# Phase Limitations of Multipliers for Nonlinearities With Monotone Bounds

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# Phase limitations of multipliers for nonlinearities with monotone bounds

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***Abstract*—We consider Lurye systems, whose nonlinear operator is characterized by a nonlinearity that is bounded above and below by monotone functions. Absolute stability can be established using a subclass of the O'Shea-Zames-Falb multipliers. We develop phase conditions for both continuous-time and discrete-time systems under which there can be no such suitable multiplier for the transfer function of a given plant. In discrete time the condition can be tested via a linear program, while in continuous time it can be tested efficiently by exploiting convex structure. Results provide useful insight into the dynamic behaviour of such systems and we illustrate the phase limitations with examples from the literature.**

***Index Terms*—Absolute stability, Lurye (or Lur'e) systems, multiplier theory, frequency domain**

## I. Introduction

We are concerned with the Lurye system (Fig. 1) where the nonlinearity $\phi$ is bounded below and above by monotone functions (Fig. 2). If the nonlinearity is itself monotone then its positivity is preserved by the OZF (O'Shea-Zames-Falb) multipliers [1]–[5]. More generally the positivity of such nonlinearities is preserved by a subclass of the OZF multipliers. If a multiplier in this subclass can be found that is suitable for the LTI component $G$ then the Lurye system is guaranteed to be absolutely stable. Cases where the bounds are odd are considered in [6]–[8]; in particular it is argued in [6] that such analysis is useful for systems with stiction. In [9] the analysis is generalised to include bounds that are not odd and applied to systems with asymmetric nonlinearities, including both an example with asymmetric deadzone that cannot have odd bounds and an example of simple saturation with asymmetric bounds.

Frequency domain criteria have played an important role in the analysis of Lurye systems [10], [11] and multipliers are best characterised by their phase properties. The phase limitations for the OZF multipliers developed in [12]–[20] allow the class of OZF multipliers to be characterised in the frequency domain.

It has been conjectured (e.g. [5]) that the existence of a suitable OZF multiplier is necessary as well as sufficient for absolute stability. The conjecture remains open, but has received considerable recent attention [5], [21]–[24].

Although the class of multipliers for nonlinearities considered in this paper is a subclass of the OZF multipliers, phase limitations of this subclass have not previously been considered. In this paper we develop phase limitations at an arbitrary number of isolated frequencies for both continuous-time and discrete-time systems. The corresponding limitations for OZF multipliers [13], [18] emerge as special cases. In discrete time the phase conditions (Theorem 7) can be tested for numerically by a linear program (cf [18], [24], [25]), while in continuous time they can be approximated by a linear program if one continuous variable over a finite interval is gridded. Further convexity properties may be exploited in a similar manner to the search for OZF multipliers proposed in [26], [27].

We consider the continuous-time multipliers in section II. After preliminaries in subsection II-A we derive the general result in subsection II-B. We consider its application to single frequencies and two frequencies in subsections II-C and II-D respectively. As with the limitations for OZF multipliers [19], there are no limitations at a single frequency save a uniform bound across frequencies; by contrast limitations at two frequencies reveal a rich behaviour provided the frequencies are rational multiples of each other. Although there is not necessarily a closed-form expression for the phase limitations at two frequencies, the limitations can be easily computed by exploiting smoothness and convexity. We illustrate these results with an example in subsection II-E.

We consider the discrete-time multipliers in section III. The structure of the Section is similar to that for continuous multipliers, but we only consider the specific application of the general result (subsection III-B) to a single frequency (subsection III-C); as with the limitations for OZF multipliers [18] limitations even at single frequencies reveal a rich behaviour and can be computed as the maximum of a finite number of closed-form expressions. We illustrate the limitations with an example in subsection III-D where guarantees of absolute stability are associated with the magnitude of any exogenous signal in steady state.

## II. Continuous-time systems

### A. Preliminaries

Let $\mathcal{L}_2$ be the space of finite energy Lebesgue integrable signals and $\mathcal{L}_{2e}$ be the corresponding extended space [28]. The Lurye system (Fig. 1) is given by

$$y_1 = \mathbf{G}u_1, y_2 = \phi u_2, u_1 = r_1 - y_2 \text{ and } u_2 = y_1 + r_2. \quad (1)$$

It is assumed to be well-posed with $\mathbf{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ linear time invariant (LTI), causal and stable, and with $\phi : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ some nonlinear causal operator. We further assume $r_1(t) = r_2(t) = 0$ for all $t < 0$. The Lurye system is said to be stable if $r_1, r_2 \in \mathcal{L}_2$ implies $y_1, y_2 \in \mathcal{L}_2$ (it then follows that $u_1, u_2 \in \mathcal{L}_2$ also).

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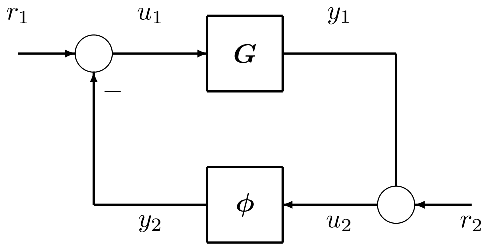


Fig. 1. Lurye system.

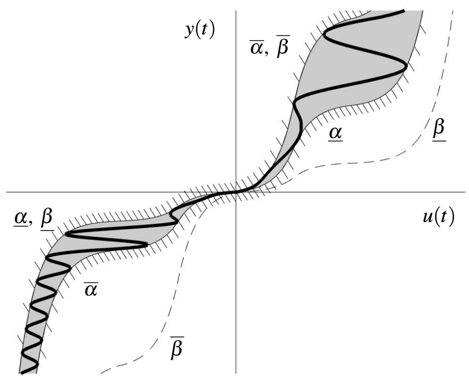


Fig. 2. (After [9].) The nonlinearity (i.e. the map from $u(t)$ to $y(t) = (\phi u)(t)$ or, in the discrete-time case, from $u[k]$ to $y[k] = (\phi u)[k]$) is bounded below and above by the monotone and bounded functions $\underline{\alpha}$ and $\overline{\alpha}$ respectively. In addition the nonlinearity is bounded below and above by the monotone, bounded and odd functions $\underline{\beta}$ and $\overline{\beta}$ respectively. For this illustration $\underline{\alpha}$ and $\underline{\beta}$ coincide in the bottom left quadrant while $\overline{\alpha}$ and $\overline{\beta}$ coincide in the top right quadrant.

