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A 2-dimensional version of holonomy

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Dedicated
to
my family

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Summary

Chapter One gives an exposition of the theory of automorphisms of crossed modules over groupoids. We introduce notions of free derivation and their Whitehead multiplication, and invertible free derivations also called coadmissible homotopies. We prove that with this multiplication the set $FDer^*(\mathcal{C})$ of all coadmissible homotopies is a group and that there is a morphism $\Delta : FDer^*(\mathcal{C}) \to Aut(\mathcal{C})$ which is a part of a pre-crossed module which gives rise to a 2-crossed module

$$M(C) \to FDer^*(\mathcal{C}) \to Aut(\mathcal{C}).$$

Chapter Two gives a detailed proof of the Brown-Spencer theorem on the equivalence between crossed modules over groupoids and double groupoids with connection. We define linear coadmissible sections for the special double groupoid corresponding to a crossed module, and we prove that the group of all linear coadmissible sections and the group of coadmissible homotopies are isomorphic.

Chapter Three generalises the notion of "locally Lie groupoid" to dimension 2 for the special double groupoid called "V-locally Lie double groupoid" and relates this to corresponding notions for crossed modules. We localise the definitions of linear coadmissible sections and coadmissible homotopies and prove that these form isomorphic inverse semigroups. We define a corresponding notion of germ, and from this obtain a holonomy groupoid as an abstract groupoid $Hol(\mathcal{D}(\mathcal{C}), W^G)$.

Chapter Four gives the Lie structure on $Hol(\mathcal{D}(\mathcal{C}), W^G)$ and gives its universal property, which shows how a V-locally Lie double groupoid give rise to its holonomy groupoid. This is the main Globalisation Theorem.

Chapter Five gives suggestions for further work in the area.

Introduction

The object of this thesis is to consider the extension to dimension 2 of some notions in the local-to-global theory of Lie groupoids, and which are important in the foundations of differential topology and its applications. We refer particularly to the notion of holonomy in the theory of foliations.

The concept of holonomy has a long and continuing history in differential geometry. However, its 2-dimensional version still needs to be investigated. The main part of this thesis is to attempt this.

0.1 Background

0.1.1 Holonomy for foliations

The notion of the holonomy groupoid was introduced by Ehresmann and Weishu in [24] and Ehresmann in [21], for a locally simple topological foliation on a topological space X (this means that X has two comparable topologies, and with respect to the finer topology on X, a cover by open sets, in each of which the two topologies coincide.) It is constructed as a groupoid of local germs of the groupoid G' of holonomy isomorphisms between the transverse spaces $\tilde{U_1}$ of simple open subsets U_1 of X such that (U_1, U_{i+1}) is a pure chain. The holonomy group at $x \in X$ is the vertex group $G(x)$ of G. This holonomy group is isomorphic to the holonomy group $G(y)$ for each $y \in X$ on the same leaf of the foliation as x.

Pradines [43] considered this holonomy groupoid G, in a wider context, with its differential structure. He took the point of view that a foliation determines an equivalence relation R by xRy if and only if x and y are on the same leaf of the foliation, and that this equivalence relation should be regarded as a groupoid R in the standard way, with multiplication $(x,y)(y,z) = (x,z)$ for $(x,y),(y,z) \in R$. The locally differential structure which gives the foliation determines, if X is paracompact, a differential structure not on R itself but 'locally' on R, that is, on a subset W of R containing the diagonal $\Delta(X)$ of X. This leads him to a definition of "un morceau différentiable de groupoide" G, for which Mackenzie [37] used the term " locally differentiable groupoid". Pradines' note [43] asserts essentially that such a (G, W) determines a differential groupoid $Q_0(G, W)$ and a homomorphism $P : Q_0(G, W) \to G$ such that the "germ" of W extends to a differential structure on G if and only if P is an isomorphism. However his statement of results assumes that the base X is paracompact and that (G, W) is α-connected, i.e., $\alpha^{-1}(x) \cap W$ is connected for each $x \in X$.

The groupoid $Q_0(G, W)$ is called *holonomy groupoid* of (G, W). Aof-Brown [1] gives full details of Pradines' construction in the topological case and the modifications for the Lie case are indicated by Brown-Mucuk [14].

One of the key motivations for the construction of the holonomy groupoid in [43] is the construction of the **monodromy groupoid** of a differential groupoid. An outline of Pradines' construction is given in [4]. Full details are given by Brown and Mucuk in [15]. Formulations and proofs of these two structures, holonomy and monodromy, in the locally trivial case, have been given in Mackenzie [37].

Following Ehresmann's work, there has long been interest in the holonomy group of a leaf of a smooth foliation, see for example [33, 34]. For the locally differential groupoid corresponding to a smooth foliation, the vertex groups of the Ehresmann-Pradines holonomy groupoid are the holonomy groups in the standard sense.

The holonomy groupoid G of a smooth foliation on a manifold X with its smooth structure was rediscovered (using a different, but equivalent, description) by Winkelnkemper [50], as the "graph of foliation". This was defined as the set S of all triples $(x, y, [\gamma])$, where $x, y \in X$ are on the same leaf L of the foliation, γ is a continuous path on L from x to y and $[\gamma]$ is the equivalence class of γ under the equivalence relation $\sim$ which is given by: for the two paths γ_1, γ_2 in L from x to y along, $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1{\gamma_2}^{-1}$ is zero.

Connes [20] has considered this differentable holonomy groupoid G of the foliation and applied to it his general theory of integration based on transerve measures on a measurable groupoid.

Phillips [42] defines the holonomy groupoid $Hol(X, F)$ of a foliated manifold (X, F) as a quotient groupoid of the monodromy groupoid $Mon(X, F)$. This develops earlier work of Winkelnkemper, Phillips who only puts a manifold structure on $Hol(X, F)$.

Also, Haefliger [27] defines a related holonomy groupoid, and consider its classifing space as a representative of the homotopy type of the transverse structure of the foliation F.

0.1.2 Local equivalence relation and "local subgroupoids"

In this subsection we mention some of our work related to that of the thesis but not included in it.

At present, it seems that only the holonomy of an equivalence relation has been extensively studied, in the form of the holonomy groups and holonomy groupoids of a smooth foliation. In this sense, Rosenthal [46, 47] has considered the concept of local equivalence relations, which was introduced by Grothendieck and Verdier [26] in a series of exercises presented as open problems concerning the construction of a certain kind of topos. A local equivalence relation is a global section of the sheaf $\mathcal{E}$ which is defined by the presheaf

$$E = \{E(U), E_{UV}, X\},$$

where $E(U)$ is the set of all equivalence relations on the open subset U of X and E_{UV} is the restriction map from $E(U)$ to $E(V)$, for $V \subseteq U$. Moreover this presheaf is not a sheaf. The key idea in this case is connectedness of the equivalence classes.

Rosenthal [47] has investigated the way a locally topological groupoid arises from a local equivalence relation. However, he esentially puts on the local equivalence relation enough conditions to ensure that it gives rise to a pair (G, W) satisfiying all the conditions for a locally topological groupoid. Brown and Mucuk [14] have verify that these conditions are satisfied for the local equivalence relation determined by a foliation on a paracompact manifold, for suitable W.

Kock and Moerdijk [31] have given alternative accounts of the theory of local equivalence relations using topoi and étendues. They prove that the category of r-sheaves is equivalent to the classifing topos $BMon(F)$ associated to the monodromy groupoid of foliations for a local equivalence relation r. They define a map of classifing (spaces or) topoi $BMon(F) \rightarrow BHol(F)$ by using the well-known groupoid homomorphism from $Mon(F)$ onto $Hol(F)$.

Brown and İçen in work in preparation [12] have considered the concept of local equivalence relation in a wider context, i.e, local subgroupoid. A local subgroupoid of a groupoid G on a topological space X is a global section of the sheaf $\mathcal{L}$ associated to the presheaf

$$L_G = \{L(U), L_{UV}, X\}$$

where $L(U)$ is the set of all wide subgroupoids of $G \mid_U$ and L_{UV} is the restriction map from $L(U)$ to $L(V)$ for $V \subseteq U$.

It is well known that an equivalence on X is a wide subgroupoid of the groupoid $X \times X$. This suggest that the well-known theory of local equivalence relations can be generalised to a theory of local subgroupoids. We show that this is indeed so far the works of Rosenthal [46, 47].

Brown and İçen [12] have obtained the holonomy groupoid of certain local subgroupoids by using the idea of a locally topological groupoid. For this reason, they define weakly

s-adaptable family, regular and strictly regular local subgroupoids and show that if s is a strictly regular local subgroupoid of the topological groupoid G on X and

$$glob(s) = H, \qquad W = \cup_{x \in X} H_x,$$

then (H, W) is a locally topological groupoid. So we get under these circumstances a holonomy groupoid of the locally topological groupoid (H, W).

At the heart of these foundations is the notion of Lie or locally Lie groupoid - the former is often called in earlier literature "differential groupoid", but the term Lie groupoid gives a better impression of the ideas and of the area of applications.

0.2 A 2-dimensional version of Holonomy

Our interest in this thesis is to test ways of extending to dimension 2 various of the above mentioned constructions in the theory of Lie groupoids.

For a 2-dimensional version, there are a number of possible choices for 2-dimensional versions of groupoids, for example double groupoids, 2-groupoids, crossed modules over groupoids. We are not able at this stage to give a version of holonomy for the most general locally Lie double groupoids. It seems reasonable therefore to restrict attention to those forms of double groupoids whose algebra is better understood, and we therefore considered the possibility of a theory for one of the equivalent categories

$$(CrsMod) \sim (2 - Grpd) \sim (DGrpd!),$$

which denote respectively the categories of crossed modules over groupoids, 2-groupoids and "special double groupoids with connection".

In this way, we hope to come nearer to 2-dimensional extensions of the notions of transport along a path. This would hopefully give ideas of, for example, transport over a surface, and pave the way for further extensions to all dimensions. It is hoped that this will lead to a deeper understanding of higher dimensional constructions and operations in differential topology.

One of the hints as to a way to procede lies in the way a group G gives rise to a crossed module $G \to Aut(G)$. The homomorphism $G \to Aut(G)$ which sends an element $x \in G$ to the inner automorphisms of $G \to G$, $k \mapsto -x + k + x$, with the standard action of $Aut(G)$ on G. This has been called (Norrie) [41] the *actor crossed module* of the group G. The notion of crossed module was introduced by Whitehead [49]. In the case of a groupoid G with base space X, we will see that the actor crossed module is of the form

$$\kappa_G: M(G) \to Aut(G)$$

where $M(G)$ is the group of coadmissible sections of G, i.e., sections s of the final map $\beta: G \to X$ such that $\alpha s: X \to X$ is a bijection. Note that κ_G, the "inner automorphism" map, is given by

$$\kappa_G(s) : a \mapsto s\alpha a + a - s\beta a$$

In the case G is a Lie groupoid, Ehresmann focussed attention first on the smooth coadmissible sections (in fact he used admissible sections), and then on the notion of local smooth coadmissible sections. From these he constructed various kinds of prolongation groupoids.

Pradines explained in 1981-85 to R.Brown the use of such sections in the case of locally Lie groupoids, and how this led to a construction of a holonomy groupoid for a large class of locally Lie groupoids. One special case of this construction is the holonomy groupoid of a foliation. This goes via a locally Lie groupoid constructed from the foliation [15].

Here we are exploring the implications of the idea that a natural generalisation to crossed modules of the notion of coadmissible section is that of **coadmissible homotopy**. This arises naturally from the work of Brown-Higgins [11] on homotopies for crossed complexes over groupoids, and also relates interestingly to important work of Whitehead [48], which was followed up by Lue [36], Norrie [41] and Brown and Gilbert [6] on automorphisms of crossed modules over groups.

There is considerable evidence to suggest that crossed modules can be thought of as a 2-dimensional version of groups. The principal argument for this is the fundamental crossed module

$$\delta: \pi_2(X, A, x) \to \pi_1(A, x)$$

of a pair of pointed spaces (X, A), where $\pi_2(X, A, x)$ is the second relative homotopy group, and $\pi_1(A, x)$ is the fundamental group [49]. This notion has also been generalised to the fundamental crossed module on a set A_0 of base points, which gives a family of groups $\{\pi_2(X, A, x)\}_{x \in A_0}$ on which the fundamental groupoid $\pi_1(A, A_0)$ acts.

There are also many algebraic examples of crossed modules, see in Chapter 1.

We now describe the Chapters in detail.

In Chapter I, we combine the notion of coadmissible sections, which is fundamental to the work of Ehresmann [21], with the notion of homotopy of morphisms of crossed modules, which occurs in Whitehead's account [48] of automorphisms of crossed modules and which is later developed by Lue, Norrie, Brown-Gilbert [36, 41, 6].

We introduce the definition of **free derivation** for a crossed module $\mathcal{C} = (C, G, \delta)$ with the base space X. A free derivation s is a pair of maps $s_0: X \to G$, $s_1 : G \to C$ which satisfy the following

$$\begin{aligned} \beta(s_0 x) &= x, \quad x \in X \\ \beta(s_1 a) &= \beta(a), \quad a \in G, \\ s_1(a+b) &= s_1(a)^b + s_1(b), \quad a, b \in G. \end{aligned}$$

Let $FDer(\mathcal{C})$ be the set of free derivations of $\mathcal{C}$.

We prove that if s is a free derivation of the crossed module $\mathcal{C} = (C, G, \delta)$ over groupoids, then the formulae

$$\begin{aligned} f_0(x) &= \alpha s_0(x), \\ f_1(a) &= s_0(\alpha a) + a + \delta s_1(a) - s_0(\beta a), \\ f_2(c) &= (c + s_1\delta c)^{-s_0\beta(c)} \end{aligned}$$

define an endomorphism $f = (f_0, f_1, f_2)$ of $\mathcal{C}$ and write $f = \Delta(s) = (f_0, f_1, f_2)$.

We prove that $FDer(\mathcal{C})$ has a monoid structure with the following multiplication.

$$(s * t)_\epsilon(z) = \begin{cases} (s * t)_0(x) = (s_0 g_0(x)) + t_0(x), \ \epsilon = 0, \quad z = x \in X, \\ (s * t)_1(a) = t_1(a) + (s_1 g_1(a))^{t_0(\beta a)}, \ \epsilon = 1, z = a \in G(x, y), \end{cases}$$

where $g = (g_0, g_1, g_2) = \Delta(t)$. This multiplication, for $\epsilon = 0$, give us Ehresmann' multiplication of coadmissible homotopies, and for $\epsilon = 1$ and $t_0(x) = 1_x$, for all $x \in X$, gives the multiplication of derivations introduced by Whitehead [48].

Let $FDer^*(\mathcal{C})$ denote the group of invertible elements of this monoid. Then each element of $FDer^*(\mathcal{C})$ is also called a **coadmissible homotopy**.

We prove the following theorems.

Theorem 1.3.5 Let $s \in FDer(\mathcal{C})$ and let $f = \Delta(s)$. Then the following conditions are equivalent.

(i) $s \in FDer^*(\mathcal{C})$, (ii) $f_1 \in Aut(G)$, (iii) $f_2 \in Aut(C)$.

Theorem 1.3.6 There is an action of $Aut(\mathcal{C})$ on $s \in FDer^*(\mathcal{C})$ given by

$$s^f(a) = \begin{cases} f_1^{-1} s_0 f_0(a), & a \in X \\ f_2^{-1} s_1 f_1(a), & a \in G. \end{cases}$$

for each $a \in G(x, y)$, which makes $\Delta\colon FDer^*(\mathcal{C}) \to Aut(\mathcal{C})$ a pre-crossed module.

The fact that $\Delta : FDer^*(\mathcal{C}) \to Aut(\mathcal{C})$ is a precrossed module is also a special case of results of Brown-Gilbert [6], which applies the monodial closed structure of the category of crossed complexes introduced by Brown-Higgins in [11]. In fact the description of $\Delta : FDer^*(\mathcal{C}) \to Aut(\mathcal{C})$ is carried out explicitly in Brown-Gilbert [6] Proposition 3.3, for the case $\mathcal{C}$ is a crossed module over a group.

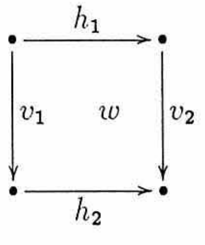
Thus, there is some overlap with the work of Brown and Gilbert [6]. However they explain in detail in their Proposition 3.3 only the case of crossed modules over groups, and this relies on the bulk of the theory on the monoidal closed structure for crossed complexes. So in our Chapter 1 we give a complete and explicit account, from the beginning, of the automorphism theory of crossed modules over groupoids.

There are results in [6] on the "2-crossed module structure"

$$M \to P \ltimes Der^*(P, M) \to Aut(M, P, \mu)$$

of a crossed module $\mu : M \to P$ over a group P. We discuss an analogue of this for a crossed module over a groupoid. However, because we have not yet developed the corresponding local theory for the 2-dimensional part, we do not give explicit verifications of all the axioms for a 2-crossed module, but rely on the general method used in [6, 11].

In Chapter II, we deal with double groupoids especially special double groupoids. A double groupoid is a groupoid object in the category of groupoids: that is, a double groupoid consists of a set $\mathcal{D}$ with two groupoid structures over H and V, which are themselves groupoids on the common set X, all subject to the compatibility condition that the structure maps of each structure on $\mathcal{D}$ are morphisms with respect to the other. Elements of $\mathcal{D}$ are pictured as squares



in which $v_1, v_2 \in V$ are the source and target of w with respect to the horizontal structure on $\mathcal{D}$, and $h_1, h_2 \in H$ are the source and target with respect to the vertical structure.

Double groupoids were introduced by Ehresmann in the early 1960's [22, 23], but primarily as instances of double categories, and as a part of a general exploration of categories with structure. Since that time their main use has been in homotopy theory. Brown-Higgins

[8] gave the earliest example of a "higher homotopy groupoid" by associating to a pointed pair of spaces (X, A) a special double groupoid with special connection $\rho(X, A)$ in the sense of Brown and Spencer (see below). In such a double groupoid, the vertical and horizontal edge structures H and V coincide. In terms of this functor ρ, [8] proved a Generalised Van Kampen Theorem, and deduced from it a Van Kampen Theorem for the second relative homotopy group $\pi_2(X, A)$, viewed as a crossed module over the fundamental group $\pi_1(A)$.

The main result of Brown-Spencer in [16] is that a special double groupoid with special connection whose double base is a singleton is entirely determined by a certain crossed module it contains; as explained above, crossed modules had arisen much earlier in the work of Whitehead [49] on 2-dimensional homotopy. This result of Brown and Spencer is easily extended to give an equivalence between arbitrary special double groupoids with special connection and crossed modules over groupoids; this is included in the result of [9].

We give this extended result as in [11] and [13], since we need the detail here. Brown and Mackenzie [13] have a more general result.

The method which is used here can be found in [16].

Let $\mathcal{D} = (D, H, V, X)$ be a double groupoid. We show that $\mathcal{D}$ determines two crossed modules over groupoids.

Let $x \in X$ and let

$$H(x) = \{a \in H : \alpha_0(a) = \beta_0(a) = x\}.$$

We define $V(x)$ similarly. We put

$$\Pi(D, H, x) = \{w \in D : \alpha_0 w = \beta_0(w) = e_x, \beta_1(w) = f_x\}$$

and

$$\Pi(D, V, x) = \{v \in D : \alpha_1(v) = \beta_1(v) = f_x, \alpha_0(v) = e_x\}$$

which have group structures induced from $+_0$, and $+_1$. Then $\Pi(D, H) = \{\Pi(D, H, x) : x \in X\}$ and $\Pi(D, V) = \{\Pi(D, V, x) : x \in X\}$ are totally intransitive groupoids over X.

Clearly maps

$$\varepsilon : \Pi(D,H) \to H \quad \text{and} \quad \partial : \Pi(D,V) \to V$$

defined by $\varepsilon(w) = \alpha_1(w)$ and $\partial(v) = \alpha_0(v)$, respectively, are homomorphism of groupoids.

Proposition 0.2.1 *Let $\mathcal{D} = (D, H, V, X)$ be a double groupoid then*

$$\gamma(D) = (\Pi(D,H), H, \varepsilon) \quad \gamma'(D) = (\Pi(D,V), V, \partial)$$

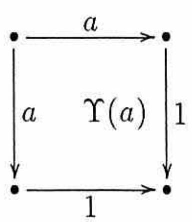
may be given the structure of crossed modules.

Clearly γ is a functor from the category of double groupoids to the category of crossed modules.

As we wrote , a *special double groupoid* is a double groupoid $\mathcal{D}$ but with the extra condition that the horizontal and vertical groupoids H and V structures coincide. These double groupoids will, from now on, be our sole concern, and for these it is convenient to denote the sets of points, edges and squares by X, G, D. The identities in G will be written 1_x or simply 1. The source and target maps $G \to X$ will be written α, β.

By a morphism $f : \mathcal{D} \to \mathcal{D}'$ of special double groupoids is meant functions $f : D \to D'$, $f : G \to G'$, $f : X \to X'$ which commute with all three groupoid structures.

Definition 0.2.2 *Let $\mathcal{D}$ be a special double groupoid. A* **special connection** *for $\mathcal{D}$ is a function $\Upsilon : G \to D$ such that if $a \in G$ then $\Upsilon(a)$ has boundaries given by the following diagram*



A morphism $f : \mathcal{D} \to \mathcal{D}'$ of special double groupoid with special connections Υ, Υ' is said to preserve the connections if $f_2\Upsilon' = \Upsilon f_1$.

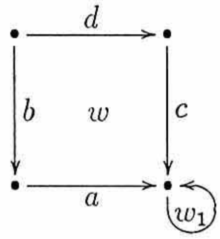
The category $DGrpd!$ has objects the pairs $(\mathcal{D}, \Upsilon)$ of a special double groupoid $\mathcal{D}$ with special connection, and arrows the morphisms of special double groupoids preserving the connection. If $(\mathcal{D}, \Upsilon)$ is an object of $DGrpd!$, then we have a crossed module $\gamma(\mathcal{D})$ by Proposition 2.3.1. Clearly γ extends to a functor from $DGrpd!$ to $CrsMod$, the category of crossed modules. The main result on double groupoids is:

Theorem 0.2.3 *The functor $\gamma : DGrpd! \to CrsMod$ is an equivalence of categories [16].*

We then show how special double groupoids arise from crossed modules over groupoids.

Let $\mathcal{C} = (C, G, \delta)$ be a crossed module over groupoids with base set X. We define a special double groupoid $\mathcal{D}(\mathcal{C})$ as follows. First, $H = V = G$ with its groupoid structure, base set X. The set $\mathcal{D}(\mathcal{C})$ of squares is to consist of quintuples

$$w = \left(w_1 : b \begin{array}{c} d \\ a \end{array} c \right)$$



such that $w_1 \in C, a, b, c, d \in G$ and

$$\delta(w_1) = -a - b + d + c.$$

The vertical and horizontal structure on the set $\mathcal{D}(\mathcal{C})$ can be defined as in [16]. Then $\mathcal{D}(\mathcal{C})$ becomes a double groupoid with these structures.

We introduce a definition of linear coadmissible section for the special double groupoid $\mathcal{D}(\mathcal{C})$ as follows.

Definition 0.2.4 *Let $\mathcal{C} = (C, G, \delta)$ be a crossed module and let $\mathcal{D}(\mathcal{C})$ be the corresponding double groupoid. A* **linear coadmissible section** *$\sigma = (\sigma_0, \sigma_1) : G \to \mathcal{D}(\mathcal{C})$ of $\mathcal{D}(\mathcal{C})$ also written*

$$\sigma(a) = \left(\sigma_1(a) : \sigma_0\alpha(a) \begin{array}{c} \\ a \end{array} \sigma_0\beta(a) \right)$$

is a pair of maps

$$\sigma_0 : X \to G, \qquad \sigma_1 : G \to C$$

such that

(i) *if* $x \in X$, $\beta\sigma_0(x) = x$, *and if* $a \in G$, *then* $\beta\sigma_1(a) = \beta a$.

(ii) *if* $a, b, a+b \in G$, *then*

$$\sigma(a+b) = \sigma(a) +_0 \sigma(b)$$

(iii) $\alpha\sigma_0 : X \to X$ *is a bijection,* $\alpha\sigma : G \to G$ *is an automorphism.*

Let $\Gamma\mathcal{D}(\mathcal{C})$ denotes the set of all linear coadmissible sections. Then a group structure on $\Gamma\mathcal{D}(\mathcal{C})$ is defined by the multiplication

$$(\sigma * \tau)_0(x) = (\sigma_0\alpha\tau_0(x)) + \tau_0(x), \quad x \in X,$$

$$(\sigma * \tau)(a) = (\sigma\alpha\tau_1(a)) +_1 \tau(a), \quad a \in G(x, y)$$

for $\sigma, \tau \in \Gamma\mathcal{D}(\mathcal{C})$.

We show in Corollary 2.4.4 that the groups of linear coadmissible section and free invertible derivation maps (coadmissible homotopies) are isomorphic.

Now we come to the main new work of this thesis.

Chapter III is aimed at the study of some local Lie structures on a special double groupoid $\mathcal{D}(\mathcal{C})$ corresponding to a crossed module $\mathcal{C} = (C, G, \delta)$– namely such a local Lie structure is given a pair of sets $(\mathcal{D}(\mathcal{C}), W^G)$ satisfying certain conditions, where $W^G \subseteq \mathcal{D}(\mathcal{C})$ has a manifold structure.

In order to cover both the topological and differentiable cases, we use the term C^r manifold for $r \geq -1$, where the case $r = -1$ deals with the case of topological spaces and continuous maps, with no local assumptions, while the case $r \geq 0$ deals as usual with C^r manifolds and C^r maps. Of course, a C^0 map is just a continuous map. We then abbreviate

C^r to *smooth*. The terms *Lie group* or *Lie groupoid* will then involve smoothness in this extended sense.

One of the key differences between the cases $r = -1$ or 0 and $r \geq 1$ is that for $r \geq 1$, the pullback of C^r maps need not be a smooth submanifold of the product, and so differentiability of maps on the pullback cannot always be defined. We therefore adopt as in Brown-Mucuk [15] the following definition of Lie groupoid. Mackenzie [37] discusses the utility of various definitions of differential groupoid.

A *Lie groupoid* is a topological groupoid G such that

(i) the space of arrows is a smooth manifold, and the space of objects is a smooth submanifold of G,

(ii) the source and target maps α, β are smooth maps and are submersions.

(iii) the domain $G \sqcap_\beta G$ of the difference map is a smooth submanifold of $G \times G$, and

(iv) the difference map d is a smooth map.

We localise the concept of the coadmissible homotopies and linear coadmissible sections. We define products on the sets of both concepts. We prove that $M_L(\mathcal{C})$, the set of all local coadmissible homotopies, and $\Gamma(\mathcal{D})(\mathcal{C})$, the set of local linear coadmissible sections are, isomorphic inverse semigroups.

We introduce the concept of a V-locally Lie double groupoid $(\mathcal{D}(\mathcal{C}), W^G)$ and related notion "locally Lie crossed module" for crossed module. Note that for the case of groupoids rather than crossed modules, Pradines stated a differential version involving germs of locally Lie groupoids in [43], and formulated this in terms of adjoint functors. A version for locally topological groupoids was given in Aof-Brown [1] and the modifications for the differential case were given in Brown-Mucuk [14]. Our general aim is to consider analogous methods for the case of crossed modules.

However, there already existed in the literature a well developed and clearly relevant theory of automorphism of crossed modules, and it therefore seemed sensible to develop a holonomy theory for forms of locally Lie crossed modules, based on "coadmissible homotopies" rather than coadmissible section.

In so doing, there arose the problem of defining "final map" on the germs of local coadmissible homotopies. It became clear that the values of such a final map had to lie in the double groupoid associated to the crossed module. This explains why our theory develops crossed modules and double groupoids in parallel. It also is sensible to keep the crossed module theory, since the algebra of crossed modules is closly related to standard algebra for group, and these show how aspects of "2-dimensional groupoid theory" are likely to prove of continuing importance.

The holonomy groupoid $Hol(\mathcal{D}(\mathcal{C}), W^G)$ is constructed in Section 3.4 as the quotient groupoid $J^r(\mathcal{D}(\mathcal{C}))/J_0$, where $J^r(\mathcal{D}(\mathcal{C}))$ is a subgroupoid of the groupoid $J(\mathcal{D}(\mathcal{C}))$ (the sheaf of germs of local coadmissible sections of the special double groupoid $\mathcal{D}(\mathcal{C})$ generated by the subsheaf $J^r(W^G)$ of germs of elements of $\Gamma(W^G)$ and J_0 is the intersection of $J^r(W^G)$ and the kernel $Ker\ \psi$ of the final map $\psi : J(\mathcal{D}(\mathcal{C})) \to \mathcal{D}(\mathcal{C})$.

