & & \boxed{\begin{array}{c} Fd_0(Fd_0(x)) = \\ Fd_1(Fd_0) \end{array}}
\end{array} \tag{8.1.1.3}$$

2. Morphisms $m\colon (x, f) \longrightarrow (y, g)$ are maps $m\colon x \longrightarrow y$ in FX_0 making the square

$$\begin{array}{ccc}
Fd_1(x) & \xrightarrow{Fd_1(m)} & Fd_1(y) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
Fd_0(x) & \xrightarrow[Fd_0(m)]{} & Fd_0(y)
\end{array} \tag{8.1.1.4}$$

commute in FX_1.

We call an object $(x, f) \in \mathrm{Des}(X_\bullet, F)$ a *descent datum* on x. If we ignore the cocycle condition then we have a pair (x, f) which we call a *gluing* of x. A descent datum (x, f) is called *normalised* if $Fs_0(f)\colon x \xrightarrow{\cong} x$ is the identity id_x, and *trivial* if f itself was an identity (thus 'trivial' implies 'normalised').

Definition 8.1.1.5. From any $X_\bullet$ and functor F as above we get a functor

$$\begin{aligned} \mathrm{des}\colon FX &\longrightarrow \mathrm{Des}(X_\bullet, F) \\ A &\longmapsto (Fp(A), \mathrm{id}_{F(d_0p)(A)}) \\ \alpha\colon A \longrightarrow B &\longmapsto Fp(\alpha) \end{aligned} \tag{8.1.1.6}$$

sending A to the trivial descent data on $Fp(A)$. (The map $Fp(\alpha)$ is a map of descent data since the simplicial identities give $F(d_0p) = F(d_1p)$, and hence

$$\begin{array}{ccc} Fd_1(Fp(A)) & \xrightarrow{Fd_1(Fp(\alpha))} & Fd_1(Fp(B)) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ Fd_0(Fp(A)) & \xrightarrow[Fd_0(Fp(\alpha))]{} & Fd_0(Fp(B)) \end{array} \tag{8.1.1.7}$$

commutes as required).

Definition 8.1.1.8. The map $p\colon X_0 \longrightarrow X$ is a *map of* 1*-descent* if the above functor des is full and faithful, and a *map of effective descent* if des is an equivalence.

A descent datum (x, f) is called *effective* if it in the essential image of des, i.e., if there is some $A \in FX$ and an isomorphism $m\colon Fp(A) \xrightarrow{\cong} x$ making

$$\begin{array}{ccc} Fd_1(Fp(A)) & \xrightarrow{Fd_1(m)} & Fd_1(x) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Fd_1(Fp(A)) & \xrightarrow[Fd_0(m)]{} & Fd_1(x) \end{array} \tag{8.1.1.9}$$

commute.

Note that applying Fs_0 to the square (8.1.1.9) gives

$$\begin{array}{ccc} Fp(A) & \xrightarrow{m} & x \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle Fs_0(f)} \\ Fp(A) & \xrightarrow[m]{} & x \end{array} \tag{8.1.1.10}$$

which means that effective descent data are automatically normalised. It is equally easy to see that the class of normalised descent data is closed under isomorphisms in $\mathrm{Des}(X_\bullet, F)$, and hence when des is essentially surjective, all descent data are normalised.

Definition 8.1.1.11. Suppose $\mathcal{C}$ has a Grothendieck topology (i.e., we have a collection of covering families). Each covering family induces a simplicial object; in fact we could define Grothendieck topologies by asking for a collection of *covering augmented simplicial objects* (+ some axioms) augmented simplicial objects $X_\bullet$. If (for each covering $X_\bullet$) des is full and faithful, we say that the functor $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ is a *prestack*. If the des functors are equivalences then we say that F is a *stack*.

8.1.2 Stacks categorify sheaves, Stacks and equalisers

For any $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$, and for any $A, B \in FX$, interpreting (8.1.1.4), we have

$$\mathrm{Hom}_{\mathrm{Des}(X_\bullet,F)}(\mathrm{des}(A), \mathrm{des}(B)) = \{\, m\colon Fp(A) \longrightarrow Fp(A)\colon Fd_1(m) = Fd_0(m) \,\} \tag{8.1.2.1}$$

hence $\mathrm{Hom}_{\mathrm{Des}(X_\bullet,F)}(\mathrm{des}(A), \mathrm{des}(B))$ is the equaliser of the diagram

$$\mathrm{Hom}_{FX_0}(Fp(A), Fp(B)) \underset{Fd_0}{\overset{Fd_1}{\rightrightarrows}} \mathrm{Hom}_{FX_1}(F(d_0p)(A), F(d_0p)(B)), \tag{8.1.2.2}$$

thus the prestack condition is equivalent to the requirement that

$$\mathrm{Hom}_{FX}(A, B) \xrightarrow{Fp} \mathrm{Hom}_{FX_0}(Fp(A), Fp(B)) \underset{Fd_0}{\overset{Fd_1}{\rightrightarrows}} \mathrm{Hom}_{FX_1}(F(d_0p)(A), F(d_0p)(B)) \tag{8.1.2.3}$$

also be an equaliser, which is exactly the condition that the functor

$$\begin{aligned}
\mathrm{Hom}_{FX}(A, B)\colon (\mathcal{C}/X)^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\
\begin{pmatrix} Y \\ \downarrow y \\ X \end{pmatrix} &\longmapsto \mathrm{Hom}_{FY}(Fy(A), Fy(B)) \\
\alpha\colon \begin{pmatrix} Y \\ \downarrow y \\ X \end{pmatrix} \longrightarrow \begin{pmatrix} Z \\ \downarrow z \\ X \end{pmatrix} &\longmapsto F\alpha
\end{aligned} \tag{8.1.2.4}$$

satisfy the sheaf condition for the cover induced by $X_\bullet$ (i.e., regard the augmented simplicial object as a simplicial object in $\mathcal{C}/X$, augmented over the terminal object id_X). So we have proved

Lemma 8.1.2.5. $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ *is a prestack for a topology iff all the functors* $\mathrm{Hom}_{FX}(A, B)$ *are sheaves for that topology.* □

Lemma 8.1.2.6. *Suppose that the functor* $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ *factors through* $\mathbf{Set}$ *(i.e., FX is always discrete). Then* $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ *is a (pre-)stack iff* $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ *is a (separated pre-)sheaf.*

Proof. $\mathrm{Des}(X_\bullet, F) \rightarrowtail FX_0$ is discrete with objects

$$\{ x \in FX_0 : Fd_0(x) = Fd_1(x) \}, \tag{8.1.2.7}$$

i.e., for any presheaf F, $\mathrm{Des}(X_\bullet, F)$ is the equaliser of the diagram

$$FX_0 \underset{Fd_1}{\overset{Fd_0}{\rightrightarrows}} FX_1. \tag{8.1.2.8}$$

For any $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ we have a commuting diagram

$$\begin{array}{ccccc} \mathrm{Des}(X_\bullet, F) & \rightarrowtail & FX_0 & \underset{Fd_1}{\overset{Fd_0}{\rightrightarrows}} & FX_1 \\ \uparrow{\scriptstyle \mathrm{des}} & \nearrow{\scriptstyle Fp} & & & \\ FX & & & & \end{array} \tag{8.1.2.9}$$

so Fp is the unique factorisation through the equaliser $\mathrm{Des}(X_\bullet, F)$.

Then $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ is a prestack iff des is a full and faithful functor between discrete categories, i.e., iff des is a monic map of sets $FX \rightarrowtail \mathrm{Des}(X_\bullet, F)$, which is exactly the condition that $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a separated presheaf.