A function $\alpha : \mathbb{R} \to \mathbb{R}$ is said to be monotone if $\alpha(x_1) \geq \alpha(x_2)$ for all $x_1 \geq x_2$. It is said to be bounded if there exists $C \geq 0$ such that $|\alpha(x)| \leq C|x|$ for all $x \in \mathbb{R}$. It is said to be odd if $\alpha(-x) = -\alpha(x)$ for all $x \in \mathbb{R}$. It is said to be slope-restricted on $[0, s]$ if $0 \leq (\alpha(x_1) - \alpha(x_2))/(x_1 - x_2) \leq s$ for all $x_1 \neq x_2$.

Following [9] we say an operator $\phi : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is bounded below by $\underline{\alpha} : \mathbb{R} \to \mathbb{R}$ and above by $\overline{\alpha} : \mathbb{R} \to \mathbb{R}$ if

$$0 \leq \frac{\underline{\alpha}(u(t))}{u(t)} \leq \frac{(\phi u)(t)}{u(t)} \leq \frac{\overline{\alpha}(u(t))}{u(t)}, \tag{2}$$

for all $u \in \mathcal{L}_2 e$ and for all $t \in \mathbb{R}$ whenever $u(t) \neq 0$.

We say $\phi \in \Phi_{A,B}$ if it is bounded below both by some monotone $\underline{\alpha}$ and by some monotone odd $\underline{\beta}$ and if it is bounded above both by $A\underline{\alpha}$ with $1 \leq A$ and by $B\underline{\beta}$ with $1 \leq A \leq B$ (and with $B$ possibly infinite).

Let $M : j\mathbb{R} \to \mathbb{C}$ and $G : j\mathbb{R} \to \mathbb{C}$. We say $M$ is suitable for $G$ if there exists some $\varepsilon > 0$ such that

$$\text{Re}\left\{M(j\omega)G(j\omega)\right\} > \varepsilon \text{ for all } \omega \in \mathbb{R}. \tag{3}$$

Define $\mathbf{H}$ as the set of generalised functions $h(\cdot)$ of the form

$$h(t) = \sum_{i=1}^{\infty} h_i \delta(t - t_i) + h_a(t), \tag{4}$$

with $t_i \neq 0$, $h_a \in \mathcal{L}_1$, $h_a(0) = 0$ and $h_i \in \mathbb{R}$ for all $i$ and $h_a(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$. In addition define the norm [29]

$$\|h\|_H = \sum_{i=1}^{\infty} |h_i| + \int_{-\infty}^{\infty} |h_a(t)|\, dt < \infty. \tag{5}$$

Define $\mathbf{H}_p$ as the subset of $\mathbf{H}$ where $h_i \geq 0$ for all $i \in \mathbb{Z}^+$ and $h_a(t) \geq 0$ for all $t \in \mathbb{R}$.

Let $\mathcal{M}_{A,B}$ with $1 \leq A \leq B$ be the class of multipliers $\mathcal{M}_{A,B} = \{M = m_0(1 - H_+ + H_-)\}$ with $m_0 > 0$ and where $H_+$ and $H_-$ are noncausal convolution operators whose respective impulse responses are $h_+ \in \mathbf{H_p}$ and $h_- \in \mathbf{H_p}$ satisfying

$$A\|h_+\|_H + B\|h_-\|_H \leq 1. \tag{6}$$

We can set $m_0 = 1$ without loss of generality.

**Theorem 1** ( [9]). *If $M \in \mathcal{M}_{A,B}$ with $1 \leq A \leq B$ then it preserves the positivity of any $\phi \in \Phi_{A,B}$ in the sense that*

$$\int_{-\infty}^{\infty} (Mu)(t)(\phi u)(t)\, dt \geq 0, \tag{7}$$

*for any $u \in \mathcal{L}_2$. Furthermore, if $M$ is suitable for $G$ then the Lurye system (1) is absolutely stable in the sense that it is input-output stable for all $\phi \in \Phi_{A,B}$.*

If we write $G(j\omega) = |G(j\omega)| \exp(j \frac{\pi}{180} \angle G(j\omega))$ with $-180^o < \angle G(j\omega) \leq 180^o$ then we call $\angle G(j\omega)$ the phase of $G$ at $\omega$. We define the phase of $M$ at $\omega$ similarly. Suppose $M \in \mathcal{M}_{A,B}$ with $1 \leq A \leq B$. Since $\mathcal{M}_{A,B} \subset \mathcal{M}_{1,1}$ we must have $-90^o \leq \angle M(j\omega) \leq 90^o$ for all $\omega$ [5]. In this case we can note that if $M$ is suitable for $G$ and $\angle G(j\omega) \leq -90^o - \theta$ for some $\theta \in [-90^o, 90^o)$ then $\angle M(j\omega) > \theta$; similarly if $\angle G(j\omega) \geq 90^o + \theta$ for some $\theta \in [-90^o, 90^o)$ then $\angle M(j\omega) < -\theta$.

The sets $\Phi_{1,\infty}$ and $\Phi_{1,1}$ are respectively the sets of monotone and monotone odd nonlinearities considered in [2]. The classes $\mathcal{M}_{1,\infty}$ and $\mathcal{M}_{1,1}$ are the corresponding OZF multipliers, discovered by O'Shea [1] and formalized by Zames and Falb [2]. In [6] the set $\Phi_{A,A}$ with $1 < A$ is used to characterise stiction and the corresponding Lurye system is analysed in [6]–[8]. In [9] we discuss an example where $\phi \in \Phi_{A,\infty}$ with $1 < A$. We also show that the class $\Phi_{1,B}$ with $1 < B < \infty$ can be used to characterise aysmmetric memoryless nonlinearities; we revisit these examples below.

In [2] loop transformations are used to admit slope-restricted nonlinearities. Loop transformations can also be used for the nonlinearities in this paper. However, the values of $A$ and $B$ are not necessarily preserved under transformation [9].

There are two approaches to the analysis of phase limitations of the classes $\mathcal{M}_{1,\infty}$ and $\mathcal{M}_{1,1}$ in the literature. In [12], [17] limitations are given over frequency intervals. By contrast, in [13] limitations are given at a finite number of distinct frequencies. In [19] we derive a more powerful condition at just two frequencies derived using the approach of [13]–[16]; the condition may also be derived as a limiting case of the frequency interval approach [20].