In order to show that the quotient groupoid is well defined, we prove:

Lemma 3.4.2 The set $J_0 = J^r(W^G) \cap Ker\ \phi$ is a wide subgroupoid of the groupoid $J^r(\mathcal{D}(\mathcal{C}))$.

Lemma 3.4.3 The groupoid J_0 is a normal subgroupoid of the groupoid $J^r(\mathcal{D}(\mathcal{C}))$.

Chapter IV, which is a main aim of this thesis, is concerned with the construction of the Lie structure on the holonomy groupoid $Hol(\mathcal{D}(\mathcal{C}), W^G)$ of a V-locally Lie double groupoid $(\mathcal{D}(\mathcal{C}), W^G)$ and we state and prove the Holonomy Theorem 4.0.1.

The aim of Section 4.1 is to construct the topology on the holonomy groupoid $Hol(\mathcal{D}(\mathcal{C}), W^G)$ such that $Hol(\mathcal{D}(\mathcal{C}), W^G)$ with this topology is a Lie groupoid. The intuition is that first of all W^G embeds in $Hol(\mathcal{D}(\mathcal{C}), W^G)$, and second that $Hol(\mathcal{D}(\mathcal{C}), W^G)$ has enough local linear coadmissible sections for it to obtain a topology by translation of the topology of W^G.

Let $s \in \Gamma(\mathcal{D}(\mathcal{C}), W^G)$. We define a partial function $\chi_s : W^G \to Hol(\mathcal{D}(\mathcal{C}), W^G)$. The domain of χ_s is the set of $w \in W^G$ such that $\alpha(w) = a \in D(s_1)$ and $\alpha(a), \beta(a) \in D(s_0)$. The value $\chi_s(w)$ is obtained as follows. Choose a local linear smooth coadmissible section θ through w. Then we set

$$\chi_s(w) = < s >_{\alpha(w)} < \theta >_{\beta(w)} = < s * \theta >_{\beta(w)} .$$

By Lemma 3.4.2, $\chi_s(w)$ is independent of the choice of the local linear smooth coadmissible section θ.

Lemma 4.1.1 χ_s is injective.

Let $s \in \Gamma(\mathcal{D}(\mathcal{C}))$. Then s defines a left translation L_s on $\mathcal{D}(\mathcal{C})$ by

$$L_s(w) = s(\alpha(w)) +_1 w.$$

This is an injective partial function on $\mathcal{D}(\mathcal{C})$. The inverse ${L_s}^{-1}$ of L_s is

$$v \mapsto -_1 s(\alpha s)^{-1}(\alpha(v)) +_1 v$$

and ${L_s}^{-1} = L_{s^{-1}}$. We call L_s the **left translation corresponding to** s.

So we have an injective function χ_s from an open subset of W^G to $Hol(\mathcal{D}(\mathcal{C}), W^G)$. By definition of $Hol(\mathcal{D}(\mathcal{C}), W^G)$, every element of $Hol(\mathcal{D}(\mathcal{C}), W^G))$ is in the image of χ_s for some s. These χ_s will form a set of charts and so induce a topology on $Hol(\mathcal{D}(\mathcal{C}), W^G)$. The compatibility of these charts results from the following lemma, which is essential to ensure that W^G retains its topology in $Hol(\mathcal{D}(\mathcal{C}), W^G)$ and is open in $Hol(\mathcal{D}(\mathcal{C}), W^G)$. As in the groupoid case, this is a key lemma.

Lemma 4.1.2 Let $s, t \in \Gamma(\mathcal{D}(\mathcal{C}), W^G)$. Then $(\chi_t)^{-1}\chi_s$ coincides with L_η, left translation by the local linear smooth coadmissible section $\eta = t^{-1} * s$, and L_η maps open sets of W^G diffeomorphicially to open sets of W^G.

Lemma 4.1.3 With the above topology, $Hol(\mathcal{D}(\mathcal{C}), W^G)$ is a Lie groupoid.

We now review definitions of Lie crossed module and double Lie groupoid. A Lie crossed module $\mathcal{C} = (C, G, \delta)$ over groupoid is a crossed module such that C and G have Lie groupoid structure and action of G on C, and $\delta : C \to G$ is smooth functor. The image of C in G is not required to be closed, see [38, 13].

In differential geometry, double Lie groupoids, but usually with one of the structure totally intransitive, have been considered in passing by Pradines [44, 45]. In general, double Lie groupoids were investigated by K.Mackenzie in [39]. A *double Lie groupoid* is a double groupoid $\mathcal{D} = (D; H, V, X)$ together with differentiable structures on D, H, V and X, such that all four groupoid structures are Lie groupoids and such that the double source map $D \to H \times_\alpha V = \{(h, v) : \alpha_H(h) = \alpha_V(v)\}$, $d \to (\tilde{\alpha_V}(d), \tilde{\alpha_H}(d))$ is surjective submersion.

We also state Theorem 4.2.7 in part of a Lie version of Brown-Spencer Theorem which occurs in [13]. Let $\mathcal{C} = (C, G, \delta)$ be a certain Lie crossed module; then the corresponding special double groupoid $\mathcal{D}(\mathcal{C})$ is a Lie double groupoid which is called a "split double groupoid" in [13],

In Section 4.2, we state and prove the main theorem of the universal property of the morphism $\psi : Hol(\mathcal{D}(\mathcal{C}), W^G) \to \mathcal{D}(\mathcal{C})$. The main idea is when we are given a V-locally Lie double groupoid $(\mathcal{D}(\mathcal{C}), W^G)$ of a double groupoid $\mathcal{D}(\mathcal{C})$ for a crossed module $\mathcal{C}$, a Lie crossed module $\mathcal{A}$ and a morphism

$$\mu : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{C})$$

with suitable conditions, we can construct a morphism

$$\mu' : \mathcal{D}(\mathcal{A}) \to Hol(\mathcal{D}(\mathcal{C}), W^G),$$

where $Hol(\mathcal{D}(\mathcal{C}), W^G))$ is the holonomy groupoid of a V-locally Lie crossed module, such that

$$\phi\mu' = \mu.$$

We prove that such a morphism μ' is well-defined, smooth and unique.

The aim of Chapter V is to give an outline of possible and interesting topics for further working this area, particulary will regard to possibility of obtaining forms of "holonomy double groupoids".

Chapter 1

Automorphisms of Crossed Modules of Groupoids

1.1 Introduction

In this chapter we are exploring the implications of the idea that a natural generalisation to crossed modules of the notion of coadmissible section is that of **coadmissible homotopy**. This arises naturally from the work of Brown-Higgins on homotopies for crossed complexes over groupoids, and also relates interestingly to important work of Whitehead [48], and followed up by Lue [36] and Norrie [41], on automorphisms of crossed modules over groups.

There is some overlap with the work of Brown and Gilbert [6]. However they explain in detail in Section 3 only the case of crossed modules over groups. So in this chapter we give a complete and explicit account, from the beginning, of automorphisms in the theory of crossed modules over groupoids.

Brown and Gilbert also relate this theory to the monoidal closed structure of crossed modules over groupoids, and indeed deduce their results from a description of this structure. The complete account of this monoidal closed structure in [11] is based on the equivalence in [11] between crossed complexes over groupoids and ω-groupoids. The latter is a cubical based theory, in which the monoidal closed structure is easy to define.

Further results in [6] are on the "2-crossed module structure"

$$M \to P \ltimes Der^*(P, M) \to Aut(M, P, \mu)$$

for a crossed module $\mu : M \to P$ over a group P. We extend this to an explicit description of a 2-crossed module equivalent to $AUT(\mathcal{C})$ in the case $\mathcal{C}$ is a crossed module over a groupoid. However, because we have not yet developed the corresponding local theory for the 2-dimensional part, we do not give explicit verification of the axioms, but rely on the general method used in [7, 6].

1.2 Crossed Modules Of Groupoids

We recall the definition of crossed modules over groupoids. The basic reference is Brown-Higgins [7].

Definition 1.2.1 Let G, C be groupoids over the same object set and let C be totally intransitive. Then an **action** of G on C is given by a partially defined function

$$C \times G \to C,$$

written $(c, a) \mapsto c^a$, which satisfies

1. c^a is defined if and only if $\beta(c) = \alpha(a)$, and then $\beta(c^a) = \beta(a)$, where α, β are respectively the source and target maps of the groupoid G.

2. $(c_1 + c_2)^a = {c_1}^a + {c_2}^a$,

3. $c_1^{a+b} = (c_1^a)^b$ and $c_1^{e_x} = c_1$

for all $c_1, c_2 \in C(x, x)$, $a \in G(x, y)$, $b \in G(y, z)$.

Definition 1.2.2 A **crossed module of groupoids** consists of a pair of groupoids C and G over a common object set such that C is totally intransitive, together with an action of G on C, together with a functor $\delta : C \to G$ which is the identity on the object set and satisfies

1. $\delta(c^a) = -a + \delta c + a$

2. $c^{\delta c_1} = -c_1 + c + c_1$

for $c, c_1 \in C(x,x)$, $a \in G(x,y)$.

A crossed module will be denoted by $\mathcal{C} = (C, G, \delta)$. A **crossed module of groups** is a crossed module of groupoids as above in which C, G are groups.

The followings are standard examples of crossed modules:

(i). Let H be a normal subgroup of a group G with $i : H \to G$ the inclusion. The action of G on the right of H by conjugation makes (H, G, i) into a crossed module.

(ii). Suppose G is a group and M is a right G-module; let $0 : M \to G$ be the constant map sending M to the identity element of G. Then $(M, G, 0)$ is a crossed module.

(iii). Suppose given a morphism

$$\eta : M \to N$$

of left G-modules and form the semi-direct product $G \ltimes N$. This is a group which acts on M via the projection from $G \ltimes N$ to G. We define a morphism

$$\delta : M \to G \ltimes N$$

by $\delta(m) = (1, \eta(m))$ where 1 denotes the identity in G. Then $(M, G \ltimes N, \delta)$ is a crossed module.

Also we can define a category $CrsMod$ of crossed modules of groupoids. Let $\mathcal{C}$, $\mathcal{C}'$ be crossed modules. A morphism $f : \mathcal{C} \to \mathcal{C}'$ consists of a pair of groupoid homomorphisms (f_1, f_2) such that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{f_2} & C' \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta'} \\ G & \xrightarrow[f_1]{} & G' \end{array} \qquad \begin{array}{ccc} C \times G & \longrightarrow & C \\ {\scriptstyle f_2 \times f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ C' \times G' & \longrightarrow & C' \end{array}$$

1.3 Free Derivations and Coadmissible Homotopies

In this section, we combine the notion of coadmissible section, which is fundamental to the work of Ehresmann [21], with the notion of homotopy of morphisms of crossed modules, which occurs in Whitehead's account [48] of automorphisms of crossed modules and which is later developed by Lue, Norrie, Brown-Gilbert [36, 41, 6].

So we are exploring the implications of the idea that a natural generalisation to crossed modules of the notion of coadmissible section is that of coadmissible homotopy. This arises naturally from the work of Brown-Higgins [11] on homotopies for crossed complexes over groupoids.

Definition 1.3.1 Let $\mathcal{C} = (C, G, \delta)$ be a crossed module over groupoids with base space X. A **free derivation** s is a pair of maps $s_0 \colon X \to G$, $s_1 : G \to C$ which satisfy the following

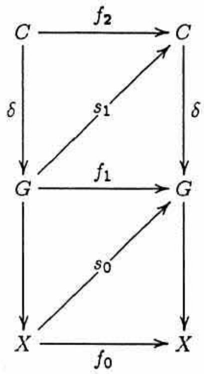
$$\begin{aligned} \beta(s_0 x) &= x, \quad x \in X \\ \beta(s_1 a) &= \beta(a), \quad a \in G, \\ s_1(a+b) &= s_1(a)^b + s_1(b), \quad a, b \in G. \end{aligned}$$

Let $FDer(\mathcal{C})$ be the set of free derivations of $\mathcal{C}$.

Proposition 1.3.2 *Let s be a free derivation of the crossed module $\mathcal{C} = (C, G, \delta)$ over groupoids. Then the formulae*

$$\begin{aligned} f_0(x) &= \alpha s_0(x), \\ f_1(a) &= s_0(\alpha a) + a + \delta s_1(a) - s_0(\beta a), \\ f_2(c) &= (c + s_1 \delta c)^{-s_0 \beta(c)} \end{aligned}$$

define an endomorphism $f = (f_0, f_1, f_2)$ of C which we write $\Delta(s) = f$.



Proof: We have to show that f_1 and f_2 are groupoid homomorphisms and $f_2(c^a) = f_2(c)^{f_1(a)}$, for $c \in C(x), a \in G(x,y)$.

$$\begin{aligned}
f_1(a+b) &= s_0(x) + a + b + \delta s_1(a+b) - s_0(z) \\
&= s_0(x) + a + b + \delta(s_1(a)^b + s_1(b)) - s_0(z) \\
&= s_0(x) + a + b - b + \delta s_1(a) + b + \delta s_1(b) - s_0(z), \quad \text{by definition of } \delta \\
&= s_0(x) + a + \delta s_1(a) - s_0(y) + s_0(y) + b + \delta s_1(b) - s_0(z) \\
&= f_1(a) + f_1(b)
\end{aligned}$$

$$\begin{aligned}
f_2(c+c') &= (c + c' + s_1\delta(c+c'))^{-s_0(x)} \\
&= (c + c' + s_1(\delta c + \delta c'))^{-s_0(x)} \\
&= (c + c' + s_1(\delta c)^{\delta c'} + s_1(\delta c'))^{-s_0(x)} \\
&= (c + c' - c' + s_1\delta c + c' + s_1\delta c')^{-s_0(x)} \\
&= (c + s_1\delta c + c' + s_1\delta c')^{-s_0(x)} \\
&= f_2(c) + f_2(c')
\end{aligned}$$

Let $c \in C(x), a \in G(x,y)$. Then $\beta(c^a) = \beta a$, $\beta c^a = y$. So

$$\begin{aligned}
f_2(c^a) &= (c^a + s_1\delta(c^a))^{-s_0(\beta c^a)} \\
&= (c^a + s_1(-a + \delta c + a))^{-s_0(y)} \\
&= (c^a + s_1(-a)^{\delta c + a} + s_1(\delta c)^a + s_1(a))^{-s_0(y)}
\end{aligned}$$

(since $-(s_1(a))^{-a+\delta c+a} = (s_1(-a))^{\delta c+a}$),

$$\begin{aligned}
&= (c^a - (s_1 a)^{-a+\delta c+a} + (s_1\delta c)^a + s_1 a)^{-s_0(y)} \\
&= (c^a - s_1 a^{\delta c^a} + (s_1\delta c)^a + s_1 a)^{-s_0(y)} \\
&= (-s_1(a) + c^a + (s_1\delta c)^a + s_1(a))^{-s_0(y)} \\
&= (-s_1(a) + (c + s_1\delta c)^a + s_1(a))^{-s_0(y)} \\
&= (-s_1(a) + (f_2(c)^{s_0(x)})^a + s_1(a))^{-s_0(y)} \qquad (1.1) \\
&= (f_2(c))^{s_0 x + a + \delta s_1 a - s_0 y} \\
&= f_2(c)^{f_1(a)}.
\end{aligned}$$

So f is an endomorphism of $\mathcal{C}$. □

Proposition 1.3.3 *Let $\mathcal{C} = (C, G, \delta)$ be a crossed module over groupoids. Then a monoid structure on $FDer(\mathcal{C})$ is defined by the multiplication*

$$(s*t)_\epsilon(z) = \begin{cases} (s*t)_0(x) = (s_0 g_0(x)) + t_0(x), & \epsilon = 0, \quad z = x \in X, \\ (s*t)_1(a) = t_1(a) + (s_1 g_1(a))^{t_0(\beta a)}, & \epsilon = 1, \quad z = a \in G(x,y) \end{cases}$$

for $s, t \in FDer(\mathcal{C})$ and $f = \Delta(s), g = \Delta(t)$. Further $\Delta(s*t) = \Delta(s)*\Delta(t), \Delta(1) = 1$.

Proof: It is clear that $\beta(s*t)_0(x) = x$ and $\beta(s*t)_1(a) = \beta(a)$ for $x \in X$, and $a \in G(x,y)$. In fact,

$$\begin{aligned}
\beta(s*t)_0(x) &= \beta(s_0(g_0(x)) + t_0(x)) \\
&= \beta t_0(x) \\
&= x.
\end{aligned}$$

Secondly ,

$$\beta(t*s)_1(a) = \beta(t_1(a) + s_1 g_1(a)^{t_0 y})$$

$$
\begin{aligned}
&= \beta(s_1 g_1(a)^{t_0 y}) \\
&= \beta s_1(g_1(a)^{t_0 y}) \\
&= \beta(g_1(a)^{t_0 y}) \\
&= \beta(t_0 y) \\
&= \beta t_0(y) \\
&= y \\
&= \beta(a).
\end{aligned}
$$

We have to show that $(s*t)_1$ is a derivation map. Let $a \in G(x,y)$, $b \in G(y,z)$. Then

$$
\begin{aligned}
(s*t)_1(a+b) &= t_1(a+b) + (s_1 g_1(a+b))^{t_0(z)} \\
&= t_1(a)^b + t_1(b) + (s_1(g_1(a) + g_1(b))^{t_0(z)} \\
&= t_1(a)^b + t_1(b) + (s_1(g_1(a))^{g_1(b)} + s_1(g_1(b))^{t_0(z)} \\
&= t_1(a)^b + t_1(b) + (s_1(g_1(a))^{t_0(y)+b+\delta t_1(b) - t_0(z)} + s_1(g_1(b))^{t_0(z)} \\
&= t_1(a)^b + t_1(b) + s_1(g_1(a))^{t_0(y)+b+\delta t_1(b)} + s_1(g_1(b))^{t_0(z)} \\
&= t_1(a)^b + t_1(b) + (s_1(g_1(a)^{t_0(y)+b})^{\delta t_1(b)} + s_1(g_1(b))^{t_0(z)} \\
&= t_1(a)^b + t_1(b) - t_1(b) + (s_1(g_1(a))^{t_0(y)+b} + t_1(b) + s_1(g_1(b)^{t_0(z)} \\
&= t_1(a)^b + (s_1(g_1(a))^{t_0(y)+b} + t_1(b) + s_1(g_1(b))^{t_0(z)} \\
&= (t_1(a) + s_1(g_1(a)^{t_0(y)})^b + t_1(b) + s_1(g_1(b)^{t_0(z)} \\
&= (s*t)_1(a)^b + (s*t)_1(b).
\end{aligned}
$$

For the associativity property, let $u, s, t \in FDer(\mathcal{C})$ and let $f = \Delta(s), g = \Delta(t), h = \Delta(u)$. Then

$$
\begin{aligned}
(u_0 * (s*t)_0)(x) &= (u_0(f_0 g_0(x)) + (t*s)_0(x) \\
&= u_0(f_0 g_0)(x) + (s_0 g_0)(x) + t_0(x) \\
&= u_0(f_0(g_0(x)) + s_0 g_0(x) + t_0(x)
\end{aligned}
$$

$$
\begin{aligned}
&= (u * s)_0(g_0(x)) + t_0(x) \\
&= ((u * s)_0 * t_0)(x)
\end{aligned}
$$

and

$$
\begin{aligned}
(u_1 * (s * t)_1)(a) &= (s * t)_1(a) + u_1(fg(a))^{(s*t)_0(x)} \\
&= t_1 a + (s_1 g a)^{t_0(y)} + (u_1 f g a)^{(s*t)_0(x)} \\
&= t_1 a + (s * u)_1(ga)^{t_0(y)} \\
&= ((u * s)_1 * t_1)(a).
\end{aligned}
$$

Let $s, t \in FDer(\mathcal{C})$ be as above and let $a \in G(x, y)$. Then

$$
\begin{aligned}
\Delta(s * t)_1(a) &= (s * t)_0(x) + a + \delta(s * t)_1(a) - (s * t)_0(y) \\
&= s_0 g_0(x) + t_0(x) + a + \delta(t_1(a) + s_1 g_1(a))^{t_0(y)} - (s_0 g_0(y) + t_0(y)) \\
&= s_0 g_0(x) + t_0(x) + a + \delta(t_1(a) + \delta(s_1 g_1(a)^{t_0(y)}) - (s_0 g_0(y) + t_0(y)) \\
&= s_0 g_0(x) + t_0(x) + a + \delta t_1(a) - t_0(y) + \delta s_1 g_1(a) + t_0(y) - t_0(y) - s_0 g_0(y) \\
&= s_0 g_0(x) + \delta_t(a) + \delta s_1 g_1(a) - s_0 g_0(y) \\
&= \Delta(s)(\Delta(t))(a) \\
&= \Delta(s) \circ \Delta(t)(a).
\end{aligned}
$$

Let $c \in C(x), a \in G(x, y)$.

$$
\begin{aligned}
\Delta(s * t)(c) &= (c + (s * t)_1(\delta(c))^{-(s*t)_0(\beta c)} \\
&= (c + t_1(\delta(c)) + s_1 g_1(\delta(c))^{t_0(x)})^{-(s*t)_0(x)} \\
&= (c + t_1(\delta(c))^{-(s*t)_0(x)} + s_1 \delta g_2(c))^{t_0(x) - (s*t)_0(x)} \\
&= (c + t_1(\delta(c))^{-t_0(x) - s_0 g_0(x)} + s_1 \delta g_2(c))^{-s_0 g_0(x)} \\
&= ((c + t_1(\delta(c))^{-t_0(x)} + s_1 \delta g_2(c))^{-s_0 g_0(x)} \\
&= (\Delta(t)(c)) + (s_1 \delta g_2(c))^{-s_0 g_0(x)}, \quad \text{since } \Delta(t)(c) = g_2(c), \\
&= \Delta(s) \circ \Delta(t)(c)
\end{aligned}
$$

So $\Delta(s * t) = \Delta(s) \circ \Delta(t)$.

Let $c = (c_0, c_1)$ be the free derivation defined by

$$c_0(x) = 1_x \quad \text{and} \quad c_1(a) = 1$$

for $x \in X$ and $a \in G$.

$$\begin{aligned} \Delta(c)(a) &= c_0(x) + a + \delta c_1(a) - c_0(y) \\ &= 1_x + a + \delta(1) - 1_y \\ &= 1(a). \end{aligned}$$

Similarly, for $c' \in C$, we have $\Delta(c)(c') = 1(c')$. □

Corollary 1.3.4 *The function Δ is a monoid morphism*

$$FDer(\mathcal{C}) \rightarrow End(\mathcal{C})$$

Let $FDer^*(\mathcal{C})$ denote the group of invertible elements of this monoid. An invertible free derivation is also called a **coadmissible homotopy**.

Theorem 1.3.5 *Let $s \in FDer(\mathcal{C})$ and let $f = \Delta(s)$. Then the following conditions are equivalent:*

(i) $s \in FDer^*(\mathcal{C})$,

(ii) $f_1 \in Aut(G)$,

(iii) $f_2 \in Aut(C)$.

Proof: That $(i) \Rightarrow (ii), (i) \Rightarrow (iii)$ follows from the fact that Δ is a morphism to $End(\mathcal{C})$. We next prove $(ii) \Rightarrow (i)$. Suppose then $f_1 \in Aut(G)$. We define $s^{-1} = ({s_0}^{-1}, {s_1}^{-1})$.

Let ${s_0}^{-1} : X \rightarrow G, {s_1}^{-1} : G \rightarrow C$ by

$${s_0}^{-1}(x) = -s_0({f_0}^{-1}(x)) \quad \text{and} \quad {s_1}^{-1}(a) = -s_1({f_1}^{-1}(a))^{{s_0}^{-1}(y)}.$$

Since $\beta {s_0}^{-1}(x) = x$ and $\alpha {s_0}^{-1}(x) = f_0(x)$, ${s_0}^{-1}$ is an inverse element of s_0. In fact,

$$\begin{aligned}(s^{-1} * s)_0(x) &= {s_0}^{-1}({f_0}^{-1})(x) + s_0(x) \\ &= -s_0({f_0}^{-1}(f_0)(x)) + s_0(x) \\ &= -s_0(x) + s_0(x) \\ &= c_0(x) = 1_x\end{aligned}$$

and also

$$\begin{aligned}(s * s^{-1})_0(x) &= s_0({f_0}^{-1})(x) + {s_0}^{-1}(x) \\ &= s_0({f_0}^{-1})(x) - s_0({f_0}^{-1}(x) \\ &= c_0(y) = 1_y.\end{aligned}$$

We have to show that ${s_1}^{-1}$ is a derivation map. Let $a, b, a+b \in G$ and let $a' = {f_1}^{-1}(a), b' = {f_1}^{-1}(b)$, $\beta b = z$. Note that ${s_0}^{-1}\beta(a+b) = {s_0}^{-1}(z)$ and $-s_0(z) = -s_0{f_0}^{-1}(z)$.

$$\begin{aligned}{s_1}^{-1}(a+b) &= -(s_1{f_1}^{-1}(a+b))^{{s_0}^{-1}\beta(a+b)}, \text{ by definition of } {s_1}^{-1} \\ &= -(s_1({f_1}^{-1}a + {f_1}^{-1}b))^{{s_0}^{-1}(z)} \\ &= -(s_1(a'+b'))^{{s_0}^{-1}(z)} \\ &= -((s_1a')^{b'} + s_1(b'))^{{s_0}^{-1}(z)}, \text{ since } s_1 \text{ is a derivation,} \\ &= (-s_1(b') - (s_1(a'))^{b'})^{-s_0(z)} \\ &= -(s_1(b') + (s_1(a'))^{b'+\delta s_1(b')})^{-s_0(z)} \\ &= -(s_1(b')^{-s_0(z)} + (s_1(a'))^{b'+\delta s_1(b') - s_0{f_0}^{-1}(z)}\end{aligned}$$

Since $f(b') = s_0{f_0}^{-1}(y) + b' + \delta s_1(b') - s_0{f_0}^{-1}(z) = b$ and $b' + \delta s_1(b') - s_0{f_0}^{-1}(z) = b - s_0{f_0}^{-1}(y)$,

$$= -(s_1(b')^{-s_0(z)} + (s_1(a'))^{b - s_0{f_0}^{-1}(y)}$$

$$\begin{aligned}
&= -(s_1(b')^{-s_0(z)} + (s_1(a'))^{b-s_0{}^{-1}(y)} \\
&= -(s_1(a'))^b)^{-s_0{}^{-1}(y)} - s_1(b')^{s_0{}^{-1}} \\
&= -(s_1(f_1{}^{-1}(a))^b)^{-s_0{}^{-1}(y)} - s_1 f_1{}^{-1}(b')^{s_0{}^{-1}} \\
&= s_1{}^{-1}(a)^b + s_1{}^{-1}(b).
\end{aligned}$$

One can easily show that $s * s^{-1} = c$ and $s^{-1} * s = c$.

$$\begin{aligned}
(s * s^{-1})_1(a) &= s_1{}^{-1}(a) + s_1(f^{-1}(a))^{s_0{}^{-1}(y)} \\
&= -s_1 f^{-1}(a)^{s_0{}^{-1}(y)} + s_1(f^{-1}(a))^{s_0{}^{-1}(y)} \\
&= c_1(a).
\end{aligned}$$

and

$$\begin{aligned}
(s^{-1} * s)_1(a) &= s_1(a) + s_1{}^{-1}(f(a))^{s_0(y)} \\
&= s_1(a) - s_1(f^{-1}(f(a))^{s_0(y)})^{s_0{}^{-1}(y)} \\
&= s_1(a) - s_1(a) \\
&= c_1(a).
\end{aligned}$$

Now we will prove $(iii) \Rightarrow (i)$. We first recalculate $(s * t)_1$ in terms of f_2. Let $\Delta(t) = g$ and let $\Delta(s) = f$, $a \in G(x, y)$ as above.

$$\begin{aligned}
(s * t)_1(a) &= t_1(a) + s_1 g_1(a)^{t_0(y)} \\
&= t_1(a) + s_1(t_0(x) + a + \delta t_1(a) - t_0(y))^{t_0(y)} \\
&= t_1(a) + s_1(t_0(x))^{a+\delta t_1(a)-t_0(y)} + s_1(a)^{\delta t_1(a)-t_0(y)} + s_1(\delta t_1(a))^{t_0(y)}) + s_1(-t_0(y))^{t_0(y)} \\
&= t_1(a) + s_1(t_0(x))^{a+\delta t_1(a)} + s_1(a)^{\delta t_1(a)} + s_1(\delta t_1(a)) + s_1(-t_0(y))^{t_0(y)} \\
&= t_1(a) + s_1(t_0(x)^a)^{\delta t_1(a)} - t_1(a) + s_1(a) + t_1(a) + s_1(\delta t_1(a)) - s_1(t_0(y)),
\end{aligned}$$

since $s_1(-t_0(y))^{t_0(y)} = -s_1t_0(y)$,

$$\begin{aligned} &= t_1(a) - t_1(a) + s_1(t_0(x)^a + t_1(a) - t_1(a) + s_1(a) + t_1(a) + s_1(\delta t_1(a)) - s_1(t_0(y)) \\ &= s_1(t_0(x))^a + s_1(a) + t_1(a) + s_1(\delta t_1(a)) - s_1(t_0(y)) \\ &= s_1(t_0(x))^a + s_1(a) + f_2(t_1(a))^{s_0(y)} - s_1(t_0(y)) \end{aligned}$$

Now, suppose that f_2 has inverse ${f_2}^{-1}$. Let $s^{-1} = ({s_0}^{-1}, {s_1}^{-1})$ be defined by

$${s_0}^{-1}(x) = -s_0{f_0}^{-1}(x), \quad x \in X$$

$${s_1}^{-1}(a) = {f_2}^{-1}(-s_1(a) - (s_1{s_0}^{-1}(x))^a + (s_1{s_0}^{-1}(y))^{-s_0(y)}, \quad a \in G(x,y)$$

We prove that s^{-1} is an inverse element of s and is a derivation map. Clearly

$$(s * s^{-1})_0(x) = c_0(x) \quad \text{and} \quad (s^{-1} * s)_0(x) = c_0(x)$$

by on argument as above.