Similarly, F is a stack iff des is an equivalence of discrete categories, i.e., iff des is an isomorphism in $\mathbf{Set}$. But this last condition says exactly that FX is also the equaliser of (8.1.2.8), which is exactly the sheaf condition for F (see [36]). □

8.1.3 Comparison of definitions of Stacks

The paper [20] defines a stack in a slightly different way. There, a stack is a groupoid fibration over $\mathcal{C}$ satisfying some additional properties. In this section we will show that (under some mild assumption) the notion of a stack in [20] is equivalent to a special case of the definition just given, namely to the case where the covering simplicial objects $X_\bullet$ came from a Grothendieck topology and the functor F factors through **Grpoids**. This result is well-known, but perhaps not written down in one place.

Covers

In [20], a 'cover' means a Grothendieck topology, i.e., a collection of *covering families* $(S_i \longrightarrow S)_{i\in I}$. In [13], a 'cover' is an (augmented) simplicial object: the link is that from a covering family $(S_i \longrightarrow S)$, we can form the following (augmented) simplicial object

$$T_\bullet = \cdots T_2 \overset{d_0,\, d_1,\, d_2}{\Rrightarrow} T_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} T_0 \xrightarrow{p} S \qquad (s_0, s_1\colon T_1 \to T_2,\ s_0\colon T_0 \to T_1) \tag{8.1.3.1}$$

Where $T_0 = \coprod_i S_i$, $T_1 = \coprod_{i,j} S_{ij}$, $T_2 = \coprod_{i,j,k} S_{ijk}$ etc., in which e.g., S_{ij} is the pullback of $S_i \longrightarrow S$ against $S_j \longrightarrow S$.

Having identified the notion of covers, we will now show that F is a stack in sense of [13] iff π was a stack in the sense of [20].

Let $\left(\begin{array}{c}\mathcal{X}\\ \downarrow \pi \\ \mathcal{C}\end{array}\right)$ be a stack over $\mathcal{C}$ in the sense of [20]. Assume that $\mathcal{C}$ has coproducts (we also need $\mathcal{C}$ to have pullbacks to make the definition in [20] work).

Form a functor $F\colon \mathcal{C}^{\text{op}} \longrightarrow \textbf{Cat}$ (this is the standard "inverse" to the Grothendieck construction):

$$F(S) = \pi^{-1}(S) \tag{8.1.3.2}$$

$$F(f\colon S \longrightarrow T) = \text{restriction to } S. \tag{8.1.3.3}$$

On morphisms F acts as 'restriction' (an object $x \in \mathcal{X}$ over T is sent to $Ff(x) = x|_S$, the codomain of 'the' (Cartesian) lift of f to a map ending at x). It is a fibration in the sense of [46]. For notational convenience, we will assume that the fibration is split, so that F is a genuine functor, rather than a pseudo-functor. Since π is a stack, every FS is a groupoid, so F factors through **Grpoids**, the category of groupoids. Conversely any $F\colon \mathcal{C}^{\text{op}} \longrightarrow$ **Grpoids** gives us a category π over $\mathcal{C}$ via the Grothendieck construction.

The Descent Category

In [13] the descent category $\text{Des}(S_\bullet, F)$ for F has objects (x, f) where $x \in FS_0$ means that x is a lift of S_0. The map $S_i \longrightarrow \coprod_i S_i$ gives us objects $x_i := F(S_i \longrightarrow \coprod_i S_i)(x) = x|_{S_i}$ over each S_i.

Now the face map $d_1\colon S_1 \longrightarrow S_0$ is defined by the following commutative diagram (in which unnamed arrows are canonical arrows corresponding to the coproduct)

$$\begin{array}{ccc} \coprod_{i,j} S_{ij} & \xrightarrow{d_1} & \coprod_i S_i \\ \nearrow & & \nearrow \\ S_{ij} & \longrightarrow & S_i \end{array} \tag{8.1.3.4}$$

And lifting this diagram to $\mathcal{X}$ gives us $x_i|_{S_{ij}} = F(S_{ij} \longrightarrow S_i)(x_i)$ over S_{ij}:

$$\begin{array}{ccc} Fd_1(x) & \longrightarrow & x \\ \nearrow & & \nearrow \\ x_i|_{S_{ij}} & \longrightarrow & x_i \end{array} \tag{8.1.3.5}$$

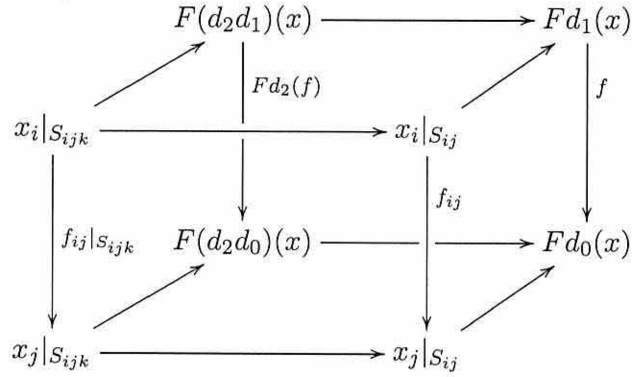
Similarly we get $x_j|_{S_{ij}}$ over S_{ij}, and hence from a glueing, f, of x, we can form isomorphisms

$$f_{ij} := F\left(S_{ij} \longrightarrow \coprod S_{ij}\right)(f)\colon x_i|_{S_{ij}} \xrightarrow{\cong} x_j|_{S_{ij}} \tag{8.1.3.6}$$

Once we have our f_{ij}'s, we can further restrict, forming

$$f_{ij}|_{S_{ijk}} := F(S_{ijk} \longrightarrow S_{ij})(f_{ij}), \tag{8.1.3.7}$$

(i.e., we have a diagram



(8.1.3.8)

lifting the diagram defining d_2).

The cocycle condition (8.1.1.3) gives us the requirement that

$$f_{ij}|_{S_{ijk}} \# f_{jk}|_{S_{ijk}} = f_{ik}|_{S_{ijk}} \tag{8.1.3.9}$$

So from a descent datum (in the sense of [13]) on x, we get a descent datum in the sense of [20] ([20] defines a descent datum as a collection of objects E_i over S_i, with vertical isomorphisms f_{ij} satisfying (8.1.3.9)). In fact this description is exactly what we find in [41], except that the extra condition '$f_{ii} = \mathrm{id}$' is required. This corresponds to the 'normalisation' condition of definition 8.1.1.1.

Maps of descent datum are not discussed in [20], but it is clear what they should be, (and indeed we can read this off from definition 8.1.1.8, or from the description in [41], since although that paper defines a stack as a (lax) functor to **Cat**, the topologies used are Grothendieck topologies, and the descent category used is exactly the $\mathrm{Des}(S, \pi)$ we are defining here): A *map* $m\colon \langle x_i, f_{ij}\rangle \longrightarrow \langle y_i, g_{ij}\rangle$ *of descent data (for* π*)* is a collection of vertical maps $m_i (= F(S_i \longrightarrow \coprod_i S_i)(m)$, so each m_i is automatically an isomorphism), making the diagram

$$\begin{array}{ccc} x_i|_{S_{ij}} & \xrightarrow{m_i|_{S_{ij}}} & y_i|_{S_{ij}} \\ {\scriptstyle f_{ij}}\downarrow & & \downarrow{\scriptstyle g_{ij}} \\ x_j|_{S_{ij}} & \xrightarrow[m_j|_{S_{ij}}]{} & y_j|_{S_{ij}} \end{array} \tag{8.1.3.10}$$

where $m_i|_{S_{ij}} = F(S_{ij} \longrightarrow S_i)(m_i) = F(S_{ij} \longrightarrow \coprod S_{ij})(Fd_1(m))$.