We will find it useful to define the scaled delay multipliers $D^-_{A,\tau}$ and $D^+_{B,\tau}$ as

$$\begin{aligned} D^-_{A,\tau}(j\omega) &= 1 - \frac{1}{A} e^{-j\tau\omega}, \\ D^+_{B,\tau}(j\omega) &= 1 + \frac{1}{B} e^{-j\tau\omega}, \end{aligned} \tag{8}$$

with $\tau \in \mathbb{R}$. Define the classes $\mathcal{D}^-_{A,I}$ and $\mathcal{D}^+_{B,I}$ for some $I \subseteq \mathbb{R}$ as

$$\begin{aligned}\mathcal{D}^-_{A,I} &= \left\{ D^-_{A,\tau} \text{ with } \tau \in I \right\},\\ \mathcal{D}^+_{B,I} &= \left\{ D^+_{B,\tau} \text{ with } \tau \in I \right\}.\end{aligned} \tag{9}$$

Note that for any $A \leq B$ and any $I \subseteq \mathbb{R}$ we have $\mathcal{D}^-_{A,I} \subset \mathcal{M}_{A,B}$ and $\mathcal{D}^+_{B,I} \subset \mathcal{M}_{A,B}$.

### B. Phase limitations at $N$ frequencies

The following result generalises Proposition 1 from [13] where $A = B = 1$.

**Theorem 2.** *Given $G : j\mathbb{R} \to \mathbb{C}$, assume there exist $0 < \omega_1 < \cdots \omega_N < \infty$ and non-negative $\lambda_1, \ldots, \lambda_N$ not all zero such that*

$$\sum_{r=1}^{N} \lambda_r \mathit{Re}\left\{ M(j\omega_r) G(j\omega_r) \right\} \leq 0, \tag{10}$$

*for all $M \in \mathcal{D}^-_{A,\mathbb{R}} \cup \mathcal{D}^+_{B,\mathbb{R}}$. Then there is no $M \in \mathcal{M}_{A,B}$ such that $Re\left\{ M(j\omega_r) G(j\omega_r) \right\} > 0$ for $r = 1, \ldots, N$ and hence no $M \in \mathcal{M}_{A,B}$ suitable for $G$.*

*Proof.* Let $M \in \mathcal{M}_{A,B}$ have impulse response

$$m(t) = \delta(t) - h_-(t) + h_+(t), \tag{11}$$

with $h_-, h_+ \in \mathbf{H}_p$ satisfying (6) and

$$\begin{aligned} h_-(t) &= \sum_{i=1}^{\infty} h_i^- \delta(t - t_i^-) + h_a^-(t),\\ h_+(t) &= \sum_{i=1}^{\infty} h_i^+ \delta(t - t_i^+) + h_a^+(t). \end{aligned} \tag{12}$$

Then

$$\begin{aligned} M(j\omega) = 1 &- \int_{-\infty}^{\infty} h_-(t) e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} h_+(t) e^{-j\omega t}\, dt,\\ =&1 - \int_{-\infty}^{\infty} h_a^-(t) e^{-j\omega t}\, dt - \sum_{i=1}^{\infty} h_i^- e^{-j\omega t_i^-}\\ &+ \int_{-\infty}^{\infty} h_a^+(t) e^{-j\omega t}\, dt + \sum_{i=1}^{\infty} h_i^+ e^{-j\omega t_i^+}. \end{aligned} \tag{13}$$

We can write (10) with $M = D^-_{A,\tau}$ as

$$\sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ G(j\omega_r) \right\} \leq \frac{1}{A} \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ e^{-j\omega_r \tau} G(j\omega_r) \right\}. \tag{14}$$

If this is true for all $\tau$ we must have

$$\begin{aligned} &\sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ G(j\omega_r) \right\}\\ &\qquad \leq \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{1}{A} \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ e^{-j\omega_r \tau} G(j\omega_r) \right\}\, d\tau,\\ &\qquad = 0, \end{aligned} \tag{15}$$

so we can write

$$\begin{aligned} &\sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ G(j\omega_r) \right\}\\ &\qquad \leq \left( A\|h_+\|_H + B\|h_-\|_H \right) \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ G(j\omega_r) \right\}, \end{aligned} \tag{16}$$

with

$$\begin{aligned} \|h_-\|_H &= \sum_{i=1}^{\infty} h_i^- + \int_{-\infty}^{\infty} h_a^-(t)\, dt,\\ \|h_+\|_H &= \sum_{i=1}^{\infty} h_i^+ + \int_{-\infty}^{\infty} h_a^+(t)\, dt. \end{aligned} \tag{17}$$

So

$$\begin{aligned} &\sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ M(j\omega_r) G(j\omega_r) \right\}\\ &\quad = \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ G(j\omega_r) \right\} - \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ H_-(j\omega_r) G(j\omega_r) \right\}\\ &\qquad + \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ H_+(j\omega_r) G(j\omega_r) \right\},\\ &\quad \leq \left( A\|h_-\|_H + B\|h_+\|_H \right) \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ G(j\omega_r) \right\}\\ &\qquad - \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ H_-(j\omega_r) G(j\omega_r) \right\}\\ &\qquad + \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ H_+(j\omega_r) G(j\omega_r) \right\}. \end{aligned} \tag{18}$$

But

$$\begin{aligned} &A\|h_-\|_H \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ G(j\omega_r) \right\} - \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ H_-(j\omega_r) G(j\omega_r) \right\}\\ &\quad = A \sum_{i=1}^{\infty} h_i^- \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ D^-_{A,t_i^-}(j\omega_r) G(j\omega_r) \right\}\\ &\qquad + A \int_{-\infty}^{\infty} h_a^-(t) \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ D^-_{A,t}(j\omega_r) G(j\omega_r) \right\}\, dt,\\ &\quad \leq 0, \end{aligned} \tag{19}$$

and similarly

$$\begin{aligned} &B\|h_+\|_H \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ G(j\omega_r) \right\} + \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ H_+(j\omega_r) G(j\omega_r) \right\}\\ &\quad = B \sum_{i=1}^{\infty} h_i^+ \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ D^+_{B,t_i^+}(j\omega_r) G(j\omega_r) \right\}\\ &\qquad + B \int_{-\infty}^{\infty} h_a^+(t) \sum_{r=1}^{N} \lambda_r \mathrm{Re}\left\{ D^+_{B,t}(j\omega_r) G(j\omega_r) \right\}\, dt,\\ &\quad \leq 0. \end{aligned} \tag{20}$$