Next we prove $(s * s^{-1})_1(a) = c_1(a)$, for $a \in G(x,y)$.

$$\begin{aligned} (s * s^{-1})_1(a) &= (s_1({s_0}^{-1}(x))^a + s_1(a) + f_2({f_2}^{-1}((-s_1(a) + (s_1s_0{f_0}^{-1}(x))^a \\ &\quad -(s_1s_0{f_0}^{-1}(y))^{-s_0(y)})^{s_0(y)}) - s_1{s_0}^{-1}(y) \\ &= (s_1(s_0{f_0}^{-1}(x))^a + s_1(a) - s_1(a) + (s_1s_0{f_0}^{-1}(x))^a \\ &\quad +(s_1s_0{f_0}^{-1}(y)) - s_1s_0{f_0}^{-1}(y) \\ &= c_1(a). \end{aligned}$$

Since $(s*s^{-1})_1 = c_1$ and also $(s^{-1}*s')_1 = c_1$. It follows that ${s_1}^{-1}*s_1 = {s_1}^{-1}*s_1*{s_1}^{-1}*{s_1}' = {s_1}^{-1}*(s_1*{s_1}^{-1})*{s_1}' = {s_1}^{-1}*{s_1}' = 1$ and so ${s_1}^{-1} = {s_1}'$, i.e.,

$$(s^{-1} * s)_1(a) = c_1(a), \quad \text{for all} \quad a \in G.$$

We have to prove that ${s_1}^{-1}$ is a derivation map. Let $a \in G(x,y), b \in G(y,z)$. We write $f_2{s_1}^{-1}(a) = (-s_1(a) - (s_1{s_0}^{-1}(x))^a + (s_1{s_0}^{-1}(y))^{-s_0(y)}$, and then

$$
\begin{aligned}
f_2(({s_1}^{-1}(a))^b + {s_1}^{-1}(b)) &= f_2({s_1}^{-1}(a))^b + f_2({s_1}^{-1}(b)) \\
&= (-s_1(b) + (f_2({s_1}^{-1}(a))^{s_0(y)})^b + s_1(b))^{-s_0(z)} + f_2({s_1}^{-1}(b)), \text{ by 1.1} \\
&= (-s_1(b)^{-s_0(z)} + (-s_1(a) - (s_1{s_0}^{-1}(x))^a + s_1{s_0}^{-1}(y))^{b-s_0(z)} \\
&\quad +s_1(b)^{-s_0(z)}(-s_1 b)^{-s_0 z} - (s_1{s_0}^{-1}(y)))^{b-s_0(z)} + s_1{s_0}^{-1}(z)^{-s_0(z)} \\
&= (-s_1(b)^{-s_0(z)} - s_1(a)^{b-s_0(z)} - (s_1{s_0}^{-1}(x))^{a+b-s_0(z)} + s_1{s_0}^{-1}(z))^{-s_0(z)} \\
&= (-s_1(b) - s_1(a)^b - (s_1{s_0}^{-1}(x))^{a+b} + s_1{s_0}^{-1}(z))^{-s_0(z)} \\
&= (-(s_1(a)^b + s_1(b)) - (s_1{s_0}^{-1}(x))^{a+b} + s_1{s_0}^{-1}(z))^{-s_0(z)} \\
&= f_2({s_1}^{-1}(a+b))
\end{aligned}
$$

Hence ${s_1}^{-1}$ is a derivation, i.e., s^{-1} is a free derivation. □

Theorem 1.3.6 *There is an action of $Aut(\mathcal{C})$ on $s \in FDer^*(\mathcal{C})$ given by*

$$
s^f(z) = \begin{cases} {f_1}^{-1} s_0 f_0(x), & z = x \in X \\ {f_2}^{-1} s_1 f_1(a), & z = a \in G. \end{cases}
$$

for each $a \in G(x,y)$, which makes

$$\Delta\colon FDer^*(\mathcal{C}) \to Aut(\mathcal{C})$$

$$
\Delta_s = \begin{cases} f_1(a) = s_0(x) + a + \delta s_1(a) - s_0(y), & a \in G \\ f_2(c) = (c + s_1\delta c)^{-s_0(\beta c)}, & c \in C. \end{cases}
$$

a pre-crossed module.

Proof: Now, we will show that

$$FDer^*(\mathcal{C}) \times Aut(\mathcal{C}) \to FDer^*(\mathcal{C})$$

$$(s,f)\mapsto s^f$$

is an action of $Aut(\mathcal{C})$ on $FDer^*(\mathcal{C})$.

$$s^f(z)=\begin{cases} {f_1}^{-1}s_0f_0(x), & z=x\in X\\ {f_2}^{-1}s_1f_1(a), & z=a\in G.\end{cases}$$

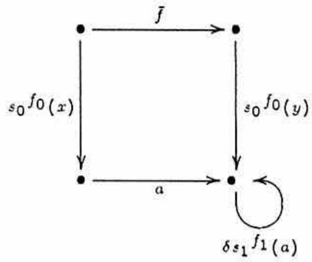
In fact this give rise to an action over groupoid:

$$s^{fg}(z)=\begin{cases} {(fg)_0}^{-1}s_0(fg)_0(x)={g_0}^{-1}{f_0}^{-1}s_0f_0g_0(x)={g_0}^{-1}{s_0}^{f_0}g_0(x)=({s_0}^{f_0})^{g_0}(x), & z=x\in X\\ {(fg)_1}^{-1}s_1(fg)_1(a)={g_1}^{-1}{f_1}^{-1}s_1f_1g_1(a)={g_1}^{-1}{s_0}^{f_1}g_1(a)=({s_1}^{f_1})^{g_1}(a), & z=a\in G\end{cases}$$

and

$$s^I(z)=\begin{cases} I^{-1}s_0I(a)=s_0(x), z=x\in X\\ I^{-1}s_1I(a)=s_1(a), z=a\in G.\end{cases}$$

Let $\Delta(s)=f$. Then $\Delta(s^f)=\bar{f}$, where $\bar{f}(a)={s_0}^{f_0}(x)+a+\delta {s_1}^f(a)-{s_0}^{f_0}(y)$. We can show s^f as the following diagram:



Is $s^f\in FDer^*(\mathcal{C})$? Clearly one can see $\beta {s_0}^{f_0}x=x$ and $\beta {s_1}^{f_1}(a)=\beta(a)$. Also we should have to show that $s^f(a+b)=s^f(a)^b+s^f(b)$. We have

$$\bar{f}(a+b)={s_0}^{f_0}(x)+(a+b)+\delta s^f(a+b)-{s_0}^{f_0}(z)$$

by definition of $\bar{f}$ and

$$\begin{aligned}\bar{f}(a)+\bar{f}(b) &= {s_0}^f(x)+a+\delta {s_1}^f(a)+b+\delta {s_1}^f(b)-{s_1}^f(z)\\ &= {s_0}^f(x)+a+b-b+\delta {s_1}^f(a)+b+\delta {s_1}^f(b)-{s_1}^f(z)\\ &= {s_0}^f(x)+a+b+\delta({s_1}^f(a)^b+{s_1}^f(b))-{s_0}^f(z)\\ &= \bar{f}(a+b)\end{aligned}$$

So $s^f(a)^b + s^f(b) = s^f(a+b)$ and also we can obtain

$$I(a) = -s_0(x) + \bar{f}(a) + s_0{}^f(y) - \delta s^f(a).$$

$$\begin{aligned}
f^{-1}\Delta s f(a) &= f^{-1}(s_0 f(x) + f(a) + \delta_s(fa) - s_0{}^f(y)) \\
&= f^{-1}s_0 f(x) + f^{-1}f(a) + f^{-1}\delta_s(fa) - f^{-1}s_0{}^f(y)) \\
&= s_0{}^f(x) + a + \delta f^{-1}s_1 f(a) - s_0{}^f(y) \\
&= s_0{}^f(x) + a + \delta s_1{}^f(a) - s_0{}^f(y) \\
&= \Delta(s^f)(a).
\end{aligned}$$

Hence $\Delta(s^f)(a) = f^{-1}\Delta s f(a)$. □

The fact that $\Delta : FDer^*(\mathcal{C}) \to Aut(\mathcal{C})$ is a precrossed module is a special case of results of Brown-Gilbert [6], which applies the monoidal closed structure of the category of crossed complexes introduced by Brown-Higgins in [11]. In fact the description of $\Delta : FDer^*(\mathcal{C}) \to Aut(\mathcal{C})$ is carried out explicitly in Brown-Gilbert [6] Proposition 3.3, but only for the case $\mathcal{C}$ is a crossed module over a group.

1.4 Braided regular crossed modules and 2-crossed modules

1.4.1 Introduction

In this section our object is to give the explicit relationship between braided regular crossed modules and 2-crossed module. This indicates a possible further context for development of work on holonomy. The following material can be found in Brown and Gilbert [6].

We begin with a review of basic facts that we need on monoidal closed categories. Let $\mathcal{C}$ be a monoidal closed category with tensor product $-\otimes-$, identity object I, and internal hom functor HOM (see [40]). Then for all objects A, B, C of $\mathcal{C}$ there exists a natural bijection

$$\theta : \mathcal{C}(A \otimes B, C) \to \mathcal{C}(A, HOM(B, C)),$$

which, together with the associativity of the tensor product, implies the existence in $\mathcal{C}$ of a natural isomorphism

$$\Theta : HOM(A \otimes B, C) \to HOM(A, HOM(B, C)).$$

Further, the bijection

$$\theta : \mathcal{C}(HOM(A, B) \otimes A, B) \to \mathcal{C}(HOM(A, B), HOM(A, B))$$

shows that there is a unique morphism

$$\epsilon_A : HOM(A, B) \otimes A \to B$$

such that $\theta(\epsilon_A)$ is the identity on $HOM(A, B)$; ϵ_A is called the *evaluation morphism*. Then for all objects A, B, C of $\mathcal{C}$, there is a morphism

$$\begin{aligned}(HOM(B, C) \otimes HOM(A, B)) \otimes A &\to HOM(B, C) \otimes (HOM(A, B) \otimes A) \\ &\to HOM(B, C) \otimes B \to C.\end{aligned}$$

This corresponds under θ to a morphism

$$\gamma_{ABC} : HOM(B, C) \otimes HOM(A, B) \to HOM(A, C)$$

which is called composition.

We write $END(C)$ for $HOM(C, C)$. There is a morphism $\eta_C : I \to END(C)$ corresponding to the morphism $\lambda : I \otimes C \to C$. The main result we need is the following [29].

Proposition 1.4.1 *The morphism η_C and the composition*

$$\mu = \gamma_{CCC} : END(C) \otimes END(C) \to END(C)$$

make $END(C)$ a monoid in $\mathcal{C}$.

1.4.2 Regular Crossed Modules and 2-crossed modules

The following definitions are due to Brown and Gilbert [6].

Let M be a monoid. A **biaction** of M on the crossed module $\mathcal{C} = (C_1, C_2, \delta)$ with point set C_0 consists of a pair of commuting left and right actions of M on the set C_0 and the groupoids C_1 and C_2 compatible with all the structure. Specifically we have functions $M \times C_i \to C_i$ and $C_i \times M \to C_i$ for $i = 0, 1, 2$, denoted by $(m, c) \mapsto m.c$ and $(c, m) \mapsto c.m$, such that

M_1 : each function $M \times C_i \to C_i$ determines a left action of M and each function $C_i \times M \to C_i$ determines a right action of M and these actions commute;

M_2: each action of M preserves the groupoid structure of C_1 over C_0 and in particular the source and target maps $\alpha, \beta : C_1 \to C_0$ are M-equivariant relative to each action;

M_3 : each action of M preserves the group operations in C_2 and if $c \in C_2(x)$ and $m \in M$ then $m.c \in C_2(m.x)$ and $c.m \in C_2(x.m)$;

M_4 : each action of M is compatible with the action of C_1 on C_2 so that if $c \in C_2(x)$, $a \in C_1(x, y)$, and $m \in M$ then

$$m.(c^a) = (m.c)^{m.a} \in C_2(m.x),$$

$$(x^a).m = (x.m)^{a.m} \in C_2(x.m);$$

M_5 : the boundary homomorphism $\delta : C_2 \to C_1$ is M-equivariant relative to each action.

The crossed module $\mathcal{C}$ is **semiregular** if the object set C_0 is a monoid and there is a biaction of C_0 on $\mathcal{C}$ in which C_0 acts on itself in its left and right regular representations. A semiregular crossed module in which C_0 is a group is said to be **regular**. Note that every crossed module of groups is regular.

Let $\mathcal{C}$ be a semiregular crossed module. We write the monoid C_0 multiplicativily with identity element e. A **braiding** on $\mathcal{C}$ is a function $C_1 \times C_1 \to C_2$, written $(a, b) \mapsto \{a, b\}$, which satisfies the following axioms (here $a, a', b, b' \in C_1, c, c' \in C_2$ and $x, y \in C_0$):

$B_1 : \{a, b\} \in C_2((\beta a)(\beta b)), \{O_e, b\} = O_{\beta b}, \{a, O_e\} = O_{\beta a};$

$B_2 : \{a, b + b'\} = \{a, b\}^{\beta a.b'} + \{a, b'\};$

$B_3 : \{a + a', b\} = \{a', b\} + \{a, b'\}^{a'.\beta b};$

$B_4 : \delta\{a, b\} = -(\beta a.b) - a.a.\alpha b + \alpha a.b + a.\beta b;$

$B_5 : \{a, \delta c'\} = -(\beta a.c') + (\alpha a.c')^{a.y}$ if $c' \in C_2(y);$

$B_6 : \{\delta c, b\} = -(c.\alpha b)^{x.b} + c.\beta b$ if $c \in C_2(p);$

$B_7 : x.\{a, b\} = \{x.a, b\},$

$\{a, b\}.x = \{a, b.x\},$

$\{a.x, b\} = \{a, x.b\}.$

Joyal and Street have defined a notion of braiding for an arbitrary monoidal category and in particular have considered *braided categorical groups*. These are equivalent to braided crossed modules, with the bracket operation in [28] given by $(a, b) \mapsto \{a^{-1}, b\}^a$. This difference is merely one of notational conventions.

The axioms $B_1, ..., B_7$ are evidently closely related to the axioms given by Conduché [19]. This relationship is given by Brown and Gilbert [6].

Recall from [19] that a **2-crossed module** consists, in the first instance, of a complex of P-groups

$$L \xrightarrow{\partial} M \xrightarrow{\partial} P$$

and P-equivariant homomorphisms, where the group P acts on itself by conjugation, such that

$$L \xrightarrow{\partial} M$$

is a crossed module, where M acts on L via P. We require that for all $l \in L, m \in M$, and $n \in P$ that $(l^m)^n = (l^n)^{m^n}$. Further, there is a function $<,>: M \times M \to L$, called a *Peiffer lifting*, which satisfies the following axioms:

$P_1 : \partial < m_0, m_1 >= {m_0}^{-1}{m_1}^{-1}m_0{m_1}^{\partial m_0},$

$P_2 : < \partial l, m >= l^{-1}l^m$,

$P_3 : < m_0, m_1m_2 >=< m_0, m_2 >< m_0, m_1 >^{{m_2}^{\partial m_0}}$

$P_4 : < m_0m_1, m_2 >=< m_0, m_2 >^{m_1}< m_1, m_2^{\partial m_0} >$,

$P_5 : < m_0, m_1 >^n =< m_0{}^n, m_1{}^n >$.

Let $2 - CrsMod$ denote the category of 2-crossed modules. Then the equivalence of Theorem 2.2 in [6], together with Conduché's equivalence [19] between the categories $2 - CrsMod$ and the category of simplicial groups with Moore complex of length 2, yields a composite equivalence between $2 - CrsMod$ and $CrsMod_{BR}$.

Let $\mathcal{C} = (C_1, C_2, \delta)$ be a regular crossed module. The 2-crossed module associated to $\mathcal{C}$ is defined to be the Moore complex of the simplicial group $S(\mathcal{C})$. Denote by K the costar in C_1 at the vertex $e \in C_0$, that is, $K = \{a \in C_1 : \beta a = e\}$. Then K is the subgroup $ker\alpha_0$ of $S(\mathcal{C})_1$ with group operation given for any $a, b \in K$ by

$$ab = b + (a.\alpha b).$$

The source map $\alpha : K \to C_0$ is a homomorphism of groups and is C_0-equivariant relative to the biaction of C_0 on C_1. Note that the new composition extends the group structure on the vertex group $C_1(e)$ so that $C_1(e)$ is a subgroup of K: it is plainly the kernel of α. Further, C_0 acts diagonally on K: for all $a \in K$ and $p \in C_0$ we set $a^p = p^{-1}.a.p$. (there should be no confusion with the given action of C_0 on C_2 which we denote in a similar way.) Then the homomorphism $\alpha : K \to C_0$ is C_0-equivariant relative to the diagonal action on K and the conjugation action of the group C_0 on itself. Now C_0 also acts diagonally on the vertex group $C_2(e)$ and so we have a complex of groups

$$C_2(e) \xrightarrow{\delta} K \xrightarrow{\alpha} C_0$$

in which δ and α are C_0-equivariant. We know that $\delta : C_2(e) \to C_1(e)$ is a crossed module: we claim that K acts on $C_2(e)$, extending the action of $C_1(e) \subseteq K$, so that $\delta : C_2 \to K$ is a crossed module.

We define an action $(c,a) \mapsto c!a$ by $c!a = (c.\alpha a)^a$ where $c \in C_2(e)$ and $a \in K$. This is indeed a group action and δ is K-equivariant. Moreover, the actions of $C_2(e)$ on itself via K and by conjugation coincide, for $\delta : C_2(e) \to C_1(e)$ is a crossed module and so for all $c, c' \in C_2(e)$,

$$c!\delta c' = (c.\alpha(\delta c'))^{\delta c'} = (c.e)^{\delta c'} = -c' + c + c'.$$

Therefore the map $\delta : C_2(e) \to K$ is a crossed module. Further the action of C_0 on $C_2(e)$ is compatible with that of K.

The final structural component of a 2-crossed module that we need is the Peiffer lifting, which is provided by the braiding. For suppose that $\mathcal{C}$ has a braiding $\{,\} : C_1 \times C_1 \to C_2$. Then the map $K \times K \to C_2(e)$ given by $(a,b) \mapsto \{a^{-1}, b\}!a =< a, b >$ is a Pieffer lifting. Therefore we have the 2-crossed module

$$C_2(e) \to K \to C_0$$

which is indeed the Moore complex of $S(\mathcal{C})$.

Then we show how a 2-crossed module give rises to a braided regular crossed module. So we begin with a 2-crossed module

$$L \xrightarrow{\partial} G \xrightarrow{\partial} P$$

and construct from it, in a functorial way, a regular, braided crossed module $\mathcal{C} = (C_1, C_2, \delta)$.

The group of object of C_0 is just the group P. The underlying set of elements of C_1 is $G \times P$ with source and target maps $\alpha(g,p) = \partial(g)p$ and $\beta(g,p) = p$. The groupoid composition in C_1 is given by $(g_1, p_1) + (g_2, p_2) = (g_1 g_2, p_2)$ if $p_1 = \partial(g_2)p_2$. The underlying set of elements of C_2 is $L \times P$ with composition $(l_1, p) + (l_2, p) = (l_1 l_2, p)$. The boundary map $\delta : C_2 \to C_1$ is given by $\delta(l,p) = (\partial l, p)$ and the action of C_1 on C_2 is given by $(l,p)^{(g,q)} = (l^g, q)$ if $p = \partial(g)q$. This does define a crossed module over (C_1, C_0) and a biaction of C_0 on $\mathcal{C}$ is obtained if we define

$$p.(g,q) = (g^{p^{-1}}, pq), (g,q).p = (g, qp),$$

$$p.(l,q) = (l^{p^{-1}}, pq), (l,q).p = (g, qp),$$

where $(g,q),(l,q) \in C_2$ and $p \in C_0 = P$ and therefore $\mathcal{C}$ with this biaction is regular. The braiding on $\mathcal{C}$ is given by

$$\{(g_1,p_1),(g_2,p_2)\} = (< {g_1}^{-1}, {g_2}^{p_1} >^{g_1}, p_1p_2)$$

where $<,>: G \times G \to L$ is the Peiffer lifting.

We do not use the notion of 2-crossed module in later work on holonomy, as the development of the local version of the theory exposed below needs further work. However we give the full theory here to show the results on automorphisms of crossed modules over groupoids corresponding to Theorem 3.4 of Brown and Gilbert [6].

1.5 f-Derivations

In this section we give an explicit description of a 2-crossed module equivalent to $AUT(\mathcal{C})$ in the case $\mathcal{C}$ is a crossed module over a groupoid.

1.5.1 END(C)

Let $\mathcal{C} = (C, G, \delta)$ be a crossed module of groupoids, regarded as a 2-truncated crossed complex with object set X. We form the crossed module $CRS(\mathcal{C})$: this is again 2-truncated and we denote it by $E : E_2 \to E_1 \to E_0$.

An explicit description of E may be extracted from [11]. The object set E_0 is just $Crs(\mathcal{C}) = End(\mathcal{C})$, the set of endomorphisms of the crossed module $\mathcal{C} = (C, G, \delta)$. We shall usually denote elements of E_0 by a single letter f and its components (f_0, f_1, f_2), where these are morphisms of X, G, C, respectively.

Now E_1 consists of all homotopies of $\mathcal{C} = (C, G, \delta)$. Such a homotopy is completely specifed by a triple (s_0, s_1, f), where $s_0 : X \to G$, $f \in E_0$, and $s_1 : G \to C$ is an f-derivation, so for all $a, b \in G$, $s_1(a + b) = s_1(a)^{f(b)} + s_1(b)$. The source and target maps are given by

$$\alpha(s_0, s_1, f) = f^0$$

and

$$\beta(s_0, s_1, f) = f$$

where f^0 is defined by

$$f^0(a) = s_0(x) + f(a) + \delta s_1(a) - s_0(y)$$

and

$$f^0(c) = (f(c) + s_1\delta(c))^{-s_0(\beta(c))}$$

$$f^0(x) = \beta(s_0(x))$$

for all $a \in G(x, y)$, $c \in C(x)$ and $x, y \in X$.

It is straightforward to check that $f^0 \in E_0$, i.e., it is a morphism $\mathcal{C} \to \mathcal{C}$.

The groupoid structure on E_1 is given by

$$(s_0, s_1, f^0) + (t_0, t_1, f) = ((s * t)_0, (s * t)_1, f)$$

where for all $a \in G$, $x \in X$.

$$(s * t)_\epsilon(z) = \begin{cases} (s * t)_0(x) = s_0(x) + t_0(x), & z = x \in X, \quad \epsilon = 0 \\ (s * t)_1(a) = t_1(a) + (s_1(a))^{t_0(\beta a)}, & z = a \in G(x, y) \quad \epsilon = 1 \end{cases}$$

An element of E_2 is a section of β. Each consists of a pair (s_2, f) where s_2 is a section and $f \in E_0$. The groupoid structure on E_2 is

$$(s_2, f) * (t_2, f) = (s_2 * t_2, f).$$

The map $\zeta : E_2 \to E_1$ is $(s_2, f) \mapsto (\delta s_2, \zeta_{s_2}, f)$ where $\zeta_{s_2}(a) = -s_2(x)^{f(a)} + s_2(y)$ for $a \in G(x, y)$. We can show that ζ_{s_2} is an f-derivation as follows:

$$\zeta_{s_2}(a + b) = (-s_2(x))^{f(a+b)} + s_2(z)$$

$$
\begin{aligned}
&= (-s_2(x))^{f(a+b)} + s_2(y)^{f(b)} - s_2(y)^{f(b)} + s_2(z) \\
&= ((-s_2(x))^{f(a)} + s_2(y))^{f(b)} - s_2(y)^{f(b)} + s_2(z) \\
&= (\zeta_{s_2}(a))^{f(b)} + \zeta_{s_2}(b)
\end{aligned}
$$

Finally, the action of E_1 on E_2 is

$$(s_2, f^0)^{(s_0,s_1,f)} = ({s_2}^{s_0}, f)$$

and, moreover,

(*i*) $\zeta((s_2, f^0)^{(s_0,s_1,f)} = -(s_0, s_1, f) + \zeta(s_2, f^0) + (s_0, s_1, f)$

(*ii*) $(s_2, f)^{\zeta(t_2,f)} = -(t_2, f) + (s_2, f) + (t_2, f)$

In fact,

$$
\begin{aligned}
\zeta((s_2, f^0)^{(s_0,s_1,f)} &= \zeta({s_2}^{s_0}, f) \\
&= (\delta({s_2}^{s_0}), \zeta_{{s_2}^{s_0}}, f)
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta_{({s_2}^{s_0})}(a) &= (-{s_2}^{s_0}(x))^{f(a)} + {s_2}^{s_0}(y), \quad a \in G(x,y) \\
&= (-s_2(x)^{s_0(x)})^{f(a)} + s_2(y)^{s_0(y)} \\
&= -s_2(x)^{s_0(x)+f(a)} + ({s_2}^{s_0})(y)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
-(s_0, s_1, f) + \zeta(s_2, f^0) + (s_0, s_1, f) &= ({s_0}^{-1}, {s_1}^{-1}, f) + (\delta(s_2), \zeta_{s_2}, f^0) + (s_0, s_1, f) \\
&= ({s_0}^{-1} * \delta(s_2) * s_0, {s_1}^{-1} * \zeta_{s_2} * s_1, f) \\
&= (\delta({s_2}^{s_0}), {s_1}^{-1} * \zeta_{s_2} * s_1, f)
\end{aligned}
$$

So we have to show that

$${s_1}^{-1} * \zeta_{s_2} * s_1 = -s_2(x)^{s_0(x)+f(a)} + ({s_2}^{s_0})(y).$$

Here

$$\begin{aligned}
\zeta(s_2, f^0) + (s_0, s_1, f) &= (\delta(s_2), \zeta_{s_2}, f^0) + (s^0, s_1, f) \\
&= (\delta(s_2) * s_0, \zeta_{s_2} * s_1, f).
\end{aligned}$$

we have

$$\zeta_{s_2} * s_1(a) = s_1(a) + \zeta_{s_2}(a) = s_1(a) + (-s_2(x)^{f^0(a)} + s_2(y))^{s_0(y)}.$$

Then

$$\begin{aligned}
{s_1}^{-1} * (\zeta_{s_2} * s_1)(a) &= (\zeta_{s_2} * s_1)(a) + ({s_1}^{-1}(a))^{(\delta(s_2)*s_0)(y)} \\
&= s_1(a) + (-s_2(x)^{f^0(a)} + s_2(y))^{s_0(y)} + ({s_1}^{-1}(a))^{(\delta(s_2)*s_0)(y)} \\
&= s_1(a) - s_2(x)^{f^0(a)+s_0(y)} + s_2(y)^{s_0(y)} + (-s_1(a)^{-s_0(y)})^{(\delta(s_2)*s_0)(y)} \\
&= s_1(a) - s_2(x)^{f^0(a)+s_0(y)} + s_2(y)^{s_0(y)} + (-s_1(a))^{-s_0(y)+(\delta(s_2))*s_0(y)} \\
&= s_1(a) - s_2(x)^{f^0(a)+s_0(y)} + s_2(y)^{s_0(y)} + (-s_1(a))^{(\delta({s_2}^{s_0})(y)} \\
&= s_1(a) - s_2(x)^{f^0(a)+s_0(y)} + s_2(y)^{s_0(y)} - s_2(y)^{s_0(y)} - s_1(a) + s_2(y)^{s_0(y)} \\
&= -s_2(x)^{f^0(a)+s_0(y)-\delta(s_1(a))} + s_2(y)^{s_0(y)} \\
&= -s_2(x)^{s_0(x)+f(a)} + {s_2}^{s_0}(y), \text{ by the definition of } f^0(a) \\
&= \zeta_{{s_2}^{s_0}}(a).
\end{aligned}$$

For the second axiom of crossed module,

$$\begin{aligned}
(s_2, f)^{\zeta(t_2, f)} &= (s_2, f)^{(\delta(t_2), \zeta_{t_2}, f)} \\
&= ({s_2}^{\delta(t_2)}, f) \\
&= ({t_2}^{-1} * s_2 * t_2, f) \\
&= ({t_2}^{-1}, f) + (s_2, f) + (t_2, f)
\end{aligned}$$

as is required. Hence $\zeta : E_2 \to E_1$ is a crossed module.