From [20], we get a category $\mathrm{Des}(S, \pi)$ of descent data for π on S, and the above argument gives us the following functor

$$\begin{aligned} \mathcal{F}\colon \operatorname{Des}(S_\bullet, F) &\longrightarrow \operatorname{Des}(S,\pi) \\ (x,f) &\longmapsto \langle x|_{S_i}, f|_{S_{ij}}\rangle \\ m &\longmapsto m|_{S_i} \end{aligned} \tag{8.1.3.11}$$

$\mathcal{F}$ is full and faithful, because any collection of maps $m_i\colon \langle x|_{S_i}, f|_{S_{ij}}\rangle \longrightarrow \langle y|_{S_i}, g|_{S_{ij}}\rangle$ give us a unique map $m = \coprod_i m_i\colon (x,f) \longrightarrow (y,g)$ with $m|_{S_i} = m_i$.

To show that $\mathcal{F}$ is essentially surjective, we need the following result

Proposition 8.1.3.12. *Let $\pi\colon \mathcal{X} \longrightarrow \mathcal{C}$ be a groupoid fibration (in the sense of [46]), and suppose that $\mathcal{C}$ and $\mathcal{X}$ have coproducts. Then π preserves coproducts.*

Proof. Let $E_i \in \mathcal{X}$ be objects over objects $S_i \in \mathcal{C}$. Form the coproducts

$$\begin{aligned} &E_i \xrightarrow{h_i} C = \textstyle\coprod_i E_i && \text{in } \mathcal{X} \\ &S_i \xrightarrow{\alpha_i} \textstyle\coprod_i S_i && \text{in } \mathcal{C} \end{aligned} \tag{8.1.3.13}$$

Applying π to the coproduct diagram defining C, we get a unique β making

$$\begin{array}{ccc} S_i & \xrightarrow{\pi(h_i)} & \pi(C) \\ {\scriptstyle \alpha_i}\downarrow & \nearrow{\scriptstyle \beta} & \\ \coprod S_i & & \end{array} \tag{8.1.3.14}$$

commute. Lift β to some map, m, ending at C; since h_i is cartesian we get a unique lifting n_i of α_i making

$$\begin{array}{ccc} E_i & \xrightarrow{h_i} & C \\ {\scriptstyle n_i}\downarrow & \nearrow{\scriptstyle m} & \\ x & & \end{array} \tag{8.1.3.15}$$

commute in $\mathcal{C}$. But C is a coproduct, so the maps n_i induce a unique $g\colon C \longrightarrow x$ with $n_i = h_i \# g$ for all i. We now have

$$h_i = n_i m = h_i g m, \tag{8.1.3.16}$$

and the h_i are jointly epi, so we have $\mathrm{id}_C = gm$, and thus $\mathrm{id}_{\pi(C)} = \pi(g)\pi(m)$.

In $\mathcal{C}$, (8.1.3.14) and the definition of g give us

$$\alpha_i \pi(m)\pi(g) = \pi(h_i)\pi(g) = \pi(n_i) = \alpha_i, \tag{8.1.3.17}$$

and since the α_i are jointly epi, we get that $\pi(m)\pi(g) = \mathrm{id}_{\coprod S_i}$, i.e., $\pi(g)$ is the required isomorphism $\pi(\coprod E_i) \xrightarrow{\cong} \coprod \pi(E_i)$. □

Corollary 8.1.3.18. *$\mathcal{F}$ is essentially surjective.*

Proof. Given $\langle x_i, f_{ij}\rangle \in \mathrm{Des}(S,F)$, let x be $\coprod x_i$. By proposition 8.1.3.12, we have (up to isomorphism) $x|_{S_i} = x_i$, and $\pi(x) = \coprod x_i$, i.e., $x \in FS_0$. To define f we take

$$\coprod_{i,j} f_{ij} \colon \coprod_{i,j} x_i|_{S_{ij}} \xrightarrow{\cong} \coprod_{i,j} x_j|_{S_{ij}} \tag{8.1.3.19}$$

and observe that

$$\begin{aligned} \pi\left(\coprod x_i|_{S_{ij}}\right) &\cong \coprod \pi(x_i|_{S_{ij}}) \\ &\cong \coprod S_{ij} \\ &\cong \pi(Fd_1(x)) \end{aligned} \tag{8.1.3.20}$$

Since π is a groupoid fibration it reflects isomorphisms and hence we have $\coprod x_i|_{S_{ij}} \cong Fd_1(x)$. Similarly we get $\coprod x_j|_{S_{ij}} \cong Fd_0(x)$, and thus $\coprod f_{ij}$ induces a unique isomorphism $f\colon Fd_1(x) \xrightarrow{\cong} Fd_0(x)$ in FS_1. Because the f_{ij} satisfied (8.1.3.9), this f satisfies the Duskin cocycle condition (8.1.1.3). So we have found $(x,f) \in \mathrm{Des}(S_\bullet, F)$ with $\mathcal{F}((x,f)) \cong \langle x_i, f_{ij}\rangle$ as required. ☐

Putting the above results together we conclude:

Proposition 8.1.3.21. *For any groupoid fibration π corresponding to $F\colon \mathcal{C}^{op} \longrightarrow$ **Cat**,*

$$\mathrm{Des}(S_\bullet, F) \simeq \mathrm{Des}(S,\pi) \tag{8.1.3.22}$$

for every covering family $S_\bullet$. ☐

The functor des

Again we do not have an explicit description of des in [20], but it is clear what the definition should be, and that definition should agree with definition 8.1.1.5 (and [41]). Thus, instantiating definition 8.1.1.5, the functor $\mathrm{des}\colon FS \longrightarrow \mathrm{Des}(S,\pi)$ should map an object A over S to the decent datum corresponding to $(Fp(A), \mathrm{id}) \in \mathrm{Des}(S_\bullet, F)$, i.e., $\mathrm{des}(A) = \langle A_i, \mathrm{id}\rangle \in \mathrm{Des}(S,\pi)$ where $A_i := A|_{S_i}$ over S_i (this works because we have a diagram

$$\begin{array}{ccc} A|_{S_i} & \longrightarrow & Fp(A) \\ & \searrow & \downarrow \\ & & A \\ \\ S_i & \longrightarrow & \coprod S_i \\ & \searrow & \downarrow{\scriptstyle p} \\ & & S \end{array} \tag{8.1.3.23}$$

where the top triangle lifts the bottom), and the map $A_i|_{S_{ij}} \xrightarrow{\cong} A_j|_{S_{ij}}$ is the identity on $A|_{S_{ij}}$. On maps $m\colon A \longrightarrow B$, we get the family $m|_{S_i}$ (which corresponds to $Fp(m)$ in the Duskin picture).

Prestacks and sheaves of isomorphisms

Proposition 8.1.3.24. *Let F and π be as in proposition 8.1.3.21. Then F is a prestack (in the sense of [13]) if and only if π satisfies the condition 'isomorphisms form a sheaf' from [20].*

Proof. The condition that F be a prestack is that whenever A and B are lifts of S, and we have isomorphisms $m_i\colon A_i \xrightarrow{\cong} B_i$, then there is a unique $m\colon A \longrightarrow B$ with $m|_{S_i} = m_i$. This is the condition 'isomorphisms form a sheaf' we find in [20]. □

Effective Descent data

Proposition 8.1.3.25. *Let F and π be as in proposition 8.1.3.21, and let $\langle E_i, \alpha_{ij}\rangle \in \mathrm{Des}(S,\pi)$ correspond to $(x, f) \in \mathrm{Des}(S_\bullet, F)$ under the equivalence* (8.1.3.22). *Then (x, f) is effective (in the sense of [13]) if and only if $\langle E_i, \alpha_{ij}\rangle$ is effective in the sense of [20].*

Proof. This is immediate from the description of $\mathrm{des}\colon FS \longrightarrow \mathrm{Des}(S,\pi)$ above. □

The Notions of Stack agree

Putting the previous sections together we get the following theorem.