So

$$\sum_{r=1}^{N} \lambda_r \text{Re}\left\{M(j\omega_r)G(j\omega_r)\right\} \leq 0, \tag{21}$$

and hence $M$ is not suitable for $G$. □

**Remark 1.** *If the frequencies $\omega_r$ are all integer multiples of some base frequency $\omega_0$ then we can exploit periodicity and restrict $\tau$ to lie in the interval $[0, 2\pi/\omega_0)$. Specifically if $\omega_r = n_r\omega_0$ with $n_r \in \mathbb{Z}^+$ for $r = 1, \cdots, N$ we find*

$$\begin{aligned} D^-_{A,\tau+2\pi/\omega_0}(j\omega_r) &= 1 - \frac{1}{A}e^{-j\omega_r\tau - j2\pi n_r}, \\ &= 1 - \frac{1}{A}e^{-j\omega_r\tau}, \\ &= D^-_{A,\tau}(j\omega_r), \end{aligned} \tag{22}$$

*and similarly*

$$D^+_{B,\tau+2\pi/\omega_0}(j\omega_r) = D^+_{B,\tau}(j\omega_r). \tag{23}$$

*Hence for this case we need only consider $M \in \mathcal{D}^-_{A,[0,2\pi/\omega_0)} \cup \mathcal{D}^+_{B,[0,2\pi/\omega_0)}$ in the statement of Theorem 2. We will find it useful to extend the interval for $\tau$ to $[0, 2\pi/\omega_0]$.*

The conditions of Theorem 2 can then be usefully expressed in terms of an infinite dimensional linear program. Specifcally, define

$$\begin{aligned} \lambda &= \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix}, \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \\ v^-(\tau) &= \begin{bmatrix} \text{Re}\left\{D^-_{A,\tau}(j\omega_1)G(j\omega_1)\right\} \\ \vdots \\ \text{Re}\left\{D^-_{A,\tau}(j\omega_N)G(j\omega_N)\right\} \end{bmatrix}, \\ v^+(\tau) &= \begin{bmatrix} \text{Re}\left\{D^+_{B,\tau}(j\omega_1)G(j\omega_1)\right\} \\ \vdots \\ \text{Re}\left\{D^+_{B,\tau}(j\omega_N)G(j\omega_N)\right\} \end{bmatrix}. \end{aligned} \tag{24}$$

Let $\mathcal{LP}$ be the infinite dimensional linear program:

$$\begin{aligned} &\max_{\lambda} \mathbf{1}^{\mathbf{T}}\lambda \text{ such that } \lambda_i \geq 0 \text{ for } i = 1, \dots N, \\ &\qquad \text{and } \lambda^T v^-(\tau) \leq 0 \text{ and } \lambda^T v^+(\tau) \leq 0, \\ &\qquad \text{for all } \tau \in [0, 2\pi/\omega_0]. \end{aligned} \tag{25}$$

Then if $\mathcal{LP}$ has a positive solution then there is no $M \in \mathcal{M}_{A,B}$ suitable for $G$.

We may approximate $\mathcal{LP}$ with a finite dimensional linear program by gridding $\tau$ over the interval $[0, 2\pi/\omega_0]$ as $\tau_l$ with $l = 1, \dots, n$ for some suffciently large $n$ (noting that $v^-$ and $v^+$ are smooth vector functions).

As an alternative, suppose we define $\phi : \mathbb{R}^N \times [-2\pi/\omega_0, 2\pi/\omega_0] \to \mathbb{R}$ as

$$\phi(\lambda, \tau) = \begin{cases} \sum_{r=1}^{N} \lambda_r \text{Re}\left\{D^-_{A,\tau}(j\omega_r)G(j\omega_r)\right\} & \text{when } \tau \geq 0, \\ \sum_{r=1}^{N} \lambda_r \text{Re}\left\{D^+_{B,-\tau}(j\omega_r)G(j\omega_r)\right\} & \text{when } \tau < 0. \end{cases} \tag{26}$$

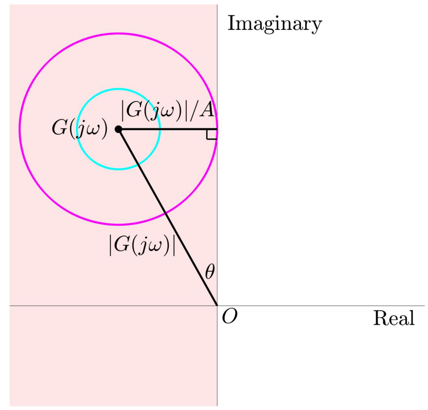


Fig. 3. Geometric interpretation of Statement 1. The locus of $M(j\omega)G(j\omega)$ with any $M \in \mathcal{D}^-_{A,(0,2\pi/\omega)}$ is a circle centre $G(j\omega)$ with radius $|G(j\omega)|/A$. If $\angle G(j\omega) \geq \theta + 90^o$ (or $\angle G(j\omega) \leq -\theta - 90^o$) with $\sin\theta = 1/A$ then Re$\{M(j\omega)G(j\omega)\} \leq 0$ for all $M \in \mathcal{D}^-_{A,(0,2\pi/\omega)}$ and hence there is no $M \in \mathcal{M}_{A,B}$ suitable for $G$ . The locus of $M(j\omega)G(j\omega)$ with $M \in \mathcal{D}^+_{B,(0,2\pi/\omega)}$ is a circle centre $G(j\omega)$ with radius $|G(j\omega|/B$. Since $B \geq A$ its value makes no difference to the Statement.

Then since $\phi(\lambda, \tau)$ is convex for every $\tau \in [-2\pi/\omega_0, 2\pi/\omega_0]$ (and since the interval is compact) then by Danskin's Theorem [30] the function

$$f(\lambda) = \max_{\tau \in [-2\pi/\omega_0, 2\pi/\omega_0]} \phi(\lambda, \tau), \tag{27}$$

is convex in $\lambda$. This allows the conditions of Theorem 2 to be tested efficiently. A similar observation is made for the computation of OZF multipliers in [26], [27].

### *C. Phase limitations at a single frequency*

When $N = 1$ we can set $\lambda_1 = 1$ without loss of generality and write Theorem 2 as:

**Statement 1.** *Given $G : j\mathbb{R} \to \mathbb{C}$, assume there exists $\omega > 0$ such that*

$$\textit{Re}\left\{M(j\omega)G(j\omega)\right\} \leq 0 \textit{ for all } M \in \mathcal{D}^-_{A,[0,2\pi/\omega)}. \tag{28}$$

*Then there is no $M \in \mathcal{M}_{A,B}$ suitable for $G$.*

**Remark 2.** *The statement makes no assumption on the value of $B$. Since $B \geq A$, if (28) holds then in addition (Fig. 3)*

$$\textit{Re}\left\{M(j\omega)G(j\omega)\right\} \leq 0 \textit{ for all } M \in \mathcal{D}^+_{B,[0,2\pi/\omega)}. \tag{29}$$

From simple geometry (Fig. 3) we can state this equivalently as:

**Statement 2.** *Given $G : j\mathbb{R} \to \mathbb{C}$, assume there exists $\omega > 0$ such that $\angle G(j\omega) \geq \theta + 90^o$ (or $\angle G(j\omega) \leq -\theta - 90^o$) with $\sin\theta = 1/A$. Then there is no $M \in \mathcal{M}_{A,B}$ suitable for $G$.*

This is in turn equivalent to the following statement about multipliers.