Proposition 1.5.1 *[6] The composition map $\gamma : E \otimes E \to E$ together with the map $\eta : 0 \to E$ adjoint to $\lambda : 0 \otimes (C, G, \delta) \to (C, G, \delta)$ make E a monoid in the category of crossed complexes.*

Proof: This is merely a special case of Proposition 1.4.1. □

So by Proposition 1.5.2, E is semiregular and braided. To determine the biaction of E_0 and the braiding we have to understand the composition map γ explicity. A direct calculation leads to the following non-trivial components for the bimorphism determining γ:

$$\begin{aligned}
E_0 \times E_0 \to E_0 &: (f_1, f_2) \mapsto f_1 f_2, \\
E_0 \times E_1 \to E_1 &: (f_1, (s_0, s_1, f)) \mapsto (f_1 s_0, f_1 s_1, f_1 f) \\
E_1 \times E_0 \to E_1 &: ((s_0, s_1, f), f_2) \mapsto (s_0 f_2, s_1 f_2, f f_2), \\
E_1 \times E_1 \to E_2 &: ((s_0, s_1, f), (t_0, t_1, f')) \mapsto (s_1 t_0, f f') \\
E_0 \times E_2 \to E_2 &: (f_1, (s_2, f)) \mapsto (f_1(s_2), f_1 f), \\
E_2 \times E_0 \to E_2 &: ((s_2, f), f_2) \mapsto (s_2, f f_2).
\end{aligned}$$

These maps give a biaction of E_0 on E and a braiding $E_1 \times E_1 \to E_2$. The monoid structure on E_0 is the usual composition of maps.

1.5.2 AUT(C) and 2-Crossed Modules

Let $A = AUT(\mathcal{C})$, the full subcrossed module of E on the object set $A_0 = Aut(\mathcal{C})$ of automorphisms of the crossed module $\mathcal{C} = (C, G, \delta)$. Thus A_0 is the group of units of E_0 and A inherits from E the structure of a regular, braided, crossed module [6]

Now an element of A_2 is a section over an automorphism of $\mathcal{C} = (C, G, \delta)$ and consists of a pair (s_2, f) where s_2 is a section and $f \in A_0$. An element of A_1 is a homotopy over an automorphism of $\mathcal{C} = (C, G, \delta)$ and consists of a triple (s_0, s_1, f) where s_0 is a section, $f \in A_0$, and s_1 is an f-derivation $G \to C$ such that the endomorphism f^0 of $\mathcal{C} = (C, G, \delta)$

which gives the source object of (s_0, s_1, f) is actually an automorphism. Clearly f^0 is an automorphism of $\mathcal{C} = (C, G, \delta)$ if and only if

$$g(a) = s_0(x) + f(a) + \delta s_1(a) - s_0(y)$$

$$g(c) = (f(c) + s_1\delta(c))^{-s_0(\beta(c)}$$

$$g(x) = \alpha s_0(x)$$

for all $a \in G(x, y)$, $c \in C(x)$, $x \in X$ defines an automorphism of $\mathcal{C} = (C, G, \delta)$.

For $f \in E_0$, denote by $Der_f(\mathcal{C})$ the set of f-derivations

Proposition 1.5.2 *If f is an automorphism of G then $Der_f(\mathcal{C})$ is a monoid with composition*

$$(s * t)_0(x) = s_0 f_0(x) + t_0 f^{-1}(x)$$

$$(s * t)_1(a) = t_1(a) + s_1(t_0\alpha(a) + a + f^{-1}\delta t_1(a) - t_0(\beta(a))^{t_0(\beta a)}$$

and identity element $c_1 : a \mapsto 1$, $c_0 : x \mapsto 1$, for all $a \in G$ and $x \in X$.

Proof: We defined a monoid structure on the set $FDer(\mathcal{C})$ of free derivations in the Section 2.3. Now if f is an automorphism of G and s is an f-derivation, then $sf^{-1} = (s_0 f_0{}^{-1}, s_1 f_1{}^{-1})$ is a derivation: hence we can use f to transport general composition on $FDer(\mathcal{C})$ defined in Proposition 1.3.3 to $Der_f(\mathcal{C})$ and the result is as stated. This general composition is of course recovered by taking $f = I$. □

Proposition 1.5.3 *Let f be an automorphism of the crossed module $\mathcal{C} = (C, G, \delta)$ and let s be an f-derivation. Then the following are equivalent.*

(i) s is a unit in the monoid $Der_f(\mathcal{C})$,

(ii) $g(a) = s_0(x) + f(a) + \delta s_1(a) - s_0(y)$ is an automorphism of G,

(iii) $g(c) = (f(c) + s_1\delta(c))^{-s_0(\beta(c)}$ is an automorphism of C.

Proof: For f equal to the identity automorphism of $\mathcal{C} = (C, G, \delta)$ this result has been given in Theorem 1.3.5. Now s is a unit in $Der(\mathcal{C})$ if and only if sf^{-1} is a unit in $Der(\mathcal{C})$ and by Theorem 1.3.5 this is equivalent to gf^{-1} being an automorphism of G or of C: since f is an automorphism of $\mathcal{C} = (C, G, \delta)$, this is in turn equivalent to g being an automorphism of G and of C. □

We write $Der^*{}_f(\mathcal{C})$ for the group of units of $Der_f(\mathcal{C})$ and s^{-1} for the inverse of $s \in Der^*{}_f(\mathcal{C})$. If f is the identity, we write $FDer^*(\mathcal{C})$ for $Der^*{}_f(\mathcal{C})$. An element of A_1 is now seen to consist of a triple (s_0, s_1, f) where $s_0 \in M(C)$, $f \in Aut(\mathcal{C})$, and s_1 is a derivation.

Theorem 1.5.4 *The regular crossed module $A = AUT(\mathcal{C})$ corresponds via the equivalence of Theorem 2.2 in [6] to the 2-crossed module*

$$M(C) \xrightarrow{\zeta} FDer^*(\mathcal{C}) \xrightarrow{\Delta} Aut(\mathcal{C})$$

in which $\zeta(s_2) = (\delta s_2, \zeta_{s_2})$, where $\zeta_{s_2}(a) = -s_2(x)^a + s_2(y)$, and $\Delta(s_0, s_1) = f$, where

$$f_1(a) = s_0(x) + a + \delta s_1(a) - s_0(y)$$

$$f_2(c) = (c + s_1\delta(c))^{-s_0(\beta(c))}$$

$$f_0(x) = \alpha s_0(x)$$

for $a \in G(x, y)$, $c \in C(y)$ and $x \in X$.

Proof: The costar in the groupoid A_1 at the identity automorphism I of $\mathcal{C}$ may be identified as a set with $FDer^*(\mathcal{C})$ and the group structure on the costar is given by $(s_0, s_1) * (t_0, t_1) = (s_0 * t_0, s_1 * t_1)$ where

$$\begin{aligned} (s * t)_1(a) &= t_1(a) + s_1 f_1(a)^{t_0(y)} \\ &= t_1(a) + s_1(t_0(x) + a + \delta t_1(a) - t_0(y))^{t_0(y)} \end{aligned}$$

$$\begin{aligned}
&= t_1(a) + s_1(t_0(x))^{a+\delta t_1(a)-t_0(y)} + s_1(a)^{\delta t_1(a)-t_0(y)} + s_1(\delta t_1(a))^{-t_0(y)})s_1 t_0(x) \\
&= t_1(a) + s_1(t_0(x))^{a+\delta t_1(a)} + s_1(a)^{\delta t_1(a)} + s_1(\delta t_1(a))^{t_0(y)} + s_1(-t_0(y)) \\
&= t_1(a) + s_1(t_0(x)^a)^{\delta t_1(a)} - t_1(a) + s_1(a) + t_1(a) + s_1(\delta t_1(a)) - s_1(t_0(y)) \\
&= t_1(a) - t_1(a) + s_1(t_0(x)^a + t_1(a) - t_1(a) + s_1(a) + t_1(a) + s_1(\delta t_1(a)) - s_1(t_0(y)) \\
&= s_1(t_0(x)^a + s_1(a) + t_1(a) + s_1(\delta t_1(a)) - s_1(t_0(y))
\end{aligned}$$

for $a \in G(x,y)$. The vertex group $A_2(I)$ is identified with the group $M(C)$ with $\zeta(s_2) = (\delta s_2, \zeta_{s_2})$ as required. Note that $Aut(\mathcal{C})$ acts on $FDer^*(\mathcal{C})$ by

$$(s_0, s_1)^{(f_0,f_1,f_2)} = ({f_1}^{-1} s_0 f_0, {f_2}^{-1} s_1 f_1)$$

proved in Theorem 1.3.6 and on $M(C)$ by ${s_2}^{(f_0,f_1,f_2)} = f^{-1} s_2 f$. The action of $FDer^*(\mathcal{C})$ on $M(C)$ is simply $s_2^{(s_0,s_1)} = s_2^{s_0}$ and the Peiffer lifting is given by

$$\begin{aligned}
< (s_0, s_1), (t_0, t_1) > &= \{(s_0, s_1)^{-1}, (t_0, t_1)\}!(s_0, s_1) \\
&= (\{({s_0}^{-1}, ({s_1}^{-1})^{{s_0}^{-1}}), (t_0, t_1)\}.\Delta(s_0, s_1))^{(s_0,s_1)} \\
&= ({s_1}^{-1})^{{s_0}^{-1}} (t_0)^{s_0} \\
&= {s_1}^{-1}({s_0}^{-1} * t_0 * s_0)
\end{aligned}$$

□

This concludes the description of the functor $2 - CrsMod \to CrsMod_{BR}$.

We give above the full definitions and proofs of the algebraic structure, because, we believe it will help the reader to see explicitly the algebra that is involved, and to make this work independent of the papers Brown and Higgins [11] and Brown and Gilbert [6]. In particular, this makes our work independent of the equivalence between crossed complexes and ω-groupoids which is used by Brown-Higgins in [11]: We quote from op.cit p:2 which discusses the formulae for the tensor product. "Given formulae (3.1), (3.11) and (3. 14), it is possible, in principle, to verify all the above facts within the category of crossed complexes, although the computations, with their numerous special cases, would be long. We prefer to prove these facts using the equivalent category ω-Grd of ω-groupoids where the formulae are simpler and have clearer geometric content".

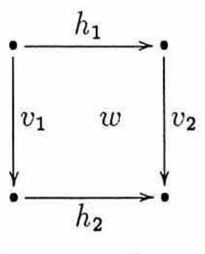
Thus we have in the above carried out a portion of this verification. For the braided part of the structure, we are however using facts from Brown and Gilbert [6]. In any case, in this thesis we will not be studying the localisation theory for $M(C)$. The extension of later theory to this 2-crossed module would be an interesting topic for further study.

Chapter 2

Special Double Groupoids and Crossed Modules

2.1 Introduction

In this chapter, we deal with double groupoids especially special double groupoids. A double groupoid is a groupoid object in the category of groupoids: that is, a double groupoid consists of a set $\mathcal{D}$ with two groupoid structures over H and V, which are themselves groupoids on the common set X, all subject to the compatibility condition that the structure maps of each structure on $\mathcal{D}$ are morphisms with respect to the other. Elements of $\mathcal{D}$ are pictured as squares



in which $v_1, v_2 \in V$ are the source and target of w with respect to the horizontal structure on $\mathcal{D}$, and $h_1, h_2 \in H$ are the source and target with respect to the vertical structure.

Double groupoids were introduced by Ehresmann in the early 1960's [22, 23], but primarily as instances of double categories, and as a part of a general exploration of categories with structure. Since that time their main use has been in homotopy theory. Brown-Higgins [8] gave the earliest example of a "higher homotopy groupoid" by associating to a pointed pair of spaces (X, A) a special double groupoid with special connection $\rho(X, A)$. In such a double groupoid, the vertical and horizontal edge structures H and V coincide. In terms of this functor ρ, [8] proved a Generalized Van Kampen Theorem, and deduced from it a Van Kampen Theorem for the second relative homotopy group $\pi_2(X, A)$, viewed as a crossed module over the fundamental group $\pi_1(A)$.

The main result of Brown-Spencer in [16] is that a special double groupoid with special connection whose double base is a singleton is entirely determined by a certain crossed module it contains; crossed modules had arisen much earlier in the work of J.H.C Whitehead [49] on 2-dimensional homotopy. This result is easily extended to give an equivalence between arbitrary special double groupoids with special connection and crossed modules over groupoids; this is included in the result of [9]. This is the result we explain in the next section as it is essential for later work.

Further, we introduce a definition of linear coadmissible section for the special double groupoid $\mathcal{D}(\mathcal{C})$. We prove that the groups of linear coadmissible sections and free invertible derivation maps are isomorphic.

2.2 Double Groupoids

In this section, we review the definition of double groupoid [17].

A double groupoid $\mathcal{D} = (D, H, V, X)$ consists of four related groupoids

$$(D, H, \alpha_1, \beta_1, +_1, 0) \quad (D, V, \alpha_0, \beta_0, +_0, 1)$$

$$(V, X, \alpha_0, \beta_0, ., e) \quad (H, X, \alpha_1, \beta_1, ., f)$$

as partially shown in the diagram

$$\begin{array}{ccc} D & \rightrightarrows & V \\ \downarrow\downarrow & & \downarrow\downarrow \\ H & \rightrightarrows & X \end{array}$$

and satisfying

(i) $\alpha_i\beta_j = \beta_j\alpha_i$, $\alpha_i\alpha_j = \alpha_j\alpha_i$, $\beta_i\beta_j = \beta_j\beta_i$ for $i, j = 0, 1$

(ii) $\alpha_0(1_a) = e_{\alpha_0(a)}$, $\beta_0(1_a) = e_{\beta_0(a)}$

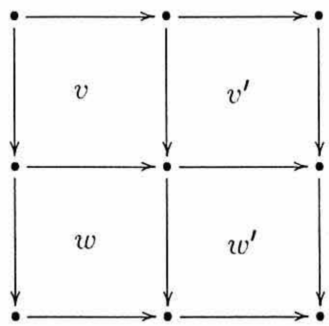
$\alpha_1(0_b) = f_{\alpha_1(b)}$, $\beta_1(0_b) = f_{\beta_1(b)}$.

(iii) $0_{e_x} = 1_{f_x}$ for $x \in X$, and this square is written O,

(iii) α_i, β_i are morphisms of groupoids for $i = 0, 1$.

(iv) (Interchange Law)

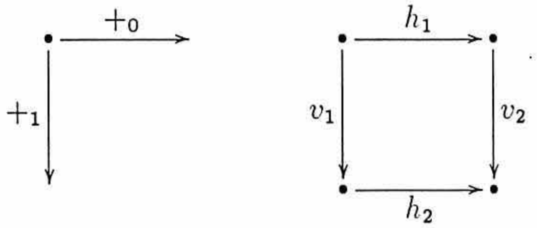
$$(v +_1 w) +_0 (v' +_1 w') = (v +_0 v') +_1 (w +_0 w')$$



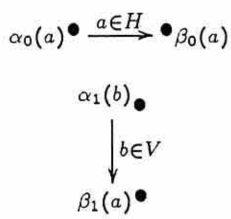
whenever $v, v', w, w' \in D$ and both sides are defined.

The element of $\mathcal{D}$ are called **squares** and the elements of H and V respectively are called horizontal and vertical edges. The elements of X are called points.

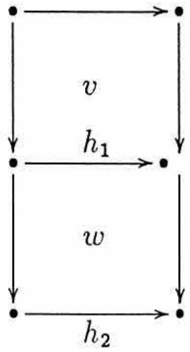
Condition (i) allows us to represent a square as having bounding edges pictured as



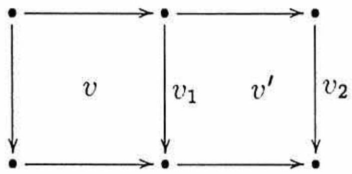
while the edges are pictured as



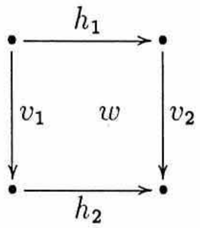
It is convenient to represent the structures $+_0, +_1$ on $\mathcal{D}$, respectively vertically and horizontally, by composition of squares as follows;



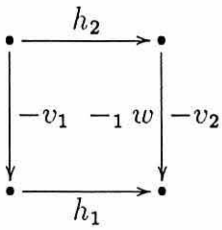
and



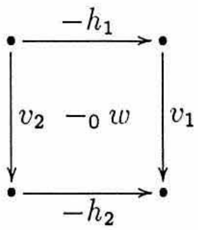
where of course $v +_1 w$ is defined if and only if $\beta_1 v = \alpha_1 w$, and $v +_0 v'$ is defined if and only if $\beta_0 v = \alpha_0 v'$. The inverse for $+_1$ on $\mathcal{D}$ is written $w \mapsto -_1 w$; the inverse for $+_0$ is written $w \mapsto -_0 w$. So if $w \in \mathcal{D}$ has faces given by



those of $-_1 w$ and $-_0 w$ are given by



and



It is convenient to use matrix notation for composition of squares. Thus if v, w satisfy $\beta_0 v = \alpha_0 w$, we write

$$\begin{bmatrix} v & w \end{bmatrix} \quad \text{for} \quad v +_0 w$$

and if $\beta_1 v = \alpha_1 u$, we write

$$\begin{bmatrix} v \\ u \end{bmatrix} \quad \text{for} \quad v +_1 u$$

More generally, we define a **subdivision** of a square w in $\mathcal{D}$ to be a rectangular array (w_{ij}), $1 \leq i \leq m, 1 \leq j \leq n$, of squares in $\mathcal{D}$ satisfying

$$\beta_0 w_{i,j-1} = \alpha_0 w_{ij}, \quad 1 \leq i \leq m, 2 \leq j \leq m$$

$$\beta_1 w_{i-1,j} = \alpha_1 w_{ij}, \quad 2 \leq i \leq m, 2 \leq j \leq n$$

such that

$$(w_{11} +_0 w_{12} +_0 \cdots +_0 w_{1n}) +_1 (w_{21} +_0 \cdots +_0 w_{2n}) +_1 \cdots +_1 (w_{m1} +_0 w_{m2} +_0 \cdots +_0 w_{mn}) = w.$$

Definition 2.2.1 A **morphism** $f : \mathcal{D} = (D, H, V, X) \to \mathcal{D}' = (D', H', V', X')$ of double groupoids consists of four functions $f_4 : D \to D', f_3 : H \to H', f_2 : V \to V', f_1 : X \to X'$ which preserve the structure.

So we have a category *DGrpd* of double groupoids.

2.3 Brown-Spencer Theorem

This section considers the relationship between crossed modules and double groupoids as given in [16].

The main result of Brown-Spencer in [16] is that a special double groupoid with special connection whose double base is a singleton is entirely determined by a certain crossed module of groups. This result is easily extended to give an equivalence between arbitrary special double groupoids with special connection and crossed modules over groupoids; this is included in the result of [9].

We here give this extended result as in [11] and [13]. The method which is used here can be found in [16].

Let $\mathcal{D} = (D, H, V, X)$ be a double groupoid. We show that $\mathcal{D}$ determines two crossed modules over groupoids.

Let $x \in X$ and let

$$H(x) = \{a \in H : \alpha_0(a) = \beta_0(a) = x\}.$$

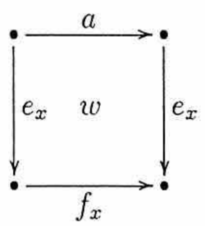
We define $V(x)$ similarly. We put

$$\Pi(D, H, x) = \{w \in D : \alpha_0 w = \beta_0(w) = e_x, \beta_1(w) = f_x\}$$

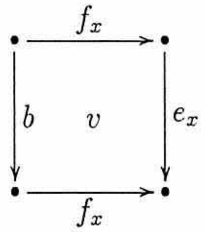
and

$$\Pi(D, V, x) = \{v \in D : \alpha_1(v) = \beta_1(v) = f_x, \beta_0(v) = e_x\}$$

so that $\Pi(D, H, x)$ and $\Pi(D, V, x)$ consist of squares with bounding edges given by



and



for some $a \in H$ and $b \in V$. $\Pi(D, H, x)$ and $\Pi(D, V, x)$ have group structures induced from $+_0$, and $+_1$. Then $\Pi(D, H) = \{\Pi(D, H, x) : x \in X\}$ and $\Pi(D, V) = \{\Pi(D, V, x) : x \in X\}$ are totally intransitive groupoids over X.

Clearly maps

$$\varepsilon : \Pi(D,H) \to H$$

and

$$\partial : \Pi(D,V) \to V$$

defined by $\varepsilon(w) = \alpha_1(w)$ and $\partial(v) = \alpha_0(v)$, respectively, are homomorphisms of groupoids.

Proposition 2.3.1 *Let $\mathcal{D} = (D, H, V, X)$ be a double groupoid then*

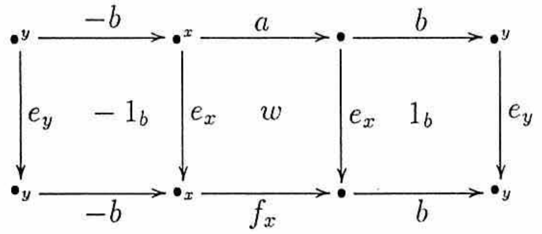
$$\gamma(D) = (\Pi(D,H), H, \varepsilon)$$

$$\gamma'(D) = (\Pi(D,V), V, \partial)$$

may be given the structure of crossed modules.

Proof: We define an action of H on $\Pi(D,H)$ as follows. Let $b \in H(x,y)$ and $w \in \Pi(D,H,x)$ and put

$$w^b = -_0 1_b +_0 w +_0 1_b$$



as in the diagram.

It easy to see that this gives an action of H on $\Pi(D,H)$. Clearly

$$\begin{aligned}
\varepsilon(w^b) &= \alpha_1(-_0 1_b +_0 w +_0 1_b), \\
&= \alpha_1(-_0 1_b) +_0 \alpha_1(w) +_0 \alpha_1(1_b), \quad \text{by linearity of } \alpha_1 \\
&= -b + \alpha_1(w) + b.
\end{aligned}$$

Suppose $v \in \Pi(D,H,x)$ and $\varepsilon(v) = b$. Then w^b and $-_0 v +_0 w +_0 v$ have common subdivision

$$\begin{bmatrix} -1_b & w & 1_b \\ -v & 0 & v \end{bmatrix}$$

and so $w^b = -v +_0 w +_0 v$.

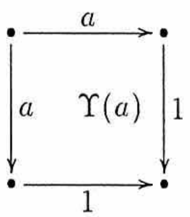
We have shown that, $\gamma(D)$ is a crossed module. A similar proof holds for $\gamma'(D)$. $\square$

Clearly γ is a functor from the category of double groupoids to the category of crossed modules.

A **special double groupoid** is a double groupoid $\mathcal{D}$ but with the extra condition that the horizontal and vertical groupoid structures H and V, on edges, coincide. These double groupoids will, from now on, be our sole concern, and for these it is convenient to denote the sets of points, edges and squares by X, G, D. The identities in G will be written 1_x or simply 1. The source and target maps $G \to X$ will be written α, β.

By a morphism $f : \mathcal{D} \to \mathcal{D}'$ of special double groupoids is meant functions $f : D \to D'$, $f : G \to G'$, $f : X \to X'$ which commute with all three groupoid structures.

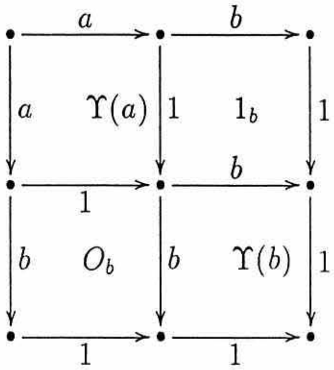
Definition 2.3.2 Let $\mathcal{D}$ be a special double groupoid. A **special connection** for $\mathcal{D}$ is a function $\Upsilon : G \to D$ such that if $a \in G$ then $\Upsilon(a)$ has boundaries given by the following diagram



Further, if $b \in G$ and $a + b$ is defined and

$$\Upsilon(a+b) = (\Upsilon(a) +_1 O_b) +_0 \Upsilon(b) \quad (*),$$

then the law $(*)$ is called **transport law** of the special connection Υ. This can be expressed as: $\Upsilon(a+b)$ is given by the diagram



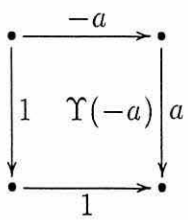
and

$$\begin{aligned}(\Upsilon(a) +_0 1_b) +_1 (O_b +_0 \Upsilon(b)) &= (\Upsilon(a) +_1 O_b) +_0 (1_b +_1 \Upsilon(b)), \quad \text{by interchange law} \\ &= (\Upsilon(a) +_1 O_b) +_0 \Upsilon(b), \\ &= \Upsilon(a+b).\end{aligned}$$

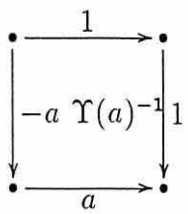
By transport law (*) we have for $x \in X$ (remembering that $1_{O_x} = O_{O_x}$ is abbreviated to O_x)

$$\Upsilon(O_x) = \Upsilon(O_x + O_x) = ((\Upsilon(O_x) +_1 O_x) +_0 \Upsilon(O_x)$$

so that $\Upsilon(O_x) = O$. Then applying transport to $\Upsilon(-a+a)$ we may obtain various identites relating $\Upsilon(-a)$ and $\Upsilon(a)^{-1}$ for example $\Upsilon(-a)$:



and $\Upsilon(a)^{-1}$:



A morphism $f : \mathcal{D} \to \mathcal{D}'$ of special double groupoid with special connections Υ, Υ' is said to preserve the connections if $f_2\Upsilon' = \Upsilon f_1$.

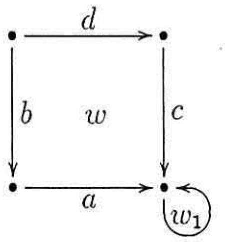
The category $DGrpd!$ has objects the pairs $(\mathcal{D}, \Upsilon)$ of a special double groupoid $\mathcal{D}$ with special connection, and arrows the morphisms of special double groupoids preserving the connection. If $(\mathcal{D}, \Upsilon)$ is an object of $DGrpd!$, then we have a crossed module $\gamma(\mathcal{D})$ by Proposition 2.3.1. Clearly γ extends to a functor from $DGrpd!$ to $CrsMod$, the category of crossed modules. The main result on double groupoids is:

Theorem 2.3.3 *The functor $\gamma : DGrpd! \to CrsMod$ is an equivalence of categories [16].*

Proof: Now, we will show how special double groupoids arise from crossed modules over groupoids.