Theorem 8.1.3.26. *Let F and π be as in proposition 8.1.3.21. Then F is a stack (in the sense of [13]) if and only if π is a stack (in the sense of [20]).* □

8.2 2-Bundles and 2-Étale spaces

We have already discussed in chapter 6 the adjunction between the category of presheaves on X and the category $\mathbf{Top}/X$ of bundles over X, and the resulting equivalence between sheaves and étale spaces on X.

We now describe a similar equivalence between fibred categories on X and what we will call 2-*bundles* on X which gives a way of moving between stacks on X and 2-*étale spaces* on X.

Throughout we work with a fixed topological space X, which is locally connected. A *bundle on* X will just mean a space over X, i.e., an object of the category $\mathbf{Top}/X$. Recall that the bundle $\begin{pmatrix} E \\ \downarrow \pi \\ X \end{pmatrix}$ is étale over X if, for every $e \in E$, we can find neighbourhoods $B \subseteq E$ for e and $U \subseteq X$ for $\pi(e)$ such that $\pi|_B\colon B \xrightarrow{\cong} U$ is a homeomorphism. The inverse for $\pi|_U$ can be considered as a section, $s\colon U \longrightarrow E$, of E over U.

8.2.1 2-Bundles over X

2-truncated simplicial bundles

We wish to consider groupoids E in $\mathbf{Top}/X$; we regard E as a 2-truncated simplicial space over X. So we have $\begin{pmatrix} E_0 \\ \downarrow \pi \\ X \end{pmatrix}$, a 'bundle of vertices' and $\begin{pmatrix} E_1 \\ \downarrow \pi \\ X \end{pmatrix}$, a 'bundle of morphisms (or 1-simplices) over X'. We require that the two face maps (i.e., codomain and domain), $d_0, d_1 \colon E_1 \longrightarrow E_0$ the degeneracy map, $s_0 \colon E_0 \longrightarrow E_1$ and composition (which is a face map d_1 from level 2 of the simplicial structure induced by E) be continuous, commute with π, and satisfy the simplicial identities. In particular, if $x_0 \xrightarrow{x} x_1 \in E_1$ then $\pi(x) = \pi(x_0) = \pi(x_1)$. We write E' for the subspace of non-identity morphisms; the identity maps are given by the subspace $s_0 E_0$. Since $d_0 s_0 = \mathrm{id}$ we have $E_1 = E' \coprod s_0 E_0$. This is a coproduct of sets, but often we will have topologies chosen so that this is a coproduct of spaces as well.

Definition of 2-Bundles

A 2-*bundle* on X is a groupoid, E, in $\mathbf{Top}/X$ in which all open sets $U \subseteq E_0$ are closed under isomorphism, i.e., if $u \in U$ $u \cong v$ then v must also be in U.

Example: 2-discrete spaces and trivial 2-bundles

As an example we generalise the notion of a discrete space to our 2-dimensional context. A topological groupoid, G, is 2-*discrete* if G' is discrete and a subset is open in G_0 iff it is closed under isomorphism. (Thus $G_1 = G' \coprod G_0$ as a topological space.)

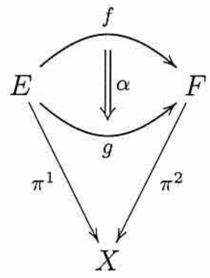
Thus G_0 is not discrete, but if we identified isomorphic points then the result becomes discrete. If we think of the open sets as providing a way to measure the distance between two points, then in a discrete space all points can be distinguished, but in a 2-discrete space isomorphic points become indistinguishable.

A *trivial* 2-*bundle* is a 2-bundle of the form $\begin{pmatrix} G \times X \\ \downarrow \mathrm{pr} \\ X \end{pmatrix}$ where G is 2-discrete. This means that as sets we have an object (f, x) for each $f \in G_0$ and $x \in X$ (and similarly for morphisms), and the open sets are products of opens from G and X.

Note that the fibre over $x \in X$ is the groupoid G, but with a not-entirely discrete topology: if we made both G_0 and G_1 discrete then we would "just" have two bundles over X, corresponding to a sheaf of groupoids. Since we want a stack of groupoids, we need the non-discrete topology.

The 2-category TwoBundles/X

We get a 2-category **TwoBundles**$/X$ of 2-bundles over X, where the picture is



(8.2.1.1)

Explicitly, a 1-cell $f\colon \pi^1 \longrightarrow \pi^2$ is a functor $f\colon E \longrightarrow F$ which is continuous (i.e., the maps on objects and morphisms are both continuous) and satisfies $\pi^2(f(x)) = \pi^1(x)$ (for all objects and maps x).

For the 2-cell, α, we want a continuous natural transformation. Let I be the 'unit groupoid' with the 2-discrete topology: $I_0 = \{0, 1\}$ with the codiscrete topology and I' has two non-identity maps, $\iota\colon 0 \longrightarrow 1$ and $\iota^{-1}\colon 1 \longrightarrow 0$, with the discrete topology. Then we can form a topological groupoid $E \times I$ in the obvious way, and we have two maps $i_0, i_1\colon E \longrightarrow E \times I$ with $i_0(e) = (e, 0)$ and $i_1(e) = (e, 1)$. These are continuous: $i_0^{-1}(U \times W)$ is empty unless $0 \in W$ in which case it is U. We can now define $\alpha\colon f \Longrightarrow g$ to be a continuous functor $\alpha\colon E \times I \longrightarrow E$ which is 'over X' in the sense that

$$\begin{array}{ccc} E \times I & \xrightarrow{\alpha} & F \\ \downarrow & & \downarrow{\scriptstyle \pi^2} \\ E & \xrightarrow[\pi^1]{} & X \end{array} \qquad (8.2.1.2)$$

commutes, and agrees with f and g in the sense that $f = (i_0 \# \alpha)$ and $g = (i_1 \# \alpha)$.

In particular if $x \in E$, we get a map $\alpha_x := \alpha(s_0 x, \iota)$ in F from $\alpha(x, 0) = fx$ to $\alpha(x, 1) = gx$, and given $m\colon x \longrightarrow y$ in E applying α to the equation $(s_0 x, \iota) \# (m, s_0 1) = (m, \iota) = (m, s_0 0) \# (s_0 y, \iota)$ gives the usual naturality square

$$\begin{array}{ccc} fx & \xrightarrow{\alpha_x} & gx \\ {\scriptstyle fm}\downarrow & & \downarrow{\scriptstyle gm} \\ fy & \xrightarrow[\alpha_y]{} & gy \end{array} \qquad (8.2.1.3)$$

If such an α exists then the 'closed under isomorphism' property tells us that if $U \subseteq E_0$ is an open set of objects, then $f^{-1}(U) = g^{-1}(U)$ (because if $f(x) \in U$ then the isomorphism $\alpha_x\colon f(x) \xrightarrow{\cong} g(x)$ forces $g(x) \in U$ and conversely)

Also the requirement (8.2.1.2) is automatically commutative because f and g are maps over X.