**Statement 3.** *The phase of any $M \in \mathcal{M}_{A,B}$ at any frequency lies in the interval $[-\theta, \theta]$ where $\sin\theta = 1/A$.*

This also has a simple geometric interpretation. The frequency response of any $h \in \mathbf{H}$ satisfies $\sup_\omega |H(j\omega)| \leq \|h\|_p$

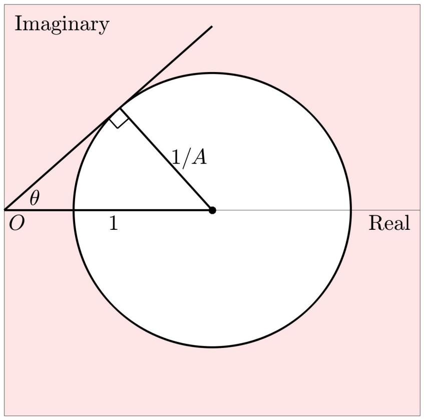


Fig. 4. The Nyquist plot of $M$ lies in a circle, centre 1 and radius $1/A$ (whether $M$ is continuous-time or discrete-time). It follows that its phase lies on the interval $[-\theta,\theta]$ where $\sin\theta = 1/A$.

( [29], p300). If follows that the Nyquist plot of any $M \in \mathcal{M}_{A,B}$ must lie in the circle centre 1, radius $1/A$ (see Fig. 4). The bound is tight in the sense that the Nyquist plot of any scaled delay multiplier $M \in \mathcal{D}^-_{A,\mathbb{R}}$ lies on the boundary of the circle. We have argued elsewhere [17] that, at least in the case $A = 1$ where the interval is $[-90^o, 90^o]$, further limitations must exist: otherwise the Kalman conjecture would be true for all systems.

### D. Phase limitations at two frequencies

When $N = 2$ we can set $\lambda_1 = \lambda$ and $\lambda_2 = 1-\lambda$ for some $\lambda \in [0,1]$. We can write Theorem 2 as:

**Theorem 3.** *Given $G : j\mathbb{R} \to \mathbb{C}$, assume there exist $0 < \omega_1 < \omega_2$ and $\lambda \in [0,1]$ such that*

$$\lambda Re\left\{M(j\omega_1)G(j\omega_1)\right\} + (1-\lambda)Re\left\{M(j\omega_2)G(j\omega_2)\right\} \leq 0, \tag{30}$$

*for all $M \in \mathcal{D}^-_{A,\mathbb{R}} \cup \mathcal{D}^+_{B,\mathbb{R}}$ . Then there is no $M \in \mathcal{M}_{A,B}$ suitable for $G$.*

**Remark 3.** *If $\omega_2$ is an irrational multiple of $\omega_1$ then Theorem 3 yields no more information than the case $N = 1$.*

*To see this, suppose given $A \geq 1$ and $0 < \omega_1 < \omega_2$ we find some $\tau_1, \tau_2 \in \mathbb{R}$ such that $Re\{D^-_{A,\tau_1}(j\omega_1)G(j\omega_1)\} > 0$ and $Re\{D^-_{A,\tau_2}(j\omega_2)G(j\omega_2)\} > 0$. If $\omega_2$ is an irrational multiple of $\omega_1$ then we can find $\tau \in \mathbb{R}$ such that $Re\{D^-_{A,\tau}(j\omega_1)G(j\omega_1)\} > 0$ and $Re\{D^-_{A,\tau}(j\omega_2)G(j\omega_2)\} > 0$. Specifically we have $Re\{D^-_{A,\tau_1+2\pi l/\omega_1}(j\omega_1)G(j\omega_1)\} > 0$ for all $l \in \mathbb{Z}$. Since $e^{j\omega_2(\tau_1+2\pi l/\omega_1)}$ with $l \in \mathbb{Z}$ is uniformly distributed on the unit circle [31] we can choose $l$ so that $|D^-_{A,\tau_1+2\pi l/\omega_1}(j\omega_2) - D^-_{A,\tau_2}(j\omega_2)|$ is arbitrarily small and hence $Re\{D^-_{A,\tau_1+2\pi l/\omega_1}(j\omega_2)G(j\omega_2)\} > 0$.*

*We can make a similar statement, given $B \geq 1$ and $0 < \omega_1 < \omega_2$, if we find some $\tau_1, \tau_2 \in \mathbb{R}$ such that $Re\{D^+_{B,\tau_1}(j\omega_1)G(j\omega_1)\} > 0$ and $Re\{D^+_{B,\tau_2}(j\omega_2)G(j\omega_2)\} > 0$.*

In the following we assume $n_2\omega_1 = n_1\omega_2$ for some $n_1, n_2 \in \mathbb{Z}^+$.

**Remark 4.** *If $n_2\omega_1 = n_1\omega_2$ with $n_1, n_2 \in \mathbb{Z}^+$ both odd then the criterion in Theorem 3 is independent of the value $B$. Specifically, given $D^+_{B,\tau} \in \mathcal{D}^+_{B,\mathbb{R}}$ we find*

$$\begin{aligned} D^-_{B,\tau+\pi n_1/\omega_1}(j\omega_1) &= 1 - \frac{1}{B}e^{-j\omega_1\tau - jn_1\pi}, \\ &= 1 + \frac{1}{B}e^{-j\omega_1\tau}, \\ &= D^+_{B,\tau}(j\omega_1), \end{aligned} \tag{31}$$

*and similarly*

$$D^-_{B,\tau+\pi n_2/\omega_2}(j\omega_2) = D^+_{B,\tau}(j\omega_2). \tag{32}$$

Given two frequencies $\omega_1$ and $\omega_2$ satisfying $n_2\omega_1 = n_1\omega_2$ with $n_1, n_2 \in \mathbb{Z}^+$ coprime, testing the condition of Theorem 3 numerically requires a search over two variables: $\lambda$ on the interval $[0,1]$ and $\tau$ on the interval $[-2\pi n_1/\omega_1, 2\pi n_1/\omega_1]$.