Let $\mathcal{C} = (C, G, \delta)$ be a crossed module over groupoids with base set X. We define a special double groupoid $\mathcal{D}(\mathcal{C})$ as follows. First, $H = V = G$ with its groupoid structure, base set X. The set $\mathcal{D}(\mathcal{C})$ of squares is to consist of quintuples

$$w = \left(w_1 : b \;\begin{matrix} d \\ a \end{matrix}\; c\right)$$



such that $w_1 \in C, a, b, c, d \in G$ and

$$\delta(w_1) = -a - b + d + c$$

The source and target maps on w yield d and a, respectively, and vertical composition is

$$\left(w_1 : b \;\begin{matrix} d \\ a \end{matrix}\; c\right) +_1 \left(w_1' : b' \;\begin{matrix} a \\ a' \end{matrix}\; c'\right) = \left(w_1' + w_1^{c'} : b + b' \;\begin{matrix} d \\ a' \end{matrix}\; c + c'\right). \quad (I)$$

For the horizontal structure, the source and target maps on w yield b and c, and the composition is

$$\left(w_1 : b \begin{smallmatrix} d \\ \\ a \end{smallmatrix} c\right) +_0 \left(v_1 : c \begin{smallmatrix} e \\ \\ i \end{smallmatrix} j\right) = \left({w_1}^i + v_1 : b \begin{smallmatrix} d+e \\ \\ a+i \end{smallmatrix} j\right). \quad (II)$$

It is straightforward to check that these operations are well-defined, i.e., that with the above data

$$\delta(w_1' + {w_1}^{c'}) = -a' - b' - b + d + c + c'$$

$$\delta({w_1}^j + v_1) = -i - a - b + d + e + j$$

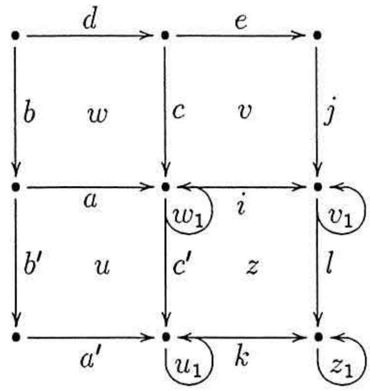
for which condition (i) of crossed module is needed. It is also easy to check that each of these operations defines a groupoid structure on $\mathcal{D}(\mathcal{C})$ with object maps,

$$a \mapsto O_a = \left(1 : a \begin{smallmatrix} 1 \\ \\ 1 \end{smallmatrix} a\right),$$

$$a \mapsto 1_a = \left(1 : 1 \begin{smallmatrix} a \\ \\ a \end{smallmatrix} 1\right)$$

for $+_0$ and $+_1$, respectively. The verification of the interchange law requires condition (ii) for a crossed module, as follows.

$$(w +_1 u) +_0 (v +_1 z) = (w +_0 v) +_1 (u +_0 z)$$



whenever $v, u, w, z \in \mathcal{D}(\mathcal{C})$ and both sides are defined.

$$\begin{aligned}
(w +_0 v) +_1 (u +_0 z) &= \left(w_1{}^i + v_1 : b \begin{matrix} d+e \\ \\ a+i \end{matrix} j\right) +_1 \left(u_1{}^k + z_1 : b' \begin{matrix} a+i \\ \\ a'+k \end{matrix} l\right) \\
&= \left(u_1{}^k + z_1 + (w_1{}^i + v_1)^l : b + b' \begin{matrix} d+e \\ \\ a'+k \end{matrix} j + l\right)
\end{aligned}$$

and

$$\begin{aligned}
(w +_1 u) +_0 (v +_1 z) &= \left(u_1 + w_1{}^{c'} : b + b' \begin{matrix} d \\ \\ a' \end{matrix} c + c'\right) +_0 \left(z_1 + v_1{}^l : c + c' \begin{matrix} e \\ \\ k \end{matrix} j + l\right) \\
&= \left((u_1 + w_1{}^{c'})^k + z_1 + v_1{}^l : b + b' \begin{matrix} d+e \\ \\ a'+k \end{matrix} j + l\right)
\end{aligned}$$

So we have to prove that

$$u_1{}^k + z_1 + (w_1{}^i + v_1)^l = (u_1 + w_1{}^{c'})^k + z_1 + v_1{}^l. \quad (1)$$

This is equivalent to

$$z_1 + w_1{}^{i+l} = w_1{}^{c'+k} + z_1.$$

But $\delta z_1 = -k - c' + i + l$ and so $i + l = c' + k + \delta z_1$. Let $x = w^{j+k}$. Then

$$z_1 + w_1{}^{(c'+k)\delta z_1} = w_1{}^{(c'+k)} + z_1. \quad (2)$$

Put $x = w_1{}^{c'+k}$, (2) is equivalent to

$$z_1 + x^{\delta z_1} = x + z_1$$

which is equivalent to

$$z_1 - z_1 + x + z_1 = x + z_1.$$

This proves that the two operations $+_0$ and $+_1$ satisfy the interchange law.

The special connection $\Upsilon : G \to \mathcal{D}(\mathcal{C})$ for $\mathcal{D}(\mathcal{C})$ is given by

$$\Upsilon(a) = \left(1 : a \begin{matrix} a \\ \\ 1 \end{matrix} 1\right)$$

Recall that by a transport law we mean that if $a, b \in G$ and $a+b$ is defined, then

$$\Upsilon(a+b) \;=\; \begin{bmatrix} \Upsilon(a) & 1_b \\ O_b & \Upsilon(b) \end{bmatrix}$$

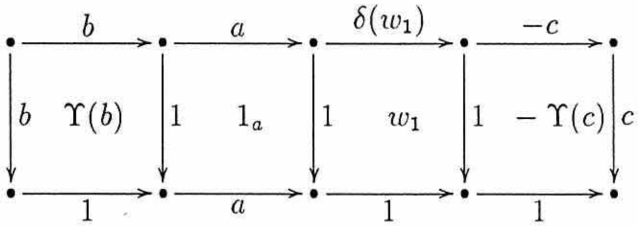
So the verification of the transport law is trivial.

This completes the description of $\mathcal{D}(\mathcal{C})$ and it is clear that Ω extends to a functor $\Omega : CrsMod \to DGrpd!$. It is immediate that $\gamma\Omega : CrsMod \to CrsMod$ is naturally equivalent to the identity. We now prove that $\Omega\gamma$ is naturally equivalent to the identity.

Let $(\mathcal{D}, \Upsilon)$ be an object of $DGrpd!$. Let $E = \Omega\gamma(D)$. Then $E_0 = D_0$, $E_1 = D_1$. We define $\phi : E \to D$ to be the identity on E_0 and E_1 and on E_2 by

$$\phi\left(w_1 : b \;\overset{d}{\underset{a}{}}\; c\right) = \Upsilon(b) +_0 1_a +_0 w_1 -_0 \Upsilon(c)$$

(for $\delta(w_1) = -a - b + d + c$) as shown in the diagram



which clearly has the correct bounding edges. Clearly ϕ is a bijection $E_2 \to D_2$, so to prove ϕ is an isomorphism it suffices to prove that ϕ preserves $+_0, +_1$ and connections.

For $+_0$ we have, by definition of w^i and using the above notation (II):

$$\begin{aligned}
\phi\left((w_1{}^i + v_1) : b \;\overset{d+e}{\underset{a+i}{}}\; j\right) &= \Upsilon(b) +_0 1_{a+i} + w_1{}^i + v_1 -_0 \Upsilon(j) \\
&= \Upsilon(b) +_0 1_a +_0 1_i -_0 1_i +_0 w_1 +_0 1_i +_0 v_1 - \Upsilon(j) \\
&= \Upsilon(b) +_0 1_a +_0 w_1 -_0 \Upsilon(c) +_0 \Upsilon(c) +_0 1_i +_0 v_1 - \Upsilon(j) \\
&= \phi\left(w_1 : b \;\overset{d}{\underset{a}{}}\; c\right) +_0 \phi\left(v_1 : c \;\overset{e}{\underset{i}{}}\; j\right)
\end{aligned}$$

For $+_1$ we have using the notation of equation (I)

$$\phi\left(w_1{}' + w_1{}^{c'} : b + b' \;\overset{d}{\underset{a'}{}}\; c + c'\right) = \Upsilon(b+b') +_0 1_{a'} +_0 (w_1{}^i + w_1{}^{c'}) -_0 \Upsilon(c + c') \quad (III)$$

while on the other hand

$$\phi\left(w_1 : b \overset{d}{\underset{a}{}} c\right) +_1 \phi\left(w_1' : b' \overset{a}{\underset{a'}{}} c'\right) = (\Upsilon(b) +_0 1_a) +_0 (w_1 -_0 \Upsilon(c)) +_0 (\Upsilon(b') +_0 1_{a'} +_0 w_1' -_0 \Upsilon(c')) \quad (VI)$$

The equality of (III) and (VI) follows from the fact that by the transport law the right hand sides of both (III) and (VI) have the common subdvision

$$\begin{bmatrix} \Upsilon(b) & 1_{b'} & -1_{b'} & 1_a & w & 1_{c'} & 1_{\alpha(w_1')} & 1_{a'} & -1_{c'} & \Upsilon(c') \\ 0_{b'} & \Upsilon(b') & -1_{b'} & 1_a & o & 1_{c'} & w_1' & 1_{a'} & \Upsilon(c') & 0_{c'} \end{bmatrix}$$

Finally ϕ preserves the connection since

$$\phi\left(O : a \overset{a}{\underset{1}{}} 1\right) = \Upsilon(a) +_0 1_1 +_0 -\Upsilon(1) = \Upsilon(a).$$

Since the naturality of ϕ in the category *DGrpd!* is clear, we have now proved that ϕ is a natural equivalence from $\Omega\gamma$ to the identity functor. This completes the proof of Theorem. □

2.4 Linear coadmissible sections

In this section, we introduce the definition of linear coadmissible section for the special double groupoid $\mathcal{D}(\mathcal{C})$ corresponding to a crossed module $\mathcal{C}$, and we prove that the group of all linear coadmissible sections and the group of coadmissible homotopies (invertible free derivations) are isomorphic.

In $\mathcal{D}(\mathcal{C})$, given $w = \left(w_1 : b \overset{d}{\underset{a}{}} c\right)$, we need only specify w_1 and three of a, b, c and d, as this determines the last side as well so for example we may write $w = \left(w_1 : b \;{}_a\; c\right)$, for such a w, where $d = b + a + \delta(w_1) - c$ and still specify the element w precisely. We use this shorthand convention below.

Definition 2.4.1 Let $\mathcal{C} = (C, G, \delta)$ be a crossed module and let $\mathcal{D}(\mathcal{C})$ be the corresponding double groupoid. A **linear coadmissible section** $\sigma = (\sigma_0, \sigma_1) : G \to \mathcal{D}(\mathcal{C})$ of $\mathcal{D}(\mathcal{C})$

written also

$$\sigma(a) = \left(\sigma_1(a) : \sigma_0\alpha(a) \;\; {}_{a} \;\; \sigma_0\beta(a)\right)$$

is a pair of maps

$$\sigma_0 : X \to G, \qquad \sigma_1 : G \to C$$

such that

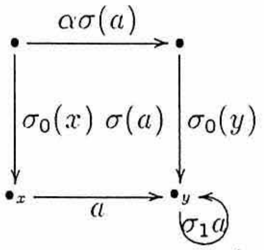
(i) if $x \in X$, $\beta\sigma_0(x) = x$, and if $a \in G$, then $\beta\sigma_1(a) = \beta a$.

(ii) if $a, b, a+b \in G$, then

$$\sigma(a+b) = \sigma(a) +_0 \sigma(b)$$

(iii) $\alpha\sigma_0 : X \to X$ is a bijection, $\alpha\sigma : G \to G$ is an automorphism.

A linear coadmissible section can be given by the following diagram



Proposition 2.4.2 *Let $\Gamma\mathcal{D}(\mathcal{C})$ denotes the set of all linear coadmissible sections. Then a group structure on $\Gamma\mathcal{D}(\mathcal{C})$ is defined by the multiplication*

$$(\sigma * \tau)_\epsilon(z) = \begin{cases} (\sigma * \tau)_0(x) = (\sigma_0\alpha\tau_0(x)) + \tau_0(x), & z = x \in X, \quad \epsilon = 0 \\ (\sigma * \tau)(a) = (\sigma\alpha\tau(a)) +_1 \tau(a), & z = a \in G(x,y) \quad \epsilon = 1 \end{cases}$$

for $\sigma, \tau \in \Gamma\mathcal{D}(\mathcal{C})$

Proof: We show that $(\sigma * \tau)$ is a linear map. i.e.,

$$(\sigma * \tau)(a+b) = (\sigma * \tau)(a) +_0 (\sigma * \tau)(b).$$

In fact,

$$\begin{aligned}(\sigma * \tau)(a+b) &= \sigma(\alpha\tau(a+b)) +_1 \tau(a+b), \quad \text{by definition } * \\ &= \sigma(\alpha\tau(a) +_0 \alpha\tau(b)) +_1 (\tau(a) +_0 \tau(b)), \quad \text{by linearity of } \alpha, \tau \\ &= (\sigma\alpha\tau(a) +_0 \sigma\alpha\tau(b)) +_1 (\tau(a) +_0 \tau(b)), \quad \text{by linearity of } \sigma \\ &= ((\sigma\alpha\tau)(a) +_1 \tau(a) +_0 (\sigma\alpha\tau(b)) +_1 \tau(b)), \quad \text{by interchange law} \\ &= (\sigma * \tau)(a) +_0 (\sigma * \tau)(b)\end{aligned}$$

An inverse element σ^{-1} of σ is defined as follows, for $a \in G, x \in X$,

$$\sigma^{-1}(a) = -\sigma((\alpha\sigma)^{-1}(a))$$

$$\sigma_0{}^{-1}(x) = -\sigma_0((\alpha\sigma_0)^{-1}(x)).$$

We have to show that σ^{-1} is linear. Let $a, b, a+b \in G$. It follows that

$$\begin{aligned}\sigma^{-1}(a+b) &= -\sigma((\alpha\sigma)^{-1}(a+b)) \\ &= -\sigma((\alpha\sigma)^{-1}(a) + (\alpha\sigma)^{-1}(b)), \quad \text{since } (\alpha\sigma)^{-1} \text{ is linear} \\ &= -\sigma(\alpha\sigma)^{-1}(a) -_0 \sigma(\alpha\sigma)^{-1}(b)) \\ &= \sigma^{-1}(a) +_0 \sigma^{-1}(b))\end{aligned}$$

Hence $\sigma^{-1} \in \Gamma\mathcal{D}(\mathcal{C})$. □

Proposition 2.4.3 *Let $\mathcal{C} = (C, G, \delta)$ be a crossed module over a groupoid and let s be an invertible free derivation with $\Delta(s) = f$. If we write*

$$\sigma_0(x) = s_0(x)$$

$$\sigma(a) = \left(s_1(a) : s_0(x) \overset{f_1(a)}{\underset{a}{}} s_0(y)\right),$$

then $\sigma = (\sigma_0, \sigma_1)$ is a linear coadmissible section of $\mathcal{D}(\mathcal{C})$.

Proof: The condition (i) and (iii) are clear from the definition of invertible free derivation.

For the linearity condition (ii), let $a \in G(x,y)$, $b \in G(y,z)$. Then

$$\begin{aligned}
\sigma(a+b) &= \left(s_1(a+b) : s_0(x) \quad \overset{f_1(a+b)}{\underset{a+b}{\quad}} \quad s_0(z) \right), \\
&= \left(s_1(a)^b + s_1(b) : s_0(x) \quad \overset{f_1(a)+f_1(b)}{\underset{a+b}{\quad}} \quad s_0(z) \right), \quad \text{by derivation map } s_1 \\
&= \left(s_1(a) : s_0(x) \quad \overset{f_1(a)}{\underset{a}{\quad}} \quad s_0(y) \right) +_0 \left(s_1(b) : s_0(y) \quad \overset{f_1(b)}{\underset{b}{\quad}} \quad s_0(z) \right), \\
&= \sigma(a) +_0 \sigma(a).
\end{aligned}$$

□

Conversely, if $\sigma = (\sigma_0, \sigma_1)$ is a linear coadmissible section of $\mathcal{D}(\mathcal{C})$ defined by

$$\sigma_0(x) = s_0(x)$$

$$\sigma(a) = \left(s_1(a) : s_0(x) \quad \overset{f_1(a)}{\underset{a}{\quad}} \quad s_0(y) \right).$$

then (s_0, s_1) is a coadmissible homotopy for the crossed module $\mathcal{C}$.

Corollary 2.4.4 *The groups of linear coadmissible sections and free invertible derivation maps are isomorphic, i.e.,*

$$FDer^*(\mathcal{C}) \cong \Gamma\mathcal{D}(\mathcal{C}).$$

Proof: Let σ be a linear coadmissible section. We define a map $\rho(\sigma) = (\sigma_0, \sigma_1)$. We have to show that $(\sigma_0, \sigma_1) \in FDer^*(\mathcal{C})$, i.e., (σ_0, σ_1) is an invertible free derivation (coadmissible homotopy).

Let $a \in G(x,y)$, we write

$$\rho\sigma(a) = (\sigma_0, \sigma_1)(a) = \left(\sigma_1(a) : \sigma_0(x) \quad \underset{a}{\quad} \quad \sigma_0(y) \right)$$

by Definition 2.4.3. Then

$$\delta\sigma_1(a) = -a - \sigma_0(x) + \alpha\sigma(a) + \sigma_0(y)$$

and

$$\alpha\sigma(a) = \sigma_0(a) + a + \delta\sigma_1(a) - \sigma_0(y)$$

is an automorphism of G. Hence $(\sigma_0, \sigma_1) \in FDer^*(\mathcal{C})$ by Proposition1.3.5.

Also we have to show that σ_1 is a derivation map, i.e.,

$$\sigma_1(a+b) = \sigma_1(a)^b + \sigma_1(b)$$

for $a \in G(x,y), b \in G(y,z)$. In fact, since $\sigma(a+b) = \sigma(a) +_0 \sigma(b)$, we have

$$\begin{aligned} \sigma(a) +_0 \sigma(b) &= \left(\sigma_1(a) : \sigma_0(x) \underset{a}{\quad} \sigma_0(y)\right) +_0 \left(\sigma_1(b) : \sigma_0(y) \underset{b}{\quad} \sigma_0(z)\right), \\ &= \left(\sigma_1(a)^b + \sigma_1(b) : \sigma_0(x) \overset{f_1(a)+f_1(b)}{\underset{a+b}{\qquad}} \sigma_0(z)\right). \end{aligned}$$

and also we have

$$\sigma(a+b) = \left(\sigma_1(a+b) : \sigma_0(x) \underset{a+b}{\qquad} \sigma_0(z)\right).$$

Hence

$$\sigma_1(a+b) = \sigma_1(a)^b + \sigma_1(b).$$

Moreover ρ is a group homomorphism, since

$$\begin{aligned} \rho(\sigma * \tau) &= ((\sigma * \tau)_0, (\sigma * \tau)_1) \\ &= (\sigma_0, \sigma_1) * (\sigma_0, \tau_1), \text{ by definition } * \\ &= \rho(\sigma) * \rho(\tau) \end{aligned}$$

Conversely, let (s_0, s_1) be an invertible free derivation (coadmissible homotopy) for a crossed module $\mathcal{C} = (C, G, \delta)$ such that $\Delta(s) = f$. We also define a map

$$\omega(s_0, s_1)(a) = s(a) = \left(s_1(a) : s_0(x) \overset{f_1(a)}{\underset{a}{\quad}} s_0(y)\right).$$

for $a \in G(x.y)$. Clearly $s(a) \in \mathcal{D}(\mathcal{C})$.

Moreover s is a linear map,i.e.,

$$s(a+b) = s(a) +_0 s(b)$$

as in Proposition 2.4.2. Hence $s \in \Gamma\mathcal{D}(\mathcal{C})$. Also we have to show that ω is a group homomorphism.

$$\begin{aligned}
\omega((s_0, s_1) * (t_0, t_1))(a) &= \omega(s_0 * t_0,\ s_1 * t_1)(a), \\
&= (s * t)(a),\ \text{by definition of } \omega \\
&= s\alpha t(a) +_1 t(a), \text{by definition of } * \\
&= \omega(s_0, s_1)(\alpha t(a)) +_1 \omega(t_0, t_1)(a), \\
&= (\omega(s_0, s_1) * \omega(t_0, t_1))(a).
\end{aligned}$$

So

$$\rho\omega(s_0, s_1) = (s_0, s_1),$$

for $(s_0, s_1) \in FDer^*(\mathcal{C})$ and

$$\omega\rho(\sigma) = \sigma,$$

for $\sigma \in \Gamma\mathcal{D}(\mathcal{C})$, i.e.,

$$FDer^*(\mathcal{C}) \cong \Gamma\mathcal{D}(\mathcal{C}).$$

□

Chapter 3

V-Locally Lie Double Groupoids

3.1 Introduction

In generalising holonomy to dimension 2, we have to show how the formal definitions and results correspond to some intuition. What we find is that the search for a formulation of a definition which works mathematically also clarifies the intuition.

Note that for the 1-dimensional case of groupoids, Pradines stated a differential version involving germs of locally Lie groupoids in [43], and formulated this in terms of adjoint functors. A version for locally topological groupoids was given in Aof-Brown [1] and the modifications for the differential case were given in Brown-Mucuk [14]. Our general aim is to consider analogous methods for the case of crossed modules and double groupoids.

The steps that are required are as follows:

(i) We need to formulate the notion of a locally Lie structure on a double groupoid $\mathcal{D}(\mathcal{C})$ that is corresponding to a crossed module $\mathcal{C} = (C, G, \delta)$ with base space X. For this reason, here (G, X) is supposed to be a Lie groupoid and that there is a smooth manifold structure on a set W such that $X \subseteq W \subseteq C$. Then $(\mathcal{D}(\mathcal{C}), W^G)$ can given as a locally Lie groupoid over G, where

$$W^G = \{w = \left(w_1 : b \overset{d}{\underset{a}{\;}} c\right) : \beta(b) = \alpha(a), \beta(a) = \beta(c) = \beta(w_1),\ d = b+a+\delta(w_1)-c, w_1 \in W\}$$

is a subset of $\mathcal{D}(\mathcal{C})$ and $a, b, c \in G$.

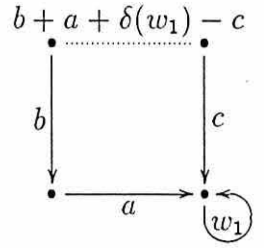
(ii) Next, we are replacing local coadmissible sections of a groupoid by local linear coadmissible sections of a special double groupoid. We define a product on the set of all local linear coadmissible sections. This easily leads to a 2-dimensional version of $\Gamma(G)$ to $\Gamma(\mathcal{D}(\mathcal{C}))$, again an inverse semigroup.

(iii) Now we form germs of $[s]_a$, where $a \in G$, $s \in \Gamma(\mathcal{D}(\mathcal{C}))$. We find this gives a groupoid $J(\mathcal{D}(\mathcal{C}))$ over G.

(iv) A key matter for decison is that of the final map ψ and its values on $[s]_a$. This is related to the question of deciding the meaning of the generalisation to dimension 2 of the term " enough local linear coadmissible sections ".

Recall that, in the groupoid case, we ask that for any $a \in G$ there is a local linear coadmissible section s such that $\beta a \in D(s)$ and $s\beta a = a$. Under certain conditions, we require s to be smooth and such that αs is a diffeomorphism of open sets. The intuition here is that $a \in G$ can be regarded as a deformation of βa, and s gives a "thickening" of this deformation.

In dimension 2, we therefore suppose given $a \in G(x, y)$ and $b \in G(z, x)$, $c \in G(w, y)$ and $w_1 \in C(y)$.



Then a local coadmissible section will be "through $w = \left(w_1 : b \;{}_{a}\; c\right)$ " if $s_0 x = b, s_0 y = c$ and $s_1 a = w_1$. Our "final map" ψ will be a morphism from $J(\mathcal{D}(\mathcal{C}))$ to a groupoid. This groupoid $\mathcal{D}(\mathcal{C})$ will be one of the groupoid structures of the double groupoid associated to the crossed module $\mathcal{C} = (C, G, \delta)$. We write

$$\psi([s]_a) = s(a) = \left(s_1(a) : s_0(x) \;\; \substack{f_1(a) \\ \\ a} \;\; s_0(y) \right),$$

so that the value of ψ on $[s]_a$ does use all the information given by $s = (s_0, s_1)$ at the arrow $a \in G$. This explains why our theory develops crossed module and double groupoids in parallel.

3.2 Local Coadmissible Homotopies

In this section, our aim is to localise the concept of the coadmissible homotopy given in Chapter 1. Note that for the 1-dimensional case, the concept of local coadmissible section is due to Ehresmann [21] and modified by Mackenzie [37].

In order to cover both the topological and differentiable cases, we use the term C^r manifold for $r \geq -1$, where the case $r = -1$ deals with the case of topological spaces and continuous maps, with no local assumptions, while the case $r \geq 0$ deals as usual with C^r manifolds and C^r maps. Of course, a C^0 map is just a continouous map. We then abbreviate C^r to *smooth*. The terms *Lie group* or *Lie groupoid* will then involve smoothness in this extended sense. By a *local diffeomorphism* $f : M \rightarrow N$ on C^r manifolds M, N we mean an injective partial function with open domain and range and such that f and f^{-1} are smooth.

One of the key differences between the cases $r = -1$ or 0 and $r \geq 1$ is that for $r \geq 1$, the pullback of C^r maps need not be a smooth submanifold of the product, and so differentiability of maps on the pullback cannot always be defined. We therefore adopt the following definition of Lie groupoid. Mackenzie [37] discusses the utility of various definitions of differential groupoid.

A **Lie groupoid** is a topological groupoid G such that

(i) the space of arrows is a smooth manifold, and the space of objects is a smooth submanifold of G,

(ii) the source and target maps α, β are smooth maps and are submersions.

(iii) the domain $G \sqcap_\beta G$ of the difference map is a smooth submanifold of $G \times G$, and

(iv) the difference map d is a smooth map.

Recall that coadmissible homotopies were defined in Chapter 1. Here we define the local version.

Definition 3.2.1 Let $\mathcal{C} = (C, G, \delta)$ be a crossed module such that (G, X) is a Lie groupoid. A **local coadmissible homotopy** $s = (s_0, s_1)$ on U_0, U_1 consists of two partial maps

$$s_0 : X \to G \quad s_1 : G \to C$$

with open domains $U_0 \subseteq X$, $U_1 \subseteq G$, say, such that $\alpha(U_1)$, $\beta(U_1) \subseteq U_0$ and

(i). If $x \in U_0$, then $\beta s_0(x) = x$.

(ii). If $a, b, a+b \in U_1$, then

$$s_1(a+b) = s_1(a)^b + s_1(b),$$

we say s_1 is a local derivation.

(iii) If $a \in U_1$ $\beta s_1(a) = \beta(a)$,

(iv) if f_0, f_1 are defined by

$f_0(x) = \alpha s_0(x), \quad x \in U_0,$

$f_1(a) = s_0\alpha(a) + a + \delta s_1(a) - s_0\beta(a), \ a \in U_1.$

then f_0, f_1 are local diffeomorphisms and $f_1{}^{-1}$, f_1 are linear.