2-Étale spaces

A 2-*étale space* on X is a 2-bundle, $\left(\begin{array}{c} E \\ \downarrow \pi \\ X \end{array}\right)$, on X with the following properties. First, $\left(\begin{array}{c} E_0 \\ \downarrow \pi \\ X \end{array}\right)$ is *homotopy-étale* over X. This means (by definition) that given $e \in E_0$ we can find neighbourhoods $B \subseteq E_0$ for e and $U \subseteq X$ for $\pi(e)$ and a map $s \colon U \longrightarrow B$ such that $s \mathbin{\#} \pi|_B = \mathrm{id}_U$ and $\pi|_B \mathbin{\#} s \simeq \mathrm{id}_B$ where the homotopy, H, is a strong homotopy, meaning that

$$\begin{array}{ccc} B \times I & \xrightarrow{H} & B \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow[\pi]{} & U \end{array} \tag{8.2.1.4}$$

commutes. On the morphisms we require the following étale-like condition: For all maps $\varphi \colon x \longrightarrow y$ in E_1 let $p = \pi(\varphi) = \pi(x) = \pi(y) \in X$. We have (using the homotopy-étale condition) sections $s_x, s_y \colon U \longrightarrow E_0$ with $s_x(p) = x$ and $s_y(p) = y$. Then there must exist an open set W and a map $s \colon W \longrightarrow E_1$ with $s(p) = \varphi$, and then the condition is that the sub-bundle $E_1^W(x, y) := \operatorname{im} s = \{\, sq \colon q \in W \,\}$ must be étale over X.

The paper [8] appears to provide a similar construction, and there they end up with an actual stack.

8.2.2 The fibred groupoid of a 2-bundle

Given a 2-bundle, $\left(\begin{array}{c} E \\ \downarrow \pi \\ X \end{array}\right)$, on X we define a fibred groupoid $\Gamma[\pi]$ on X (i.e., a functor $\Gamma[\pi] \colon \mathcal{O}(X)^{\mathrm{op}} \longrightarrow \mathbf{Grpoids}$ as follows. On an open set U we have a groupoid $\Gamma[\pi](U)$ whose objects are the sections of $\left(\begin{array}{c} E_0 \\ \downarrow \pi \\ X \end{array}\right)$ over U, and whose maps are sections of E_1. Note that on maps we are doing the 'classical' construction of a presheaf of sections from a bundle.

Explicitly, an object is a continuous map $a \colon U \longrightarrow E_0$ with $\pi(a(x)) = x$ for all $x \in U$; a morphism $a \longrightarrow b$ is $\alpha \colon U \longrightarrow E_1$ over p with $\alpha \mathbin{\#} d_1 = a$ and $\alpha \mathbin{\#} d_0 = b$, i.e., for each $u \in U$ we have a 1-simplex (i.e., a morphism) $a(u) \xrightarrow{\alpha(u)} b(u)$ in E. Composition of morphisms happens pointwise (this is why we required 2-bundles to have E be a groupoid, rather than a general simplicial set). Restriction functors $\Gamma[\pi](U) \longrightarrow \Gamma[\pi](V)$ just pre-compose with the inclusion $V \hookrightarrow U$. (Thus we always get that $\Gamma[\pi]$ is a strict functor $\mathcal{O}(X)^{\mathrm{op}} \longrightarrow \mathbf{Grpoids}$.)

Descent data

A descent datum for a cover, $(U_\alpha \colon \alpha \in A)$ of U, is given by maps $\varphi_\alpha \colon U_\alpha \longrightarrow E_0$ and isomorphisms $\varphi_{\alpha\beta}(u) \colon \varphi_\alpha(u) \xrightarrow{\cong} \varphi_\alpha(u)$ (which are 'continuous in u' meaning that $u \mapsto \varphi_{\alpha\beta}(u)$ is a continuous map from U to E_1) plus the condition that whenever u lies in a triple intersection $U_{\alpha\beta\gamma}$, $\varphi_{\alpha\beta}(u) \# \varphi_{\beta\gamma}(u) = \varphi_{\alpha\gamma}(u)$.

The descent datum is effective if we have a collation of the $\varphi_\alpha(u)$, consisting of a continuous map $\varphi \colon U \longrightarrow E_0$ and isomorphisms (again forming a function continuous in u) $\psi_\alpha(u) \colon \varphi(u) \xrightarrow{\cong} \varphi_\alpha(u)$ which satisfy $\psi_\alpha(u) \# \varphi_{\alpha\beta}(u) = \psi_{\beta(u)}$ on $U_{\alpha\beta}$.

Gluing conditions in $\Gamma(\pi)$

We would like $\Gamma(\pi)$ to be a stack, so let us examine how true this is.

Let $(U_\alpha \colon \alpha \in A)$ be an open cover of U, and let there be a descent datum for this cover given by maps $\varphi_\alpha \colon U_\alpha \longrightarrow E_0$ and $\varphi_{\alpha,\beta} \colon U_{\alpha\beta} \longrightarrow E_1$.

Choose a section $s \colon U \longrightarrow \coprod_\alpha U_\alpha$ for the map $\coprod_\alpha U_\alpha \longrightarrow U$ coming from the coproduct (on the underlying sets: s need not be continuous), i.e., we have $s(u) = (su, u)$ where $su \in A$ is an index for which $u \in U_{su}$. This is used to define φ with

$$\begin{aligned} \varphi \colon U &\longrightarrow E_0 \\ u &\longmapsto \varphi_{su}(u) \end{aligned} \tag{8.2.2.1}$$

Claim: φ is continuous

Proof of claim. If $V \subseteq E_0$ is open then $\varphi^{-1}(V) = \{ w \colon \varphi_{sw}(w) \in V \}$. For $w \in \varphi^{-1}(V)$ we have that $B_w = \varphi_{sw}^{-1}(V) \subseteq U_{sw}$, is an open neighbourhood of w, so we just need to show that $B_w \subseteq \varphi^{-1}(V)$. But if $x \in B_w$ then $\varphi_{sw}(x) \in V$ and the isomorphism $\varphi_{sw,sx}(x) \colon \varphi_{sw}(x) \longrightarrow \varphi_{sx}(x) = \varphi(x)$ shows $\varphi(x) \in V$, i.e., $B_w \subseteq \varphi^{-1}(V)$. □

Now $\varphi|_{U_\alpha}$ still sends $u \in U_\alpha$ to $\varphi_{su}(u)$, but we have $u \in U_{s(u),\alpha}$ and an isomorphism $\varphi_{s(u),\alpha}(u) \colon \varphi_{s(u)}u \xrightarrow{\cong} \varphi_\alpha u$.

But now we have a problem as the above is almost the ψ_α we need (the condition on ψ_α being compatible with $\varphi_{\alpha\beta}$ over double intersections $U_{\alpha,\beta}$ is exactly the condition on $\varphi_{su,\alpha,\beta}$ over triple intersections $U_{su,\alpha,\beta}$) but ψ_α need not be a continuous map $U_\alpha \longrightarrow E_1$. We conjecture that there is a way to repair this, perhaps by a better choice of s, for at least some 'nice' contexts. In other words, we define a nice 2-bundle to be one for which $\Gamma(\pi)$ is an actual stack, and call the resulting category **nTwoBundles**.

This would give us a map $\Gamma \colon \mathbf{nTwoBundles}/X \longrightarrow [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Grpoids}]$ whose image was inside $\mathbf{Stacks}(X)$. Even if the 2-bundle is not 'nice', $\Gamma(F)$ acts on morphisms just like the functor from (6.1.2.4), so we do have image inside the (2-)category of prestacks, and there is a 'stackification' map from **PreStacks**(X) to **Stacks**(X) (which is left adjoint to the obvious forgetful functor)

8.2.3 The 2-stalk of a fibred category

The above generalised sections of a bundle, and the stalk of a presheaf can be generalised as follows.

We define a functor Stalk: $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Grpoids}] \longrightarrow \mathbf{nTwoBundles}$ called a 2-*stalk*. Let $F\colon \mathcal{O}(X)^{\mathrm{op}} \longrightarrow \mathbf{Grpoids}$, then $\mathrm{Stalk}(F)$ is a generalisation of the usual stalk construction.