Specifically we can write condition (30) as

$$\min_{\lambda\in[0,1]} f(\lambda) \leq 0, \tag{33}$$

where $f(\lambda)$ is defined by (27) with our slight change of notation ($\lambda_1 = \lambda; \lambda_2 = 1-\lambda$). For a given $\lambda$ it is straightforward to find $f(\lambda)$ as $\phi(\lambda,\tau)$ has at most $4(n_1+n_2)$ turning points when viewed as a function of $\tau$ on the interval $[-2\pi n_1/\omega_1, 2\pi n_1/\omega_1]$. Then since $f(\lambda)$ is convex we can use a golden-section search [32] to test condition (33).

If $A = 1$ and if $B = 1$, $B = \infty$ or if $n_1$ and $n_2$ are both odd, then a more efficient test is available. Specifically:

**Theorem 4** ( [19], [20]). *Let $n_2\omega_1 = n_1\omega_2$ with $n_1, n_2 \in \mathbb{Z}^+$ coprime and let $M \in \mathcal{M}_{1,\infty}$. Then*

$$\left|\frac{n_2\angle M(j\omega_1) - n_1\angle M(j\omega_2)}{n_1/2 + n_2/2 - p}\right| \leq 180^o, \tag{34}$$

*with $p = 1$.*

**Remark 5.** *If in addition $n_1$ and $n_2$ are both odd then the statement holds for any $M \in \mathcal{M}_{1,B}$ by Remark 4.*

**Theorem 5** ( [19], [20]). *Let $n_2\omega_1 = n_1\omega_2$ with $n_1, n_2 \in \mathbb{Z}^+$ coprime and not both odd and let $M \in \mathcal{M}_{1,1}$. Then inequality (34) holds with $p = 1/2$.*

Fig. 5 illustrates Theorems 3, 4 and 5 for the case $n_1 = 1$, $n_2 = 2$, $A = 1$ and various values of $B$. Similarly Fig. 6 illustrates Theorems 3 and 4 for the case $n_1 = 1$, $n_2 = 3$ and various values of $A$. The values for Theorem 3 are obtained numerically as described above.

### E. Example

In [9] we considered a Lurye system with LTI plant

$$G(s) = e^{-ds}\frac{1}{s^2+\xi s+1} \text{ with } d = 0.2 \text{ and } \xi = 0.3, \tag{35}$$

and a bounded nonlinearity with asymmetric deadzone. After loop transformation, the bounds on the nonlinearity could be characterised by the values $A = 1.5$ and $B = \infty$. We constructed a suitable multiplier for $1 + G$. Fig. 7 shows the phase of $1 + G$; it can be seen to lie above the bound $-90^o - \arctan\frac{1}{\sqrt{A^2-1}}$. The figure also shows the phase of

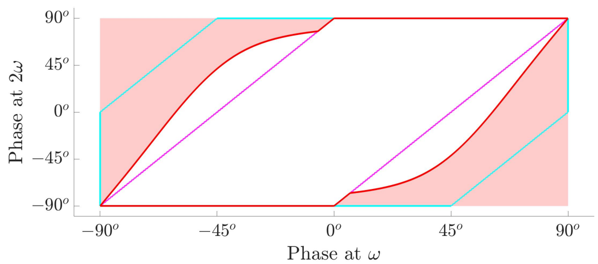


Fig. 5. Phase constraint with $n_1 = 1$ and $n_2 = 2$ (red: $A = 1$, $B = 1.4$, Theorem 3). When $B = 1$ (cyan) or $B = \infty$ (magenta) the constraint boundary is a parallelogram, determined respectively by Theorem 4 and Theorem 5.

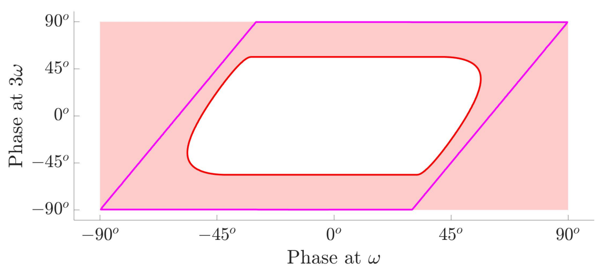


Fig. 6. Phase constraint with $n_1 = 1$ and $n_2 = 3$ (red: $A = 1.2$, Theorem 3). When $A = 1$ (magenta) the constraint boundary is a parallelogram determined by Theorem 4. Because $n_1$ and $n_2$ are both odd, the constraint boundaries are independent of the value of $B$.

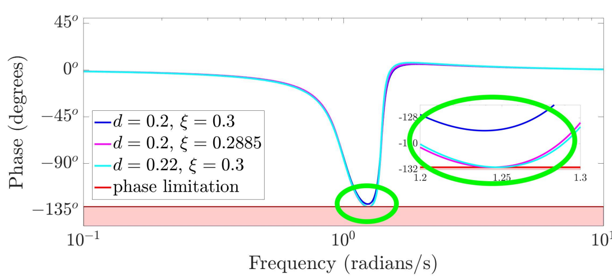


Fig. 7. Phase of $1+G$ when $G$ is given by (35). The phase does not exceed the bounds given by Statement 2 when $A = 1.5$. The phase touches the bounds with small perturbations to either the damping ratio $\xi$ or the delay $d$.

$1+G$ for the two cases $d = 0.2$, $\xi = 0.2885$ and $d = 0.22$, $\xi = 0.3$ . In each of these latter cases, the phase touches the bound. We conclude there is no suitable multiplier when $d > 0.22$ or $\xi < 0.2885$.

Suppose instead the LTI plant is O'Shea's example [1]

$$G(s) = \frac{s^2}{(s^2 + 2\xi s + 1)^2}. \tag{36}$$

Fig. 8 shows the phase of $1 + G$ when $\xi = 0.1$. It does not exceed the bounds given by Statement 2 when $A = 1.5$. Nevertheless Fig. 9 shows that there are frequencies where the phase exceeds the bounds of Theorem 3. Specifically the phases at $\omega_1$ and $\omega_2 = 2\omega_1$ exceed the limits given by Theorem 3 when $\omega_1 = 0.74805$, $A = 1.5$ and $B = \infty$. We conclude there is no suitable multiplier in this case.

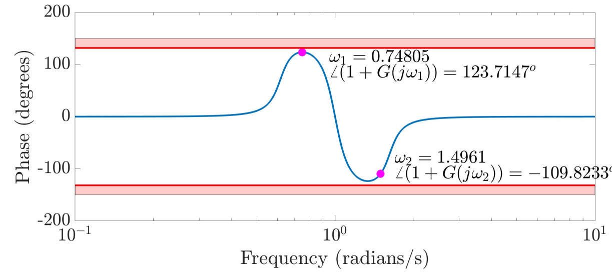


Fig. 8. Phase of $1+G$ for O'Shea's example. The phase does not exceed the bounds given by Statement 2 when $A = 1.5$.