A local coadmissible homotopy s defined as above will be denoted by $s : f \simeq I$ and, we will write $U_0 = D(s_0)$, $U_1 = D(s_1)$ and called them jointly the domains of s, this can be illustrated by the following diagram.

$$\begin{array}{ccc} C & \longrightarrow & C \\ \delta \downarrow & \overset{s_1}{\nearrow} & \downarrow \delta \\ U_1 & \longrightarrow & G \\ \downarrow & \overset{s_0}{\nearrow} & \downarrow \\ U_0 & \longrightarrow & X \end{array}$$

Suppose given open subsets $V_0 \subseteq X$ and $V_1 \subseteq G$ such that $\alpha(V_1), \beta(V_1) \subseteq V_0$. Let $t : (V_0, V_1) \to (C, G)$ be a local coadmissible homotopy on V_0, V_1 with $t : g \simeq I$. Let $s : f \simeq I$ be as above. Let $D(s*t)_0 = V_0 \cap {g_0}^{-1}(U_0)$ and $D(s*t)_1 = V_1 \cap {g_1}^{-1}(U_1) \cap \beta^{-1}(V_0)$. Now we can define a multiplication of s and t in the following way

$$(s*t)_1(a) = t_1(a) + s_1 g_1(a)^{t_0(\beta a)}, \quad a \in D(s*t)_1$$

$$(s*t)_0(x) = s_0 g_0(x) + t_0(x), \quad x \in D(s*t)_0$$

Lemma 3.2.2 *The product function $s*t$ is a local coadmissible homotopy.*

Proof:

We will prove that the domain of $s*t$ is open. In fact, if $a \in V_1$, $g_1(a) \in U_1$, $\beta(a) \in V_0$, then $a \in V_1 \cap {g_1}^{-1}(U_1) \cap \beta^{-1}(V_0)$ is an open set in G and also if $x \in V_0$ and $g_0(x) \in U_0$ then $x \in V_0 \cap g_0^{-1}(U_0)$ is open in X, so the domain of $(s*t)$ is open. One can show that

$$\beta(s*t)_0(x) = \beta(x), \text{ for } x \in D(s*t)_0$$

$$\beta(s*t)_1(a) = \beta(a), \text{ for } a \in D(s*t)_1$$

and $(s*t)_1$ is a derivation map as in Proposition 1.3.3. i.e.,

$$(s*t)_1(a+b) = (s*t)_1(a)^b + (s*t)_1(b)$$

for $a, b, a+b \in D(s*t)_1$. We define maps h_0, h_1 as follows:

$$h_0(x) = f_0 g_0(x) = \alpha(s*t)_0(x) \quad \text{for } x \in D(s*t)_0$$

$$h_1(a) = f_1 g_1(a) = (s*t)_0(\alpha a) + a + \delta(s*t)_1(a) - (s*t)_0(\beta a), \quad \text{for } a \in D(s*t)_1.$$

Since h_0, h_1 are compositions of local diffeomorphisms, they are local diffeomorphisms.

□

Proposition 3.2.3 *Let $M_L(\mathcal{C})$ denotes the set of all local coadmissible homotopies of a crossed module $\mathcal{C} = (C, G, \delta)$ such that (G, X) is a Lie groupoid. For each $s, t \in M_L(\mathcal{C})$, $s * t \in M_L(\mathcal{C})$ and for each $s \in M_L(\mathcal{C})$, let $s : f \simeq I$ and let*

$$s^{-1}(z) = \begin{cases} s_1{}^{-1}(a) = -s_1(f_1{}^{-1}(a))^{s_0{}^{-1}(y)}, & z = a \in U_1(x, y) \\ s_0{}^{-1}(x) = -s_0(f_0{}^{-1}(x)), & z = x \in U_0 \end{cases} \tag{3.1}$$

Then $s^{-1} \in M_L(\mathcal{C})$, and with this product and inverse element the set $M_L(\mathcal{C})$ of local coadmissible homotopies becomes an inverse semigroup.

Proof: The proof is very similar to that for the groupoid case given in the Appendix. □

3.3 Local linear coadmissible sections

Recall that linear coadmissible sections were defined in the previous chapter. Here we define the local version.

Definition 3.3.1 Let $\mathcal{C} = (C, G, \delta)$ be a crossed module of groupoids with (G, X) a Lie groupoid, and let $\mathcal{D}(\mathcal{C})$ be the corresponding special double groupoid.

A **local linear coadmissible section** $\sigma = (\sigma_0, \sigma_1) : G \to \mathcal{D}(\mathcal{C})$, written

$$\sigma(a) = \left(\sigma_1(a) : \sigma_0\alpha(a) \underset{a}{\quad} \sigma_0\beta(a)\right)$$

consists of two partial maps

$$\sigma_0 : X \to G \qquad \sigma_1 : G \to C$$

with open domains $U_0 \subseteq X$, $U_1 \subseteq G$, say, such that $\alpha(U_1)$, $\beta(U_1) \subseteq U_0$ and

(i). If $x \in U_0$, then $\beta\sigma_0(x) = x$, and if $a \in U_1$, then $\beta\sigma_1(a) = \beta a$.

(ii). If $a, b, a + b \in U_1$, then

$$\sigma(a + b) = \sigma(a) +_0 \sigma(b)$$

we say σ is local linear,

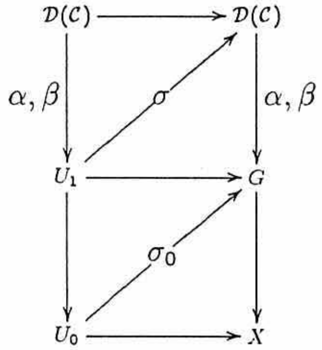
(iv) if f_0, f_1 are defined by

$$f_0(x) = \alpha\sigma_0(x), \quad x \in U_0,$$

$$f_1(a) = \alpha\sigma(a), \ a \in U_1.$$

Then f_0, f_1 are local diffeomorphisms and $f_1, {f_1}^{-1}$ are linear.

A local linear coadmissible section can be illustrated by the following diagram.



Given open subsets $V_0 \subseteq X$ and $V_1 \subseteq G$ such that $\alpha(V_1), \beta(V_1) \subseteq V_0$, let τ be a local linear section with domain V_0 and V_1. Let σ be as above and let $D(\sigma*\tau)_0 = V_0 \cap (\alpha\tau_0)^{-1}(U_0)$, $D(\sigma * \tau)_1 = a \in V_1 \cap (\alpha\tau)^{-1}(U_1) \cap \beta^{-1}(V_0)$. Now we can define a multiplication of σ and τ in the following way

$$(\sigma * \tau)(a) = \sigma(\alpha\tau)(a) + \tau(a), \quad a \in D(\sigma * \tau)$$

$$(\sigma * \tau)_0(x) = \sigma_0(\alpha\tau_0)(x) + \tau_0(x), \quad x \in D(\sigma * \tau)_0$$

Lemma 3.3.2 *The product function $\sigma * \tau$ is a local linear coadmissible section.*

Proof: The key point is to prove that the domain of $\sigma * \tau$ is open. In fact, if $a \in V_1$, $\alpha\tau(a) \in U_1$, $\beta(a) \in V_0$, then $a \in V_1 \cap (\alpha\tau)^{-1}(U_1) \cap \beta^{-1}(V_0)$ is an open set in G and also if $x \in V_0$ and $\alpha\tau_0(x) \in U_0$ then $x \in V_0 \cap (\alpha\tau_0)^{-1}(U_0)$ is open in X, so domain of $(\sigma * \tau)$ is open.

The remaining part is easily done as follows.

We show that $(\sigma * \tau)$ is a linear map as in Proposition 2.4.2. i.e.,

$$(\sigma * \tau)(a+b) = (\sigma * \tau)(a) +_0 (\sigma * \tau)(b)$$

for $a, b, a+b \in D(\sigma * \tau)$. Also we have to show that if $x \in D(\sigma * \tau)_0$

$$\beta(\sigma * \tau)_0(x) = \beta(x).$$

In fact,

$$\begin{aligned} \beta(\sigma * \tau)_0(x) &= \beta(\sigma_0 \alpha \tau_0(x) + \tau_0(x)) \\ &= \beta \tau_0(x) \\ &= x \end{aligned}$$

and if $a \in D(\sigma * \tau)$, then

$$\begin{aligned} \beta(\sigma * \tau)(a) &= \beta(\sigma \alpha \tau(a) +_1 \tau(a)) \\ &= \beta \tau(a) \\ &= a \end{aligned}$$

□

Proposition 3.3.3 *Suppose $\mathcal{C}$ is a crossed module (C, G, δ) such that (G, X) is a Lie groupoid. Let $\Gamma_L(\mathcal{D}(\mathcal{C}))$ denote the set of all local linear coadmissible sections of $\mathcal{D}(\mathcal{C})$. For each $\sigma, \tau \in \Gamma_L(\mathcal{D}(\mathcal{C}))$, $\sigma * \tau \in \Gamma_L(\mathcal{D}(\mathcal{C}))$ and for each $\sigma \in \Gamma_L(\mathcal{D}(\mathcal{C}))$, let*

$$\sigma^{-1}(z) = \begin{cases} \sigma^{-1}(a) = -\sigma(\alpha\sigma)^{-1}(a), & z = a \in U_1 \\ \sigma_0{}^{-1}(x) = -\sigma_0(\alpha\sigma_0)^{-1}(x)), & z = x \in U_0 \end{cases} \tag{3.2}$$

Then with this product and inverse operation, the set $\Gamma_L(\mathcal{D}(\mathcal{C}))$ of local linear coadmissible sections becomes an inverse semigroup.

Proof: Inverse of σ is a linear coadmissible section, because it is a composition of the linear map σ and $(\alpha\sigma)^{-1}$. $\square$

Proposition 3.3.4 *Let (s_0, s_1) be a local coadmissible homotopy for a crossed module $\mathcal{C} = (C, G, \delta)$ and let $\mathcal{D}(\mathcal{C})$ be the corresponding double groupoid. A partial map s is defined by*

$$s = (s_0, s_1) : G \to \mathcal{D}(\mathcal{C})$$

$$s'(a) = (s_0, s_1)(a) = \left(s_1(a) : s_0(x) \overset{f_1(a)}{\underset{a}{}} s_0(y) \right).$$

Then (s_0, s), shortly s' is a local linear coadmissible section.

Proof: It is easy to see from the definitions of local coadmissible sections, local coadmissible homotopies and Proposition 2.4.3 $\square$

Corollary 3.3.5 *The inverse semigroups of local coadmissible homotopies and local linear sections are isomorphic.*

Proof: Proof as in Corollary 2.4.4 $\square$

Throughout the next two chapters, we will deal with the linear coadmissible sections rather than coadmissible homotopies.

3.4 V-locally Lie Double Groupoid

Let $\mathcal{C} = (C, G, \delta)$ be a crossed module such that (G, X) is a Lie groupoid. Let $\mathcal{D}(\mathcal{C})$ be the corresponding double groupoid. Let $\Gamma(\mathcal{D}(\mathcal{C}))$ be the set of all local linear coadmissible sections and let W be a subset of C such that W has the structure of a manifold and $\beta : W \to X$ is a smooth surmersion.

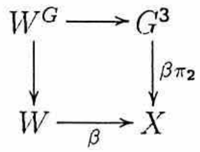
Let

$$W^G = \{w = \left(w_1 : b \overset{d}{\underset{a}{}} c\right) : \beta(b) = \alpha(a), \beta(a) = \beta(c) = \beta(w_1), d = b+a+\delta(w_1)-c, w_1 \in W\} \quad (*)$$

where $a, b, c \in G$. Here the set W^G can be considered as a repeated pullback, i.e., if

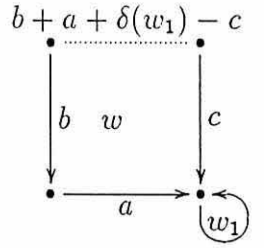
$$G^3 = \{(b, a, c) : \alpha(a) = \beta(b), \beta(a) = \beta(c)\}$$

is a pullback, then



is a pullback, so W^G has a manifold structure on it, because β and $\beta\pi_1$ are smooth and surmersions.

We can represent an element $w \in W^G$ by the following diagram:



Clearly $W^G \subseteq W \times G \times G \times G$ and $W^G \subseteq \mathcal{D}(\mathcal{C})$.

A local linear coadmissible section $(s_0, (s_0, s_1))$ as given in Proposition 3.3.4, or s' for short is said to be **W-smooth** if $Im(s_1) \subseteq W$ and s_0, s_1 are smooth. Let $\Gamma^r(W^G)$ be the set of local linear W-smooth coadmissible sections. We say that the triple (α, β, W^G) has **enough smooth local linear coadmissible sections** if for each $w = \left(w_1 : b \overset{d}{\underset{a}{}} c\right) \in W^G$, there is a local linear smooth coadmissible section $s : f \simeq I$ with domains (U_0, U_1) such that

(i). $s\beta(w) = w$, $\alpha(w) = f_1(a)$; $s_1\beta(w) = w_1 = s_1(a)$, $s_0\beta(a) = c, s_0\alpha(a) = b$.

(ii). the values of s lie in W^G

(iii). s is smooth as a pair of function $U_0 = D(s_0) \to G$ and $U_1 = D(s_1) \to W^G$.

We call such an s a **local linear smooth coadmissible section through** w.

Definition 3.4.1 Let $\mathcal{C} = (C, G, \delta)$ be a crossed module over a groupoid with base space X and let $\mathcal{D}(\mathcal{C})$ be the corresponding double groupoid. A V**-locally Lie double groupoid structure** $(\mathcal{D}(\mathcal{C}), W^G)$ on $\mathcal{D}(\mathcal{C})$ consists of a smooth structure on G, X making (G, X) a Lie groupoid and a smooth manifold W, $W \subseteq C$ such that if

$$W^G = \{w = \left(w_1 : b \overset{d}{\underset{a}{}} c\right) : \beta(b) = \alpha(a), \beta(a) = \beta(c) = \beta(w_1), d = b + a + \delta(w_1) - c, w_1 \in W\},$$

as in $*$, then

S_1). $W^G = -_1 W^G$.

S_2). $G \subseteq W^G \subseteq \mathcal{D}(\mathcal{C})$,

S_3). the set $(W^G \sqcap_\beta W^G) \cap \mathsf{d}^{-1}(W^G) = W_{\mathsf{d}}{}^G$ is open in $(W^G \sqcap_\beta W^G)$ and the restriction to $W_{\mathsf{d}}{}^G$ of the difference map

$$\mathsf{d} : \mathcal{D}(\mathcal{C}) \sqcap_\beta \mathcal{D}(\mathcal{C}) \to \mathcal{D}(\mathcal{C})$$

$$(w, v) \mapsto w -_1 v,$$

is smooth.

S_4). the restriction to W^G of the source and target maps α and β are smooth and the triple (α, β, W^G) has enough local linear smooth coadmissible sections,

S_5). W^G generates $\mathcal{D}(\mathcal{C})$ as a groupoid with respect to $+_1$.

Also one can define locally Lie crossed module structure on a crossed module by considering the above Definition 3.4.1.

Let $\mathcal{C} = (C, G, \delta)$ be a crossed module over a groupoid with base space X. A **locally Lie crossed module structure** (C, W, δ) on $\mathcal{C}$ consists of a Lie groupoid structure (G, X) and a subset W of C with a smooth structure on W such that W is G-equivariant and

C_1) (C, W) is a locally Lie groupoid,

C_2) $I(G) \subseteq W \subseteq C$,

C_3) the restriction to W of the map $\delta : C \to G$ is smooth,

C_4) the set $W_A = A^{-1}(W) \cap (W \sqcap_\beta G)$ is open in $W \sqcap_\beta G$ and the restriction to $A_W : W_A \to W$ of the action $A : C \sqcap_\beta G \to C$ is smooth.

C_5) Let

$$\underline{W} = \{(w; b, a, c) : w \in W, a, b, c \in G, \beta(a) = \beta(w), \alpha(a) = \beta(b), \beta(c) = \beta(a)\}.$$

We say that $\underline{W}$ has enough local smooth coadmissible homotopies if for all $(w; b, a, c) \in \underline{W}$ there exists a local smooth coadmissible homotopy (s_0, s_1) such that $s_1(a) = w, s_0\beta(a) = c, s_0\alpha(a) = b$.

Let us compare the above two definitions.

First of all, in the definition of locally Lie crossed module, conditions C_3 and C_4 gives rise to the difference map

$$\mathsf{d} : \mathcal{D}(\mathcal{C}) \sqcap_\beta \mathcal{D}(\mathcal{C}) \to \mathcal{D}(\mathcal{C})$$

$$(w, v) \mapsto w -_1 v,$$

which is smooth. In fact,

$$\begin{aligned}
\mathsf{d}\left(\left(w_1 : b \overset{d}{\underset{a}{\;}} c\right), \left(v_1 : b' \overset{d'}{\underset{a}{\;}} c'\right)\right) &= \left(w_1 : b \overset{d}{\underset{a}{\;}} c\right) -_1 \left(v_1 : b' \overset{d'}{\underset{a}{\;}} c'\right) \\
&= \left(w_1 : b \overset{d}{\underset{a}{\;}} c\right) +_1 \left(-v_1{}^{-c'} : -b' \overset{a}{\underset{d'}{\;}} -c'\right) \\
&= \left((-v_1')^{-c'} + w_1{}^{-c'} : b - b' \overset{d}{\underset{d'}{\;}} c - c'\right)
\end{aligned}$$

Since δ_W, $+$, A_W are smooth, d is smooth. This is equivalent to the two smooth conditions C_3 , C_4 for locally Lie crossed module, because the formulae for d involves $+$ and the action A_W.

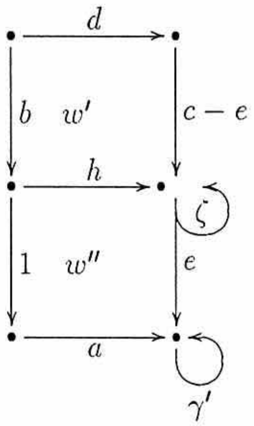
The condition C_1) that (C, W) is a locally Lie groupoid, which includes W generates C. The other equivalent condition can be stated as follows: We first prove that if W generates C and is G-equivariant, then W^G generates $\mathcal{D}(\mathcal{C})$ with respect to $+_1$.

Let $w = \left(\gamma : b \; {d \atop a} \; c\right) \in \mathcal{D}(\mathcal{C})$. We prove by induction that if γ can be expressed as a word of length n in conjugates of elements of W then w can be expressed as

$$w = w^1 +_1 \cdots +_1 w^n$$

where $w^i = \left(\gamma_i : b_i \; {d_i \atop a_i} \; c_i\right) \in W^G$, for $i = 1, ..., n$, and $\gamma_i \in W$.

This is certainly true for $n = 1$, since $w \in W^G$ if and only if $\gamma \in W$.



Suppose $\gamma = \gamma' + \zeta^e$ where γ' can be expressed as a word of length n in conjugates of elements of W and $\zeta \in W$.

Let $h = a + \delta\gamma' - e$, and let $w'' = \left(\gamma' : 1 \; {h \atop a} \; e\right)$. Then $w'' \in \mathcal{D}(\mathcal{C})$ and so w'' can be expressed as a word of length n in elements of W^G, by the inductive assumption.

Let $w' = \left(\zeta : b \; {d \atop h} \; c - e\right)$. Then $w' \in \mathcal{D}(\mathcal{C})$, since

$$\begin{aligned}
\delta\zeta &= \delta(-\gamma' + \gamma)^{-e} \\
&= -\delta(\gamma')^{-e} + \delta(\gamma)^{-e} \\
&= e - \delta\gamma' - e + e + \delta\gamma - e \\
&= e - (-a + h + e) - a - b + d + c - c \\
&= e - e - h + a - a - b + d + c - e \\
&= -h - b + d + c - e
\end{aligned}$$

and $\zeta \in W$. Clearly $w = w' +_1 w''$, and so w can be expressed as a word of length $n+1$ in conjugates of elements of W^G.

Conversely, suppose W^G generates $\mathcal{D}(\mathcal{C})$ with respect to $+_1$.

Let $\gamma \in C$ and let $w = \left(\gamma : 1 \ \begin{smallmatrix} 1 \\ \\ 1 \end{smallmatrix} \ 1\right)$. Then $w \in \mathcal{D}(\mathcal{C})$. Since W^G generates $\mathcal{D}(\mathcal{C})$, we can write

$$w = w^1 +_1 \cdots +_1 w^n$$

where $w^i = \left(\gamma_i : b_i \ {}_{a_i} \ c_i\right)$, $\gamma_i \in W$, for $i = 1, \cdots, n$ and $w^i \in W_G$. Then

$$w = \left(\gamma_1{}^{c_1} + \gamma_2{}^{c_2} \cdots + \gamma_n{}^{c_n + \cdots + c_1} : b_1 + \cdots + b_n \ \begin{smallmatrix} d \\ \\ a \end{smallmatrix} \ c_1 + \cdots + c_n\right) = \left(\gamma : 1 \ \begin{smallmatrix} 1 \\ \\ 1 \end{smallmatrix} \ 1\right)$$

We get

$$\gamma = \gamma_1{}^{c_1} + \gamma_2{}^{c_1 + c_2} \cdots + \gamma_n{}^{c_n + \cdots + c_1}, \quad \gamma_i \in W$$

So W generates C.

In the definition of V-locally Lie double groupoid, condition S_4 transfers as follows: Let (α, β, W^G) have enough local linear smooth coadmissible sections. Then for each $w = \left(\gamma : b \ {}_a \ c\right) \in W^G$ there exists a local linear smooth coadmissible section s such that $s\beta(w) = w$,. i.e., there exits (s_0, s_1) that is a local coadmissible homotopy for the crossed module $\mathcal{C} = (C, G, \delta)$. So for $(w; b, a, c)$ defined as above, there exists a local smooth coadmissible homotopy $s = (s_0, s_1)$ such that $s_1(a) = w, s_0\alpha a = b, s_0\beta a = c$.

Lemma 3.4.2 *Suppose $s, t \in \Gamma(W^G)$, $a \in G$ and $s(a) = t(a)$. Then there is a pair of neighbourhoods (U_0, U_1), where U_0 is a neighbourhood both of $\alpha(a)$ and $\beta(a)$ and U_1 is a neighbourhood of a such that the restriction of $s * t^{-1}$ to (U_0, U_1) lies in $\Gamma(W^G)$.*

Proof: Since s and t are smooth and $s(a) = t(a)$, then $(s(a), t(a)) \in W^G \sqcap_\beta W^G$. This gives rise to maps

$$(s_0, t_0) : D(s_0) \cap D(t_0) \to G \sqcap_\beta G \quad \text{and} \quad (s, t) : D(s_1) \cap D(t_1) \to W^G \sqcap_\beta W^G$$

which are smooth. But by condition (S_3) of Definition 3.4.1, $(W^G \sqcap_\beta W^G) \cap \mathsf{d}^{-1}(W^G)$ is open in $W^G \sqcap_\beta W^G$ and (G, X) is a (globally) Lie groupoid. Hence there exist open neighbourhoods U_1 of a in G, U_0 of $\alpha(a)$, $\beta(a)$ in X such that $(s,t)(U_1) \subseteq (W^G \sqcap_\beta W^G) \cap \mathsf{d}^{-1}(W^G)$ and $(s_0, t_0)(U_0) \subseteq (G \sqcap_\beta G) \cap \mathsf{d}^{-1}(G)$. Hence $\mathsf{d}(s,t)(U_1)$ is contained in W^G and $\mathsf{d}(s_0, t_0)(U_0)$ is contained in G. This gives $(s * t^{-1})(U_1) \subseteq W^G$ and $(s * t^{-1})_0(U_0) \subseteq G$. So $s * t^{-1} \in \Gamma(W^G)$. □

3.5 Germs

Let s, t be two local linear smooth coadmissible sections with domains, respectively, (U_0, U_1) and (U_0', U_1') and let $a \in U_1 \cap U_1'$. We will define an equivalence relation as follows: set $s \sim_a t$ if and only if $U_1 \cap U_1'$ contains an open neighbourhood V_1 of a such that

$$s_1 |_{V_1} = t_1 |_{V_1}, \quad s_0 |_{V_0} = t_0 |_{V_0}$$

and $\alpha(V_1), \beta(V_1) \subseteq V_0$.

Let $J_a(\mathcal{D}(\mathcal{C}))$ be the set of all equivalence classes of $\sim_a$ and let

$$J(\mathcal{D}(\mathcal{C})) = \cup\{J_a(\mathcal{D}(\mathcal{C})) : a \in G\}.$$

Each element of $J_a(\mathcal{D}(\mathcal{C}))$ is called a **germ** at a and is denoted by $[s]_a$ for $s \in \Gamma(\mathcal{D}(\mathcal{C}))$, and $J(\mathcal{D}(\mathcal{C}))$ is called the sheaf of germs of local linear smooth coadmissible sections of the double groupoid $\mathcal{D}(\mathcal{C})$.

Proposition 3.5.1 *Let $J(\mathcal{D}(\mathcal{C}))$ denote the set of all germs of local linear smooth coadmissible sections of the double groupoid $\mathcal{D}(\mathcal{C})$. Then $J(\mathcal{D}(\mathcal{C}))$ has a natural groupoid structure over G.*

Proof: Let $s, t \in \Gamma(\mathcal{D}(\mathcal{C}))$ and $s : f \simeq I$, $t : g \simeq I$. The source and target maps are

$$\alpha([s]_a) = f_1(a)$$

$$\beta([s]_a) = a$$

and the object map is $a \mapsto [c]_a$, the multiplication $*$ is

$$[s]_{g_1 a} * [t]_a = [(s * t)]_a$$

and inversion map is

$$[s]_a^{-1} = [s^{-1}]_{f_1 a}.$$

□

Remark: One can give a sheaf topology on $J(\mathcal{D}(\mathcal{C}))$ defined by taking as basis the sets $\{[s]_a : a \in U_1\}$ for $s \in \Gamma(\mathcal{D}(\mathcal{C}))$, U_1 open in G. With this topology $J(\mathcal{D}(\mathcal{C}))$ is a topological groupoid. We do not use the sheaf topology since this will not give W^G embedded as an open set.

Suppose now that $(\mathcal{D}(\mathcal{C}), W^G)$ is a V-locally Lie double groupoid. Let $\Gamma^r(W^G)$ be the subset of $\Gamma_L(\mathcal{D}(\mathcal{C}))$ consists of local linear coadmissible sections with values in W^G and which are smooth. Let $\Gamma^r(\mathcal{D}(\mathcal{C}), W^G)$ be the sub inverse semigroup of $\Gamma_L(\mathcal{D}(\mathcal{C}))$ generated by $\Gamma^r(W^G)$. Then $\Gamma^r(\mathcal{D}(\mathcal{C}), W^G)$ is again an inverse semigroup. If $s \in \Gamma^r(\mathcal{D}(\mathcal{C}), W^G)$, then there are $s^i \in \Gamma^r(W^G)$, $i = 1, \cdots, n$ such that

$$s = s^n * \cdots * s^1.$$

So let $J^r(\mathcal{D}(\mathcal{C}))$ be the subsheaf of $J(\mathcal{D}(\mathcal{C}))$ of germs of elements of $\Gamma^r(\mathcal{D}(\mathcal{C}), W^G)$. Then $J^r(\mathcal{D}(\mathcal{C}))$ is generated as a subgroupoid of $J(\mathcal{D}(\mathcal{C}))$ by the sheaf $J^r(W^G)$ of germs of element of $\Gamma^r(W^G)$. Thus an elements of $J^r(\mathcal{D}(\mathcal{C}))$ is of the form

$$[s]_a = [s^n]_{a_n} * \cdots * [s^1]_{a_1}$$

where $s = s^n * \cdots * s^1$ with $[s^i]_{a_i} \in J^r(W^G)$, $a_{i+1} = f_i(a_i), i = 1, \ldots, n$ and $a_1 = a \in D(s^1)$.

Let $\psi : J(\mathcal{D}(\mathcal{C})) \to \mathcal{D}(\mathcal{C})$ be the final map defined by

$$\psi([s]_a) = s(a) = \left(s_1(a) : s_0(x) \quad \begin{matrix} f_1(a) \\ a \end{matrix} \quad s_0(y) \right),$$

where s is a local linear coadmissible section. Then ψ is a groupoid morphism. In fact, let $s : f \simeq I$, $t : g \simeq I$, then

$$\begin{aligned} \psi([s]_{g_1(a)} * [t]_a) &= \psi([s * t]_a) \\ &= (s * t)(a) \\ &= s\alpha t(a) +_1 t(a) \\ &= s(g_1(a)) +_1 t(a) \\ &= \psi[s]_{g_1(a)} +_1 \psi[t]_a. \end{aligned}$$

Then

$$\psi(J^r\mathcal{D}(\mathcal{C})) = \mathcal{D}(\mathcal{C}),$$

from the axiom S_4 of a V-locally Lie double groupoid on $\mathcal{D}(\mathcal{C})$ in Definition 3.4.1.

Let $J_0 = J^r(W^G) \cap Ker\ \psi$, where as usual

$$Ker\ \psi = \{[s]_a : \psi[s]_a = 1_a\}$$

We will prove that J_0 is a normal subgroupoid of $J^r(\mathcal{D}(\mathcal{C}))$.

Lemma 3.5.2 *The set $J_0 = J^r(W^G) \cap Ker\ \psi$ is a wide subgroupoid of the groupoid $J^r(\mathcal{D}(\mathcal{C}))$.*

Proof: Let $a \in G$. Recall that $c : I \simeq I$ is the constant linear section. Then $[c]_a$ is the identity at a for $J^r(\mathcal{D})(\mathcal{C})$ and $[c]_a \in J_0$. So J_0 is wide in $J^r(\mathcal{D}(\mathcal{C}))$.