For objects, $\mathrm{Stalk}(F)_0$ consists of (equivalence classes of) elements (p, U, x) where x is an object of FU and $p \in U \in \mathcal{O}(X)$ with the equivalence relation $(p, U, x) \sim (p, V, x|_V)$ whenever $V \subseteq U$.

For the maps, we essentially apply the usual stalk construction to the arrows of F, i.e., $\mathrm{Stalk}(F)((p, U, x), (p, V, y))$ is obtained from

$$\{\, (p, W, \varphi)\colon \varphi\colon x|_W \longrightarrow y|_W, p \in W \in \mathcal{O}(U \cap V) \,\} \tag{8.2.3.1}$$

where we have the usual equivalence relation $(p, W, \varphi) \sim (p, W, \varphi)$ iff $\varphi|_\Omega = \varphi'|_\Omega$ where $p \in \Omega \in \mathcal{O}(W \cap W')$. This defines all the maps, including the identities, in the same way, however the to define the topology we treat the identities differently. The equivalence relation on the objects ensures that domain and codomain are well-defined.

We are not yet done because we have not defined a topology on $\mathrm{Stalk}(F)$, for which it is easiest to introduce the map η in the next section.

8.2.4 $\eta\colon F \longrightarrow \Gamma\,\mathrm{Stalk}(F)$

Ignoring the topology on $\mathrm{Stalk}(F)_0$ for the moment, we have a map $\eta_F(U)\colon FU \longrightarrow \Gamma\,\mathrm{Stalk}(F)(U)$, where on an open set U we define $\eta_F(U)(x) = \left(\frac{x}{U}\right)^\bullet$, with

$$\begin{aligned} \left(\tfrac{x}{U}\right)^\bullet : U &\longrightarrow \mathrm{Stalk}(F) \\ p &\longmapsto (p, U, x) \end{aligned} \tag{8.2.4.1}$$

If x is an arrow, this is the usual unit for the $\mathrm{Stalk} \dashv \Gamma$ adjunction and we choose basic open sets in $\mathrm{Stalk}(F)'$ to be images $\left(\frac{x}{U}\right)^\bullet(U) = \{\, (q, W)\colon q \in U \,\}$ (we only do this for the non-identity arrows as the basic opens at an identity $s_0 x$ are just $s_0 V$ where V is a basic open at the object x).

If x is an object, then we choose our basic opens in $\mathrm{Stalk}(F)_0$ to be 'essential images',

$$\begin{aligned} \operatorname{ess\,img}\left(\frac{x}{U}\right)^\bullet &= \left\{ (q, V, y)\colon (q, V, y) \cong \left(\frac{x}{U}\right)^\bullet(p) \quad \text{for some } p \in U \right\} \\ &= \{\, (q, V, y)\colon (q, V, y) \cong (q, U, x),\ q \in U \,\} \end{aligned} \tag{8.2.4.2}$$

(the second line follows because we can only have an isomorphism if $p = q$).

If x is a morphism, then $\left(\frac{x}{U}\right)^\bullet$ is continuous—this is the result for the stalk of a sheaf plus the lemma below.

Lemma 8.2.4.3. *For $x \in \operatorname{Ob} FU$, $\left(\frac{x}{U}\right)^\bullet : U \longrightarrow \operatorname{Stalk}(F)_0$ is continuous*

Proof. We must show that $((\frac{x}{U})^\bullet)^{-1}(\operatorname{ess\,img}((\frac{y}{V})^\bullet)) = B$ is open in X.

If $p \in B$ then $p \in U$ and there is an isomorphism $(p, W, \varphi) \colon (p, U, x) \xrightarrow{\cong} (p, V, y)$. We will show that $W \subseteq B$ which shows B is open.

If $q \in W$ then we have isomorphisms

$$\left(\frac{y}{V}\right)^\bullet(q) = (q, V, y) = (q, W, y|_W) \xrightarrow{(q,W,\varphi)} (q, W, x|_W) = (q, U, x) = \left(\frac{x}{U}\right)^\bullet(q) \quad (8.2.4.4)$$

therefore $q \in B$. □

Lemma 8.2.4.5. $\pi\colon \operatorname{Stalk}(F)_0 \longrightarrow X$ *is continuous.*

Proof. If U is open in X, and $(p, U, x) \in \pi^{-1}(U)$ then $\operatorname{ess\,img}\left(\frac{x}{U}\right)^\bullet \subseteq \pi^{-1}(U)$. □

Lemma 8.2.4.6. $\operatorname{Stalk}(F)$ *is a 2-étale space over X.*

Proof. Open sets in $\operatorname{Stalk}(F)_0$ are closed under isomorphism by definition. The maps s_0, d_0, d_1 are continuous. (so we have a 2-bundle over X).

$\operatorname{Stalk}(F)_0$ is homotopy-étale: if $(p, U, x) \in \operatorname{Stalk}(F)_0$ then we have an open neighbourhood $B = \operatorname{ess\,img}((\frac{x}{U})^\bullet)$ of (p, U, x) and an open neighbourhood U for p for which $\pi|_B$ and $(\frac{x}{U})^\bullet$ form a homotopy equivalence with $(\frac{x}{U})^\bullet \mathbin{\#} \pi = \mathrm{id}$. (The homotopy $H\colon B \times [0,1] \longrightarrow B$ can be given by $H(b, 0) = (q, U, x)$ and $H(b, t) = b$ if $t > 0$. This is continuous because all functions with codomain a codiscrete space are continuous)

The condition on $\operatorname{Stalk}(F)_1$: Given a non-identity map $(p, W, \varphi)\colon (p, U, x) \longrightarrow (p, V, y)$ we can restrict x and y to W, so without loss of generality we can assume $U = V = W$, then let $s_x = (\frac{x}{W})^\bullet$, $s_y = (\frac{y}{W})^\bullet$ and $s = (\frac{\varphi}{W})^\bullet$. Then $E_1^W(x, y) = \operatorname{im} s = \{\, (q, W, \varphi) \colon q \in W \,\}$ is étale by the argument showing the stalk of a sheaf is étale. Similarly if $s_0(p, U, x)$ is an identity map then $s_x = (\frac{u}{U})^\bullet$ and $s = s_x \mathbin{\#} s_0$ gives $E_1^U(x, x) = \{\, s_0(q, U, x) \colon q \in U \,\}$ is again clearly étale.

□

We get a (2-)functor $\operatorname{Stalk}\colon [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Grpoids}] \longrightarrow \mathbf{TwoBundles}/X$ whose image is inside $\mathbf{TwoEtale}/X$.

8.2.5 The counit, $\varepsilon\colon \operatorname{Stalk}\Gamma(\pi) \longrightarrow \pi$

If π is a 2-bundle over X, we have a map (over X)

$$\begin{aligned} \varepsilon_\pi\colon \operatorname{Stalk}\Gamma(\pi) &\longrightarrow \pi \\ (p, U, x) &\longmapsto x(p) \end{aligned} \quad (8.2.5.1)$$

on level 1 this is the usual counit of $\operatorname{Stalk} \dashv \Gamma$, and hence continuous for the same reasons. On level 0 it is continuous because we required that open sets in E_0 be closed under isomorphisms: If $B \subseteq E_0$ is open, then $\varepsilon^{-1}(B) = \{\, (p, U, x\colon U \longrightarrow E_0) \colon x(p) \in B \,\}$.

If $(p, U, x) \in \varepsilon^{-1}(B)$ then let $V = x^{-1}(B)$, which is open in X and we claim that $\operatorname{ess\,img}\left(\frac{x|_V}{V}\right)^\bullet \subseteq \varepsilon^{-1}(B)$. To prove this we must show that if $(q, W, Y) \in \operatorname{ess\,img}\left(\frac{x|_V}{V}\right)^\bullet$, then $y(q) \in B$. To do this, just observe that the isomorphism $(q, W, y) \xrightarrow{\cong} (q, V, x|_V)$ gives us $y(q) \xrightarrow{\cong} x(q)$. Since $q \in V$ we have $x(q) \in B$, and since B is closed under isomorphism, $y(q) \in B$ also.