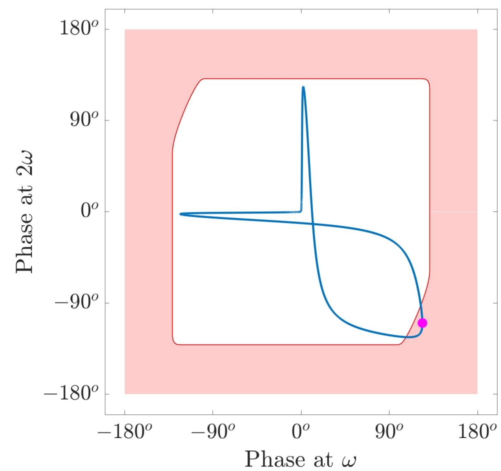


Fig. 9. Phase of $1+G$ at $2\omega$ versus phase of $1+G$ at $\omega$ for O'Shea's example. When $\omega_1 = 0.74805$ the bounds given by Theorem 3 with $A = 1.5$ and $B = \infty$ are exceeded.

## III. Discrete-time systems

### A. *Preliminaries*

Let $\ell$ be the space of all sequences $h : \mathbb{Z}^+ \to \mathbb{R}$ and $\ell_2$ be the space of all square-summable sequences $h : \mathbb{Z}^+ \to \mathbb{R}$. We consider the Lurye system (1), Fig. 1, once again assumed to be well-posed, with $\mathbf{G} : \ell \to \ell$ LTI, causal and stable, and with $\phi : \ell \to \ell$ some nonlinear causal operator. We further assume $r_1[k] = r_2[k] = 0$ for all $k < 0$. The Lurye system is said to be stable if $r_1, r_2 \in \ell_2$ implies $u_1, u_2, y_1, y_2 \in \ell_2$.

We say an operator $\phi : \ell \to \ell$ is bounded below by $\underline{\alpha} : \mathbb{R} \to \mathbb{R}$ and above by $\overline{\alpha} : \mathbb{R} \to \mathbb{R}$ in an analogous fashion to the continuous-time case. Similarly we define the class $\Phi^d_{A,B}$ analogously to the continuous-time case.

Let $\mathbb{D}$ by the unit circle in the complex plane. Let $M : \mathbb{D} \to \mathbb{C}$ and $G : \mathbb{D} \to \mathbb{C}$. We say $M$ is suitable for $G$ if

$$\text{Re}\left\{M(e^{j\omega})G(e^{j\omega})\right\} > 0 \text{ for all } \omega \in [0, 2\pi). \tag{37}$$

Define $\mathbf{h}_p$ as the set of sequences in $\ell$ where $h[k] \geq 0$ for all $k \in \mathbb{Z}$ and $h[0] = 0$. Let $\mathcal{M}^d_{A,B}$ with $1 \leq A \leq B$ be the class of multipliers $\mathcal{M}^d_{A,B} = \{M = 1 - H_- + H_+\}$ where $H_-$ and $H_+$ are noncausal convolution operators whose respective impulse responses are $h_- \in \mathbf{h_p}$ and $h_+ \in \mathbf{h_p}$ satisfying

$$A\|h_-\|_1 + B\|h_+\|_1 \leq 1. \tag{38}$$

**Theorem 6** ( [9]). *If $M \in \mathcal{M}^d_{A,B}$ with $1 \leq A \leq B$ then it preserves the positivity of any $\phi \in \Phi^d_{A,B}$ in the sense that*

$$\sum_{k=-\infty}^{\infty} (Mu)[k](\phi u)[k] \geq 0, \tag{39}$$

*for any $u \in \ell_2$. Furthermore, if $M$ is suitable for $G$ then the Lurye system (1) is absolutely stable in the sense that it is input-output stable for all $\phi \in \Phi^d_{A,B}$.*

We call $\angle G(e^{j\omega})$ the phase of $G$ at $\omega$ with $-180^o < \angle G(e^{j\omega}) \leq 180^o$.

The classes $\mathcal{M}^d_{1,\infty}$ and $\mathcal{M}^d_{1,1}$ are the discrete-time counterparts of the OZF multipliers [3], [4].

We define the discrete-time counterparts of the scaled delay multipliers as

$$\begin{aligned} D^{d-}_{A,k}(e^{j\omega}) &= 1 - \frac{1}{A}e^{-jk\omega} \text{ with } k \in \mathbb{Z}, \\ D^{d+}_{B,k}(e^{j\omega}) &= 1 + \frac{1}{B}e^{-jk\omega} \text{ with } k \in \mathbb{Z}. \end{aligned} \tag{40}$$

Define the classes $\mathcal{D}^{d-}_{A,I}$ and $\mathcal{D}^{d+}_{B,I}$ for some $I \subseteq \mathbb{Z}$ as

$$\begin{aligned} \mathcal{D}^{d-}_{A,I} &= \left\{ D^{d-}_{A,k} \text{ with } k \in I \right\}, \\ \mathcal{D}^{d+}_{B,I} &= \left\{ D^{d+}_{B,k} \text{ with } k \in I \right\}. \end{aligned} \tag{41}$$

### *B. Phase limitations at $N$ frequencies*

The following result generalises Theorem 2 in [18] where $A = 1$ and either $B = 1$ or $B = \infty$.

**Theorem 7.** *Given $G : \mathbb{D} \to \mathbb{C}$, assume there exist $0 < \omega_1 < \cdots \omega_N < \pi$ and non-negative $\lambda_1, \ldots, \lambda_N$ not all zero such that*

$$\sum_{r=1}^{N} \lambda_r Re\left\{ M(e^{j\omega_r})G(e^{j\omega_r}) \right\} \leq 0 \tag{42}$$

*for all $M \in \mathcal{D}^{d-}_{A,\mathbb{Z}} \cup \mathcal{D}^{d+}_{B,\mathbb{Z}}$. Then there is no $M \in \mathcal{M}^d_{A,B}$ such that $Re\left\{ M(e^{j\omega_r})G(e^{j\omega_r}) \right\} > 0$ for $r = 1, \ldots, N$ and hence no $M \in \mathcal{M}^d_{A,B}$ suitable for $G$.*

*Proof.* Similar to that for Theorem 2 □

Theorem 2 in [18] restricts each $\omega_r$ to take the value $\omega_r = \frac{r}{N+1}\pi$. If we similarly define each $\omega_r$ then we may exploit periodicity and consider only $M \in \mathcal{D}^{d-}_{A,I} \cup \mathcal{D}^{d+}_{B,I}$ with $I = \{0, \ldots, 2N+1\}$. In this case, as in [18], the condition may be expressed in terms of a finite dimensional linear program. Its construction is similar to that of the infinite dimensional linear program $\mathcal{LP}$ (25).