Let $[s]_a, [t]_a \in J_0(a, a)$, where s and t are local linear smooth coadmissible sections with $a \in D(s_1) \cap D(t_1)$ and $\alpha(a), \beta(a) \in D(s_0) \cap D(t_0)$.

Since $J_0 = J^r(W^G) \cap Ker\ \psi$, then we have that

i) $[s]_a, [t]_a \in J^r(W^G)$ and so we may assume that the images of s and t are both contained in W^G and s, t are smooth by definition of germs of local linear smooth coadmissible sections.

ii) $[s]_a, [t]_a \in Ker\ \psi$ and this implies that $\psi([s]_a) = \psi([t]_a) = 1_a \in \mathcal{D}(\mathcal{C})$ which gives $s(a) = t(a) = 1_a$ by definition of the final map.

Therefore $(s(a), t(a)) \in W^G \sqcap_\beta W^G$ and $\mathsf{d}(s(a), t(a)) = s(a) -_1 t(a) = 1_a \in W^G$ which implies that

$$(s(a), t(a)) \in (W^G \sqcap_\beta W^G) \cap \mathsf{d}^{-1}(W^G).$$

Since s and t are smooth, then the induced maps

$$(s_0, t_0) : D(s_0) \cap D(t_0) \to G \sqcap_\beta G \text{ and } (s, t) : D(s_1) \cap D(t_1) \to W^G \sqcap_\beta W^G$$

are smooth. But, by condition (S_3) of definition 3.4.1 , $(W^G \sqcap_\beta W^G) \cap \mathsf{d}^{-1}(W^G)$ is open in $W^G \sqcap_\beta W^G$. Since (G, X) is a globally Lie groupoid, there exist open neighbourhoods U_1 of a in G, U_0 of $\alpha(a)$, $\beta(a)$ in X and $\alpha(U_1), \beta(U_1) \subseteq U_0$ such that

$$(s, t)(U_1) \subseteq (W^G \sqcap_\beta W^G) \cap \mathsf{d}^{-1}(W^G)$$

$$(s_0, t_0)(U_0) \subseteq (G \sqcap_\beta G) \cap \mathsf{d}^{-1}(G)$$

which implies that $(s, t)(U_1) \subseteq \mathsf{d}^{-1}(W^G)$ and $(s_0, t_0)(U_0) \subseteq \mathsf{d}^{-1}(G)$. Thus $(s * t^{-1})(U_1) \subseteq W^G$ and $(s * t^{-1})_0(U_0) \subseteq G$, and hence $[s * t^{-1}]_a \in J^r(W^G)$. Since $s(a) = t(a)$, then $[s * t^{-1}]_a \in Ker\ \psi$. Therefore $[s * t^{-1}]_a \in J_0(a, a)$ and this completes the proof. □

Lemma 3.5.3 *The groupoid J_0 is a normal subgroupoid of the groupoid $J^r(\mathcal{D}(\mathcal{C}))$.*

Proof: Let $[k]_a \in J_0(a, a)$ and let $[s]_a \in J_0(b, a)$, $s : g \simeq I$ where k, s are local smooth coadmissible sections with $b = f_1(a)$ and $\beta k(a) = \alpha k(a) = \beta s(a) = a$. Moreover $k(a) = 1_a$. Since $J^r(\mathcal{D}(\mathcal{C}))$ is generated by $J^r(W^G)$, then

$$[s]_a = [s^n]_{a_n} * \cdots * [s^1]_{a_1}, \quad s^i \in \Gamma^r(W^G)$$

where $a_1 = a$, $a_{i+1} = f_i(a_i)$, $i = 1, \cdots, n$. $[s^i]_{a_i} \in J^r(W^G)$, where we may assume that the images of the $s^i, i = 1, \ldots, n$ are contained in W^G and are smooth.

$$\begin{aligned}
[s]_a[k]_a[s]_a^{-1} &= [s^n]_{a_n} * \cdots * [s^1]_{a_1} * [k]_a * ([s^n]_{a_n} * \cdots * [s^1]_{a_1})^{-1} \\
&= [s^n]_{a_n} * \cdots * [s^1]_{a_1} * [k]_a * [s^1]_{a_1}^{-1} * \cdots * [s^n]_{a_n}^{-1} \\
&= [s^n]_{a_n} * \cdots * [s^1]_{a_1} * [k]_a * [(s^1)^{-1}]_{f_1 a_1} * \cdots * [(s^n)^{-1}]_{f_1 a_n = b} \\
&= [s * k * s^{-1}]_b \in J_0(b, b).
\end{aligned}$$

In fact, now, since $k^{-1}(a) = -k(I^{-1}(a)) = -k(a)$, then $k^{-1}(a) = -k(a)$. But $k(a) = 1_a$, by definition of J_0; hence $k^{-1}(a) = 1_a \in -_1W^G$.

Since, by condition S_1 of definition 3.4.1, $W^G = -_1W^G$, then $k(a) \in W^G$. Since $[s]_a \in J_0(b, a)$, then we may assume that the image of s is contained in W^G and s is a local linear smooth coadmissible section. So $s(a) \in W^G$, and therefore

$$(s(a), -_1k(a)) \in W^G \sqcap_\beta W^G, \quad (s_0(x), -k_0(x)) \in G \sqcap_\beta G$$

and $\mathsf{d}(s(a), -_1k(a)) = s(a) +_1 k(a) = (s * k)(a) = s(a)$. Also $\mathsf{d}(s_0(x), -k_0(x)) = s_0(x) + k_0(x) = (s_0 * k_0)(x) = s_0(x)$. Hence $(s(a), -_1k(a)) \in {W^G}_{\mathsf{d}}$ and $(s_0(x), -k_0(x)) \in G_{\mathsf{d}}$, for $y \in X$, By the smoothness of k^{-1} and s, induced maps

$$(s, k^{-1}) : D(s_1) \cap D({k_1}^{-1}) \to W^G \sqcap_\beta W^G \text{ and } (s_0, {k_0}^{-1}) : D(s_0 \cap D({k_0}^{-1}) \to G \sqcap_\beta G$$

are smooth. Hence there exists a pair of open neighbourhoods (U_0, U_1) where $\alpha(a), \beta(a) \in U_0$ in X, and $a \in U_1$ in G such that

$$(s, k^{-1})(U_1) \subseteq {W^G}_\delta, \quad (s_0, {k_0}^{-1})(U_0) \subseteq G_\delta$$

$$(s(U_1) -_1 k^{-1}(U_1)) \subseteq W^G, \quad (s_0(U_0) - {k_0}^{-1}(U_0)) \subseteq G.$$

Therefore $[s * k]_a \in J_0(W^G)$.

Thus we may assume that the image of $s * k$ is contained in W^G and $s * k$ is a local linear smooth coadmissible section. Since $\beta(s * k)(a) = \beta s(a) = \beta k(a) = a$ and $(s * k)(a) = s(a)$.

Then $((s*k)(a), s(a)) \in W^G \sqcap_\beta W^G$ and so $\mathsf{d}((s*k)(a), s(a)) = (k*s) -_1 s(a) = 1_a \in W^G$. and $((s*k)(a), s(a)) \in W_{\mathsf{d}}{}^G$. Similarly $a \in G(x,y)$, for $x, y \in X$, $((s*k)_0(x), s_0(x)) \in G_{\mathsf{d}}$. Since s and $s*k$ are smooth, then they induce a smooth map

$$((s*k), s) : D(k_1) \cap D(s_1) \to W^G \sqcap_\beta W^G$$

$$((s*k)_0, s_0) : D(k_0) \cap D(s_0) \to G \sqcap_\beta G.$$

But $W^G{}_{\mathsf{d}}$ and G_{d} are open in $W^G \sqcap_\beta W^G$ and $G \sqcap_\beta G$, respectively. Hence there exists a pair of neighbourhoods (U'_0, U'_1) of $\alpha(a), \beta(a) \in U'_0$ in X and $a \in U'_1$ in G such that

$$((s*k), s)(U'_1) \subseteq W^G \sqcap_\beta W^G \quad ((s*k)_0, s_0)(U'_0) \subseteq G \sqcap_\beta G$$

which implies that

$$(s*k)(U'_1) -_1 s(U'_1) \subseteq W^G \text{ and } (s*k)_0(U'_0) - s_0(U'_0) \subseteq G.$$

Therefore $[s*k]_a * [s]_a{}^{-1} = [s*k]_a * [s^{-1}]_{g_1(a)=b} = [s*k*s^{-1}]_b \in J(b,b)$. But $[s*k*s^{-1}]_b \in (Ker\phi)(b,b)$, since $(s*k*s^{-1})(b) = 1_b$. Hence $[s*k*s^{-1}]_b \in J_0(b,b)$ and so J_0 is a normal subgroupoid of $J^r(\mathcal{D}(\mathcal{C}))$. □

We define the quotient groupoid

$$Hol(\mathcal{D}(\mathcal{C}), W^G) = J^r(\mathcal{D}(\mathcal{C}))/J_0$$

and call this the **holonomy groupoid** of the V-locally Lie double groupoid $(\mathcal{D}(\mathcal{C}), W^G)$ on $\mathcal{D}(\mathcal{C})$. Let $p : J^r(\mathcal{D}(\mathcal{C})) \to Hol(\mathcal{D}(\mathcal{C}), W^G)$ be the quotient morphism, and write $< s >_a$ for $p[s]_a$. Then the final map $\psi : J(\mathcal{D}(\mathcal{C})) \to \mathcal{D}(\mathcal{C})$ induces a surjective morphism $\phi : Hol(\mathcal{D}(\mathcal{C}), W^G) \to \mathcal{D}(\mathcal{C})$ such that $\phi(< s >_a) = s(a)$.

Chapter 4

The Holonomy Groupoid of $(\mathbf{D(C)}, \mathbf{W^G})$

4.1 Introduction

In this chapter, we deal with some local Lie structures on a special double groupoid $\mathcal{D}(\mathcal{C})$ corresponding to a crossed module $\mathcal{C} = (C, G, \delta)$ –namely such a local Lie structure is a Lie groupoid structure on the groupoid (G, X) of $\mathcal{D}(\mathcal{C})$, and a manifold structure on a certain subset W^G of the set of squares, satisfing certain conditions. This Lie groupoid $Hol(\mathcal{D}(\mathcal{C}), W^G)$ is called the holonomy groupoid of the V-locally the Lie double groupoid $(\mathcal{D}(\mathcal{C}), W^G)$. Further, we state a universal property of Lie groupoid $Hol(\mathcal{D}(\mathcal{C}), W^G)$ in Theorem 4.2.8.

4.2 Lie Crossed Modules and Double Lie groupoid

We devote this section to a brief survey of Lie crossed modules and Double Lie groupoids.We state a part of a Lie version of Brown-Spencer Theorem given in Brown-Mackenzie [13].

It is reasonable to recall the definition of Lie groupoid in this section. A *Lie groupoid* (G, X) is a topological groupoid such that

(i) the space G of arrows is a smooth manifold, and the space X of objects is a smooth submanifold of G,

(ii) the source and target maps α, β are smooth maps and are submersions,

(iii) the domain

$$G \sqcap_\beta G = \{(a,b) \in G \times G : \beta a = \beta b\}$$

of the difference map

$$\mathsf{d} : G \sqcap_\beta G \to G$$

$$(a,b) \mapsto a - b$$

is a smooth submanifold of $G \times G$, and

(iv) the difference map d is a smooth map.

Moreover, the anchor map, i.e., the map

$$[,] : G \to X \times X$$

$$a \mapsto (\alpha a, \beta a)$$

is a Lie groupoid morphism of G to the coarse groupoid $X \times X$. Similiarly, the manifold $G \sqcap_\beta G$ is a wide subgroupoid of the coarse groupoid $(G \times G, G)$, and the difference map

$$\mathsf{d} : G \sqcap_\beta G \to G$$

is a Lie groupoid morphism over $\beta : G \to X$.

4.2.1 Lie Crossed Module

Definition 4.2.1 Let G, C be two Lie groupoids over the same object set and let C be totally intransitive. Then a **Lie action** of G on C is given by a partially defined smooth function, written $(c, a) \mapsto c^a$, which satisfies

1. c^a is defined if and only if $\beta(c) = \alpha(a)$, and then $\beta(c^a) = \beta(a)$, where α, β are respectively the source and target maps of the groupoid G.

2. $(c_1 + c_2)^a = {c_1}^a + {c_2}^a$ and $(e_x)^a = e_y$

3. ${c_1}^{a_1+a_2} = {c_1}^{(a_1)^{a_2}}$ and ${c_1}^{e_x} = c_1$

for all $c_1, c_2 \in C(x,x)$, $a_1 \in G(x,y)$, $a_2 \in G(y,z)$.

Further, the domain

$$C \sqcap G = \{(c,a) : \beta c = \alpha a\}$$

of the action

$$C \sqcap G \to C$$

is a smooth manifold. Image of C in G is not required to be closed, see [38, 13].

Definition 4.2.2 A **Lie crossed module of groupoids** consists of a pair of Lie groupoids C, G over a common object set with a Lie action of G on C, together with a smooth functor $\delta : C \to G$ which is the identity on the object set and satisfies

1. $\delta(c^a) = -a + \delta c + a$
2. $c^{\delta c_1} = -c_1 + c + c_1$

for $c, c_1 \in C(x,x)$, $a \in G(x,y)$.

Note that $\beta\delta(c) = \beta(c) = \alpha(c)$, since δ is a functor over the identity and $\alpha = \beta$ on C. Image of C in G is not required to be closed, see [38, 13].

Example 4.2.3 Every Lie groupoid G give rise to a Lie crossed module over groupoids, with G acting on its inner group bundle. In fact, let G be a Lie groupoid over X and let IG be the inner group bundle of G, i.e., $IG = \cup_{x\in X} G(x)$. Then clearly IG is a Lie subgroupoid of G. The inclusion map $i : IG \to G$ is a smooth homomorphism, and G acts on IG smoothly by conjugation:

$$IG \sqcap G \to IG$$

$$(c,a) \mapsto c^a = -a + c + a.$$

Hence $\mathcal{C} = (IG, G, \delta)$ is a Lie crossed module over a groupoid.

A **Lie crossed module of groups** is a Lie crossed module of groupoids as above in which C, G are Lie groups. Examples given in Chapter 1 can be stated as examples of Lie crossed modules of groups.

Also we can define a category of Lie crossed modules of groupoids. Let $\mathcal{C} = (C, G, \delta)$, $\mathcal{C}' = (C', G, \delta')$ be two Lie crossed modules with same base space X. A morphism $f : \mathcal{C} \to \mathcal{C}'$ consists of a pair of smooth Lie groupoid homomorphism (f_1, f_2) such that the following diagrams commute:

$$\begin{array}{ccc} C \sqcap G & \longrightarrow & C \\ {\scriptstyle f\times I}\downarrow & & \downarrow{\scriptstyle f} \\ C' \sqcap G & \longrightarrow & C' \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\ f\ } & C' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

The monoid of all morphisms from a crossed module (C, G, δ) to itself is called the endomorphism monoid of (C, G, δ), and denoted by $End(C, G, \delta)$. Its maximal subgroup is the group $Aut(C, G, \delta)$ of automorphisms of $\mathcal{C} = (C, G, \delta)$.

4.2.2 Double Lie Groupoid

In differential geometry, double Lie groupoids, but usually with one of the structure totally intransitive, have been considered in passing by Pradines [44, 45]. In general, double Lie groupoids were investigated by K.Mackenzie in [39] and Brown and Mackenzie [13].

Recall that a **double groupoid** consists of a quadruple of sets (D, H, V, X), together with groupoid structures on H and V, both with base X, and two groupoid structure on D, a horizontal with base V, and a vertical structure with base H, such that the structure maps (source, target, difference map, and identity maps) of each structure on D are morphisms with respect to the other.

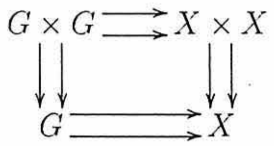
$$\begin{array}{ccc} D & \rightrightarrows & V \\ \downarrow\downarrow & & \downarrow\downarrow \\ H & \rightrightarrows & X \end{array}$$

Definition 4.2.4 A **double Lie groupoid** is a double groupoid $\mathcal{D} = (D; H, V, X)$ together with differentiable structures on D, H, V and X, such that all four groupoid structures are Lie groupoids and such that the double source map $D \to H \times_\alpha V = \{(h,v) : \alpha_h(h) = \alpha_v(v)\}$, $d \to (\alpha_V(d), \alpha_H(d))$ is a surjective submersion, where α_V, α_H are source and target maps on D.

A **morphism of double Lie groupoids** $(\Phi, \Phi_H, \Phi_V, \Phi_X) : (D'; H', V' : X') \to (D; H, V : X)$ is a quadruple of smooth maps, $\Phi : D' \to D$, $\Phi_H : H' \to H$, $\Phi_V : V' \to V$, $\Phi_X : X' \to X$ such that (Φ, Φ_H), (Φ, Φ_V), (Φ_H, Φ_X), (Φ_V, Φ_X) are morphism of their respective groupoids.

We give two examples which are found in Brown-Mackenzie [13]. Later, these will be used in the proof of Theorem 4.2.7.

Example 4.2.5 For any manifold X, the product manifold $X \times X$ has a natural Lie groupoid structure, where (x,y) has source x, target y, and the composition is $(x,y)(z,u) = (x,u)$, defined if $y = z$. This is known as the *pair* or *coarse groupoid* on X. If (G, X) is a Lie groupoid, then $G \times G$ can be considered both as the Cartesian product groupoid on base $X \times X$, and as the pair groupoid on base G. These two structures constitute a double Lie groupoid.



Given any double Lie groupoid $\mathcal{D} = (D, H, V, X)$, the anchor $[,]_V : D \to H \times H$ together with $id : H \to H$, $[,]_v : V \to X \times X$, $id : X \to X$ is a morphism of double groupoids $\mathcal{D} = (D, H, V, X) \to (H \times H, H, X \times X, X)$. Similarly, the vertical morphism is $\mathcal{D} = (D, H, V, X) \to (V \times V, V, X \times X, X)$.

Example 4.2.6 Let H and V be Lie groupoids on the same base X, and suppose that the two anchors $[,]_h : H \to X \times X$ and $[,]_v : V \to X \times X$ are transversal as smooth maps;

that is , the tangent bundle of $X \times X$ is generated, at each point, by the images of the tangent maps to $[,]_h$ and $[,]_v$. Then the pullback of

$$\begin{array}{ccc} & & V \times V \\ & & \downarrow{\scriptstyle [,]_v \times [,]_v} \\ H \times H & \xrightarrow[{[,]_h \times [,]_h}]{} & X^4 \end{array}$$

may be regarded as defining either the pullback groupoid $[,]_h^{**}(V \times V)$ on base H or the pullback groupoid $[,]_v^{**}(H \times H)$ on V. These two structures constitute a double Lie groupoid which is denoted by $\square(H, V)$, and whose elements are squares

$$\begin{array}{ccc} \bullet & \xrightarrow{h_1} & \bullet \\ {\scriptstyle v_1}\downarrow & w & \downarrow{\scriptstyle v_2} \\ \bullet & \xrightarrow[h_2]{} & \bullet \end{array}$$

with $h_1, h_2 \in H$, $v_1, v_2 \in V$ and source and targets matching as shown. If $H = V$ we write $\square H$ for $\square(H, H)$. Taking $H = X \times X$, the pair groupoid on X, we obtain the double groupoid (X^4, X^2, X^2, X) in which all four groupoid structures are pair groupoids.

Theorem 4.2.7 *[16] Let $\mathcal{C} = (C, G, \delta)$ be a Lie crossed module with base space X and let the maps $k : [,]^{**}(G \times G) \to G \ltimes IG$ and $id \ltimes \delta : G \ltimes C \to G \ltimes IG$ are transversal (see [13], p.29). Then the corresponding special double groupoid $\mathcal{D}(\mathcal{C})$ is a double Lie groupoid.*

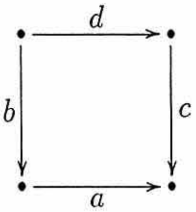
Proof: Let $\mathcal{C} = (C, G, \delta)$ be a fixed Lie crossed module. Let $IG = \cup_{x \in X} G_x$ be the inner group bundle of G (sometimes called the gauge group bundle). Form the semi-direct product group $G \ltimes IG$ on base X; this consists of all pairs (a, c) with $\beta(a) = \beta(c)$, and composition

$$(a, c_1) + (b, c_2) = (a + b, {c_1}^b + c_2)$$

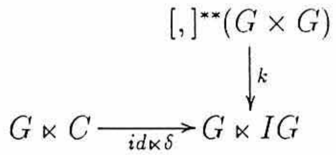
defined if $\beta(a) = \alpha(b)$. Next, form the pullback Lie groupoid $[,]^{**}(G \times G)$ of the Cartesian square groupoid over its own anchor; this admits the double groupoid structure $\square G$ as given in Example 4.2.6 but we are here considering it merely as an ordinary groupoid. Define a map

$$k : [,]^{**}(G \times G) \to G \ltimes IG, \quad (b, a, c, d) \mapsto (a, -a - b + d + c)$$

where a, b, c, d are arranged as



with our usual orientation; in particular, d is the source and a the target. Now k is a regular fibration over $\alpha : G \to X$, and δ is base-preserving, so we can take the pullback in the category of Lie groupoids of the diagram



We obtain a groupoid $\mathcal{D}(\mathcal{C})$ whose element are 5-tuples (w, b, a, c, d) such that $(b, a, c, d) \in [,]^{**}(G \times G)$ and $w \in C$ with $\beta(w) = \beta(a)$ and $\delta(w) = -a - b + d + c$. To keep the notation clear, we rewrite (w, b, a, c, d) as

$$\left(w : b \begin{matrix} d \\ \\ a \end{matrix} c \right)$$

The source and target of this element are d and a, respectively, and the composition is defined in Chapter 2. Now $\mathcal{D}(\mathcal{C})$ becomes a double groupoid by defining a horizontal structure as in Chapter 2. □

We now start with a statement of the theorem, the proof of which then occupies this and the next two sections.

Theorem 4.2.8 *Let $\mathcal{C} = (C, G, \delta)$ be a crossed module and let $\mathcal{D}(\mathcal{C})$ be the corresponding double groupoid. Let $(\mathcal{D}(\mathcal{C}), W^G)$ be a V-locally Lie double groupoid for the double groupoid*

$\mathcal{D}(\mathcal{C})$. Then there is a Lie groupoid $Hol(\mathcal{D}(\mathcal{C}), W^G)$, a morphism

$$\psi : Hol(\mathcal{D}(\mathcal{C}), W^G) \to \mathcal{D}(\mathcal{C})$$

of groupoids, and an embedding

$$i : W^G \to Hol(\mathcal{D}(\mathcal{C}), W^G)$$

of W^G to an open neighbourhood of $Ob(Hol(\mathcal{D}(\mathcal{C}), W^G)) = G$, such that

(i) ψ is the identity on G, $\psi i = I_{W^G}$, $\psi^{-1} \mid_{(W^G)}$ is open in $Hol(\mathcal{D}(\mathcal{C}), W^G)$, and the restriction $\psi_{W^G} : \psi^{-1}(W^G) \to W^G$ of ψ is smooth.

(ii) if $\mathcal{A} = (A, B, \delta')$ is a Lie crossed module with object set X and $\mu : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{C})$ is a morphism of groupoids such that

(a) μ is the identity on objects;

(b) the restriction $\mu_{W^G} : \mu^{-1}(W^G) \to W^G$ is smooth and $\mu^{-1}(W^G)$ is open in $\mathcal{D}(\mathcal{A})$ and generates $\mathcal{D}(\mathcal{A})$ as a groupoid.

(c) the triple $(\alpha, \beta, \mathcal{D}(\mathcal{A}))$ has enough local linear smooth coadmissible sections;

then there is a unique morphism $\mu' : (\mathcal{D}(\mathcal{A}), B, +_1) \to Hol(\mathcal{D}(\mathcal{C}), W^G)$ of Lie groupoids such that $\psi\mu' = \mu$ and $\mu'(w) = (i\mu)(w)$ for $w \in \mu^{-1}(W^G)$.

Lemma 4.2.9 *Let $w \in W^G$, and let s and t be local linear smooth coadmissible sections through w. Let $a = \beta w$. Then $< s >_a = < t >_a$ in $Hol(\mathcal{D}(\mathcal{C}), W^G)$.*

Proof: By assumption $sa = ta = w$. Let $b = \alpha w$. Without loss of generality, we may assume that s and t have the same domain (U_0, U_1) and have image contained in W^G and G, respectively. By Lemma 3.4.2, $s * t^{-1} \in \Gamma(W^G)$. So $[s * t^{-1}]_b \in J_0$. Hence

$$< t >_a = < s * t^{-1} >_b < t >_a = < s * t^{-1} * t >_a = < s >_a$$

□

4.3 Lie groupoid structure on $Hol(\mathcal{D}(\mathcal{C}), W^G)$

The aim of this section is to construct a topology on the holonomy groupoid $Hol(\mathcal{D}(\mathcal{C}), W^G)$ such that $Hol(\mathcal{D}(\mathcal{C}), W^G)$ with this topology is a Lie groupoid. In the next section we verify that the universal property of Theorem 4.2.8 holds. The intuition is that first of all W^G embeds in $Hol(\mathcal{D}(\mathcal{C}), W^G)$, and second that $Hol(\mathcal{D}(\mathcal{C}), W^G)$ has enough local linear coadmissible sections for it to obtain a topology by translation of the topology of W^G.

For our construction of the topology on $Hol(\mathcal{D}(\mathcal{C}), W^G)$, we remind the reader of the following well-known facts in the theory of differential topology see [5, 18, 3, 2].

Let W be a topological space and let X be a set. A W-chart (U, χ) on X is an injective partial function $\chi : W \to X$ with open domain $U \subseteq W$, and a W-atlas on X is a family $\{(U_i, \chi_i) : i \in I\}$ of W-charts on X such that the family $\{\chi_i(U_i) : i \in I\}$ covers X and if $i, j \in I$ is such that $\chi_i(U_i) \cap \chi_j(U_j)$ is non-empty, then the change of coordinates ${\chi_i}^{-1}\chi_j : W \to W$ is a partial diffeomorphism of an open subset of W onto an open subset of W. It is easy to prove that X can be given a topology in a unique way, which is the initial topology on X with respect to all the W-charts $\{(U_i, \chi_i) : i \in I\}$ such that each U_i is open and any W-chart on X is a homeomorphism.

Let $s \in \Gamma(\mathcal{D}(\mathcal{C}), W^G)$. We define a partial function $\chi_s : W^G \to Hol(\mathcal{D}(\mathcal{C}), W^G)$. The domain of χ_s is the set of $w \in W^G$ such that $\alpha(w) = a \in D(s_1)$ and $\alpha(a), \beta(a) \in D(s_0)$. The value $\chi_s(w)$ is obtained as follows. Choose a local linear smooth coadmissible section θ through w. Then we set

$$\chi_s(w) = < s >_{\alpha(w)} < \theta >_{\beta(w)} = < s * \theta >_{\beta(w)} .$$

By Lemma 3.4.2, $\chi_s(w)$ is independent of the choice of the local linear smooth coadmissible section θ.

Lemma 4.3.1 *χ_s is injective.*

Proof: Suppose $\chi_s v = \chi_s w$. Then $\beta w = \beta v = a$, say and $\alpha s \alpha v = \alpha s \alpha w$. By definition of s, $\alpha v = \alpha w = d$, say. Let θ, θ' be local linear smooth coadmissible sections through w and v respectively. Then we now obtain from $\chi_s v = \chi_s w$ that

$$< s >_d < \theta >_a = < s >_d < \theta' >_a$$

and hence, since $Hol(\mathcal{D}(\mathcal{C}), W^G)$ is a groupoid, that $< \theta >_a = < \theta' >_a$. Hence $v = \theta(a) = \theta'(a) = w \in W^G$. □

Let $s \in \Gamma(\mathcal{D}(\mathcal{C}))$. Then s defines a left translation L_s on $\mathcal{D}(\mathcal{C})$ by

$$L_s(w) = s(\alpha(w)) +_1 w.$$

This is an injective partial function on $\mathcal{D}(\mathcal{C})$. The inverse ${L_s}^{-1}$ of L_s is

$$v \mapsto -_1 s(\alpha s)^{-1}(\alpha(v)) +_1 v$$

and ${L_s}^{-1} = L_{s^{-1}}$. We call L_s the **left translation corresponding to** s.