8.2.6 The adjunction Stalk $\dashv \Gamma$

In section 8.2.4 we defined a map $(\eta_F)(U)$ for each open set U. These form a natural transformation $\eta_F \colon F \longrightarrow \Gamma\,\mathrm{Stalk}(F)$, which is natural in F, i.e., they fit together to form $\eta \colon \mathrm{id} \Longrightarrow \Gamma\,\mathrm{Stalk}$. Similarly the map ε_π from section 8.2.5 is natural in π, i.e., we have a natural transformation $\varepsilon \colon \mathrm{Stalk}\,\Gamma \Longrightarrow \mathrm{id}$. These are the unit and counit of an adjunction

$$[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Grpoids}] \underset{\Gamma}{\overset{\mathrm{Stalk}}{\rightleftarrows}} \perp\ \mathbf{nTwoBundles}/X \tag{8.2.6.1}$$

as can be easily verified by checking the triangle equalities.

We have seen that the adjunction restricts to one between $\mathbf{Stacks}(X)$ and $\mathbf{TwoEtale}/X$. We now show that this new, restricted, adjunction is actually a 2-equivalence, i.e., that η and ε are equivalences when restricted to stacks and 2-étale spaces.

Proposition 8.2.6.2. *If F is a stack on X then η_F is an equivalence.*

Proof. We must show that each $\eta_F(U)$ is an equivalence of categories, i.e., full, faithful and essentially surjective. On arrows, $\eta_F(U)$ is full and faithful because the arrows of F form a sheaf.

Let $\alpha \in \Gamma\,\mathrm{Stalk}(F)(U)$. We need $Z \in FU$ with $\alpha \cong \left(\frac{Z}{U}\right)^\bullet$.

α has the form $\alpha(p) = (p, V_p, x_p)$ where $x_p \in F(V_p)$. α is continuous, so we have open sets

$$\begin{aligned} K_p &:= \alpha^{-1}(\operatorname{ess\,img}\left(\frac{x_p}{V_p}\right)^\bullet) \\ &= \{\, q \in U \colon \alpha(q) \cong (q, V_p, x_p) \,\} \end{aligned} \tag{8.2.6.3}$$

with $p \in K_p \subseteq U$. Let $L_p = V_p \cap K_p$, then we have

$$\begin{aligned} \alpha|_{L_p} &\cong \left(\frac{x_p}{V_p}\right)^\bullet |_{K_p \cap V_p} \\ &\cong \left(\frac{x_p}{K_p \cap V_p}\right)^\bullet \\ &= \left(\frac{x_p}{L_p}\right)^\bullet. \end{aligned} \tag{8.2.6.4}$$

Let $Z_p = x_p|_{L_p} \in F(L_p)$ with $\varphi_p\colon \left(\frac{Z_p}{L_p}\right)^{\bullet} \longrightarrow \alpha|_{L_p}$. We get φ_{pq} making the following diagram commute:

$$\begin{array}{ccccc} \left(\frac{Z_p|_{L_p\cap L_q}}{L_p\cap L_q}\right)^{\bullet} & \xrightarrow{\cong} & \left(\frac{Z_p}{L_p}\right)^{\bullet}|_{L_p\cap L_q} & \xrightarrow[\varphi_p|_{L_p\cap L_q}]{\cong} & \alpha|_{L_p\cap L_q} \\ {\scriptstyle \varphi_{pq}}\downarrow{\scriptstyle \cong} & & & & \Vert \\ \left(\frac{Z_p|_{L_p\cap L_q}}{L_p\cap L_q}\right)^{\bullet} & \xrightarrow[\cong]{} & \left(\frac{Z_q}{L_p}\right)^{\bullet}|_{L_p\cap L_q} & \xrightarrow[\cong]{\varphi_q|_{L_p\cap L_q}} & \alpha|_{L_p\cap L_q} \end{array} \tag{8.2.6.5}$$

Using the diagram (8.2.6.5) it is easy to show that $\varphi_{pq} \# \varphi_{qr} = \varphi_{pr}$ over triple intersections: we simply put the diagram for φ_{pq} above the diagram for φ_{qr}, and notice that the outside of the resulting rectangle is the diagram for φ_{pr}.

Also note that because η_F is full and faithful, $\varphi_{pq} = \left(\frac{\psi_{pq}}{L_{pq}}\right)^{\bullet}$ for a unique map ψ_{pq} in $F(L_{pq})$. Since $\cup_p L_p = U$ and F is a stack, the descent datum (Z_p, ψ_{pq}) thus glues to give $Z \in FU$ with $Z|_{L_p} \cong Z_p$.

We now have

$$\begin{aligned} \alpha(p) &= \alpha|_{L_p}(p) \\ &\cong \left(\frac{Z_p}{L_p}\right)^{\bullet}(p) \\ &\cong \left(\frac{Z|_{L_p}}{L_p}\right)^{\bullet}(p) \\ &\cong \left(\frac{Z}{U}\right)^{\bullet}|_{L_p}(p) \\ &= \left(\frac{Z}{U}\right)^{\bullet}(p) \end{aligned} \tag{8.2.6.6}$$

therefore $\alpha \cong \eta_F(U)(Z)$ as required. □

Proposition 8.2.6.7. *If* $\left(\begin{array}{c} E \\ \downarrow{\scriptstyle \pi} \\ X \end{array}\right)$ *is 2-étale over X then ε_π is a homotopy equivalence*

$\varepsilon\colon \operatorname{Stalk}\Gamma(\pi) \Longrightarrow \pi$.

Proof. Given $e \in E$ we have a homotopy inverse $s\colon U \longrightarrow B$ for some U and B, and mapping e to $(\pi(e), U, s)$ gives a homotopy inverse to ε. □

Corollary 8.2.6.8. Stacks(X) *is 2-equivalent to* **TwoEtale**$/X$

8.2.7 Trivial and Locally trivial stacks

Since we know what a trivial 2-bundle is, we should be able to see what a trivial stack is: it should be the stackification of the constant fibred category which maps all U to some fixed groupoid G.

We should be able to define locally constant 2-bundles, and thus locally constant stacks.

8.3 $Z(\lambda)$ as a stack

We turn $Z(\lambda)$ into a locally trivial stack using the 2-equivalence in section 8.2 and the fact that $\mathbf{Stacks}(-)$ is a 2-stack on Top. This is a generalisation of a process from [5]: essentially all we are doing is replacing the crossed module $G \xrightarrow{\partial} \mathrm{Aut}(G)$ used there with M.

8.3.1 Over U_i

On the open set U_i we place the object $Z_i = \left(\begin{array}{c} U_i \times \mathrm{Ner}\,\mathcal{X}(M)_v \\ \downarrow \\ U_i \end{array} \right)$. where now on $\mathrm{Ner}\,\mathcal{X}(M)_v$ we use the 2-discrete topology from section 8.2.1, i.e., we have a constant stack on U_i, where the fibre is the (2-dimensional part of the) simplicial set $\mathrm{Ner}\,\mathcal{X}(M)_v$.