### *C. Phase limitations at a single frequency*

Let $a, b \in \mathbb{Z}^+$ be coprime with $a < b$. We can find the phase limitation at the frequency $\omega = \frac{a}{b}\pi$ by applying Theorem 7 with $b = N + 1$ and setting $\lambda_r = 0$ for $r \neq a$. Some manipulation leads to:

**Theorem 8.** *Given $1 \leq A \leq B$ and $\omega = \frac{a}{b}\pi$ for some integers $0 < a < b$, there is no $M \in \mathcal{M}^d_{A,B}$ with*

$$\angle M(e^{j\omega}) > \arctan \rho \text{ or } \angle M(e^{j\omega}) < -\arctan \rho \tag{43}$$

*where*

$$\rho = \begin{cases} \displaystyle\max_{k=1,\ldots,b-1} \frac{\sin(k\pi/b)}{A - \cos(k\pi/b)} \text{ when } a \text{ is odd,} \\ \displaystyle\max_{k=1,\ldots,b-1} \max\left\{ \frac{\sin(2k\pi/b)}{A - \cos(2k\pi/b)}, \frac{\sin(k\pi/b)}{B - \cos(k\pi/b)} \right\} \\ \qquad\qquad\qquad \text{when } a \text{ is even,} \end{cases} \tag{44}$$

*Proof.* The result follows from Theorem 7 in a similar manner to Corollary 1 and Theorem 3 in [18]. □

The condition requires evaluation at a finite number of values of $k$. In the special case where $A = 1$ and either $B = 1$ or $B = \infty$ the condition is even more straightforward, as the maximum occurs when $k = 1$; in this case the result agrees with [18]. Taking the limit as $b \to \infty$ allows us to state the following discrete-time counterpart to Statement 3.

**Statement 4.** *The phase of any $M \in \mathcal{M}^d_{A,B}$ at any frequency lies in the interval $[-\theta, \theta]$ where $\sin\theta = 1/A$.*

### *D. Example*

Consider the discrete-time Lurye system (1) where $\phi$ is characterised by a memoryless saturation function and $G$ is given by

$$G(z) = \frac{2z + 0.92}{z^2 - 0.5z}. \tag{45}$$

This is known to satisfy the Kalman conjecture when nonlinearity is symmetric but not necessarily otherwise [33]. In [9] we argued that behaviour with exogenous signals whose steady state is non-zero can be interpreted as asymmetry in the nonlinearity. Specifically: there is an OZF multiplier when $A = 1$ and $B = 1$ [33]; in [9] we constructed a multiplier for the case $B = 1.467$ which corresponds to $r_2$ with steady state 1.295; there is a three-period limit cycle when $B = 436/275 \approx 1.586$ [33] which corresponds to $r_2$ with steady state 1.55. Consider the phase of $1 + G(e^{j\omega})$ (Fig. 10). We find

$$1 + G(e^{j\omega}) = -\frac{48}{175} - j\frac{31\sqrt{3}}{175} \text{ at } \omega = \frac{2\pi}{3}. \tag{46}$$

Hence any suitable multiplier must have

$$\angle M(e^{j\omega}) \geq \arctan\frac{48}{31\sqrt{3}} \text{ at } \omega = \frac{2\pi}{3}. \tag{47}$$

By Theorem 8 there is no suitable multiplier when $B > \frac{47}{32} \approx 1.469$. This corresponds to an exogenous signal $r_2$ with steady state greater than $\frac{513}{395} \approx 1.299$.

These results are summarised in Table I. Although absolute stability is guaranteed via multiplier analysis when $r_2$ has steady state value less than 1.295 [9], by Theorem 7 there can be no multiplier when $r_2$ has steady state value greater than 1.299.

## IV. Conclusion

We have developed phase limitations of multipliers that preserve the positivity of wider classes of nonlinearity than

TABLE I
SUMMARY OF RESULTS FOR DISCRETE-TIME EXAMPLE

| $B$ | $r_2$ | Comment |
|---|---|---|
| 1.467 | 1.295 | Stability guaranteed by construction of OZF multiplier [9] |
| 1.469 | 1.299 | Phase limitation (here) |
| 1.586 | 1.55 | Existence of 3 term limit cycle [33] |

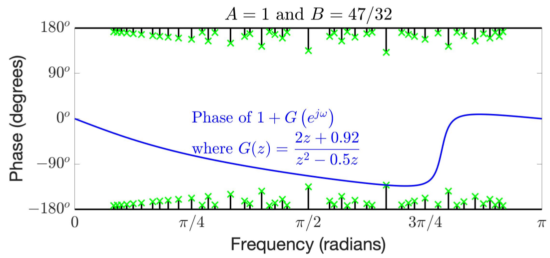


Fig. 10. Phase of $1+G$ for the example of subsection III-D. The phase limitations given by Theorem 8 are shown when $B=\frac{47}{32}$. Specifically the green crosses show values of $90^o+\arctan\rho$ and $-90^o-\arctan\rho$ where $\rho$ is determined by (44).

those addressed by the classical OZF multipliers. The examples illustrate that the limitations are insightful for closed-loop behaviour and give a useful frequency domain interpretation.

The limitations are applicable to an arbitrary number of isolated frequencies, and generalise the results (for OZF multipliers) of [13]–[16], [18], [19]. Convexity properties ensure they are straightforward to compute.

We have not generalised the results (for OZF multipliers) of [12], [17] which are applicable across frequency intervals. Although such generalisations are simple to derive, they do not appear to add additional insight over and above the results here. It is also possible to derive phase limitations at isolated frequencies in the limit via this approach (c.f. [20]) and there may be cases where this brings numerical advantages. Nevertheless we omit further discussion for the sake of concision.

It remains an open question whether the existence of a suitable OZF multiplier is necessary for the absolute stability of a Lurye system with monotone or slope-restricted nonlinearity The conjecture remains open, but has received considerable recent attention [23], [24]. Similarly it remains open whether the existence of a phase limitation developed in this paper negates absolute stability for the corresponding Lurye system.

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