So we have an injective function χ_s from an open subset of W^G to $Hol(\mathcal{D}(\mathcal{C}), W^G)$. By definition of $Hol(\mathcal{D}(\mathcal{C}), W^G)$, every element of $Hol(\mathcal{D}(\mathcal{C}), W^G))$ is in the image of χ_s for some s. These χ_s will form a set of charts and so induce a topology on $Hol(\mathcal{D}(\mathcal{C}), W^G)$. The compatibility of these charts results from the following lemma, which is essential to ensure that W^G retains its topology in $Hol(\mathcal{D}(\mathcal{C}), W^G)$ and is open in $Hol(\mathcal{D}(\mathcal{C}), W^G)$. As in the groupoid case [1], this is a key lemma.

Lemma 4.3.2 *Let $s, t \in \Gamma^r(\mathcal{D}(\mathcal{C}), W^G)$. Then $(\chi_t)^{-1}\chi_s$ coincides with L_η, left translation by the local linear smooth coadmissible section $\eta = t^{-1} * s$, and L_η maps open sets of W^G diffeomorphicially to open sets of W^G.*

Proof: Suppose $v, w \in W^G$ and $\chi_s v = \chi_t w$. Choose local linear smooth coadmissible sections θ and θ' through v and w respectively such that the images of θ and θ' are contained in W^G. Since $\chi_s v = \chi_t w$, then $\beta v = \beta w = a$ say. Let $\alpha v = b$, $\alpha w = c$.

Since $\chi_s v = \chi_t w$, we have

$$< s * \theta >_a = < t * \theta' >_a$$

Hence there exists a local linear smooth coadmissible section ζ with $a \in D(\zeta)$ such that $[\zeta]_a \in J_0$ and

$$[s * \theta]_a = [t * \theta']_a [\zeta]_a$$

Let $\eta = t^{-1} * s$. Then in the semigroup $\Gamma(\mathcal{D}(\mathcal{C}), W)$ we have from the above that $\eta * \theta = \theta' * \zeta$ locally near a. So we get $w = (\theta' * \zeta)(a) = \theta'(a) +_1 \zeta(a) = \theta'(a) +_1 1_a = (\eta * \theta)a = \eta \alpha v +_1 v$. This shows that $(\chi_t)^{-1} \chi_s = L_\eta$, left translation by $\eta \in \Gamma(\mathcal{D}(\mathcal{C}))$, i.e.,

$$\begin{aligned}
(\chi_t)^{-1}(\chi_s)(v) &= (\chi_t)^{-1}(< s * \theta >_{\beta v = a}) \\
&= (t^{-1} * s * \theta)(a), \\
&= (\eta * \theta)(a), \quad \text{since } \eta = t^{-1} * s \\
&= \eta(\alpha(\theta(a)) +_1 \theta(a), \quad \text{by definition of } * \\
&= \eta(\alpha(v)) +_1 v, \quad \text{since } \theta(a) = v \\
&= L_\eta(v), \quad \text{by definition of } L_\eta.
\end{aligned}$$

However, we also have $\eta = \theta' * \zeta * \theta^{-1}$ near αv. Hence $L_\eta = L_{\theta'} L_\zeta L_{\theta^{-1}}$ near v. Now $L_{\theta^{-1}}$ maps v to 1_a, L_ζ maps 1_a to 1_a, and $L_{\theta'}$ maps 1_a to w. Namely,

$$\begin{aligned}
L_{\theta^{-1}}(v) &= \theta^{-1}(\alpha(v)) +_1 v \\
&= -_1\theta(\alpha\theta)^{-1}(\alpha v) +_1 v, \quad \text{by definition of } \theta^{-1} \\
&= -_1\theta(\beta(v)) +_1 \theta(\beta v), \quad \text{since } \theta(\beta v) = v \\
&= 1_a,
\end{aligned}$$

$$L_\zeta(1_a) = \zeta(\alpha(1_a)) +_1 1_a, \quad \text{by definition of } L_\zeta$$

$$
\begin{aligned}
&= \zeta(a) +_1 1_a, \quad \text{since} \quad \zeta \in J_0 \\
&= 1_a +_1 1_a \\
&= 1_a
\end{aligned}
$$

and

$$
\begin{aligned}
L_{\theta'}(1_a) &= \theta'(\alpha(1_a)) +_1 1_a, \quad \text{by definition of} \quad L_{\theta'} \\
&= \theta'(a) +_1 1_a, \quad \text{since} \quad \theta'(a) = w \\
&= w +_1 1_a \\
&= w.
\end{aligned}
$$

So these left translations are defined and smooth on open neighbourhoods of v, 1_a and 1_a respectively. Hence L_η is defined and smooth on an open neighbourhood of v. □

We now impose on $Hol(\mathcal{D}(\mathcal{C}), W^G)$ the initial topology with respect to the charts χ_s for all $s \in \Gamma(\mathcal{D}(\mathcal{C}), W^G)$. In this topology each element $h \in Hol(\mathcal{D}(\mathcal{C}), W^G)$ has an open neighbourhood diffeomorphic to an open neighbourhood of $1_{\beta h}$ in W^G.

Lemma 4.3.3 *With the above topology, $Hol(\mathcal{D}(\mathcal{C}), W^G)$ is a Lie groupoid.*

Proof: Source and target maps are smooth: In fact, for $w \in W^G$,

$$\beta_H(\chi_s(w)) = \beta(w), \quad \alpha_H(\chi_s(w)) = \alpha(s\alpha(w)).$$

It follows that α_H and β_H are smooth.

Now we have to prove that

$$\mathrm{d}_H : Hol(\mathcal{D}(\mathcal{C}), W^G) \sqcap_\beta Hol(\mathcal{D}(\mathcal{C}), W^G) \to Hol(\mathcal{D}(\mathcal{C}), W^G)$$

is smooth. Let $< s >_a, < t >_a \in Hol(\mathcal{D}(\mathcal{C}), W^G)$. Then $\chi_s(1_a) =< s >_a, \chi_t(1_a) =< t >_a$, and if $\eta = s * t^{-1}$, then $\chi_\eta(1_b) =< s * t^{-1} >_b$ where $b = \beta t(a)$. Let $v \in D(\chi_s), w \in D(\chi_t)$, with $\beta v = \beta w = c$, say and let θ and θ' be elements of $\Gamma(W^G)$ through v and w respectively. Let $d = \beta(t * \theta')(c)$. Then

$$\begin{aligned}
{\chi_\eta}^{-1}\mathsf{d}_H(\chi_s \times \chi_t)(v,w) &= {\chi_\eta}^{-1}\mathsf{d}_H(\chi_s(v), \chi_t(w)) \\
&= {\chi_\eta}^{-1}\mathsf{d}_H(< s * \theta >_c, < t * \theta' >_c), \text{ by definition of } \chi_s, \chi_t \\
&= \chi_\eta^{-1}(< (s * \theta) * (t * \theta')^{-1} >_d), \text{ by definition of } \mathsf{d}_H \\
&= (\eta^{-1}) * (s * \theta) * (t * \theta')^{-1}(d), \text{ by definition of } {\chi_\eta}^{-1} \\
&= ((s * t^{-1})^{-1}) * (s * \theta) * (t * \theta')^{-1}(d), \text{ since } \eta = (s * t^{-1}) \\
&= (t * s^{-1} * s * \theta) * (t * \theta')^{-1})(d) \\
&= ((t * \theta) * (t * \theta')^{-1})(d) \\
&= ((t * \theta)(\alpha(t * \theta')^{-1}(d) +_1 (t * \theta')^{-1}(d) \\
&= (t * \theta)_1(c) -_1 (t * \theta')(\alpha(t * \theta')^{-1})(d), \text{ since } \alpha(t * \theta')^{-1}(d) = c \\
&= t(\alpha\theta_1(c) +_1 \theta(c) -_1 (t(\alpha\theta'(c) +_1 \theta'(c)) \\
&= (t(\alpha(v)) +_1 v -_1 (t(\alpha(w)) +_1 w) \\
&= L_t(v) -_1 L_t(w) \\
&= \Omega(v, w),
\end{aligned}$$

say. The smoothness of this map Ω at $(1_a, 1_a)$ is now easily shown by writing $t = t_n * \cdots * t_1$ where $t_i \in \Gamma^r(W^G)$ and using induction and a similar argument to that of Lemma 3.5.3.

□

4.4 The Universal Property of Hol(D(C), W^G)

In this section we state and prove the main theorem of the universal property of the morphism $\psi : Hol(\mathcal{D}(C), W^G) \to \mathcal{D}(C)$. Note that for the case of groupoids rather than crossed modules, Pradines [43] stated a differential version involving germs of locally Lie groupoids in [43], and formulated the theorem in terms of adjoint functors. No information was given on the construction or proof. A version for locally topological groupoids was given in Aof-Brown [1], with complete details of the construction and proof, based on conversations of Brown with Pradines. The modifications for the differential case were given in Brown-Mucuk [14].

The main idea is when we are given a V-locally Lie double groupoid $(\mathcal{D}(\mathcal{C}), W^G)$ of a double groupoid $\mathcal{D}(\mathcal{C})$ for a Lie crossed module $\mathcal{C}$ and given a Lie crossed module $\mathcal{A}$ and a morphism

$$\mu : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{C})$$

with suitable conditions, we can construct a morphism

$$\mu' : \mathcal{D}(\mathcal{A}) \to Hol(\mathcal{D}(\mathcal{C}), W^G),$$

where $Hol(\mathcal{D}(\mathcal{C}), W^G))$ is a holonomy groupoid of a locally Lie crossed module, such that

$$\psi\mu' = \mu.$$

We prove that such a morphism μ' is well-defined, smooth and unique. Now let $(\mathcal{D}(\mathcal{C}), W^G)$ be a V- locally Lie double groupoid as above.

Theorem 4.4.1 *If $\mathcal{A} = (A, B, \delta')$ is a Lie crossed module and $\mu : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{C})$ is a morphism of groupoids such that*

i). μ is the identity on objects;

ii).the restriction $\mu_{W^G} : \mu^{-1}(W^G) \to W^G$ of μ is smooth, $\mu^{-1}(W^G)$ is open in $\mathcal{D}(\mathcal{A})$ and generates $\mathcal{D}(\mathcal{A})$ as a groupoid.

iii). the triple $(\alpha, \beta, \mathcal{D}(\mathcal{A}))$ has enough local smooth coadmissible sections.

Then there exists a unique morphism

$$\mu' : \mathcal{D}(\mathcal{A}) \to Hol(\mathcal{D}(\mathcal{C}), W^G)$$

of Lie groupoids such that $\psi\mu' = \mu$ and $\mu'(w) = i\mu(w)$ for $w \in \mu^{-1}(W^G)$.

Proof: Since, by condition (i), $\mu_1 = 1_G$, then $G = B$ and $X = X'$ which implies that $\mu(G) = G$, $\mu(X) = X$. But $G \subseteq W^G \subseteq \mathcal{D}(\mathcal{C})$, by condition (S_2) of Definition 3.4.1. Hence $\mu(G) \subseteq W^G \subseteq \mathcal{D}(\mathcal{C})$. So it follows that $G \subseteq \mu^{-1}(W^G) \subseteq \mathcal{D}(\mathcal{A})$.

Let $w \in \mathcal{D}(\mathcal{A})$. The aim is to define $\mu'(w) \in Hol(\mathcal{D}(\mathcal{C}), W^G)$.

But, by condition (ii), $\mu^{-1}(W^G)$ is an open set of $\mathcal{D}(\mathcal{A})$. Hence $\mu^{-1}(W^G)$ is an open neighbourhood of G in $\mathcal{D}(\mathcal{A})$. Since $\mu^{-1}(W^G)$ generates $D(\mathcal{A})$, we can write $w = w^n +_1 \cdots +_1 w^1$, where $\mu(w^i) \in W^G$, $i = 1, \ldots, n$.

Since $(\alpha, \beta, \mathcal{D}(\mathcal{A}))$ has enough local linear smooth coadmissible sections, by condition (iii), we can choose local linear smooth coadmissible sections θ_i through w_i, $i = 1, \ldots, n$, such that they are composable and their images are contained in $\mu^{-1}(W^G)$.

Because of the condition (ii), the smoothness of μ on $\mu^{-1}(W^G)$ implies that $\mu\theta_i$ is a local linear smooth coadmissible section through $\mu(w_i) \in W^G$ whose image is contained in W^G. Therefore $\mu\theta \in \Gamma^r(\mathcal{D}(\mathcal{C}), W^G)$. Hence we can set

$$\mu'(w) = < \mu\theta >_{\beta(w)}$$

and prove the following lemmas.

Lemma 4.4.2 *$\mu'(w)$ is independent of the choices which have been made.*

Proof: Let $w = v_m +_1 \cdots +_1 v_1$, where $\mu v_j \in W^G$ and $j = 1, \cdots, m$, $\beta(w) = c$. Choose a set of local linear smooth coadmissible sections θ'_j through v_j such that the θ'_j are composable and their images are contained in $\mu^{-1}(W^G)$.

Let $\theta' = \theta'_m * \cdots * \theta'_1$. Then $\mu\theta' \in \Gamma^r(\mathcal{D}(\mathcal{C}), W^G)$, and so $< \mu\theta' >_c \in Hol(\mathcal{D}(\mathcal{C}), W^G)$. Since by assumption, $\theta(c) = \theta'(c) = w \in \mathcal{D}(\mathcal{A})$, then $(\theta(c), \theta'(c)) \in \mathcal{D}(\mathcal{A}) \sqcap_\beta \mathcal{D}(\mathcal{A})$ and $\mathsf{d}_A(\theta(c), \theta'(c)) = \theta(c) -_1 \theta'(c) = 1_c$. Hence $(\theta(c), \theta'(c)) \in {\mathsf{d}_A}^{-1}\mu^{-1}(W^G)$ because $1_c \in \mu^{-1}(W^G)$.

Because $\mathcal{A}$ is a Lie crossed module and the corresponding double groupoid $\mathcal{D}(\mathcal{A})$ is a double Lie groupoid, the difference map $\mathsf{d}_A : \mathcal{D}(\mathcal{A}) \sqcap_\beta \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is smooth. Since

$\mu^{-1}(W^G)$ is open in $\mathcal{D}(\mathcal{A})$, by condition (ii), then ${\sf d}_A{}^{-1}\mu^{-1}(W^G)$ is open in $\mathcal{D}(\mathcal{A}) \sqcap_\beta \mathcal{D}(\mathcal{A})$.

But, by the smoothness of θ and θ', the induced map $(\theta, \theta') : D(\theta) \cap D(\theta') \to \mathcal{D}(\mathcal{A}) \sqcap_\beta \mathcal{D}(\mathcal{A})$ is smooth. Hence there exists open neighbourhoods N of c in G and N_0 of $\alpha(c), \beta(c)$ such that $(\theta, \theta')(N) \subseteq ({\sf d}_A{}^{-1}\mu^{-1})(W^G)$. This implies that $\theta * \theta'^{-1}(\alpha\theta' N) \subseteq \mu^{-1}(W^G)$, and so, after suitably restricting θ, θ', which we may suppose done without change of notation, we have that $\theta * \theta'^{-1}$ is a local linear smooth coadmissible section through $1_d \in \mathcal{D}(\mathcal{A})$ and its image is contained in $\mu^{-1}(W^G)$. So $\mu(\theta * \theta'^{-1})$ is a local linear smooth coadmissible section through $1_d \in W^G$, and its image is contained in W^G. Therefore $[\mu(\theta * \theta'^{-1})]_d \in J^c(W^G)$.

Since $\theta(c) = \theta'(c)$, then $\psi[\mu\theta]_c = \psi[\mu\theta']_c$. But ψ and μ are morphisms of groupoids; hence $\psi[\mu(\theta * \theta'^{-1})]_d = 1_d$, and so $[\mu(\theta * \theta'^{-1})]_d \in Ker\psi$. Therefore $[\mu(\theta * \theta'^{-1})]_d \in J^c(W) \cap Ker\psi = J_0$. Since μ is a morphism of groupoids, we have $[\mu(\theta * \theta'^{-1})]_d \in J^c$. Hence $< \mu(\theta * \theta'^{-1}) >_d = 1_d \in Hol(\mathcal{D}(C), W^G))$, and so

$$< \mu\theta >_c = < \mu\theta >_c < \mu(\theta * \theta'^{-1}) >_d = < \mu\theta' >_c$$

which shows that $\mu' w$ is independent of the choices made. □

Lemma 4.4.3 *μ' is a morphism of groupoids.*

Proof: Let $u = w +_1 v$ be an element of $\mathcal{D}(\mathcal{A})$ such that $w = w_n +_1 \ldots +_1 w_1$ and $v = v_m +_1 \ldots +_1 v_1$, where $w_i, v_j \in \mu^{-1}(W^G)$, $i = 1, \cdots, n$ and $j = 1, \cdots, m$. Then $u = w_n +_1 \cdots +_1 w_1 +_1 v_m +_1 \cdots +_1 v_1$.

Let θ_i, θ'_j be local linear smooth coadmissible section through w_i and v_j respectively such that they are composable and their images are contained in $\mu^{-1}(W^G)$. Let $\theta = \theta_n * \cdots * \theta_1$ and $\theta' = \theta'_m * \cdots * \theta'_1$, $\kappa = \theta * \theta'$. Then κ is a local linear smooth coadmissible section through $u \in \mathcal{D}(\mathcal{A})$, and $\mu\theta$, $\mu\theta'$, $\mu\kappa \in \Gamma^r(\mathcal{D}(C), W^G)$, and $\mu\kappa = \mu\theta * \mu\theta'$, since μ is a morphsim of groupoids.

Let $a = \beta w$, $b = \beta v$. Then $< \mu\kappa >_a = < \mu\theta >_a < \mu\theta >_b$ and so μ' is a morphism. □

Lemma 4.4.4 *The morphism μ' is smooth, and is the only morphism of groupoids such that $\psi\mu' = \mu$ and $\mu' w = (i\mu)(w)$ for all $a \in \mu^{-1}(W^G)$.*

Proof:

Since $(\alpha_A, \beta_A, \mathcal{D}(\mathcal{A}))$ has enough local linear smooth coadmissible section, it is enough to prove that μ' is smooth at 1_a for all $a \in G$. Let $\mathbf{c}$ denote the linear coadmissible section $\mathbf{c} : G \to \mathcal{D}(\mathcal{C})$, $a \mapsto 1_a$ and $c_0 : X \to G$, $x \mapsto 1_x$.

Let $a \in G$. If $w \in \mu^{-1}(W^G)$ and s is a local linear smooth coadmissible section through w, then $\mu' w =< \mu s >_{\beta w}= \chi_{\mathbf{c}}\mu(w)$. Since μ is smooth, it follows that μ' is smooth.

The uniqueness of μ' follows from the fact that μ' is determined on $\mu^{-1}(W^G)$ and that this set generates $\mathcal{D}(\mathcal{A})$.

□

This completes the proof of our main result, Theorem 4.2.8. □

Chapter 5

Conclusions and suggestions for further work

The way of proceeding further has been discussed by the writer and Ronald Brown.

5.1 2-Groupoids

5.1.1 Introduction

For a 2-dimensional version of holonomy, there are a number of possible choices. It seems reasonable therefore to attend to those whose algebra is better understood. There are at least six categories equivalent to that of crossed modules over groupoids. We consider the possibility of a theory for one of the equivalence categories

$$CrsMod \sim DGrpd! \sim 2 - Grpd.$$

The equivalence of 2-groupoids and crossed modules over groupoids is a 2 dimensional case of a result due to Brown and Higgins [10].

5.1.2 2-Groupoids

2-groupoids are special cases of the so-called 2-categories originally due to Ehresmann [23] and see also Kelly and Street [30]. The 2-categories with invertible 1-cells and 2-cells are called 2-groupoids. In another way of defining it, a 2-groupoid may be thought of as a double groupoid in which all the vertical edge arrows are identities. So a 2-groupoid consists of a set H with groupoid structures over H_1 and H_0 and H_1 is also a groupoid on H_0 all subject to the compatibility condition that the structure maps of each structure on H are morphisms with respect to the other.

Full details of the following material has been written down seperately from the thesis.

5.1.3 Equivalence of crossed modules and 2-groupoids

As said earlier, the equivalence given in the title of this section is a 2-dimensional case of Brown and Higgins [10]. We give this equivalence briefly as follows.

Let H be a 2-groupoid. Then it has a groupoid structure $(H_i, \alpha_i, \beta_i, +_i)$ for $i = 0, 1$, satisfying the usual compatibility conditions. We obtain a corresponding crossed module $\mathcal{C} = \lambda H$, by $X = H_0$, $G = H_1$ and $C = \{C(x)\}_{x \in X}$, where $C(x) = \{n \in H : \alpha_0 n = \beta_0 n = x, \beta_1 n = 1_x\}$. Then $\mathcal{C} = (C, G, \delta)$ becomes a crossed module with boundary $\delta(n) = \alpha_1(n)$, $n \in C$.

Conversely, let $\mathcal{C} = (C, G, \delta)$ be a crossed module over a groupoid. We can obtain a 2-groupoid $H = \theta(\mathcal{C})$, with 2-cells forming the set

$$G \ltimes C = \{(a, c) : a \in G, c \in C(\beta(a))\}$$

with a 2-groupoid structure. The following theorem can be stated.

Theorem 5.1.1 *The functors*

$$\lambda : 2 - Grpd \to CrsMod$$

$$\theta : CrsMod \to 2 - Grpd$$

indicated above are inverse equivalences [10].

5.1.4 Homotopy for 2-groupoids

The notion of homotopy for morphism of crossed modules over groups (groupoids) has been well known for many years, Whitehead [48], Brown and Higgins [11], Brown and Gilbert [6] and also see the first chapter of this thesis. This was put in the general context of a monoidal closed structure on the category of crossed complexes in Brown and Higgins [11]. The notion of homotopies for 2-groupoids is essentially a special case of the notion of 2-natural transformation due to Gray in [25].

The relation between homotopies for crossed modules over groupoids and homotopies for 2-groupoids can be explained by extending Theorem 5.1.1 to an equivalence of 2-categories.

We can add to this theory an analogue of Ehresmann's product of (co)-admissible sections. In the groupoid case, the latter can be consider as homotopies $\sigma : f \simeq 1, \theta : g \simeq 1 : G \to G$, and the product $\sigma * \theta$ is a homotopy $gf \simeq 1$, where f, g are here automorphisms. The same formulation holds in the 2-groupoid case, i.e., so that we have a product of coadmissible homotopies.

Corresponding Lie and locally Lie notions may be developed, analogous to previous work.

However, it turned out, in working with the appropriate sheaf of germs of local coadmissible 2-homotopies, that we have to consider also the double groupoids associated to the 2-groupoid and the exposition becomes closely related to that given above for crossed modules. Thus there is a natural holonomy theory in the context of 2-groupoids.

5.2 2-Crossed Modules

Brown and Higgins proved that the category of crossed modules has a monoidal closed structure. Then, for any crossed module $\mathcal{C}$ over a groupoid we can determine crossed modules $END(\mathcal{C})$ and $AUT(\mathcal{C})$ as in Chapter 1. Brown and Gilbert proved that $AUT(\mathcal{C})$

is a braided regular crossed module over the group $Aut(\mathcal{C})$. We have studied this in detail in Chapter 1. But material on the braiding is relevant to studying the larger structure. We already have constructed

$$Hol(\mathcal{D}(\mathcal{C}), W^G) \to \mathcal{D}(\mathcal{C}),$$

where $Hol(\mathcal{D}(\mathcal{C}), W^G)$ is a Lie groupoid on G and $\mathcal{D}(\mathcal{C})$ is the corresponding double groupoid for the crossed module $\mathcal{C} = (C, G, \delta)$. This uses free derivations or linear coadmissible sections. We have not used the inner derivations

$$M(C) \to FDer^*(\mathcal{C}),$$

described in Section 1.5. In the rest of Chapter 1 we did get as far as describing explicitly the maps and morphisms

$$M(C) \to FDer^*(\mathcal{C}) \to Aut(\mathcal{C}).$$

Also, the Peiffer lifting structure which makes this a 2-crossed module is not used in the localisation part and we have therefore been content to show that the verification of the axioms follows from Brown and Gilbert [6]. The development of the local theory using the full structure of 2-crossed modules requires further work.

5.3 Double Holonomy Groupoids

A natural question on globalisation is: Does there exist a Lie crossed module $Hol(C, W)$ with a universal property related to the diagram:

$$\begin{array}{ccc} Hol(C,W) & \xrightarrow{\phi} & C \\ \downarrow & & \downarrow \\ (G,X) & \longrightarrow & (G,X) \end{array}$$

We started with the idea of globalising a locally Lie crossed module (C, W, δ) for a crossed module $\mathcal{C} = (C, G, \delta)$. The latter is a concept not difficult to define. In carrying out such a conjectured globalisation project, it became clear that our methods did not produce a global crossed module. Instead, we have produced from (C, W, δ) a holonomy Lie groupoid $Hol(\mathcal{D}(\mathcal{C}), W^G)$, using all the information in (C, W, δ). It seems possible that this Lie groupoid is a part of a Lie 2-crossed module. Thus the examination of local theory and crossed modules seems to lead outside crossed modules and to some more complex structure. This is probably related to work of L. Breen on stacks of groupoids.

So the area requires considerably more work to develop and reveal the underlying structures. This thesis is intended as a start in this important direction.

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Appendix

Inverse semigroup

It is standard that inverse semigroups are natural generalisations of groups, encoding information about partial rather than global symmetries. See for more information, *"Inverse Semigroup Theory"*, by M.V. Lawson [35].

We give the verification that our semigroups of local coadmissible sections form an inverse semigroup.

Definition .0.1 A **semigroup** S is said to be **inverse** if for each $s \in S$ there exists a unique element called the inverse of s, denoted by s^{-1}, satisfying $s = ss^{-1}s$ and $s^{-1} = s^{-1}ss^{-1}$.

Example .0.2 Let X be a C^r-manifold and let $M(X)$ denote the set of all diffeomorphisms between open subsets of X. We define a multiplication on $M(X)$ as follows: Let $f : U \to V$, and let $g : V' \to W'$ be two diffeomorphisms, where U, V, U', W' are open subsets of X. Then we define a composition,

$$gf : f^{-1}(V \cap V') \to g(V \cap V').$$

With this composition, $M(X)$ becomes an inverse semigroup, i.e., for each $(f : U \to V) \in M(X)$ there exist $(f^{-1} : V \to U) \in M(X)$ such that $f = ff^{-1}f$ and $f^{-1} = f^{-1}ff^{-1}$. Note the key point that $f^{-1}f = I_U, ff^{-1} = I_V$.

Definition .0.3 A **local coadmissible section** of a groupoid G with base space X is a function $s : U \to G$ from an open subset U of X such that s satisfies;

(i) $\beta sx = x$, for $x \in X$,

(ii) $\alpha s(U)$ is open in X, and

(iii) αs maps U homeomorphically to $\beta s(U)$.

The first point of this Appendix is to show that the set $M(G)$ of local coadmissible sections of a groupoid G such that $X = Ob(G)$ is a topological space has the structure of inverse semigroup under the $*$ multiplication of Ehresmann.

Proposition .0.4 *The set $M(G)$ of local coadmissible section of a groupoid G is an inverse semigroup.*

Proof: We can easily verify that if s^{-1} is as given earlier, then $s^{-1} * s * s^{-1} = s^{-1}$, $s * s^{-1} * s = s$. Then we have only to verify uniqueness. So suppose s' satisfies $s = s * s' * s$ and $s' = s' * s * s'$. We have to show that $s' = s^{-1}$.

Let us start with $s * s' * s = s$, where $\alpha s : U \to V$. We need to prove that $D(s') = V$. Let $y \in V$. Then there is an x such that $\alpha s(x) = y$. If $s'(y)$ not defined, then $s * s' * s(x)$ is not defined and so $s * s' * s \neq s$. This proves $V \subseteq D(s')$.

Suppose $s'(y)$ is defined. Let $x = \alpha s'(y)$. Then $s(x)$ is defined, since $s' * s * s' = s'$. So $x \in U$, and so $y \in V$, i.e., $D(s') \subseteq V$. Hence $D(s') = V$.

Finally, $y = \beta s(x), s(x') + s'(y) + s(x) = s(x)$ implies $s(x') + s'(y) = 1_y$, so $s(x') = s'(y)^{-1}$.

□