8.3.2 Over U_{ij}

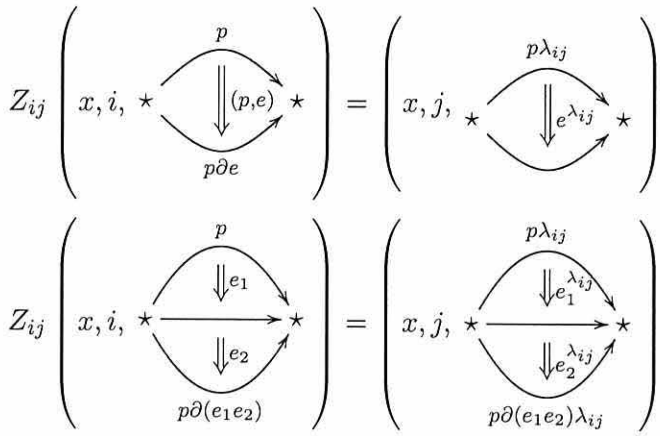
We can restrict Z_i to a subset of U_i by pullback; the result is another constant stack:

$$\begin{array}{ccc} V \times \mathrm{Ner}\,\mathcal{X}(M)_v & \longrightarrow & U_i \times \mathrm{Ner}\,\mathcal{X}(M)_v \\ {\scriptstyle Z_i|_V}\downarrow & & \downarrow{\scriptstyle Z_i} \\ V & \longrightarrow & U_i \end{array} \tag{8.3.2.1}$$

We have a map $Z_{ij} \colon Z_i|_{U_{ij}} \longrightarrow Z_j|_{U_{ij}}$ induced by the action of M on itself by multiplication and the formal map λ: $Z_{ij} = - \#_0 \lambda_{ij}$.

Explicitly this looks like

$$Z_{ij}(x, i, p) = (x, j, p\lambda_{ij})$$



(8.3.2.2)

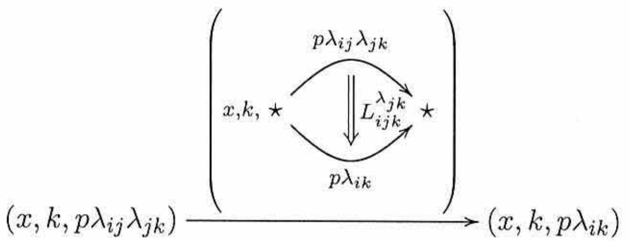
etc.

8.3.3 Over U_{ijk}

Restricting to a triple intersection, U_{ijk}, the maps $Z_i|_{U_{ijk}} \# Z_j|_{U_{ijk}}$ and $Z_k|_{U_{ijk}}$ become homotopic, the homotopy begin given by multiplication by the two-cell λ_{ijk} (referring to section 7.3.2 we are here writing λ_{ijk} for the composition of the diagram (7.3.2.7), i.e., $\lambda_{ijk} = L_{ijk} \#_0 \lambda_{ik} = (\lambda_{ij}\lambda_{jk}, L_{ijk}^{\lambda_{jk}})$).

We can use section 2.4.2 to visualise the homotopy Z_{ijk} as follows.

On a 0-simplex $(x, i, p) \in Z_i|_{U_{ijk}}$ is assigned the 1-simplex



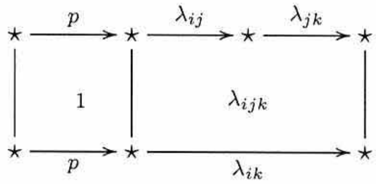
(8.3.3.1)

in $Z_k|_{U_{ijk}}$. (The codomain of the 2-cell is $p\lambda_{ij}\lambda_{jk}\partial\left(L_{ijk}^{\lambda_{jk}}\right)$ which is $p\lambda_{ik}$ by (7.3.2.8))

To save space in the diagrams we could abbreviate the above to

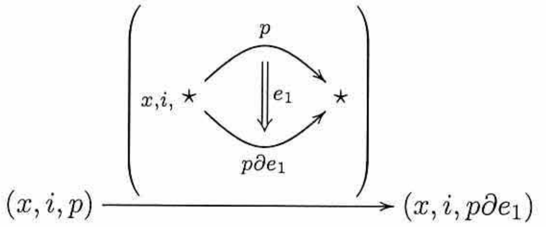
$$(x, k, p\lambda_{ij}\lambda_{jk}) \xrightarrow{L_{ijk}^{\lambda_{jk}}} (x, k, p\lambda_{ik}) \tag{8.3.3.2}$$

The picture is



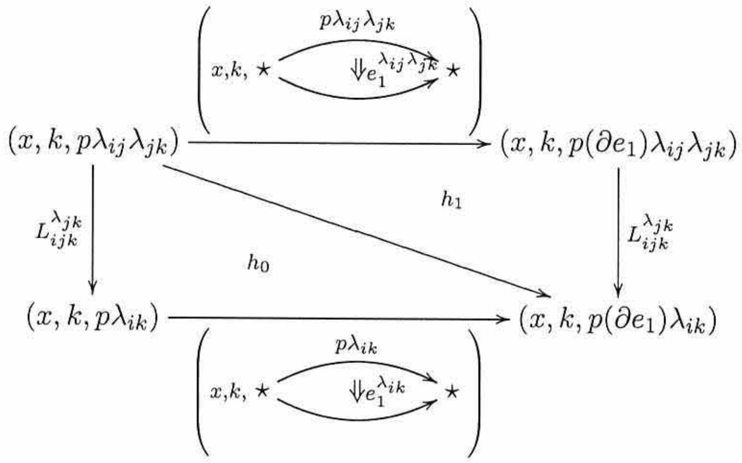
(8.3.3.3)

Looking at the next dimension, a 1-simplex, $(x, i, (p, e_1))$,



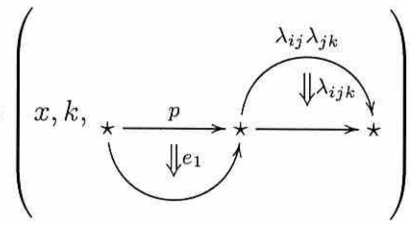
(8.3.3.4)

is assigned a square

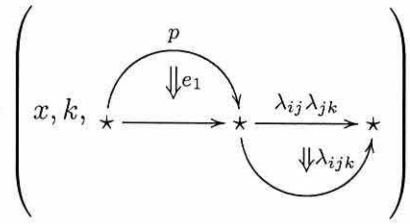


(8.3.3.5)

The common d_1-face of h_0 and h_1 is given by composing the two-cells in the other two faces (and by the interchange law in the 2-category M the two composites are equal), i.e., $d_1 h_0 = d_1 h1 = (p, e_1) \#_0 \lambda_{ijk}$ (this is $(x, i, (p, e_1))^{Z_{ijk}}$ in the 'action' notation of section 2.4.2). Explicitly we have

$h_0(x, i, (p, e_1)) =$ 

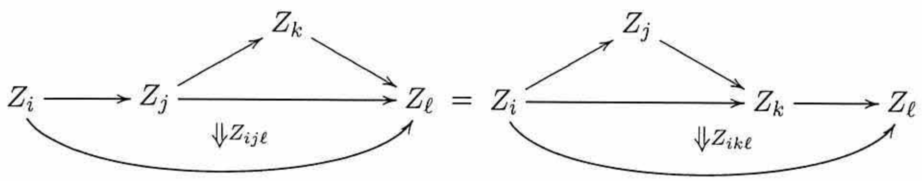
(8.3.3.6)

$h_1(x, i, (p, e_1)) =$ 

(8.3.3.7)

8.3.4 On $U_{ijk\ell}$

Over quadruple intersections $U_{ijk\ell}$ the equation (7.3.2.16) induces an equation between the Z_{ijk}:



(8.3.4.1)

8.3.5 Getting a stack

Hence the $Z_\bullet$ form a 2-descent-datum so we can glue to get a stack, Z, on X. Z is locally trivial in the sense that over the open cover $\mathcal{U}$ it trivialises.

More generally we can replace the regular action of M on itself by any action of M on a groupoid, and we conjecture that similar construction should work if $\mathcal{U}$ is a hypercover as in [5].

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